OPTIMAL PUBLIC DEBT MANAGEMENT AND LIQUIDITY PROVISION

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NBER Working Paper No. 18800
February 2013
JEL No. E4,E6,H6

<u>ABSTRACT</u>

We study the Ramsey policy problem in an economy in which firms face a collateral constraint. Issuing more public debt alleviates this friction by increasing the aggregate quantity of collateral. In so doing, however, the issuance of more debt also raises interest rates, which in turn increases the tax burden of servicing the entire outstanding debt. We first document how this trade-off upsets the optimality of tax smoothing and, in contrast to the standard paradigm, helps induce a unique and stable steady-state level of debt in the deterministic version of the model. We next study the optimal policy response to fiscal and financial shocks in the stochastic version. We finally show how the results extend to a variant model in which the financial friction afflicts consumers rather than firms.

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1 Introduction

The recent financial crisis has been associated with a large increase in the demand for public debt. For example, the quantity of treasury (and agency) securities in commercial bank portfolios increased from 1.11 USD trillion in July 2008 to 1.79 trillion in July 2012. At the same time, interest rates across all maturities of public debt have fallen sharply—with the yield, for instance, on newly issued 10-year Treasuries going down from 4% to 1.53%, a historically low.[1]

These patters are consistent with a class of models in which the "natural rate of interest" decreases when credit gets tighter. Guerrieri and Lorenzoni (2012), Eggertsson and Krugman (2012) and Buera and Nicolini (2012) have used such models to explain why the zero lower bound on monetary policy has become binding and aggregate demand is accordingly depressed. These models, however, also predict that the issuance of more public debt helps alleviate the credit friction, in effect permitting private agents to borrow against future income (Woodford, 1990; Aiyagari and McGrattan, 1994). In a similar vein, Holmstrom and Tirole (1998) argue that public-debt issuance increases the aggregate collateral in the economy, while Stein (2011) and Krishnamurthy and Vissing-Jorgensen (2012) claim that government bonds command a premium against other assets because they serve as a special form of collateral, or medium of exchange, in financial markets.[2]

These considerations raise an elementary policy question: what is the optimal dynamic pattern of taxes, deficits, and public-debt issuance if public debt facilitates private liquidity?

To address this question, we modify the standard Ramsey policy paradigm (Barro, 1979, Lucas and Stokey, 1983) by introducing a financial friction, which in turn can be amended by public-debt issuance.[3] We then study how this modification matters for some quintessential predictions of the Ramsey paradigm, such as those regarding the optimality of tax smoothing and the determinacy of the steady-state level of public debt. We finally study the optimal policy response to two types of shocks that are relevant in the context of the great recession as well as more broadly: fiscal shocks (higher government speeding) and financial shocks (tighter private credit).

Our baseline model assumes that the liquidity role of public debt emerges in the production side of the economy, in the spirit of Holmstrom and Tirole (1998). In particular, the economy is populated by a large number of agents, each running a firm. Importantly, firms are subject to idiosyncratic productivity shocks. Consequently, capital must be reallocated in each period from the relatively less productive firms to the relatively more productive ones. This reallocation, however, is hindered by a financial friction.

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[1]Similar increases in the demand for safe, public debt were seen throughout the world. For instance, nominal, short term rates on new issues of Swiss public debt even became briefly negative in 2011.

[2]What is more, Krishnamurthy and Vissing-Jorgensen (2012) provide evidence that this premium is positive, as well as that the government is effectively facing a downward slopping demand for the liquidity services of public debt.

[3]Like much of the Ramsey literature, our framework abstracts from sticky prices, Keynesian multipliers, and the like. Whether such effects are relevant in practice or not, they are not central to our contribution.

To rent additional capital, the more productive firms must raise external funds in the form of collateralized borrowing. Both own capital and holdings of government bonds can serve as collateral, but the entrepreneur's future income cannot be pledged. This implies that the collateral constraint is binding as long as the agent's net worth is low enough. When this is the case, an agent's valuation of public debt embeds the shadow value of relaxing her collateral constraint. In equilibrium, this means that interest rate on public debt is lower than the underlying discount rate in preferences. We henceforth refer to this wedge as the "liquidity premium" on public debt.[4]

As long as the financial friction binds, the allocation of capital is suboptimal, output and employment are below potential, and interest rates are depressed. At the same time, aggregate savings may be higher because of a particular type of precautionary motive: agents find it optimal to save more than what it would have been save in the absence of the friction in order to relax the likely bite of future collateral constraints. The financial friction therefore distorts not only the intra-temporal allocation of resources but also inter-temporal consumption and saving decisions.

Issuing more public debt helps mitigate all these distortions by increasing the aggregate collateral that firm owners can hold and pledge in the future. Relative to the standard Ramsey problem, this represents a novel welfare gain from issuing more debt. There is, however, a novel welfare cost as well: by relaxing the bite of the financial friction, the issuance of more debt also reduces the liquidity premium, which in turn raises interest-rate costs on the *entire* outstanding (inframarginal) quantity of public debt. The properties of optimal taxation and debt management in our model are pinned down by the interaction of this novel trade off with the familiar tax-smoothing considerations.

We start by studying the deterministic version of the model and showing that the aforementioned trade off induces a unique and globally stable steady state: no matter the initial conditions, the level of debt converges to a unique long-run value. This is in contrast to the standard paradigm (Lucas and Stokey, 1983, Chari, Christiano and Kehoe, 1988) where optimality requires tax distortions to be perfectly smoothed across periods, implying that the long-run level of debt, as well as the associated taxes and allocations, move one-to-one with the initial level of debt. In our model, by contrast, the Ramsey planner deviates from tax smoothing and chooses the long-run level of debt so as to strike an optimal, long-run balance between the premium he receives for the collateral value of public debt and the level of the financial friction he tolerates in the economy.

In particular, suppose that the initial level of public debt happens to be sufficiently high so that the collateral constraint does not bind and the liquidity-driven demand for public debt is saturated. Starting from this point, the Ramsey planner finds it optimal to gradually reduce the level of debt and eventually bring the economy to a region where the collateral constraint binds. By doing so, he aggravates the financial friction, but he also reduces the interest-rate costs of serving the debt.

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[4]Note that our notion of the liquidity premium is no necessarily the same as the one often used in finance: what matters for our result is the wedge between the interest rate on public debt and the discount rate that shows up in welfare, not the wedge between, say, government bonds and corporate bonds.

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Conversely, if the initial level of public debt is lower than the steady-state target, the welfare cost of the financial friction dominates the tax friction, and makes it optimal for the planner to issue more public debt in order to relax the tightness of the financial friction.

We then study a stochastic version of our model in which the government can issue only non-contingent debt. The relevant benchmark here is Aiyagari, Marcet, Sargent and Sepala (2002), henceforth AMSS. This is essentially a micro-founded version of Barro (1978) or, equivalently, an incomplete-markets version of Lucas and Stokey (1983). In that model, optimal tax smoothing gives rise to a martingale component in the optimal allocations and optimal taxes. At the same time, a precautionary motive emerges, in the sense that the government asymptotically accumulates a large asset position (i.e., debt becomes negative). In our model, by contrast, the aforementioned trade off introduces mean reversion in the optimal dynamics; in the long run, debt hovers around the steady-state level of the deterministic version of the model. In short, the empirically unappealing long-run properties of AMSS disappear once public debt has a liquidity function.

We next analyze the optimal policy response to fiscal shocks. Consider, for instance, a permanent increase in government spending. In the standard paradigm, it is optimal to finance this increase with an immediate and equally-sized increase in taxes, so that the level of deficits and debt remain unchanged. In our setting, instead, it is optimal to front load the financing of the shock, run surpluses in the short run, and converge to a lower level of public debt in the long run. By doing so, the planner effectively "starves the economy for liquidity" in order to raise the liquidity premium, reduce interest-rate costs, and mitigate the overall tax burden of a bigger government.

Finally, consider a shock that seems relevant from the perspective of the recent financial crisis, namely, a shock that tightens the collateral constraint.[5] This shock depresses economic activity, raises liquidity premia, and lowers interest rates, much in line with the the typical financial shocks employed in the recent macroeconomic literature on financial frictions. The optimal policy response is to lower taxes, run a deficit, and issue more public debt. This, however, is not simply because it is "cheap to borrow" during the crisis: when lower interest rates reflect changes in the underlying intertemporal MRS, they do not justify deviations from tax smoothing. Rather, it is because these lower rates signal an increase in the tightness of the financial friction and an increase in the associated wedge between the underlying intertemporal MRS and the market price of public debt. Issuing more debt is the means to increase the aggregate collateral and mitigate the financial shock. This clarifies the precise way in which counter-cyclical debt management is optimal.

We conclude the paper by studying a variant model in which the financial friction operates on the consumer side of the economy. The key friction here is the same as in Bewley-type models (e.g., Aiyagari, 1994, Guirrieri and Lorenzoni, 2012): consumers face idiosyncratic shocks and borrowing

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[5]This shock can be modelled either an an exogenous reduction in private collateral, as in, say, Buera and Moll (2012), Guerreri and Lorenzoni (2012), and Kahn and Thomas (2011), or as an increase in firm heterogeneity, as in Arellano, Bai, and Kehoe (2012). We follow the former approach but the latter gives similar results.

constraints. We only follow a similar modeling trick as in Lagos and Wright (2005) in order to shut down the role of the wealth distribution as a state variable and simplify the planner's problem.

This variant model changes the theoretical micro-foundations the liquidity services of public debt. Yet, it produces no essential differences with regard to our earlier policy conclusions. This Bewley-like variant therefore helps illustrate the broader applicability of our insights; it also facilitates a sharper connection to the literature on the Friedman rule, which we discuss below.

Related literature. Our paper can be seen as a hybrid of AMSS, Woodford (1990), and Holmostrom and Tirole (1998). We share with AMSS the same methodological backbone, namely the Ramsey policy paradigm with linear labor-income taxation and short-term non-contingent debt. At the same time, we adapt the insight of Woodford (1990) and Holmostrom and Tirole (1998) that public-debt issuance can facilitate private liquidity. To the best of our knowledge, our paper is the first to embed this insight in the Ramsey paradigm and study its implications for optimal policy.

By adding capital to AMSS, our paper also connects to Farhi (2010). But whereas that paper studies optimal capital taxation, our paper abstracts from it and instead introduces a friction in the allocation of capital. This permits us to focus on our preferred policy questions, while also keeping the analysis more directly comparable to AMSS and Barro (1979). For similar reasons, we also preclude the government from issuing multiple maturities. Allowing it to do so could help the government hedge against shocks to its budget, as in Angeletos (2002) and Buera and Nicolini (2004), but would not affect the essence of our results. Finally, if we were to allow the planner to make positive lump-sum transfers, we could also reinterpret our results in terms of the optimal dynamic pattern of transfers, in a manner akin to Golosov and Sargent (2012).

When government bonds can be posted as collateral or otherwise facilitate private liquidity, they effectively serve as a form of "money". At an abstract level, our paper is thus closely connected to the literature on the Friedman rule (e.g., Chari, Christiano and Kehoe, 1996, Correia and Teles, 1999). To make this relation most transparent, we design our Bewley-like variant in such a way that the planner's problem is formally equivalent to the one in a monetary economy in which real money balances enter the household's utility. This permits us to isolate the key modelling differences that are responsible for the violation of the Friedman rule in our setting. As we explain in due course, these differences concern the micro-foundations of the demand for liquidity; the details of the available tax instruments; and, most crucially, the fact that *all* government liabilities provide liquidity services in our model, while in the aforementioned literature only a subset do.

Guerrieri and Lorenzoni (2012) and Eggertsson and Krugman (2012) study the impact of adverse financial shocks in Bewley-like models, documenting how such shocks depress interest rates and aggregate demand. Our analysis has similar positive properties, but it also establishes how optimal debt management ought to respond to such shocks, and how this alleviates their macroeconomic effects. Challe and Ragot (2011) also explore the liquidity role of public debt in a Bewley-type

model, but do not characterize the optimal policy.

The analysis of optimal policy also differentiates ours from work that, motivated by the recent crisis, studies the impact of a credit crunch on the allocation of capital across firms (e.g., Arellano, Bai and Kehoe, 2012, Buera and Moll, 2012, Khan and Thomas, 2011). This literature often allows for rich heterogeneity and rich firm-level dynamics in order to deliver more on the quantitative front. By contrast, we simplify in these dimensions in order to build a bridge to the Ramsey paradigm and shed new light on the policy questions of interest.

Layout. The rest of the paper is organized as follows. Section 2 sets up our baseline model, in which the financial friction afflicts firms. Section 3 characterizes the equilibrium. Section 4 studies the Ramsey problem and characterizes optimal policy. Section 6 elaborates on the key trade-off that shapes the optimal policy. Section 6 examines the optimal policy response to fiscal and financial shocks. Section 7 extends the analysis to a model variant in which the financial friction afflicts consumers and elaborates on the relation of our paper to the literature on the Friedman rule. Section 8 concludes.

2 Baseline Model

The economy is populated by a large number of ex-ante identical households and there is a single final good that can be used either for consumption or investment purposes. Time is indexed by $t \in \mathbb{N}$ and households by $i \in [0, 1]$. Each household contains a consumer, a worker and a producer (or entrepreneur). The consumer chooses how to allocate the household income between consumption, investment in physical capital, and purchases of government bonds. The worker supplies labor in a competitive labor market. The producer runs a firm, that produces the final good out of capital and labor. We index firms by the identity of their owner and we use the term "agent i" interchangeably for either household i or firm i.

To simplify the exposition, we abstract for the time being from aggregate uncertainty (we introduce aggregate shocks in Section 6). We let firms be subject to idiosyncratic TFP shocks. In particular, the technology that firm i has access to during period t is given by

$$y_{it} = a_{it} F(k_{it}, n_{it}),$$

where a_{it} is the idiosyncratic TFP shock, k_{it} is the capital input, n_{it} is the labor input, and F is a neoclassical production function, with $F(k, n) = k^\alpha n^\theta$, $\alpha, \theta \in (0, 1)$, and $\alpha + \theta < 1$. To simplify the analysis, we assume that the TFP shock is i.i.d. across both i and t and can take two values, $a_{it} = a^L = 1$ or $a_{it} = a^H = A > 1$, with respective probabilities $(1 - \pi)$ and π, where $\pi \in (0, 1)$.

Firms can adjust their inputs after observing their realized productivity. To attain first-best efficiency, capital would thus have to constantly flow from the rest of the economy to the firms that

happen to be have received the high-productivity shock at any given point. A financial friction, however, inhibits the efficient allocation of capital.

In particular, we specify the market structure in the model as follows. Each period is split in two stages. Employment and production take place in the "afternoon" (second stage), alongside consumption and saving decisions. A capital market operates in the "morning" (first stage) in order to facilitate the re-allocation of capital from low-productivity to high-productivity firms. In this market, agents rent capital in exchange for private IOUs. Let p_t denote the price that clears this market and z_{it} the position taken by agent i; that is, agent i borrows z_{it} units of capital in the morning and promises to pay $p_t z_{it}$ units of the consumption good in the afternoon.[6] Once the afternoon arrives, however, a borrower can always renege and "run away". If he does so, his creditors cannot force repayment against his future labor income.[7] They can nevertheless confiscate a fraction $\xi \in (0, 1]$ of any capital installed in his firm, as well as all of his bond holdings. It follows that agent i faces the following borrowing, or collateral, constraint:

$$p_t z_{it} \leqslant \xi k_{it} + b_{it},$$

where k_{it} is his capital stock and b_{it} are his bond holdings (measured in terms of consumption-good receipts during the afternoon).[8]

Let κ_{it} be the capital that agent i has invested in his firm by the end of period $t-1$. Then, the total amount of capital employed in his firm during period t and that can be confiscated in case of default is given by $k_{it} = \kappa_{it} + z_{it}$. Using this fact and anticipating that in equilibrium $p_t > \xi$ this constraint can be rewritten as follows:

$$k_{it} \leqslant \frac{p_t \kappa_{it} + b_{it}}{p_t - \xi}. \tag{1}$$

This highlights how the financial friction may inhibit the efficient allocation of capital. In particular, the left hand side of condition (1) gives an upper bound on the amount of capital that can be installed in firm i. This amount is a multiple of the net worth, $p_t \kappa_{it} + b_{it}$, of the firm's owner: for any extra unit of net worth, the agent can leverage up and increase his capital input by ℓ_t units, where $\ell_t = 1/(p_t - \xi)$ represents the equilibrium leverage ratio. Naturally, this ratio increases with ξ, the fraction of capital that can be posted as collateral. This permit us to interpret ξ as a proxy of the severity of financial frictions: The smaller the value of ξ, the tighter the financial frictions.

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[6]Agent i is a borrower if $z_{it} > 0$ and a creditor if $z_{it} < 0$.

[7]This is the assumption of the "inalienability of human capital" introduced by Hart and Moore (1994) and followed by Kiyotaki and Moore (1997), Holmstrom and Tirole (1998), and much of the literature on financial frictions.

[8]The assumption that creditors can confiscate the entity, rather than a fraction, of an agent's bond holdings is inessential for the qualitative properties of our model. For instance, if we let creditors confiscate only a fraction ξ' of i's bond holdings, then the borrowing constraint becomes $p_t z_{it} \leqslant \xi k_{it} + \xi' b_{it}$. Nothing essential then changes in our subsequent analysis, provided of course that $\xi' > 0$, so that government bonds can still serve as collateral.

These features of our model are similar to Kiyotaki and Moore (1997) and a vast macroeconomic literature on financial frictions. But whereas much of this literature is concerned with the endogeneity and the cyclical properties of the leverage ratio, the key feature for our purposes is that the cross-sectional of the right-hand-side term of (1) increases with the level of public debt. Our model thereby captures the idea that issuance of more public debt can increase the "aggregate collateral" in the economy and can thereby alleviate the financial friction.

As already mentioned, production takes place in stage 2, along with consumption and saving decisions. At this point, the value of the firm run by agent i, defined as non-depreciated capital plus sales minus labor and borrowing costs, is given by

$$\pi_{it} = (1 - \delta)k_{it} + y_{it} - w_t n_{it} - p_t z_{it},$$

where $\delta \in [0, 1]$ is the physical depreciation rate. The end-of-period budget constraint of the agent can thus be written as

$$c_{it} + x_{it} + q_t b_{it+1} = b_{it} + \pi_{it} + (1 - \tau_t)w_t h_{it},$$

where c_{it} is consumption expenditure, κ_{it+1} is savings in capital, b_{it+1} is purchases of government bonds, q_t is the price of these bonds, w_t is the wage rate, and τ_t is the tax rate on labor income.

The government's budget constraint is given by

$$q_t B_{t+1} = B_t + g_t - \tau_t w_t H_t,$$

where B_{t+1} is the level of debt issued in period t, q_t is their price, g_t is the (exogenous) level of government spending, and $H_t \equiv \int_i h_{it}$ is total labor supply. The government bond market therefore clears if and only if $\int_i b_{it} = B_t$. Similarly, the IOU/capital market clears if and only if $\int_i z_{it} = 0$, and the labor market clears if and only if $\int_i n_{it} = \int_i h_{it}$.

We complete the description of the model by specifying household preferences as follows:

$$\mathcal{U} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_{it}, h_{it}) \right] \tag{2}$$

where $\beta \in (0, 1)$ is the discount factor, and $c_{it} \in \mathbb{R}$ is consumption, $h_{it} \in \mathbb{R}_+$ is labor supply. The per-period utility function is given by $U(c, h) = c - v(h)$, with $v(h) = \frac{1}{1+\epsilon}h^{1+\epsilon}$ and $\epsilon > 0$.

Remarks. As we will see, the specification of shocks and preferences mimics the modeling strategy of Lagos and Wright (2005) in that it guarantees that the equilibrium wealth distribution is degenerate (all agents enter any period with the same level of wealth). If one were interested in steady-state allocations for time-invariant policies, this modeling strategy would not be necessary. For the purposes of our paper, however, it is essential to compute the equilibrium dynamics and the associated welfare for arbitrary policy paths. Clearly, this would be a taunting task if one had to keep the entire wealth distribution as a state variable, which explains the necessity of the aforementioned modeling strategy. An additional benefit is that our preference specification is exactly the same as the one in the baseline model of AMSS, thus providing a clean benchmark for reference.

3 Equilibrium

We first solve the typical agent's decision problem. We then proceed to show that the aggregate equilibrium behavior of our economy can be represented as the equilibrium behavior of a fictitious representative-agent model with aggregate TFP distortions. We finally summarize the set of equilibrium conditions by a smaller, and more convenient, set of implementability constraints faced by the Ramsey planner.

The agent's decision problem. The assumption that preferences are linear in consumption and that the productivity shock is i.i.d. guarantees that any heterogeneity in income is absorbed by consumption and that all agents choose the same labor supply and the same savings at the end of each period. We thus let $h_{it} = h_t$, $b_{it} = b_t$, and $\kappa_{it} = \kappa_t$ for all i. Furthermore, borrowing and production decisions differ across agents only in so far as they have different productivity levels. We thus let $z_{it} = z_t^s$, $k_{it} = k_t^s$, $n_{it} = n_t^s$, $y_{it} = y_t^s$, and $\pi_{it} = \pi_t^s$, with $s = L$ if $a_{it} = a^L$ and $s = H$ if $a_{it} = a^H$. That is, we effectively have only two types of agents, those in state $s = L$ (low productivity, or low type) and those in state $s = H$ (high productivity, or high type). Agents transit across these two types in an i.i.d. fashion.

Let $\omega_t^s \equiv \pi_t^s - p_t \kappa_t = y_t^s - w_t n_{it} - p_t k_{it}$, for $s \in \{L, H\}$. Building on the preceding observations, the household's problem can be reduced to the following:

$$\max \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[c_t^s - v(h_t) \right] \right\} \tag{3}$$

subject to

$$c_t^s + \kappa_{t+1} + q_t b_{t+1} = p_t \kappa_t + b_t + \omega_t^s + (1 - \tau_t) w_t h_t \quad \forall s \in \{L, H\} \tag{4}$$

$$\omega_t^s = a^s F(k_t^s, n_t^s) + (1 - \delta) k_t^s - w_t n_t^s - p_t k_t^s \quad \forall s \in \{L, H\} \tag{5}$$

$$k_t^s \leqslant \frac{p_t \kappa_t + b_t}{p_t - \xi} \quad \forall s \in \{L, H\} \tag{6}$$

Let μ_t^s be the Lagrange multiplier on the collateral constraint in state s, for $s \in \{L, H\}$. It is easy to verify that, in any equilibrium in which the quantities of capital and public debt are positive ($\kappa_t, B_t > 0$), the borrowing constraint can bind at most for the high type.[9] Furthermore, note that output and employment would have been zero if capital were zero, which implies that the

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[9]To see this, note that, for any given wages, the high type demands more capital than the low type, simply because his productivity is higher. Therefore, if the borrowing constraint binds for the low type, it must also bind for the high type. But if the constraint binds for both types, then it must be that $k_t^s = \frac{p_t \kappa_t + b_t}{p_t - \xi}$ for both s and hence

$$(1 - \pi) k_t^L + \pi k_t^H = \frac{p_t \kappa_t + b_t}{p_t - \xi}.$$

At the same time, market clearing in the capital and bond markets requires $(1 - \pi) k_t^L + \pi k_t^H = \kappa_t$ and $b_t = B_t$.

restriction $\kappa_t > 0$ is without any loss of generality. Finally, the restriction $B_t > 0$ is also without loss of generality at least in the neighborhood of the Ramsey steady state, where we will find that public debt is strictly positive. With these observations in mind, we take henceforth for granted that $\kappa_t > 0$ and $B_t > 0$. It follows that we can drop the borrowing constraint for the low type (or, equivalently, anticipate that $\mu_t^L = 0$).

Note that the (k_t^s, n_t^s) matter for the agent's optimization problem only through constraints (5) and (6). It follows that the optimal input choices of the low and the high type maximize, respectively, ω_t^L and $\omega_t^H - \mu_t^H k_t^H$. Their demands for labor and capital are therefore pinned down by the following:

$$F_n(k_t^L, n_t^L) = w_t \qquad\qquad AF_n(k_t^H, n_t^H) = w_t$$
$$1 - \delta + F_k(k_t^L, n_t^L) = p_t \qquad 1 - \delta + AF_k(k_t^H, n_t^H) = p_t + \mu_t^H$$

That is, both types equate their MPL with the wage rate, but only the high type equates his MPK with the rental rate of capital. The low type, instead, faces a wedge between his MPL and the rental rate of capital due to the financial friction.[10]

Now let ν_t be the Lagrange multiplier on budget constraint (4). The agent's first-order condition for labor supply is then given by

$$v'(h_t) = \nu_t(1 - \tau_t)w_t;$$

the one for bonds is given by

$$q_t \nu_t = \beta \nu_{t+1} \left\{ 1 + \pi \frac{1}{p_{t+1} - \xi} \mu_{t+1}^H \right\}.$$

and the one for capital by

$$\nu_t = p_{t+1} \beta \nu_{t+1} \left\{ 1 + \pi \frac{1}{p_{t+1} - \xi} \mu_{t+1}^H \right\}$$

Finally, the first-order condition for consumption together with the assumption that U is linear in c imply that $\nu_t = 1$.

In order to interpret the conditions for bonds and capital, note first that, in the absence of financial frictions ($\mu_{t+1} = 0$), they reduce to $q_t = 1/p_{t+1} = \beta$, meaning that the returns to bonds and capital are equated with each other and they are both equal to the inter-temporal MRS in consumption (which is here fixed at β because of the linearity of U in c). In the presence of financial frictions, the two rates of return are still equated with one another, because both bonds and capital can serve equally well as collateral. Nevertheless, as long as the borrowing constraint

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Combining these conditions gives

$$\kappa_t = \frac{p_t \kappa_t + B_t}{p_t - \xi},$$

which cannot possibly hold as long as $\kappa_t, B_t > 0$.

[10]If the high-type had to borrow in order not only to rent extra capital but also to employ labor, then the wedge would show up in his demand for labor as well. This, however, would not affect the essence of our results.

binds in the low state, the associated Lagrange multiplier introduces a wedge between these two rates of return and the inter-temporal MRS. In this sense, we have that both capital and government bonds are priced at a premium relative to any other asset that is an equally good form of saving but cannot serve as collateral, as well as at a premium relative to their prices in the frictionless benchmark. The equilibrium value of this premium in period t is given by $\pi \frac{1}{p_{t+1}-\xi}\mu_{t+1}^H$. It is thus increasing in the probability that the constraint binds, (π), in the wedge between the marginal product of capital of the high and the low type, (μ_{t+1}^H), and in the leverage ratio $(\frac{1}{p_{t+1}-\xi})$.

The aggregate TFP distortion. Let Y_t and C_t denote aggregate output and consumption:

$$Y_t \equiv \pi y_t^H + (1-\pi)y_t^L, \qquad C_t \equiv \pi c_t^H + (1-\pi)c_t^L.$$

We can show that aggregate output can be expressed as follows:

$$Y_t = \Gamma(x_t)\kappa_t^\alpha h_t^\theta$$

where $x_t \equiv k_t^H/\kappa_t$ is the fraction of aggregate capital allocated to the high type and

$$\Gamma(x) \equiv \left\{ \pi A^{\frac{1}{1-\theta}}(x)^{\frac{\alpha}{1-\theta}} + (1-\pi)\left(\frac{1-\pi x}{1-\pi}\right)^{\frac{\alpha}{1-\theta}} \right\}^{1-\theta}. \tag{7}$$

The quantity $\Gamma_t = \Gamma(x_t)$ therefore identifies aggregate TFP as a function of the allocation of capital between the low and the high type.

Clearly, if there were no financial friction, the equilibrium would feature $x_t = x^\star \equiv \arg\max_x \Gamma(x)$, that is, the allocation of capital would maximize aggregate TFP. The same remains true in our model as long as the borrowing constraint does not bind. But when this constraints binds, $x_t < x^\star$ and aggregate TFP is less than maximal.

The magnitude of the TFP distortion is manifested in the shadow cost of the collateral constraint and thereby in the liquidity premium. Indeed, it can be shown that the Lagrange multiplier, μ_t^H, which measures the shadow cost of the collateral constraint, satisfies

$$\pi \mu_t^H = \Gamma'(x_t)\kappa_t^{\alpha-1}h_t^\theta.$$

From the household's Euler conditions for capital and the bond we have that $p_t = 1/q_{t-1}$ and $q_{t-1} = \beta\left\{1 + \frac{1}{p_t-\xi}\pi\mu_t^H\right\}$. Combining these results we obtain the following relation between the price of public debt and the TFP distortion:

$$q_{t-1} = \beta\left\{1 + \frac{q_{t-1}}{(1-\xi q_{t-1})}\Gamma'(x_t)\kappa_t^{\alpha-1}h_t^\theta\right\}.$$

It is evident that $q_{t-1} > \beta$ if and only if $\Gamma'(x_t) > 0$, or equivalently $x_t < x^\star$. Furthermore, q_{t-1} increases with the magnitude of $\Gamma'(x_t)$ for any given κ_t and h_t. The Ramsey planner can therefore

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alleviate the TFP distortion by inducing a higher bond price—or, conversely, the planner can enjoy a lower cost of servicing the public debt by accommodating a higher TFP distortion. This trade off between TFP distortions and interest rates appears to be endemic to many other models in which credit frictions affect the allocation of capital and the incentives to save,[11] and is central to the nature of the policy results we obtain in the next section.

The following result, which expresses the equilibrium capital allocation and the associated TFP level as functions of the bond price, summarizes the observations made above.

Lemma 1. *There exist decreasing functions* $X : [\beta, \xi^{-1}) \to \mathbb{R}_+$ *and* $\Psi : [\beta, \xi^{-1}) \to \mathbb{R}_+$ *such that the equilibrium value of the ratio* $x_t \equiv k_t^H/\kappa_t$ *and of aggregate TFP are given by*

$$x_t = X(q_{t-1}) \qquad and \qquad \Gamma_t = \Psi(q_{t-1}).$$

These functions satisfy $X(\beta) = x^\star \equiv \arg\max_x \Gamma(x) > 1$ *and* $\Psi(\beta) = \Gamma^\star \equiv \max_x \Gamma(x)$, *and are given by*

$$X(q) \equiv \frac{\Phi(q)}{1 - \pi + \pi\Phi(q)}, \quad with \quad \Phi(q) \equiv \left(\frac{\beta\pi A^{\frac{1}{1-\theta}}(1 - (1-\delta)q)}{\beta\pi(1 - (1-\delta)q) + (q - \beta)(1 - \xi q)} \right)^{\frac{1-\theta}{1-\alpha-\theta}}$$

and $\Psi(q) \equiv \Gamma(X(q))$, *with* $\Gamma(x)$ *defined as in (7).*

The implementability constraints. Using the preceding results, we can state the following proposition that summarizes the key implementability constraints faced by the Ramsey planner.

Proposition 1. *Let* $\tilde{s} = (\tilde{s}^1, \tilde{s}^2)$, *with* $\tilde{s}^1 \equiv \{Y_t, C_t, h_t, \kappa_t, b_t, q_{t-1}\}_{t=0}^{\infty}$ *and* $\tilde{s}^2 \equiv \{p_t, w_t, \tau_t, c_t^H, c_t^L, y_t^H,$ $y_t^L, k_t^H, k_t^L, n_t^H, n_t^L\}_{t=0}^{\infty}$. *The sequence* $\tilde{s}$ *can be implemented as an equilibrium if and only if*

(i) *given* $\tilde{s}^1$, $\tilde{s}^2$ *satisfies the preceding characterization results; and*

(ii) $\tilde{s}^1$ *satisfies the following system:*

$$C_t + g_t + \kappa_{t+1} = Y_t + (1-\delta)\kappa_t \tag{8}$$

$$Y_t = \Psi(q_{t-1})\kappa_t^\alpha h_t^\theta \tag{9}$$

$$q_t b_{t+1} = b_t + g_t - \theta Y_t + v'(h_t)h_t \tag{10}$$

$$q_{t-1} = \beta \left\{ 1 + \frac{q_{t-1}}{(1 - \xi q_{t-1})} \Gamma'(x_t)\kappa_t^{\alpha-1}h_t^\theta \right\} \tag{11}$$

$$(1 - \xi q_{t-1})X(q_{t-1})\kappa_t \leqslant \kappa_t + q_{t-1}b_t \quad and \quad q_{t-1} \geqslant \beta \quad wcs. \tag{12}$$

The interpretation of the above conditions is straightforward: (8) is the resource constraint; (9) is the effective aggregate production function, with the equilibrium TFP distortion captured by the term $\Psi(q_{t-1})$; (10) is the budget constraint of the government, with the equilibrium condition for labor already included; (11) is the Euler condition that pins down the equilibrium price of public debt; and finally (12) is the equilibrium collateral constraint.

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[11]For example, it is also present in Holmstrom and Tirole (1998) and Buera and Nicolini (2012).

Before proceeding to solve the Ramsey problem, let us note that a higher κ_t has two opposing effects on the collateral constraint. On the one hand, for any given $x_t = X(q_{t-1})$, a higher κ_t means a higher investment in the high-productivity technology, which tightens the constraint. On the other hand, a higher κ_t also increases the available collateral, which loosens the constraint. Which effect dominates depends on the capital allocation $x_t \equiv \frac{k_t^H}{\kappa_t}$ and the leverage ratio $\ell_t \equiv \frac{1}{p_t - \xi} = \frac{q_{t-1}}{1 - \xi q_{t-1}}$. We henceforth impose $\xi < \bar{\xi} \equiv \frac{x^\star - \beta}{\beta x^\star}$, which is necessary and sufficient for the first-best allocation to violate the collateral constraint whenever $b_t = 0$. Under this parameter restriction, if the planner wishes to induce the first-best use of capital, he can do so only by issuing a sufficiently positive level of debt. Furthermore, the aforementioned tightening effect of a higher κ_t dominates at least as long as x_t and q_{t-1} are near their first-best levels.

4 Ramsey optimum

Proposition 1 implies that the optimal policy and the optimal allocation can be identified by selecting the sequence $\{Y_t, C_t, h_t, \kappa_t, b_t, q_{t-1}\}_{t=0}^\infty$ that maximizes ex-ante welfare subject to the constraints (8)-(12). Before solving this problem, we first consider the benchmark without financial frictions. This is not only a useful benchmark for comparison purposes, but also a useful building step towards the Ramsey problem with frictions.

The Ramsey problem without frictions. Suppose there is no financial friction. In this case, the Ramsey optimum satisfies $x_t = x^\star$, $\Gamma_t = \Gamma^\star$, and $q_t = \beta$ for all t. Furthermore, using the fact that $C_t = Y_t + (1 - \delta)\kappa_t - \kappa_{t+1} - g_t$, we can express the present value of consumption as

$$\sum_{t=0}^\infty \beta^t C_t = Y_0 + (1 - \delta)\kappa_0 - g_0 + \sum_{t=1}^\infty \beta^t \left\{ Y_t + (1 - \delta)\kappa_t - \frac{1}{\beta}\kappa_t - g_t \right\},$$

where $Y_t = \Gamma^\star \kappa_t^\alpha h_t^\theta$. It follows that the optimal κ_t is given by

$$\kappa_t = \tilde{\kappa}_t \equiv \max_\kappa \left\{ \Gamma^\star \kappa^\alpha h_t^\theta + (1 - \delta)\kappa - \frac{1}{\beta}\kappa \right\} = \chi h_t^{\frac{\theta}{1-\alpha}},$$

where $\chi \equiv \left(\frac{\alpha\beta\Gamma^\star}{1 - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$. That is, we can think of κ_t as an intermediate input, whose price is pinned down by β, and whose optimal use increases with employment. Let us now define the following transformations of the output and consumption series:

$$\tilde{y}_t \equiv Y_t + (1 - \delta)\tilde{\kappa}_t - \frac{1}{\beta}\tilde{\kappa}_t \quad \text{and} \quad \tilde{c}_t \equiv C_t + \tilde{\kappa}_{t+1} - \frac{1}{\beta}\tilde{\kappa}_t,$$

with $\tilde{\kappa}_t$ being the aforementioned optimal capital choice. To simplify the exposition, we assume that the initial κ_0 happens to be the optimal one ($\kappa_0 = \tilde{\kappa}_0$).[12] Letting $\Delta \equiv (1 - \alpha)\frac{1 - \beta(1-\delta)}{\alpha\beta}\chi > 0$,

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[12]If we let $\kappa_0 \neq \tilde{\kappa}_0$, nothing essential changes in the result we present below. The only difference is that the very first period of the Ramsey problem becomes special: optimal employment and taxes are the same from $t = 1$ and onwards, but they are different at $t = 0$ because the productivity of labor is different.

$\eta \equiv \frac{\theta}{1-\alpha} \in (0,1)$, and $f(h) \equiv \Delta h^\varsigma$, we then reach the following reduced-form representation of the Ramsey problem.

Lemma 2. *Let the financial friction be removed. The allocation $\{Y_t, C_t, h_t\}$ is optimal if and only if the associated sequence $\{\tilde{y}_t, \tilde{c}_t, h_t\}$ solves the following problem:*

$$\max \sum_{t=0}^{\infty} \beta^t \{\tilde{c}_t - v(h_t)\}$$

$$s.t. \quad \tilde{c}_t + g_t = f(h_t) \ \forall t \geqslant 0$$

$$b_0 = \sum_{t=0}^{\infty} \beta^t \{\eta \tilde{c}_t - v'(h_t)h_t - (1-\eta)g_t\}$$

This problem is essentially the same as the standard Ramsey problem under the restriction of linear utility, as in AMSS. The first constraint is the resource constraint and the second constant is the familiar implementability constraint.[13]

To obtain the solution to this problem, let λ be the Lagrange multiplier on the implementability constraint (the shadow value of tax revenue) and θ_t the Lagrange multiplier on the period-t resource constraint (the shadow value of resources). The first-order conditions give

$$\theta_t = 1 + \eta\lambda$$

and

$$\theta_t f'(h_t) = v'(h_t) + \lambda \left[v'(h_t) + v''(h_t)h_t\right]$$

It follows that $h_t = h^\star$ for all t, where $h^\star$ is the unique solution to

$$(1 + \eta\lambda)f'(h) = (1 + \lambda + \epsilon\lambda)v'(h)$$

and $\epsilon \equiv v''(h)h/v'(h)$ is the inverse of the Frisch elasticity. We thus arrive at the following result.

Proposition 2. *In the absence of frictions, the Ramsey optimum features identical taxes and identical employment, investment, and output levels across all periods:*

$$\tau_t = \tau_0, \ h_t = h_0, \ \kappa_t = \kappa_0, \ y_t = y_0, \quad \forall t \geqslant 0.$$

The frictionless variant of our model features a perfect form of tax smoothing, as in the deterministic backbones of Barro (1979) and AMSS. The rate of taxation and the level of debt stay constant over time, and so do aggregate employment and output. By the same token, the model does not feature a determinate steady state: the long-run values of all policy instruments and all allocations move one-to-one with the initial level of debt. When there is aggregate uncertainty but the government can issue (or replicate) a complete set of state-contingent debt instruments, the analogue to this property is the presence of a unit-root component, as established in AMSS. As we will see shortly, these properties cease to hold once we give public debt a liquidity role.

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[13]The only formal difference is the presence of the scalars η and $(1-\eta)$ in front of, respectively, $\tilde{c}_t$ and g_t.

The Ramsey problem with frictions We now return to the case of interest, the Ramsey problem in the presence of financial frictions. The characterization of this problem is somewhat complex, because the dynamics of the economy alternate between two regimes, depending on whether the collateral constraint binds or not, and also because the dynamics are highly non-linear whenever the constraint binds. In what follows we only sketch the procedure we follow in order to implement a low-dimensional numerical solution.

Let z_t be the period-t level of debt minus the contemporaneous tax revenue:

$$z_t \equiv b_t - \tau_t w_t h_t.$$

We can think of z_t as a monotone transformation of b_t, and in this sense, as a relevant state variable. We can then show the following.

Lemma 3. *There exists a function $\mathcal{Z}$ such that the collateral constraint binds in period t if and only if $z_t < \mathcal{Z}(\tau_t)$.*

To prove this result, assume that the constraint does not bind. In this case, $x_t = x^\star$, $\Gamma_t = \Gamma^\star$, $q_t = \beta$, and $\kappa_t = \tilde{\kappa}_t \equiv \chi h_t^{\frac{\theta}{1-\alpha}}$, implying that the collateral constraint reduces to

$$\beta b_t \geqslant \left[(1 - \xi\beta)x^\star - 1\right]\chi h_t^{\frac{\theta}{1-\alpha}}.$$

It follows that the collateral constraint does not bind only if

$$b_t \geqslant B(h_t),$$

where $B(h) \equiv \frac{((1-\xi\beta)x^\star-1)\chi}{\beta} h^{\frac{\theta}{1-\alpha}}$. Next, noting that the wage rate satisfies $w_t = f'(h_t)$ and the labor supply $v'(h_t) = (1 - \tau_t)w_t$, we infer that, as long as the collateral constraint does not bind, employment and tax revenue are given by

$$h_t = H^\star(\tau_t) \equiv \Phi_h(1 - \tau_t)^{\frac{1-\alpha}{(1-\alpha)(1+\nu)-\theta}}$$

$$\tau_t w_t h_t = T^\star(\tau_t) \equiv \Phi_\tau \tau_t(1 - \tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}.$$

for some constants $\Phi_h, \Phi_t > 0$. Using these facts together with the definition of z_t, we conclude that $b_t \geqslant B(h_t)$ (and the constraint ceases to bind) if and only if

$$z_t \geqslant \mathcal{Z}(\tau_t),$$

where $\mathcal{Z}(\tau) \equiv B\left(H(\tau)\right) - T(\tau) = (\Phi_b - \Phi_\tau\tau)(1 - \tau)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$, for some constant Φ_b. Note also that $\mathcal{Z}(\tau)$ is a decreasing function of τ. This is because a higher rate of taxation discourages labor supply, which in turn depresses the marginal product of capital and reduces the minimal level of debt that is consistent with a non-binding collateral constraint.

Consider now the case in which $z_t < \mathcal{Z}(\tau_t)$, or equivalently $b_t < B(h_t)$. In this case, labor supply is still pinned down by $v'(h_t) = (1 - \tau_t)w_t$, but now the wage rate is a function of q_{t-1}, because the latter determines the TFP distortion in the economy. Accordingly, the equilibrium allocations and the resulting tax revenue and utility flows can be expressed as functions of τ_t and q_{t-1}.

Lemma 4. *There exist continuous functions $H, K, Y, T,$ and U such that*

$$h_t = H(\tau_t, q_{t-1}), \quad \kappa_t = K(\tau_t, q_{t-1}), \quad Y_t = Y(\tau_t, q_{t-1}), \tag{13}$$

$$\tau_t w_t h_t = T(\tau_t, q_{t-1}), \quad C_t - v(h_t) = U(\tau_t, q_{t-1}). \tag{14}$$

The functions $H, T, Y,$ and K are spelled out in the appendix. Not surprisingly, their values at $q = \beta$ coincide with those in the unconstrained regime, since $q = \beta$ if and only if the collateral constraint does not bind. The above lemma therefore applies irrespectively of whether the constraint binds or not; the only difference is that $q_{t-1} = \beta$ if $z_t \geqslant \mathcal{Z}(\tau_t)$, whereas $q_{t-1} > \beta$ if $z_t < \mathcal{Z}(\tau_t)$.

Given these results and the definition of z_t, we can express the collateral constraint as

$$q_{t-1}\left(z_t + T(\tau_t)\right) = \left[(1 - \xi q_{t-1})X(q_{t-1}) - 1\right]K(\tau_t, q_{t-1}),$$

which in turn can be solved for q_{t-1} as a function of z_t and τ_t. We can thus find a (continuous) function Q such that, irrespectively of whether the collateral constraint binds or not, the price of government debt can be expressed as

$$q_{t-1} = Q(z_t, \tau_t). \tag{15}$$

The function Q captures the liquidity premium on public debt as a function of the levels of debt and the tax distortion. It is described in the appendix and is such that it satisfies $Q(z, \tau) = \beta$ if $z \geqslant \mathcal{Z}(\tau)$ and $Q(z, \tau) > \beta$ otherwise.

Using the above results, we can express the expected period-t utility as

$$W(\tau_t, z_t) \equiv U(\tau_t, Q(z_t, \tau_t)).$$

Next, letting $\mathcal{B}(z, \tau) \equiv z + T(\tau, Q(z, \tau))$, we can write the level of debt as a monotone function of z_t and τ_t:

$$b_{t+1} = \mathcal{B}(z_t, \tau_t) \tag{16}$$

We can also represent the implementability constraint as a constraint on (z_{t-1}, z_t, τ_t):

$$Q(z_t, \tau_t)\mathcal{B}(z_t, \tau_t) = z_{t-1} + g_{t-1}.$$

This means that z_t serves as the key endogenous state of the Ramsey problem, while τ_t is the key control variable.

Proposition 3. *The sequence $\{Y_t, C_t, h_t, \kappa_t, b_t, q_{t-1}\}_{t=0}^{\infty}$ is optimal if and only if the following properties hold.*

(i) The sequence $\{\tau_t, z_t\}_{t=0}^{\infty}$ solves the following reduced-form Ramsey problem:

$$\max_{\{\tau_t, z_t\}} \sum_{t=0}^{\infty} \beta^t W(\tau_t, z_t) \tag{17}$$

$$s.t. \quad Q(z_t, \tau_t)\mathcal{B}(z_t, \tau_t) = z_{t-1} + g_{t-1}. \tag{18}$$

(ii) With the sequence $\{\tau_t, z_t\}_{t=0}^{\infty}$ determined as above, the sequence $\{Y_t, C_t, h_t, \kappa_t, b_t, q_{t-1}\}_{t=0}^{\infty}$ satisfies conditions (13)-(16).

This result provides us with a low-dimension dynamic program that can be solved to obtain the optimal policy and the optimal allocation. The details of the numerical algorithm are explained in the appendix.

Parametrization. Our model is too stylized to permit a serious quantitative exploration.[14] We thus use a plausible parametrization of the model, as reported in Table 1, only to illustrate the shape of the optimal policy and its sensitivity to the underlying primitives.

Table 1: Parameters

Parameter		Value	Parameter		Value
capital elasticity	α	0.30	labor elasticity	θ	0.60
discount factor	β	0.96	inverse labor supply elasticity	ε	1.00
labor disutility	ϑ	1.00	depreciation rate	δ	0.06
high TFP probability	π	0.50	high-to-low TFP ratio	A	2.25
share of confiscated capital	ξ	0.38	government share	g/y	0.22

The discount rate is set to 4% per year. The labor elasticity is set to 0.6 and the capital share to 0.3, so that the overall returns to scale are 0.9. The labor disutility is assumed to be quadratic, meaning that the Frisch elasticity is 1. The depreciation rate is set to 6% per year. The high TFP level is set to 2.25, with a probability of occurrence of 0.5. The government share is set to 22%. The severity of the financial friction, which is captured by the fraction of capital, ξ, that is confiscated in the event of default is calibrated so that the steady-state interest rate is 2% (equivalently, the liquidity premium is also 2%); this implies $\xi = 0.38$. While the numerical results reported below are specific to this parametrization, the qualitative patterns that emerge are quite general.

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[14]Such an exploration would have to allow, at the very least, for realistic firm-level heterogeneity and firm-level dynamics, which would undermine the tractability of our framework and give rise to severe computational challenges.

Figure 1: The Optimal Policy Rules and the Price of Debt
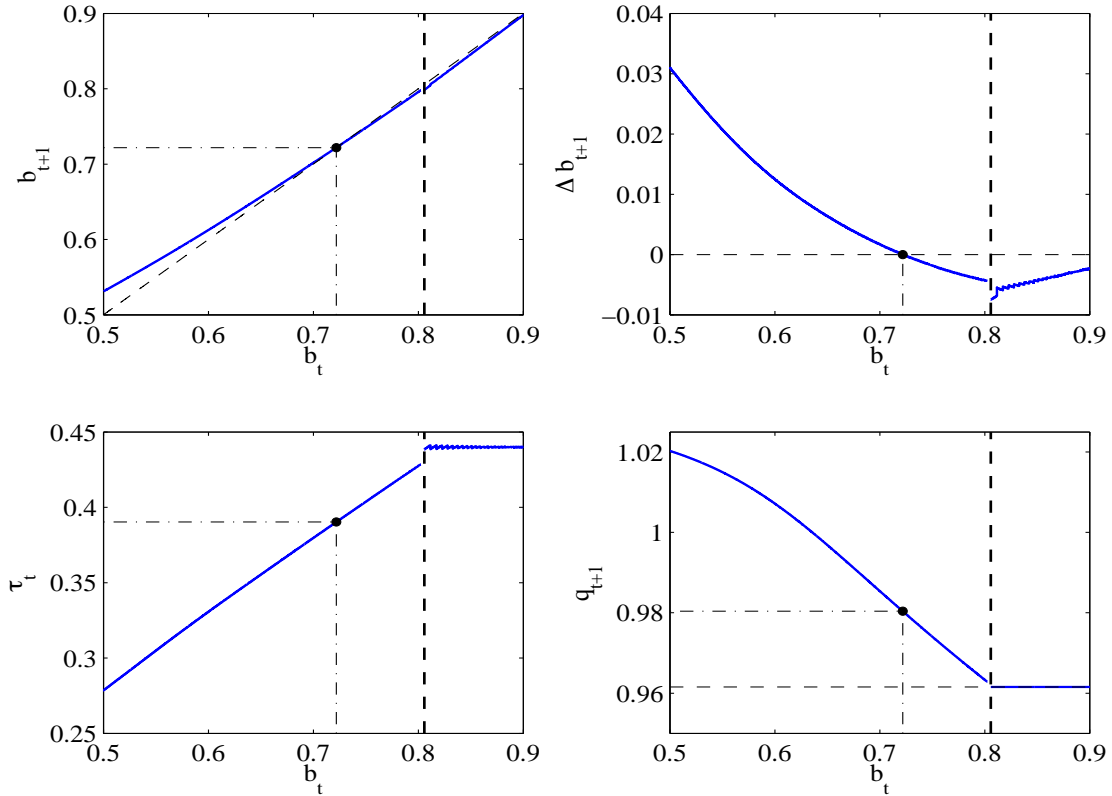
Note: The vertical dashed line indicates the level of debt above which the borrowing constraint ceases to bind and the liquidity premium vanishes (the "satiation point"). The dot indicates the steady state.

The optimal policy. Figure 1 illustrates the optimal policy. The top-left and top-right panels depict, respectively, the optimal gross debt issuance (b_{t+1}) and the optimal deficit ($\Delta b_{t+1} \equiv b_{t+1} - b_t$) as a function of the inherited level of debt (b_t). The bottom-left panel gives the optimal level of taxation (τ_t). Finally, the bottom right panel gives the equilibrium price of debt (q_{t+1}).[15]

Four properties are worth highlighting.

First, there exists a unique steady state. The steady state is identified by the intersection of the policy rule for debt with the 45-degree line and is indicated with a bullet in the figure. In our numerical example, the steady-state debt-to-GDP ratio is about 72%.

Second, the steady-state level of debt is below the threshold level at which the borrowing constraint ceases to bind. This threshold level is indicated with the dashed vertical line in the figure and can also be interpreted as the point at which the economy's demand for liquidity is satiated (the liquidity premium is zero). In this sense, the analogue of the Friedman rule does not hold.

Third, the policy rules are continuous and monotonic on the left of the aforementioned thresh-

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[15]We solve the model in terms of z_t for computational reasons, but report the policy rules in terms of b_t to facilitate the interpretation. Also, we normalize the scale of b_t by the steady state level of GDP.

old, but exhibit multiple discontinuities, and patterns that resemble "wiggles" on the right of this threshold. These wiggles are the manifestation of the non-convexity of the planner's problem, which in turn emerges because, and only because, the price of debt is a function of the level of debt.[16]

Fourth, the steady state is globally stable: no matter the initial level of debt, the economy eventually gets closer and closer to it. By the same token, perfect tax smoothing is not optimal. Instead, there is a positive drift in the optimal levels of taxation and debt when the economy starts below the steady state, and a negative drift when it starts above.

These results reflect the trade off between the two key distortions confronted by the planner. By lowering the level of public debt, the planner tightens the agents' borrowing constraint, which exacerbates the inefficiency in the allocation of capital and reduces output. But this tightening also increases the agents' willingness to hold public debt as collateral, which in turn reduces the interest rate the government has to pay on its debt, mitigating the tax distortion. The steady-state debt level is determined by the balancing of these two effects. When the initial level of debt is below the steady state, the benefit of improving efficiency in production outweighs the cost of raising interest rates and suffering higher taxes, so the planner finds it optimal to issue more debt. The converse is true when the initial level of debt is above the steady state.

These properties are further illustrated in Figure 2, which depicts the transitional dynamics when the economy starts with an initial level of debt that is either below the steady state (solid red lines) or above it (dashed blue lines).[17]

In the first scenario, the initial level of debt is sufficiently below the the steady state, and hence the bite of the financial friction is sufficiently severe, that the interest rate is negative.[18] The planner then finds it optimal to accumulate more debt, not only because it wishes to alleviate the financial friction, but also because it can effectively reduce the present value of taxes by raising a bit more debt. As more and more debt is issued, however, the friction gets relaxed, the interest rate turns positive, and issuing debt becomes costly. Debt nevertheless continues to increases monotonically towards the steady state, as the benefit of facilitating a more efficient allocation of capital outweighs any other consideration. Along this transition, taxes also increase monotonically. Finally, output increases despite the increase in taxes, thanks to the more efficient allocation of capital.

In the second scenario, the initial level of debt is sufficiently that the economy actually starts

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[16]When we mechanically shut down this effect (which we do below in order to build intuition), the value function of the planner is strictly concave and the policy rules are continuous and monotone.

[17]We choose the initial level of debt to be 12.5% below/above the steady state. Within our numerical example, this choice helps us best illustrate the properties we discuss below.

[18]This can be seen in the bottom-left panel of Figure 2: in the early phase of the transition (solid red line), the liquidity premium exceeds 4%, which is the discount rate, and hence the interest rate is negative. Also note that this possibility illustrates the mechanism that is at the core of Eggertsson and Krugman (2012) and Guirrieri and Lorenzoni (2012), namely that the natural rate of interest can turn negative when financial constraints are sufficiently tight. We revisit this point in Section 6, when we study the optimal policy response to financial shocks.

Figure 2: Transitional Dynamics
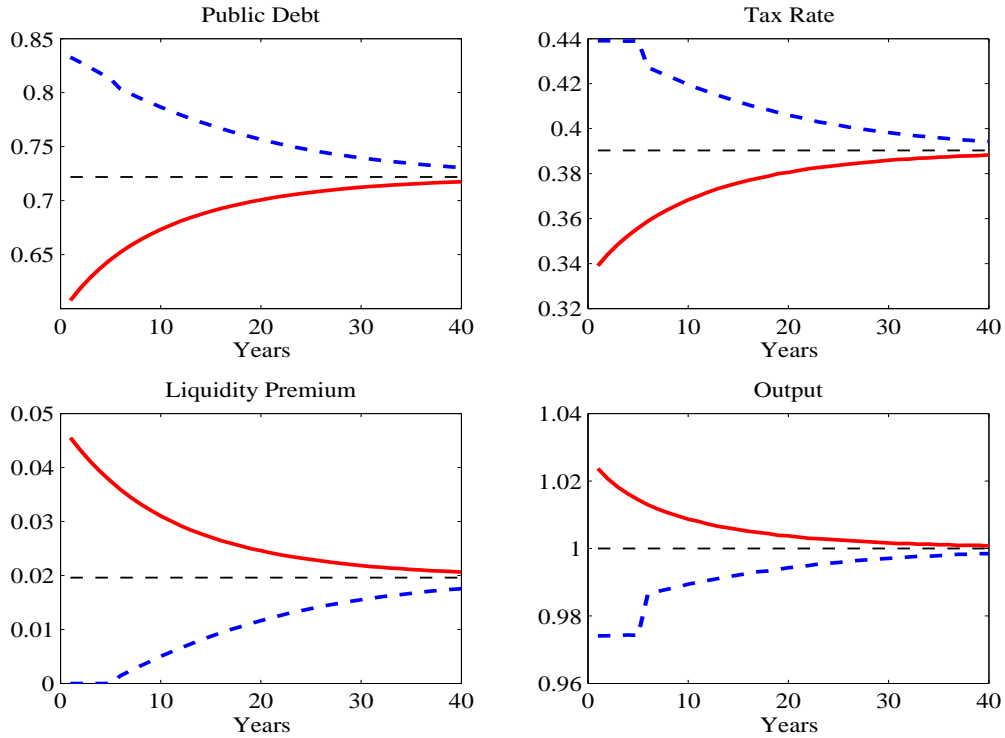
Note: This figure illustrates the optimal transition to the steady state when the economy starts either below the steady state (——) or above it (– –).

above the the satiation threshold: the borrowing constraint is not binding and the liquidity premium is zero. As long as the economy is above this point, the tax rate is kept constant, as in Barro (1979), but at a level that is higher than the level required to keep the level of debt constant. The planner thus runs surpluses, gradually withdrawing debt and eventually crossing the satiation threshold. At that point, the borrowing constraint begins to bind, the liquidity premium becomes positive and the planner starts enjoying the benefit of a lower interest rate. Public debt is further reduced until this benefit is balanced by the production inefficiency created by the tighter borrowing constraints.

5 Anatomy of the Optimal Policy

In this section we elaborate on the key trade-off behind our results. We next illustrate how this trade-off shapes the comparative statics of the steady state. We finally clarify the role that interest-rate variation plays in the Ramsey problem once interest rates manifest the bite of financial frictions.

The key trade-off and tax smoothing.   To reveal how the trade-off between liquidity provision and interest-rate manipulation shapes the optimal policy, we consider two variants of the Ramsey problem. In the first variant, we mechanically block the effect of public debt on production and

Figure 3: Isolating the liquidity-provision and the interest-rate-manipulation effects.
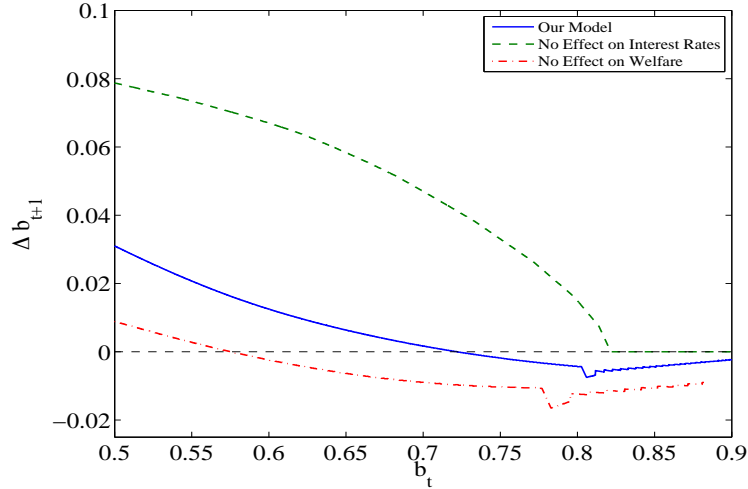
welfare. Formally, we maintain the budget constraint unchanged as in (1), but replace the welfare objective $W(\tau_t, z_t)$ in (17) with $\hat{W}(\tau_t) \equiv W(\tau_t, \hat{z})$, where $\hat{z}$ is exogenously fixed. In the second variant, we mechanically shut down the interest-rate manipulation. Formally, we now leave the welfare objective unchanged as in (17), but replace the price of debt $q_t = Q(\tau_t, z_t)$ in the budget constraint (1) with $\hat{q}_t = \beta$. By solving these two variant Ramsey problems, we can thus isolate the policy implications of each of the two sides of the aforementioned trade-off.

The results of this exercise are reported in Figure 3. When we block the impact of debt on welfare (dotted red line in the figure), the policy rule for debt-issuance shifts down: the planner chooses to run smaller deficits and issue less debt than in our model in order to decrease the interest-rate costs and tax distortion. When we take away, instead, his ability to manipulate interest rates (dashed blue line), the policy rule shifts up: the planner opts to run bigger deficits and issue more debt, relative to our model, in order to further alleviate the impact of the financial friction on production and welfare. What is more, in the long run the planner actually finds it optimal to satiate the economy's demand for public debt: the steady state of this variant obtains exactly at the point where the borrowing constraint ceases to bind.[19]

To develop more intuition, it is useful to think about these observations in the context of the familiar analogy between tax smoothing and consumption smoothing. It is well known from Barro

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[19]Note that the satiation point is not the same across the three Ramsey problems under consideration because (i) the planner chooses different taxes in each problem and (ii) the rate of taxation affects labor supply, which in equilibrium affects the internal return to capital and thereby the threshold level of debt needed for the borrowing constraint not to bind. If modify the model so that labor is used in a different sector than capital (say, the "entrepreneurial sector" employs only capital, and the "agricultural sector" employs only labor), the aforementioned interaction between the rate of taxation and the bite of the financial friction disappears, and the satiation level of debt becomes the same across the three Ramsey problems.

(1979) and AMSS that the standard Ramsey policy problem is formally similar to a consumption-saving problem; one only has to reinterpret the relevant control and state variables, from consumption and wealth in the consumer problem to taxes (or surpluses) and debt in the Ramsey problem. Our model deviates from the standard Ramsey problem by effectively making the level of debt enter both the planner's welfare objective and the interest rate in the planner's budget constraint.

With this in mind, consider a modification of the canonical consumer problem that allows the consumer's wealth to enter his preferences, as well as to affect the interest rate he faces:

$$\max_{\{c_t, w_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[U(c_t) + V(w_t) \right]$$

$$s.t. \quad w_{t+1} = R(w_{t+1})(w_t + y_t - c_t)$$

where c is consumption, w is wealth, y is (exogenous) income, U measures the utility from consumption (with $U' > 0 > U''$), V measures the utility from wealth (with $V' \geq 0 \geq V''$), and R gives the interest rate as a function of wealth.

In the standard case, where $V(w) = 0$ and $R(w) = \beta^{-1}$ for all w, the consumer's Euler condition gives $U'(c_t) = U'(c_{t+1})$ and therefore $c_t = c_{t+1}$, irrespectively of the time path of income. This explains the optimality of consumption smoothing in the textbook treatment of the consumer problem—and, by analogy, the optimality of tax smoothing in Barro (1979) and AMSS. But now suppose that there exists some $\bar{w}$ such that $V'(w) > 0$ if $w < \bar{w}$ and $V'(w) = 0$ otherwise, meaning that $\bar{w}$ identifies the satiation point of wealth. If we maintain the assumption $R = \beta^{-1}$, the Euler condition gives $U'(c_t) = U'(c_{t+1}) + V'(w_{t+1})$ and therefore $c_t < c_{t+1}$ as long as $w_{t+1} < \bar{w}$. For levels of wealth below the satiation point, the consumer thus give up consumption smoothing for the sake of getting closer to that satiation point. By analogy, in the variant of the Ramsey problem considered above where we maintained the effect of debt on welfare but exogenously imposed $q_t = \beta$, the planner found it optimal to give up tax smoothing and accumulate more debt, up to the point that the economy's demand for liquidity was satiated. Finally, consider the more general case where wealth also affects the interest rate. In this case, the consumer may find it optimal to keep his wealth below $\bar{w}$ in the long run in order to manipulate the interest rate he faces—just like the planner in our model finds it optimal not to satiate the economy's demand for liquidity in the long run in order to manipulate the interest-rate cost of servicing the public debt.

These observations help also clarify the driving forces behind the violation of the Friedman rule in the short (transitional dynamics) vs the long run (steady state). Both our model and the variant considered above (where debt enters welfare but the interest is fixed) imply a violation of the Friedman rule in the short run, in the sense that the optimal level of public debt in any given period falls short of the satiation level when the initial level of debt is not too high. However, this variant implies that the planner will find it optimal to satisfy the Friedman rule in the long run, which is not the case in our model. Since this variant shuts down the planner's ability to manipulate interest

rates but otherwise maintains the distortionary effects of taxation, this proves that the violation of the Friedman rule in the long run of our model hinges on the interest-rate manipulation channel, not merely on distortionary taxation. By contrast, in the short run the Friedman rule could be violated even if this manipulation channel were dormant.

At this point, it also worth clarifying the role of precluding the government from making non-negative transfers to the agents. In a recent paper, Golosov and Sargent (2012) study a heterogeneous-agent variant of AMSS that allows for richer tax schemes than our framework, and prove a certain debt-irrelevance result: allocations that are implemented by a particular debt policy can often be replicated with a particular sequence of non-negative lump-sum transfers and no public debt.[20] Seen through these lens, our analysis pins down a unique path for public debt only by ruling out the possibility of replicating the liquidity services of public debt with lump-sum transfers. The same qualification applies to, inter alia, Woodford (1990), Aiyagari and McGrattan (1997) and Holmstrom and Tirole (1998). That been said, the key trade-off between liquidity provision and interest-rate manipulation that we have documented above remains present irrespectively of whether liquidity is provided by debt or transfers. In this sense, our analysis could be recast as a theory of optimal liquidity provision, without necessarily taking a stand on whether the optimal allocation is ultimately implemented with debt or transfers. At the same time, we do favor the debt-based interpretation on grounds of empirical plausibility and potential robustness to extensions of the theory that restrict the ability, or desirability, of replicating debt with transfers.[21]

The optimal long-run quantity of public debt. To illustrate how the trade-off between liquidity provision and interest-rate manipulation shapes the optimal long-run level of public debt, we now explore how the steady state varies with the underlying primitives of the economy.

Figure 4 considers the comparative statics of the steady-state level of public debt with respect to the severity of the financial friction, as parameterized by ξ. An increase in ξ, which means that a larger fraction of capital can be seized by creditors in the event of default, relaxes the bite of the borrowing constraint, facilitating a more efficient allocation of capital and leading to an increase in output. By the same token, public debt becomes less valuable as collateral, which in turn explains why the increase in ξ causes a reduction in the optimal quantity of debt.

To understand why the effects of ξ on the liquidity premium and on the price of debt can be non-monotone, note that there are two opposing forces at work. As ξ becomes larger, capital becomes

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[20]A similar point is made in Werning (2007) for the complete-markets case.

[21]For example, suppose that firms (or banks) have private information about their liquidity needs. Letting them trade public debt may be a more practical, or more robust, way of implementing an efficient allocation of liquidity than, say, a direct mechanism that tries to solicit this information and then send transfers to these agents. Alternatively, suppose that politicians are not entirely benevolent and different groups can lobby for group-specific transfers. Then substituting debt for transfers can help mitigate both the financial and the political-economy frictions. We leave a formal exploration of these ideas for future work.

Figure 4: Comparative statics: ξ
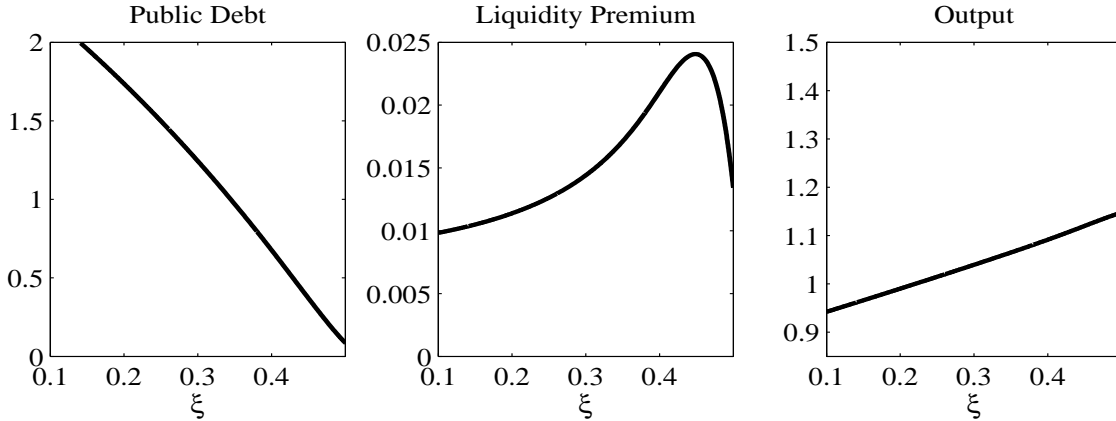
more valuable as collateral, so the demand for capital increases, pushing up the liquidity value and price of debt. But an increased supply of capital means more capital owned by the high types, which dampens their borrowing needs (and hence their demand for debt). At sufficiently high levels of ξ, the latter effect appears to dominate and liquidity (i.e., public debt) is priced at a lower rate in spite of its reduced supply.

The effects of a change in the share of high types in the population, π, are quite similar to those described above for changes in ξ. An increase in π means a larger share of high types. Having more of them implies that a larger share of the capital stock is owned by the borrowers and thus more collateral and a looser financial constraint (much like a higher ξ). At the same time, more high types means higher borrowing needs and thus a tighter constraint.

In Figure 5 we turn attention to the comparative statics of the steady state with respect to the level of government spending, which can be thought more broadly of as a parameter that controls the shadow cost of the government budget and thereby the benefit of manipulating interest rates. Other things equal, an increase in G/Y raises the tax rate that is needed to balance the budget in steady state and makes the tax distortion more costly at the margin. To alleviate this additional cost, public debt declines, causing a liquidity squeeze in the economy. This squeeze depresses economic activity, but it also raises the liquidity premium on public debt, which in turn reduces interest-rate costs and alleviates the aforementioned additional tax distortion.

Finally, it is worth contrasting our characterization of the optimal long-run level of public debt to those in Woodford (1990) and Aiyagari and McGrattan (1998). These papers do not characterize the optimal Ramsey plan. Instead, they study the steady state of their economies for an exogenously fixed level of debt, and then identify the level of debt that maximizes the steady-state level of welfare. In other words, they study the simple "golden rule", which ignores the welfare effects of the transitional dynamics. By contrast, we study the "modified golden rule", which takes into account

Figure 5: Comparative statics: G/Y
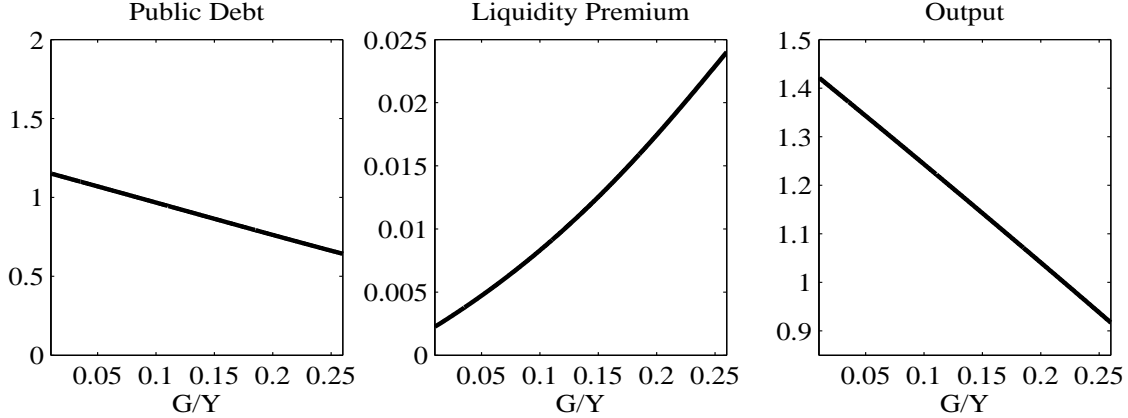
the entire transitional dynamics. Depending on the parametrization, the quantitative implications of this distinction can be either large or small.[22] In any event, the marginal contribution of our paper vis-a-vis these papers rests primarily in documenting the sub optimality of tax smoothing, which we did above, and exploring the optimal dynamic response to shocks, which we do next.

Interest rates versus wedges. We conclude this section by clarifying the distinct role that the equilibrium variation in interest rates starts playing in the Ramsey problem once there are financial frictions. To this goal, it helps to draw again on the analogy to the standard consumer problem. In that context, it is trivial to check that consumption smoothing remains optimal if the consumer's subjective discount rates varies over time and the interest rate happens to covary perfectly with it. In a similar manner, the optimality of tax smoothing that we documented in Proposition 2 remains intact if we let the discount factor of the private agents to very over time; the interest rate faced by the planner now varies over time, but it continues to equal his discount factor. Importantly, this not a mere coincidence; it is a necessary implication of the general equilibrium. What drives our results is therefore, not the overall variation in interest rates, but rather the extent to which this variation reflects variation in the bite of the financial friction, as captured by the wedge μ_t.

Apart from clarifying the mechanics of our theory, the above observation also has practical consequences. In the ongoing policy debate about US fiscal policy, it is often argued that running deficits and issuing more debt is desirable, not only because of Keynensian considerations, but also

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[22]In our baseline parametrization, where the steady-state value of the liquidity premium is 1%, the steady-state optimal debt output ratio is 0.84. The golden-rule steady state debt output ratio amounts to 0.77. Although the difference between the two ratios is rather small in this benchmark parametrization, it may be much greater under alternative parameterizations that raise the liquidity demand for public debt. For instance if ξ is 0.2, which means a liquidity premium equal to 1.71%, the golden rule debt/output ratio is 1.68, while the optimal Ramsey policy gives a steady-state debt/output ratio of 1.75. Likewise, if the share of government expenditures in GDP rises to 25%, the golden-rule ratio is 0.66, while the Ramsey optimum is 0.71.

because the interest rate on public debt is at a historically low level and hence it is "cheap" for the government to borrow. Unfortunately, this argument appears to make no sense within the standard Ramsey paradigm. In this framework, the equilibrium interest rate is perfectly tied to the discount rate that the planner uses to discount future welfare and thereby future tax distortions, which in turn means that observing a low interest rate in the present does not justify transferring the tax burden to the future. In effect, public debt can be "cheap" in terms of the current budget only if tax distortions are "expensive" in terms of future welfare, which invalidates the popular argument. It is thus possible that the popular argument is merely a fallacy. The resolution, however, we propose is that the popular argument makes sense if the observed variation in interest rates reflects variation in the bite of financial frictions, which seems relevant in the context of the recent crisis. We revisit this issue in Section 6, when we study the optimal policy response to a shock that increases the bite of the borrowing constraint in our model.

6 Optimal Response to Fiscal and Financial Shocks

In this section we extend the analysis to a stochastic variant in order to explore the dynamic response of the optimal policy to two kinds of exogenous disturbances: shocks to the level of government spending; and shocks to the severity of the financial friction, as parameterized by ξ. These shocks are assumed to follow independent Markov processes with finite supports and complete mixing. Finally, as in Barro (1979) and AMSS, we restrict the government to issuing only one-period risk-free debt, thus focusing on the realistic case where the government can not insulate its budget from these shocks.

If the support of the underlying shocks is small enough (say, the shock can take only 2 values), we can solve the full, non-linear, dynamic program for the planner's problem in a similar manner as in the deterministic benchmark. This approach, however, becomes computationally costly once we let the support of the shock become large enough. We have thus opted to study a linear approximation of the stochastic dynamics around the deterministic steady state.[23]

The key qualitative lessons of the deterministic benchmark survive: the optimal policy now varies with the shocks, but for any given realization of the shock the policy rules have the same shape as in Figure 1. Our earlier result regarding the determinacy of the steady state thus translates to an invariant long-run distribution. In the remainder of this section we thus focus on documenting the impulse responses of the optimal policy to the aforementioned two types of shocks, under the same parametrization as in Table 1.

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[23]Note that this approach is *not* valid in Ramsey policy exercises such as AMSS because the underlying deterministic steady state is indeterminate and, accordingly, the stochastic dynamics have a unit root. By contrast, this approach is valid in our case because the financial friction induces, in effect, a mean-reverting force.

Fiscal shocks. We first consider a permanent change in government spending: we let g_t follow a random walk and study the impulse response of the optimal policy to a positive innovation in g_t.

In the standard paradigm, tax smoothing requires that taxes increase immediately, permanently, and by essentially the same amount as the increase in government spending.[24] Consequently, the deficit stays at zero, and the level of public debt remains at its initial value. Finally, employment and output fall because of the increase in taxes, but they stay constant throughout time.

The aforementioned responses are illustrated by the dashed lines in Figure 6. The solid lines, by contrast, illustrate the dynamic responses of the optimal policy and the optimal allocation in our model. As is evident from this figure, the optimal response is notably different in our model.

Figure 6: Permanent fiscal shock
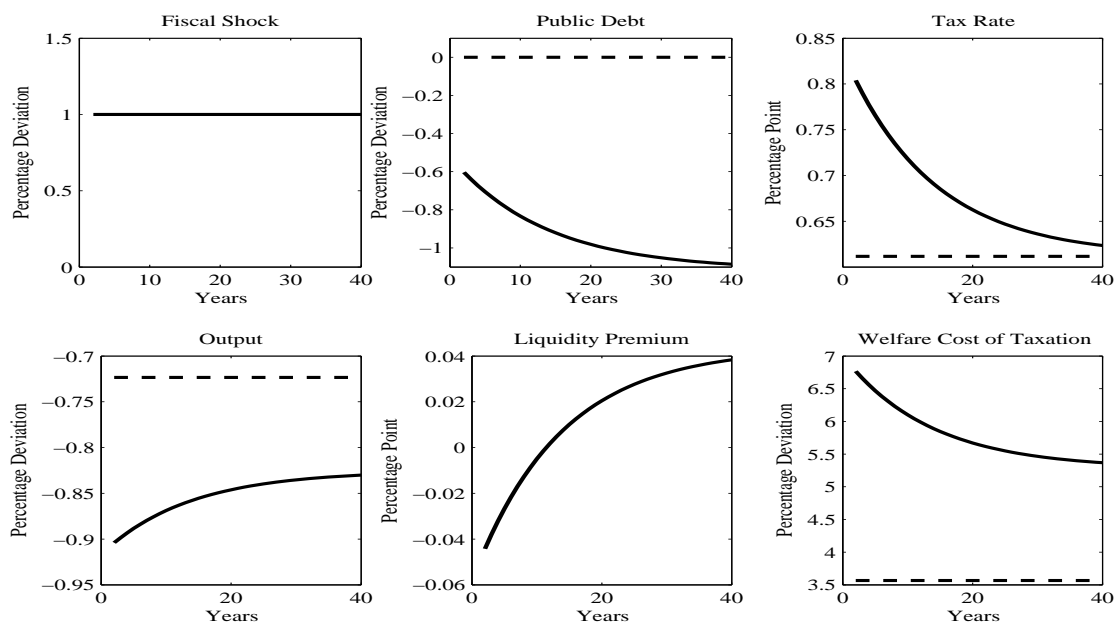
Note: —— Our model, – – Standard Paradigm.

The planner reacts by initially raising taxes more than one-to-one, that is, by running a surplus and withdrawing public debt. By doing so, the planner reduces the aggregate quantity of collateral, which in turn increases the bite of the collateral constraint. The allocation of capital is therefore distorted and aggregate TFP falls, which, other things equal, means a reduction in welfare as well as in tax revenue. However, by increasing the bite of the financial friction, the planner also raises liquidity premia, which helps reduce interest-rate costs. In effect, the planner is "starving the economy from liquidity" in order to push down interest rates and thereby limit the tax burden of the shock.

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[24]To be precise, these properties are exact in Barro (1979) but only approximately true in AMSS. The dashed lines in Figures 6 and are drawn on the basis of Barro (1979) only to sharpen the illustration.

Figure 7 repeats the exercise for the case of a transitory fiscal shock, namely for the case that g_t follows an $AR(1)$ with autocorrelation coefficient equal to 0.86 (as in US data). In the standard paradigm, tax smoothing requires that taxes increase permanently and by as much as the innovation in the annuity value of government spending. Since g_t is mean-reverting, this translates to the government running a deficit early on, accumulating a higher level of debt, and running surpluses later on to pay for the interest-rate cost of the extra debt. Here, by contrast, the government restrains the initial increase in deficits and public debt in order to induce higher liquidity premia and lower interest-rate costs. In the long-run, public debt returns to its original steady-state value, underscoring the mean-reversion force that the financial friction introduces in the Ramsey problem.

Figure 7: Transitory fiscal shock
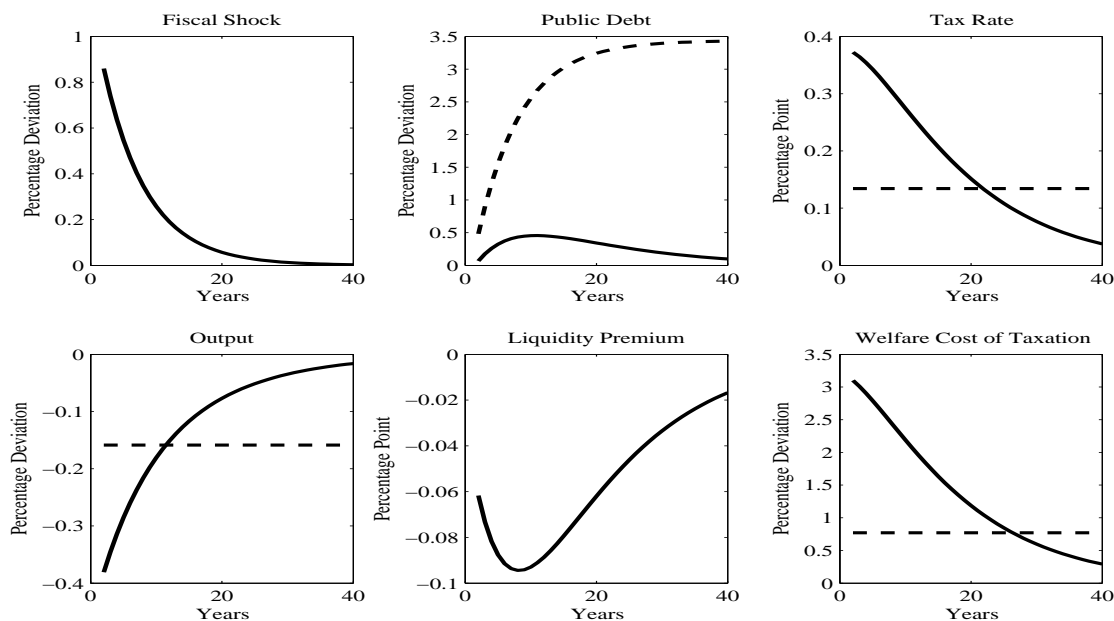
Note: —— Our model, – – Standard Paradigm.

In conclusion, whether the fiscal shock is permanent or transitory, there is considerably less tax smoothing than in the standard model, and the tax burden of the shock is front-loaded: taxes increase more in the present so as to to run smaller deficits than in the standard paradigm, and thereby lower interest rates and enjoy lower taxes in the future.

Financial shocks. Figure 8 describes the effects of an adverse financial shock. We let ξ_t follow an $AR(1)$ process with autocorrelation coefficient 0.90 and study the response of optimal policy and of the optimal allocation to a negative innovation in ξ_t. Note that this shock tightens credit, depresses economic activity, raises liquidity premia, and pushes down interest rates—thus mimicking many of the key stylized facts of the recent financial crisis.

Figure 8: Adverse Financial Shock (ξ)
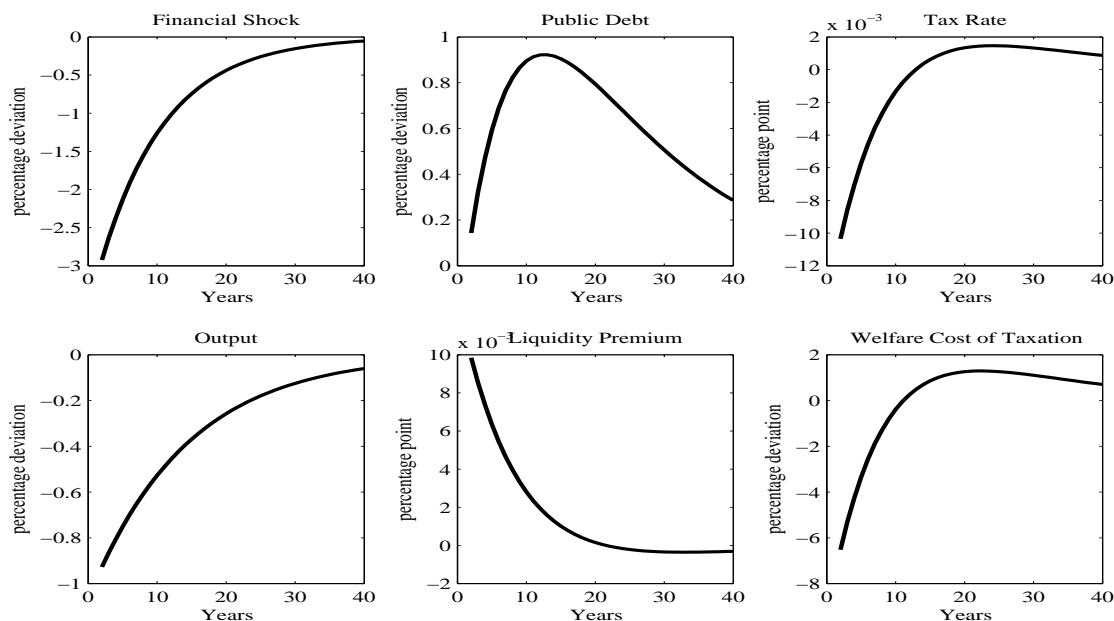
When this shock occurs, the demand for public debt increases. The planner accommodates the higher demand but not fully (as can be seen from the increase in the liquidity value of debt) because of the future tax burden of higher debt; and she does so in a gradual fashion for reasons of tax smoothing. With the level of government spending fixed, debt issuance is accompanied by a reduction in the tax rate. As can be seen in the graph for the welfare cost of taxation (λ), the financial shock represents "good news" for the government budget, as it makes debt more valuable to the agents, suppressing interest-rate costs, and thus making a tax cut attractive. The partial amelioration of the financial friction and the reduction in taxes imply that the recession triggered by the adverse ξ shock is not as deep as it would have otherwise been. But it is bigger than what it would have been if there were no concern for the servicing requirements of debt, in which case enough public debt would have been issued so as to offset entirely the negative effect of the ξ shock on the liquidity premium.

Figure 9 demonstrates these points by comparing the response of the economy under three different policy regimes. The thick line corresponds to the case above where there is a trade off between issuing new debt and facing higher servicing costs in the future due to distortionary taxation. The thin line corresponds to the case where neither debt nor the tax rate are adjusted following the shock and use of lump sum taxes is made in order to satisfy the government budget constraint. The dotted line corresponds to the case where lump sum taxes can be used to finance any obligations arising from *new* public debt issuance. In this case, new debt issuance is not hindered by the prospect of higher tax rates and the government has no reason to hold back and not issue as much

Figure 9: Decomposing the Response to Adverse Financial Shock

Note: ——— Benchmark Experiment; ━ ━ ━ Constant Welfare Cost of Taxation;
━ ∘ ━ Constant Policy;
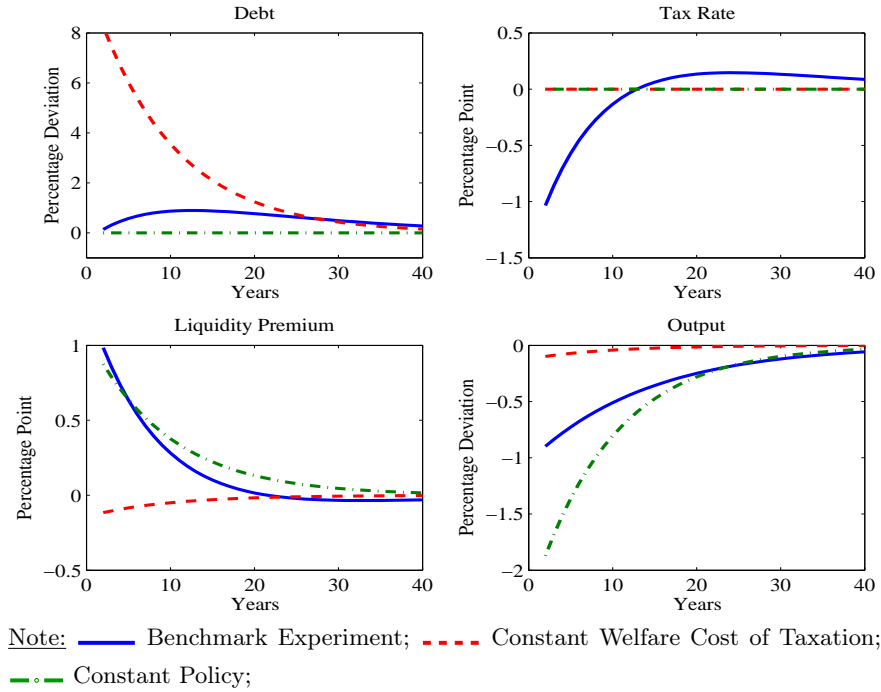
as it is needed in order to fully eliminate the effect of the financial shock on the liquidity premium (that is, keep the premium at its steady state level). As can be seen, the severity of the recession varies negatively with the level of public debt issued, with the most debt expansionary policy regime resulting in negligible output losses relative to the other regimes. Consequently, our model's implication regarding the relationship between public debt issuance and the level of economic activity agrees with that of the old Keynesian models, but the mechanism is different.

Remark. While the financial shock has been represented as an exogenous change in ξ, we expect alternative formalizations to deliver similar results. For example, consider Arellano, Bai and Kehoe (2012) and Di Tella (2012). In these papers, a financial crisis is modeled as the byproduct of an exogenous increase in the level of idiosyncratic risk rather than an exogenous tightening of the collateral constraint. We can mimic this approach in our setting by keeping ξ fixed and introducing instead a shock that increases the TFP difference between the two types of firms. In equilibrium, this manifests itself as an increase in the wedge between the internal and the external return of investment for the high-type firms. This kind of shock therefore triggers almost identical positive and normative responses as the ξ shock we have studied here.

Alternatively, one may extend the model so as to allow the private sector to generate some form of liquidity that is an imperfect substitute for public debt—think, e.g., of high-grade corporate bonds—and model a financial shock as an adverse shock in the liquidity-producing capacity of that sector. While we have not considered this possibility, we expect similar results as in the case of an

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exogenous shock to ξ. Such an extension also raises the following possibility. Stein (2012) argues that the planner may wish to regulate and/or crowd out such privately-produced liquidity because of fire-sale externalities. Our results indicate that the planner may sometimes wish to do so even in the absence of such externalities, in order to increase his monopoly power in the market for liquidity and thereby reduce the tax burden of servicing the public debt.

7   A model with frictions on the consumer side

The preceding analysis has focused on frictions on firms. We now show how our analysis extends directly to a variant model where the liquidity premium on public debt originates from frictions on the consumer side of the economy. This serves three purposes. First, it brings our framework closer to Bewley (1977), Woodford (1990), Aiyagari (1994), and Aiyagari and McGrattan (1998). Second, it connects our work to recent work that argues that the "great recession" is driven in part by tighter borrowing constraints depressing consumer demand; see, e.g., Mian and Sufi (2011a,b), Eggertsson and Krugman (2012), and Guerrieri and Lorenzoni (2012). Last but not least, it facilitates a more direct comparison of our results to the literature on the Friedman rule.

Model set-up.   The economy is populated by a large number of ex-ante identical households (consumers). Time is indexed by $t \in \mathbb{N}$. Each period t is split in two sub-periods, the "morning" and the "afternoon". There is no aggregate uncertainty, but households are subject to idiosyncratic income and/or taste shocks in the morning of each period.

   The household's preferences are given by

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_{it}, x_{it}, h_{it}; \zeta_{it}) \right] \tag{19}$$

where $\beta \in (0, 1)$ is the discount factor, $x_{it} \in \mathbb{R}_+$ and $c_{it} \in \mathbb{R}$ denote consumption in, respectively, the morning and the afternoon of period t, $h_{it} \in \mathbb{R}_+$ denotes labor supply, ζ_{it} is an idiosyncratic taste shock, and

$$U(c, x, h; \zeta) \equiv c + \zeta u(x) - v(h) \tag{20}$$

The household faces two budget constraints per period, one in the morning and one in the afternoon. The morning constraint is

$$x_{it} + p_t z_{it} = e_{it}$$

where z_{it} is the quantity of private loans bought in the morning of period t, p_t is their price, and e_{it} is an exogenous endowment shock. The afternoon constraint is

$$c_{it} + q_t b_{it+1} = (1 - \tau_t) w_t h_{it} + b_{it} + z_{it},$$

where b_{it+1} is the quantity of bonds purchased in period t, q_t is their price, w_t is the wage rate, and τ_t is the tax rate.[25]

The endowment and taste shocks are i.i.d. across agents and periods, and can take two possible values, $\zeta_{it} \in \{1, \theta\}$ and $e_{it} \in \{1, \delta\}$, where $\delta \leqslant 1$, $\theta \geqslant 1$, and $\delta \neq \theta$. These shocks are proxies for idiosyncratic variation in borrowing and saving attitudes: other things being equal, an agent who receives the high taste shock and/or the low endowment shock in the morning has a higher desire to borrow (or, a lower desire to save) than an agent who receives the opposite kind of shocks. To simplify the exposition, we assume that the two shocks are perfectly correlated with one another, so that there are only two types of consumers: "borrowers", who receive $(e_{it}, \zeta_{it}) = (\delta, \theta)$, and "savers", who receive $(e_{it}, \zeta_{it}) = (1, 1)$. The corresponding probabilities of these two types are π and $1 - \pi$, with $0 < \pi < 1$.

If the amounts of borrowing and saving were unrestricted, the economy could attain first-best outcomes (full insurance). We introduce financial frictions by assuming that the household faces a ceiling $\phi \geqslant 0$ on the amount of its future labor income that it can pledge as collateral. This implies the following borrowing constraints:

$$-z_{it} \leqslant b_{it} + \phi \qquad \text{and} \qquad -b_{it+1} \leqslant \phi. \tag{21}$$

Note that the first constraint indicates that households can borrow in the morning by using their bond holdings as collateral. Alternatively we could have assumed the households sell their bond holdings during the morning in order to finance early consumption. The two specifications are equivalent in our model, and we choose to talk of z_{it} as collateralized borrowing only for exposition purposes.

Finally, the government budget is given by

$$B_t + g_t = q_t B_{t+1} + \tau_t Y_t$$

where B_t is the level of debt inherited from period $t - 1$, g_t is the (exogenous) level of government spending, Y_t is aggregate income, and B_{t+1} is newly issued debt.[26]

Remark. Our modeling strategy follows once again Lagos and Wright (2005) and Guerrieri and Lorenzoni (2009) and avoids having the wealth distribution as a relevant state variable for the equilibrium dynamics. At the same time, the introduction of idiosyncratic income and/or taste shocks and the borrowing constraint embedded in (21) allows us to capture the essence of the role of public debt in incomplete-market models such as Bewley (1977), Woodford (1990), Aiyagari (1994), Aiyagari and McGrattan (1995), Challe and Ragot (2011), and Guerrieri and Lorenzoni (2012): As

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[25]Note that only income in the afternoon is assumed to be taxable; as long as ones limits the extent of lump-sum taxation, this assumption's sole purpose is to simplify the exposition.

[26]The assumption that the government operates (i.e., buys goods, collects taxes, and trades in financial markets) only in the afternoon is made only for expositional simplicity.

in these papers, issuing more public debt in our model helps consumers smooth consumption by permitting them, in effect, to borrow against their future labor income.

Equilibrium. Solving the morning budget constraint for z_{it} and substituting the result into the afternoon constraint, we obtain the following "integrated" budget constraint for period t:

$$\tilde{c}_{it} + q_t b_{it+1} = (1 - \tau_t)h_{it} + b_{it}, \tag{22}$$

where $\tilde{c}_{it} \equiv c_{it} + \frac{1}{p_t}(x_{it} - e_{it})$. Condition (22) is a familiar-looking budget constraint. The borrowing constraint can be similarly restated as

$$\frac{1}{p_t}(x_{it} - e_{it}) \leqslant b_{it} + \phi$$

which highlights that the amount of early consumption a household can enjoy is restricted by the bond holdings the household has accumulated by the beginning of the period.

The household's optimization problem can thus be summarized by the following Lagrangian:

$$\mathcal{L} \equiv \mathbb{E}_0 \sum_t \beta^t \left\{ [c_{it} + \zeta_{it} u(x_{it}) - v(h_{it})] + \mu_{it} \left[b_{it} + \phi - \frac{1}{p_t}(x_{it} - e_{it}) \right] \right.$$
$$\left. + \nu_{it} \left[b_{it} + (1 - \tau_t)h_{it} - c_{it} - \frac{1}{p_t}(x_{it} - e_{it}) - q_t b_{it+1} \right] \right\}$$

Here, ν_{it} identifies the marginal value of wealth (the Lagrange multiplier on the integrated budget constraint), while μ_{it} identifies the shadow value of liquidity (the multiplier on the morning borrowing constraint).

Due to the linearity of preferences, the first-order condition with respect to c_{it} gives $\nu_{it} = 1$, The first-order condition for b_{it+1} is then

$$q_t = \beta(1 + \mathbb{E}_{it}\mu_{it+1}), \tag{23}$$

which requires that the price of public debt be equated with the discount factor adjusted by the shadow value of the borrowing constraint next period. Note that this is formally the same pricing condition as in our baseline model. The nature of the financial friction is different—the borrowing constraint inhibits consumption smoothing rather than capital reallocation—but its implication for the pricing of public debt is the same.

The first-order conditions for x_{it} and h_{it} are:

$$p_t \zeta_{it} u'(x_{it}) = 1 + \mu_{it}$$

$$v'(h_{it}) = 1 - \tau_t$$

where $\mu_{it} \geqslant 0$, with $\mu_{it} = 0$ whenever the borrowing constraint does not bind. The first condition captures how the severity of the financial friction distorts consumption, whereas the second one

reveals that the friction does not affect labor-supply decisions.[27]

Note that, because of the linearity of preferences in afternoon consumption, we can easily guess and verify that the labor-supply and morning consumption choices of any given household in any given period are independent of the history of this household prior to that period. Furthermore, without any loss of generality, we impose $b_{it} = b_t$ for all i.[28]

Next, note that the morning borrowing constraint cannot bind for both types of households: if that were the case, the resource constraint of the economy during the morning would be violated. It is then straightforward to show that the constraint cannot bind for the first type of agents (the patient, high-income ones), while it may or may not bind for constraint for the second type of agents (the impatient, low-income ones). This is because agents of the first type necessarily save, while those of the second type necessarily borrow.

With these facts in mind, we henceforth refer to the first type of agents as "savers" and the latter as "borrowers". We accordingly index by "s" all the variables for the savers and by "b" those for the borrowers, and denote with μ_t the shadow cost of the borrowing constraint for the borrowers. We can then reach the following characterization of the equilibrium.

Lemma 5. *A sequence of tax and debt policies $\{\tau_t, b_t\}_{t=0}^{\infty}$, of bond prices $\{q_t\}_{t=0}^{\infty}$, and of allocations $\{x_t^s, x_t^b, c_t^s, c_t^b, h_t\}_{t=0}^{\infty}$ constitutes a competitive equilibrium if and only if there exists a non-negative sequence $\{\mu_t\}_{t=0}^{\infty}$ such that the following hold:*

$$\theta u'(x_t^b) = (1 + \mu_t)u'(x_t^s), \tag{24}$$

$$u'(x_t^s)(x_t^b - \delta) \leqslant \phi + b_t, \quad \mu_t \geqslant 0, \quad [u'(x_t^s)(x_t^b - \delta) - (\phi + b_t)]\mu_t = 0 \tag{25}$$

$$v'(h_t) = 1 - \tau_t, \qquad q_t = \beta(1 + \pi\mu_{t+1}), \tag{26}$$

$$\pi x_t^s + (1 - \pi)x_t^b = \bar{e}, \qquad \pi c_t^s + (1 - \pi)c_t^b + g_t = h_t, \qquad q_t b_{t+1} = b_t + g_t - \tau_t h_t \tag{27}$$

where $\bar{e} \equiv 1 - \pi + \pi\delta$.

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[27]This last property hinges on two simplifying assumptions that have been hard-wired in the model. Namely, that all of the labor income is received in the afternoon and that none of it is pledgeable in the morning. Were we to relax either of these assumptions then tighter borrowing constraints would boost labor supply. Note then that our baseline model contains the opposite effect on the demand for labor: There, tighter constraints, by distorting the allocation of capital, contributed to a reduction in the aggregate demand for labor and thereby in equilibrium employment. These observations extend to richer macroeconomic models with financial frictions: whether tighter credit contributes to higher or lower employment depends on whether the friction afflicts the firms or the consumers. In any event, while this issue may be essential for understanding the positive effects of financial frictions, it is not essential for our results.

[28]Similarly to our baseline model, the linearity of preferences implies that the dynamic pattern of bond holdings and afternoon consumption is indeterminate at the individual level: at the equilibrium interest rate, each agent is indifferent between consuming in the afternoon of one day or any other day. Restricting attention to symmetric equilibria where $b_{it} = b_t$ for all i is then without any loss of optimality.

Condition (24) identifies μ_t as the wedge between the marginal utility of borrowers and savers or, equivalently, as the multiplier on the collateral constraint of the borrowers. Condition (25) states this constraint and the complementary slackness condition between this constraint and its multiplier. Finally, condition (26) contains the optimality condition for labor and the Euler condition, while condition (27) contains the resource and budget constraints.

Note that, whenever the collateral constraint binds ($\mu_{t+1} > 0$), the price of public debt in the previous period, q_{t-1}, exceeds β by an amount proportional to μ_t. This implies a reduced cost of borrowing for the government. As in our baseline model, this reflects the interaction of public debt and financial frictions: Public debt commands a liquidity premium because it relaxes the bite of borrowing constraints.

The Ramsey problem. Let $x^{b\star}$ and $x^{s\star}$ be the first-best levels of morning consumption. These are given by the solution to

$$\theta u'(x^{b\star}) = u'(x^{s\star}) \qquad \text{and} \qquad \pi x^{b\star} + (1-\pi)x^{s\star} = \bar{e}$$

Next, let $V^\star \equiv \beta(\pi\theta u(x^{b\star}) + (1-\pi)u'(x^{s\star}))$ be the discounted expected utility of morning consumption at the first best. Finally, define $b^\star$ as the minimal amount of debt holdings that is needed in order for the first-best allocation not to violate the collateral constraint:

$$b^\star \equiv u'(x^{s\star})(x^{b\star} - \delta) - \phi$$

Now, pick an arbitrary period t and suppose that $b_{t+1} \geqslant b^\star$, meaning that the collateral constraint will be slack in the next period. It follows that $x_{t+1}^s = x^{s\star}$, $x_{t+1}^b = x^{b\star}$, $\mu_{t+1} = 0$, and $q_t = \beta$. If instead $b_{t+1} < b^\star$, the morning allocation in period $t+1$ is distorted away from the first best, and the distortion is higher the lower b_{t+1}. The associated premium μ_t and the resulting price of debt q_t can be expressed as increasing functions of b_{t+1}: a lower b_{t+1} implies a bigger gap between the marginal utilities of a borrower and a saver, and therefore a higher liquidity premium and a higher price of public debt. Based on these observations, we can derive the following result.

Lemma 6. *There exist continuous functions Q and V such that*

$$q_t = Q(b_{t+1}) \qquad and \qquad \beta\pi u(x_{t+1}^s) + \beta(1-\pi)\theta u(x_{t+1}^b)) = V(b_{t+1})$$

Furthermore, these functions satisfy $Q(b) = \beta$ and $V(b) = V^\star$ whenever $b \geqslant b^\star$, while $Q(b) > \beta$, $Q'(b) > 0$, $V(b) < V^\star$, and $V'(b) > 0$ whenever $b < b^\star$.

Let c_t denote aggregate consumption: $c_t \equiv \pi c_t^s + (1-\pi)c_t^b$. The resource constraint can then be restated as $c_t + g_t = h_t$. Now, take the government budget constraint and replace g_t from the resource constraint and τ_t from the optimality condition for labor. We then get

$$[c_t - v'(h_t)h_t] + q_t b_{t+1} = b_t,$$

which represents the implementability constraint in recursive form. Combining these facts with Lemma 6, we conclude that the Ramsey problem can be stated as follows.

Proposition 4. *The Ramsey plan solves*

$$\max \sum_{t=0}^{\infty} \beta^t \left[c_t - v(h_t) + V(b_{t+1}) \right] \tag{28}$$

$$s.t. \quad Q(b_{t+1})b_{t+1} = b_t + [v'(h_t)h_t - c_t] \tag{29}$$

$$c_t + g_t = h_t \tag{30}$$

The interpretation of Lemma 6 and Proposition 4 is simple. The function V identifies the indirect utility of holding public debt; the function Q captures the associated liquidity premium; the objective in (28) measures ex-ante welfare; (29) is the implementability constraint in recursive form; and (30) is the resource constraint.

The Ramsey problem of the present model is therefore formally similar to the Ramsey problem in models with money in the utility function. Alternatively, and complementarily, one can think of our "morning" and "afternoon" consumptions as, respectively, the "cash" and "credit" goods in models with a cash-in-advance constraint. Following this line of analogies, one can perhaps also think of our baseline model as a model with money in the production function, in the sense that public debt affects the allocation of capital and thereby aggregate output in our baseline model. This last analogy, however, is not exact, in part because the planner's problem in Proposition 3 has the price rather than the quantity of debt showing up in aggregate TFP. The variant of this section, by contrast, permits an exact isomorphism to the pertinent literature on the Friedman rule, thus helping sharpen the comparison between our results and those of that literature. We return to this issue below.

Results. Let us now concentrate on the Ramsey problem in Proposition 4. This problem can be solved similarly to that in our baseline model. See the appendix for details. In fact, the only formal difference is that we can now work directly with b_t as the relevant state variable, instead of the transformed variable z_t that we used in the baseline model.

The resulting policy rules are illustrated in Figure 10 for a particular parametrization.[29] As it is evident from this figure, the economy exhibits a unique and globally stable steady state. As in the baseline model, this steady state obtains at a lower level of debt than the threshold level, $\hat{b}$,

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[29]The discount factor is $\beta = 0.96$. We set the curvature parameter of the utility function to $\sigma = 2$ and the inverse Frisch elasticity is set to 1. The labor dis-utility parameter ϑ is set such that hours worked are 1 in an equilibrium, while the utility parameter accounting for the relative impatience of borrowers, θ, is set to 3. The borrowing constraint parameter is set to 0. We assume that there are as many patient as impatient agents ($\pi = 0.5$). The total endowment in the morning good is set to 1 and government expenditures are 20% of steady state GDP.

at which the demand for liquidity would have been satiated. In other words, the Friedman rule does not apply. Moreover, one can see, by comparing Figure 10 to Figure 1, that the shape of the policy rules and the associated transitional dynamics are nearly indistinguishable from those in our baseline model.

Figure 10: Decision Rules
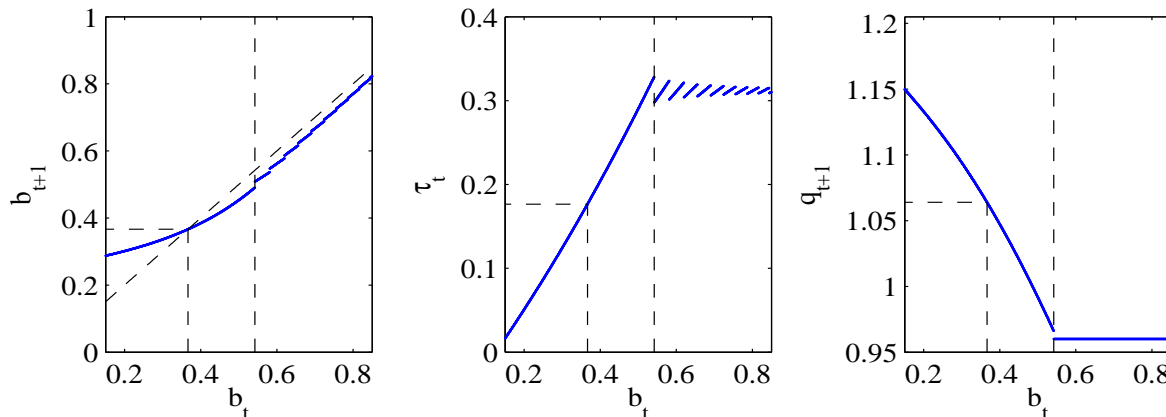
A similar pattern emerges when we consider the comparative statics of the steady state, or when we add shocks and consider the associated impulse responses. We conclude that the two models behave very similarly. And although both models are highly stylized, this near-equivalence suggests more generally that the insights offered in this paper need not be overly sensitive to the precise details of the specification of the micro-foundations of the liquidity demand for public debt. That is, whether the friction is on the firm side or the consumer side of the economy, the same qualitative properties of the policy responses obtain.

On the Friedman Rule We now return to the comparison of our results to those obtained in the literature on the Friedman rule. As noted earlier, a key result in this literature is that the Friedman rule is optimal in a large class of monetary economies, irrespectively of whether taxation is distortionary or not. In the models we have studied, by contrast, the analogue of the Friedman rule does not apply, despite the fact that one can reinterpret public debt as a form of money. The explanation behind this finding builds on two observations.

First and foremost, the pertinent literature on the Friedman rule assumes that only a certain subset of government liabilities can offer liquidity services: in that literature, money offers liquidity services, but government bonds do not. This gives the planner a profitable "arbitrage" opportunity: by substituting money for debt the planner can meet the economy's demand for liquidity without affecting his overall liabilities and, therefore, without interfering with tax smoothing. This in turn guarantees that, for the class of economies for which the Friedman rule holds, the planner finds it optimal to satiate the economy's demand for liquidity, not only in the long run, but also in every

period after the very first one. In particular, the planner floods the economy with lots of *nominal* money in the first period in order to collect a lot of seigniorage, but thereafter sets a zero nominal interest rate so as to satiate the economy's demand for *real* money balances. What is more, the first-period seigniorage is saved in the form of a lower—possibly negative—quantity of governments bonds in order to pay for lower taxes in the future, as well for the continuous money withdrawal (negative inflation) that is required in order to implement the Friedman rule.

In this paper, by contrast, we have effectively assumed that *all* government liabilities offer liquidity services. The aforementioned arbitrage-like argument is therefore no longer applicable. Instead, if the planner wishes to supply more liquidity to the economy, he can do so only by raising his overall debt obligation. By itself, this guarantees that, even if the planner finds it optimal to satiate the economy's demand for liquidity in the long-run, he may not do so in the short-run.[30]

Second, even in the pertinent literature, the validity of the Friedman rule ultimately hinges on special assumptions about the micro-foundations of the demand for money (final vs intermediate good etc) and on the menu of available tax instruments. The analogues to these assumptions are violated in our model, which help us understand why the analogue of the Friedman rule is violated in our model, not only in the short run, but also in the long run.

To elaborate on this point, consider the class of models with money in the utility function. Suppose that per-period utility is given by $U(c) + V(m)$, where c denotes consumption and m denotes real money balances, and let both U and V take a power (or isoelastic) form, with respective elasticities σ_c and σ_m. Then, it is well known (e.g., Chari, Christiano, and Kehoe, 1996) that the Friedman rule holds whenever $\sigma_c < \sigma_m$ but fails whenever $\sigma_c > \sigma_m$. This helps explain why the analogue of the Friedman rule does not apply in the Bewley-like variant we studied in the previous section: this model essentially has $\sigma_c = \infty > \sigma_m$.

Alternatively, consider the class of models in which money is an intermediate input. In this case, the optimality of the Friedman rule hinges on whether the government can appropriately tax firm profits. A similar property holds in our baseline model. Recall that our model can be roughly interpreted as a model in which the quantity of debt affects the aggregate production function.[31]

To summarize the preceding discussion, one can think of variants of our models in which the planner finds it optimal to satiate the economy's demand for public debt in the *long run*. It nevertheless remains an open question whether this possibility is empirically relevant, or whether

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[30]This observation is consistent with the results obtained when we studied the modification of our baseline model that blocked the effect of liquidity on interest rates. As shown in Figure 3, this modification forced the Friedman rule to hold in the long run (steady state) of our model. Yet, the Friedman rule failed to be satisfied along the transition path to the steady state, precisely because of the reasons offered above.

[31]Note that there are two possible definitions of profits in our model, depending on whether the cost of capital is evaluated at its market price (external return) or its shadow value (internal return). Allowing for the appropriation of all profits under the first definition, which seems the most natural in practice, does not restore the optimality of the Friedman rule. But it does under the second definition.

it applies to models with rich heterogeneity, such as Aiyagari (1994). In any event, for the reasons explained above, the theory appears to predict robustly that the planner does not find it optimal to satiate the economy's demand for public debt in the *short run*. Consequently, the qualitative properties of our results regarding the optimal policy response to fiscal and financial shocks apply independent of the debt satiation properties of the steady state.[32]

8 Conclusion

The great recession has revived interest in the macroeconomic implications of financial frictions and the potential role of fiscal policies in alleviating such frictions. In this paper we have explored the idea that the issuance of more public debt can help on this front by increasing the aggregate quantity of collateral or enhancing private liquidity.

To do so, we revisited the Ramsey policy paradigm in the presence of a financial friction on either the production or the consumption side of the economy. Our analysis produced three key results relative to the standard Ramsey paradigm. First, the steady-state level of debt is no longer indeterminate (in the sense of moving one-to-one with initial conditions); rather, it is pinned down by balancing the inefficiencies caused by the financial friction with the budgetary benefit of lower interest rates on public debt. Second, perfect tax smoothing is no longer optimal; rather, it is optimal to front-load the tax burden of fiscal shocks so as to induce the desirable movement in liquidity premia and interest rates. Third, optimal debt management helps reduce the overall tax burden of adverse fiscal shocks, as well as stabilize economic activity against financial shocks.

There are several possibilities regarding extensions of this work. First, the framework used here is highly stylized. Embedding our qualitative insights in richer incomplete-markets models, which allow for more realistic heterogeneity and more realistic dynamics, could yield useful quantification of the relevant trade offs.

Second, in this paper we have abstracted from the optimal maturity structure of public debt. Long-term debt appears to possess a smaller liquidity premium and a lower price than short-term debt, but it also provides the government with more insurance against short-term interest-rate fluctuations. Furthermore, at the same time that the financial crisis has raised the demand for US-issued debt and has kept the cost of US borrowing low, concerns are accumulating about the country's long-term fiscal prospects. Extensions of our model may provide a tractable framework for studying how these considerations affect the optimal size and maturity structure of public debt.

Finally, one could embed our analysis into a new-Keynesian model so as to study how optimal debt management (or, the optimal timing of transfers) can substitute for monetary policy when

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[32]To be precise, this presumes, of course, that the planner does not start in the short run with a level of debt already above the satiation level.

the zero-lower bound binds because of financial frictions. This would complement Krugman and Eggertsson (2012), Guerrieri and Lorenzoni (2012), and Buera and Nicolini (2012), who have argued that the existence of borrowing constraints is the key reason why the zero-lower bound became binding during the great recession, but have not studied the possibility that public debt or transfers might provide a more direct remedy to the underlying problem.[33]

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[33]This is related to Correia, Farhi, Nicolini, and Teles (2012), who have abstracted from borrowing constraints but have sought to identify fiscal policies that can substitute for monetary policy when the zero-lower bound binds.

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Appendix A: The Baseline Model

This appendix gives a detailed derivation of the equilibrium and the planner's problem for our baseline model.

Equilibrium characterization (Lemma 1 and Proposition 1). From the fact that both types of firms equate the marginal product of labor with the wage rate and the market-clearing condition for labor, we can express the equilibrium labor inputs as follows:

$$n_t^L = \frac{(k_t^L)^{\frac{\alpha}{1-\theta}}}{u_t} h_t \qquad \text{and} \qquad n_t^H = \frac{A^{\frac{1}{1-\theta}}(k_t^H)^{\frac{\alpha}{1-\theta}}}{u_t} h_t$$

where

$$u_t \equiv \pi A^{\frac{1}{1-\theta}}(k_t^H)^{\frac{\alpha}{1-\theta}} + (1-\pi)(k_t^L)^{\frac{\alpha}{1-\theta}}.$$

Letting R_t^H and R_t^L denote the marginal products of capital (aka internal returns) in the two types of firms and $Y_t = \pi y_t^H + (1-\pi)y_t^L$ the aggregate output, we obtain

$$R_t^H = \alpha A^{\frac{1}{1-\theta}}(k_t^H)^{\frac{\alpha}{1-\theta}-1}\left(\frac{h_t}{u_t}\right)^{\theta} \tag{31}$$

$$R_t^L = \alpha(k_t^L)^{\frac{\alpha}{1-\theta}-1}\left(\frac{h_t}{u_t}\right)^{\theta} \tag{32}$$

$$w_t = \theta\left(\frac{u_t}{h_t}\right)^{1-\theta} \tag{33}$$

$$Y_t = u_t^{1-\theta}h_t^{\theta} \tag{34}$$

We thus obtain aggregate TFP as

$$TFP_t \equiv \frac{Y_t}{\kappa_t^{\alpha}h_t^{\theta}} = u_t^{1-\theta}\kappa_t^{-\alpha} = \left\{\pi A^{\frac{1}{1-\theta}}x_t^{\frac{\alpha}{1-\theta}} + (1-\pi)\left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}}\right\}^{1-\theta} = \Gamma(x_t)$$

where the function Γ the same as in (7),

$$x_t \equiv k_t^H/\kappa_t, \quad k_t^L/\kappa_t = \frac{1-\pi x_t}{1-\pi}, \quad \text{and} \quad \kappa_t = \pi k_t^H + (1-\pi)k_t^L$$

Note that

$$\Gamma'(x_t) = \alpha\left\{\pi A^{\frac{1}{1-\theta}}x_t^{\frac{\alpha}{1-\theta}-1} - \pi\left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}-1}\right\}\Gamma(x_t)^{-\frac{\theta}{1-\theta}}$$

Aggregate output and the marginal product of capital (the internal return) in the two types of firms are the given as follows:

$$y_t = u_t^{1-\theta}h_t^{\theta} = \Gamma(x_t)\kappa_t^{\alpha}h_t^{\theta}$$

$$w_t = \theta\Gamma(x_t)\kappa_t^{\alpha}h_t^{\theta-1}$$

$$R_t^H = \alpha A^{\frac{1}{1-\theta}}x_t^{\frac{\alpha}{1-\theta}-1}\Gamma(x_t)^{-\frac{\theta}{1-\theta}}\kappa_t^{\alpha-1}h_t^{\theta}$$

$$R_t^L = \alpha\left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}-1}\Gamma(x_t)^{-\frac{\theta}{1-\theta}}\kappa_t^{\alpha-1}h_t^{\theta}$$

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By implication, the wedge between the internal returns of the two types of firms is given by

$$R_t^H - R_t^L = \frac{1}{\pi} \Gamma'(x_t) \kappa_t^{\alpha-1} h_t^\theta.$$

For a given output-to-capital ratio, the wedge is therefore a one-to-one transformation of the TFP distortion, as measured by the derivative Γ evaluated at the underlying allocation of capital: the bigger the gap between x_t and x^*, the bigger the aforementioned wedge and the associated TFP distortion.

We now proceed to express the equilibrium allocation as a function of the price of debt. Using the fact that $(p_t - \xi)\mu_t = R_t^H - R_t^L$, we get

$$(p_t - \xi)\mu_t = \frac{1}{\pi} \Gamma'(x_t) \kappa_t^{\alpha-1} h_t^\theta = \alpha \left\{ A^{\frac{1}{1-\theta}} x_t^{\frac{\alpha}{1-\theta}-1} - \left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}-1} \right\} \Gamma(x_t)^{-\frac{\theta}{1-\theta}} \kappa_t^{\alpha-1} h_t^\theta$$

Solving $\beta(1 + \pi\mu_t) = q_{t-1}$ for μ_t and substituting the solution into $(p_t - \xi)\mu_t$ along with the fact that $p_t = 1/q_{t-1}$, we get

$$(p_t - \xi)\mu_t = \frac{(1 - \xi q_{t-1})(q_{t-1} - \beta)}{\beta\pi q_{t-1}}.$$

Combining the last two equations we infer

$$\frac{(1 - \xi q_{t-1})(q_{t-1} - \beta)}{\beta\pi q_{t-1}} = \alpha \left\{ A^{\frac{1}{1-\theta}} x_t^{\frac{\alpha}{1-\theta}-1} - \left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}-1} \right\} \Gamma(x_t)^{-\frac{\theta}{1-\theta}} \kappa_t^{\alpha-1} h_t^\theta.$$

At the same time, using the facts that $p_t = 1/q_{t-1}$ and $p_t = 1 - \delta + R_t^L$, we get

$$1 - (1-\delta)q_{t-1} = R_t^L q_{t-1} = q_{t-1}\alpha \left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}-1} \Gamma(x_t)^{-\frac{\theta}{1-\theta}} \kappa_t^{\alpha-1} h_t^\theta.$$

Combing the last two equations we conclude

$$\frac{A^{\frac{1}{1-\theta}} x_t^{\frac{\alpha}{1-\theta}-1} - \left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}-1}}{\left(\frac{1-\pi x_t}{1-\pi}\right)^{\frac{\alpha}{1-\theta}-1}} = \frac{(1 - \xi q_{t-1})(q_{t-1} - \beta)}{\beta\pi(1 - (1-\delta)q_{t-1})}$$

or equivalently

$$\frac{(1-\pi)x_t}{1-\pi x_t} = \Phi(q_{t-1}) \equiv \left(\frac{\beta\pi A^{\frac{1}{1-\theta}}(1 - (1-\delta)q_{t-1})}{\beta\pi(1 - (1-\delta)q_{t-1}) + (q_{t-1} - \beta)(1 - \xi q_{t-1})}\right)^{\frac{1-\theta}{1-\alpha-\theta}}$$

Combining with the fact that $\pi x_t + (1-\pi)\left(\frac{1-\pi x_t}{1-\pi}\right) = 1$, we conclude that

$$x_t = X(q_{t-1}) \equiv \frac{\Phi(q_{t-1})}{1 - \pi + \pi\Phi(q_{t-1})}$$

The above is defined as long as $q_{t-1} \geqslant \beta$. Furthermore, $\Phi(\beta) = A^{\frac{1}{1-\alpha-\theta}}$ and

$$X(\beta) = x^\star = \arg\max_x \Gamma(x)$$

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That is, $X(\beta)$ gives the first-best allocation of capital. For all $q > \beta$, $X(q) < x^\star$.

Given the above result, we have that

$$y_t = \Psi(q_{t-1})\kappa_t^\alpha h_t^\theta$$

where $\Psi(q) = \Gamma(X(q))$. This completes the proof of Lemma 1.

Similarly, from $(p_t - \xi)\mu_t = \frac{1}{\pi}\Gamma'(X(q_{t-1}))\kappa_t^{\alpha-1}h_t^\theta$, we get

$$\mu_t = M(q_{t-1})\kappa_t^{\alpha-1}h_t^\theta$$

where

$$M(q) \equiv \frac{q\Gamma'(X(q))}{\pi(1-\xi q)}$$

Proposition 1 then follows from the above results and the analysis in the main text. ∎

The welfare objective. The welfare function takes the form

$$W_0 = \sum_{t=0}^{\infty}\beta^t\left\{c_t - \vartheta\frac{h_t^{1+\nu}}{1+\nu}\right\}$$

Using the aggregate resource constraint, we obtain

$$W_t = \sum_{t=0}^{\infty}\beta^t\left\{y_t - g_t - \kappa_{t+1} + (1-\delta)\kappa_t - \vartheta\frac{h_t^{1+\nu}}{1+\nu}\right\}$$

$$= \left\{y_0 + (1-\delta)\kappa_0 - g_0 - \vartheta\frac{h_0^{1+\nu}}{1+\nu}\right\} + \sum_{t=1}^{\infty}\beta^t\left\{y_t - g_t - \frac{1-\beta(1-\delta)}{\beta}\kappa_t - \vartheta\frac{h_t^{1+\nu}}{1+\nu}\right\}$$

$$= \left\{y_0 + (1-\delta)\kappa_0 - g_0 - \vartheta\frac{h_0^{1+\nu}}{1+\nu}\right\} + \beta\Omega_1$$

where Ω_1 is given recursively by the following:

$$\Omega_t = \left\{y_t - g_t - \frac{1-\beta(1-\delta)}{\beta}\kappa_t - \vartheta\frac{h_t^{1+\nu}}{1+\nu}\right\} + \beta\Omega_{t+1} \qquad (35)$$

subject to the the equilibrium conditions we derived in the previous section. Our characterization in the main text focuses on the maximization of Ω_1, putting aside the (trivial) period-0 allocation.

The Ramsey problem without frictions (Lemma 2 and Proposition 2). In the absence of a financial friction, we have $q_t = \beta$ and $\Gamma(x_t) = \Gamma^\star$. Furthermore, the optimal savings are given by the level of κ_t that maximizes

$$\Gamma^\star\kappa_t^\alpha h_t^\theta - \frac{1-\beta(1-\delta)}{\beta}\kappa_t.$$

It follows that $\kappa_t = \chi h_t^{\frac{\theta}{1-\alpha}}$, where $\chi \equiv \left(\frac{\alpha\beta\Gamma^\star}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}$.

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Defining the transformed variables $\widetilde{c}_t \equiv c_t + k_{t+1} - \frac{1}{\beta}k_t$ and $\widetilde{y}_t \equiv y_t + (1-\delta)k_t - \frac{1}{\beta}k_t$, the resource constraint reduces to

$$\widetilde{c}_t + g_t = \widetilde{y}_t$$

At the optimal capital decision, $\widetilde{y}_t$ is given by

$$\widetilde{y}_t = \Gamma^\star \chi^\alpha h_t^{\frac{\theta}{1-\alpha}} - \frac{1-\beta(1-\delta)}{\beta}\chi h_t^{\frac{\theta}{1-\alpha}} = f(h_t)$$

where $f(h) \equiv \Delta h^\eta$, $\Delta \equiv \Gamma^\star \chi^\alpha - \frac{1-\beta(1-\delta)}{\beta}\chi = (1-\alpha)\frac{1-\beta(1-\delta)}{\alpha\beta}\chi$, and $\eta \equiv \frac{\theta}{1-\alpha}$.

Finally, since $q_t = \beta$, the intertemporal government budget constraint writes

$$b_0 = \sum_{t=0}^\infty \beta^t (\tau_t w_t h_t - g_t)$$

Using the labor supply condition $v'(h_t) = (1-\tau_t)w_t$ and, from eq. (35), the labor demand $w_t = \theta y_t/h_t$, this can be rewritten as

$$b_0 = \sum_{t=0}^\infty \beta^t (\theta y_t - v'(h_t)h_t - g_t)$$

Note $y_t = \frac{\chi^\alpha \Gamma^\star}{\Delta}\widetilde{y}_t$ and $\frac{\chi^\alpha \Gamma^\star}{\Delta} = \frac{1}{1-\alpha}$, so that $\theta y_t = \eta \widetilde{y}_t$, while by the resource constraint $\widetilde{y}_t = \widetilde{c}_t + g_t$. Using these facts, the implementability constraint can finally be stated as

$$b_0 = \sum_{t=0}^\infty \beta^t (\eta \widetilde{c}_t - v'(h_t)h_t - (1-\eta)g_t)$$

where $\eta \equiv \frac{\theta}{1-\alpha}$. The Ramsey problem therefore reduces to the following:

$$\max \sum_{t=0}^\infty \beta^t (\widetilde{c}_t - v(h_t))$$

$$s.t. \quad b_0 = \sum_{t=0}^\infty \beta^t (\eta \widetilde{c}_t - v'(h_t)h_t - (1-\eta)g_t)$$

$$\widetilde{c}_t + g_t = f(h_t)$$

where $v(h_t) \equiv \vartheta \frac{h_t^{1+\nu}}{1+\nu}$ in our benchmark case. This completes the proof of Lemma 2.

The set of first order conditions is then given by

$$\theta_t = 1 + \eta\lambda$$

$$\theta_t f'(h_t) = v'(h_t) + \lambda[v'(h_t) + h_t v''(h_t)]$$

Therefore, $h_t = h_0$ solves

$$(1 + \eta\lambda)f'(h_t) = v'(h_t) + \lambda[v'(h_t) + h_t v''(h_t)]$$

Therefore, $\tau_t = \tau_0 = 1 = \frac{v'(h_0)}{f'(h_0)}$, $y_t = y_0 = f(h_0)$, $k_t = k_0 = \chi h_0^\zeta$ and $b_t = b_0$. This establishes Proposition 2. ∎

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The Ramsey problem with frictions (Lemmas 3-4 and Proposition 3). We now explain the details behind Proposition 3. First, we characterize the subset of implementable allocations for which the borrowing constraint does not bind; we call this the "unconstrained regime". Next, we characterize the complementary subset in which the constraint binds; we call this the "constrained regime". Finally, we complete the transformation of the overall Ramsey problem in terms of the low-dimension program in Proposition 3.

The unconstrained regime. In the unconstrained regime, $\mu_t = 0$, $q_{t-1} = \beta$ and $p_t = \frac{1}{\beta}$. This implies that $R_t^H = R_t^L$. The allocation of both capital and labor is therefore first-best efficient, and is given by

$$n_t^H = \frac{A^{\frac{1}{1-\alpha-\theta}}}{1 + \pi A^{\frac{1}{1-\alpha-\theta}}} h_t \qquad n_t^L = \frac{1}{(1-\pi)(1 + \pi A^{\frac{1}{1-\alpha-\theta}})} h_t$$

$$k_t^H = \frac{A^{\frac{1}{1-\alpha-\theta}}}{1 + \pi A^{\frac{1}{1-\alpha-\theta}}} \kappa_t \qquad k_t^L = \frac{1}{(1-\pi)(1 + \pi A^{\frac{1}{1-\alpha-\theta}})} \kappa_t$$

Plugging these results in u_t, y_t, w_t and $R_t = R_t^H = R_t^L$ we obtain

$$u_t = (1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{1-\frac{\alpha}{1-\theta}} \kappa_t^{\frac{\alpha}{1-\theta}}$$

$$y_t = (1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{1-\alpha-\theta} \kappa_t^\alpha h_t^\theta$$

$$w_t = \theta(1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{1-\alpha-\theta} \kappa_t^\alpha h_t^{\theta-1}$$

$$R_t = \alpha(1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{1-\alpha-\theta} \kappa_t^{\alpha-1} h_t^\theta$$

Aggregate labor and capital are then given by the solution to

$$\frac{1}{\beta} = \alpha(1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{1-\alpha-\theta} \kappa_t^{\alpha-1} h_t^\theta + 1 - \delta$$

$$\vartheta h_t^\nu = (1 - \tau_t)\theta(1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{1-\alpha-\theta} \kappa_t^\alpha h_t^{\theta-1}$$

which leads to

$$\kappa_t = \Gamma_k (1 - \tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$$

$$h_t = \Gamma_h (1 - \tau_t)^{\frac{1-\alpha}{(1-\alpha)(1+\nu)-\theta}}$$

where

$$\Gamma_k = \left(\frac{\alpha\beta}{1 - \beta(1-\delta)}\right)^{\frac{1+\nu-\theta}{(1-\alpha)(1+\nu)-\theta}} (1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{\frac{(1+\nu)(1-\alpha-\theta)}{(1-\alpha)(1+\nu)-\theta}} \left(\frac{\theta}{\vartheta}\right)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$$

$$\Gamma_h = \left(\frac{\alpha\beta}{1 - \beta(1-\delta)}\right)^{\frac{\alpha}{(1-\alpha)(1+\nu)-\theta}} (1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{\frac{1-\alpha-\theta}{(1-\alpha)(1+\nu)-\theta}} \left(\frac{\theta}{\vartheta}\right)^{\frac{1-\alpha}{(1-\alpha)(1+\nu)-\theta}}$$

From this solution, all variables entering the utility can be obtained as functions of the tax rate alone—the level of debt per se is not affecting welfare because the collateral constraint is not binding.

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Furthermore, tax revenue can be obtained as

$$\tau_t w_t h_t = \Gamma_{wh} \tau_t (1 - \tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$$

where

$$\Gamma_{wh} = \theta \left(\frac{\alpha\beta}{1-\beta(1-\delta)} \right)^{\frac{\alpha(1+\nu)}{(1-\alpha)(1+\nu)-\theta}} (1 + \pi A^{\frac{1}{1-\alpha-\theta}})^{\frac{(1+\nu)(1-\alpha-\theta)}{(1-\alpha)(1+\nu)-\theta}} \left(\frac{\theta}{\vartheta} \right)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$$

Hence, with the transformed state variable defined as $z_t \equiv b_t - g_t - \tau_t w_t h_t$, we get

$$b_t = z_t - g_t + \Gamma_{wh} \tau_t (1 - \tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$$

The threshold debt level that makes the borrowing constraint bind is given by solving

$$k_t^H = \frac{\kappa_t - \beta b_t}{1 - \xi\beta}$$

with respect to b_t. This gives the threshold level as

$$b_t^\star = \left(\frac{1-\xi\beta}{\beta} \frac{A^{\frac{1}{1-\alpha-\theta}}}{1 + \pi A^{\frac{1}{1-\alpha-\theta}}} - \frac{1}{\beta} \right) \Gamma_k (1-\tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$$

The unconstrained case then applies if and only if $b_t \geqslant b_t^\star$, or equivalently $z_t \geqslant z_t^\star$, with

$$z_t^\star \equiv g_t + \left[\left(\frac{(1-\xi\beta)A^{\frac{1}{1-\alpha-\theta}}}{\beta(1 + \pi A^{\frac{1}{1-\alpha-\theta}})} - \frac{1}{\beta} \right) \Gamma_k - \Gamma_{wh}\tau_t \right] (1-\tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$$

Note that since $\nu \geqslant 0$ and $0 < \alpha + \theta < 1$ we have $(1-\alpha)(1+\nu) - \theta >$. Therefore, $z_t^\star$ is a decreasing function of τ_t. The intuition is that, when the tax rate is higher, the supply of labor is lower, implying that the marginal product of capital and the efficient capital input are also lower; but then the level of debt needed in order for the constraint not to bind is also lower.

Finally, the government budget constraint can be rewritten in terms of z as

$$q_{t-1} \left(z_t - g_t + \Gamma_{wh}\tau_t(1-\tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}} \right) = z_{t-1}$$

Given that $q_t = \beta$ in the unconstrained case, the above can be restated as

$$\mathcal{T}(\tau_t) = \frac{z_{t-1} - \beta z_t + \beta g_t}{\beta \Gamma_{wh}}$$

where $\mathcal{T}(\tau_t) \equiv \tau_t(1-\tau_t)^{\frac{\theta}{(1-\alpha)(1+\nu)-\theta}}$ identifies the Laffer curve, that is, the equilibrium tax revenue as a function of the tax rate. Since $(1-\alpha)(1+\nu) - \theta > 0$, we have $\mathcal{T}'(\tau_t) > 0$ if and only if $\tau_t < \tau^\star \equiv 1 - \frac{\theta}{(1-\alpha)(1+\nu)}$. The set of solutions for the preceding equation is therefore non-empty if and only

$$z_t \geqslant z_{t-1} + \beta g_t - \beta \Gamma_{wh} \mathcal{T}(\tau^\star)$$

For any value below this threshold, the equation admits no real solution. Above it, the equation admits two solutions, but the relevant one is the one that lies on the upward sloping part of the Laffer curve, that is, the unique solution that is less than $\tau^\star$. By using this solution in all other quantities, we express all quantities in the unconstrained case as a function of (z_{t-1}, z_t).

The Constrained Regime. In the constrained regime, $\mu_t > 0$ and the borrowing constraint binds. Using the equation for the wage and the labor supply decision, one obtains

$$h_t = \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{1}{1+\nu-\theta}} u_t^{\frac{1-\theta}{1+\nu-\theta}}$$

Plugging this result in the wage , output, tax revenues and interest rates, we obtain

$$w_t = \theta \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta-1}{1+\nu-\theta}} u_t^{\frac{\nu(1-\theta)}{1+\nu-\theta}}$$

$$y_t = \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{\frac{(1+\nu)(1-\theta)}{1+\nu-\theta}}$$

$$\tau_t w_t h_t = \theta\tau_t \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{\frac{(1+\nu)(1-\theta)}{1+\nu-\theta}}$$

$$R_t^H = \alpha A^{\frac{1}{1-\theta}} k_t^{H\frac{\alpha}{1-\theta}-1} \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{-\frac{\nu\theta}{1+\nu-\theta}}$$

$$R_t^L = \alpha k_t^{L\frac{\alpha}{1-\theta}-1} \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{-\frac{\nu\theta}{1+\nu-\theta}}$$

Then, from the definition of the Lagrange multiplier μ_t and the price of borrowings, p_t, we get

$$\mu_t = \frac{R_t^H - R_t^L}{p_t - \xi} = \frac{\alpha q_{t-1}}{1 - \xi q_{t-1}} \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{-\frac{\nu\theta}{1+\nu-\theta}} \left(A^{\frac{1}{1-\theta}} k_t^{H\frac{\alpha}{1-\theta}-1} - k_t^{L\frac{\alpha}{1-\theta}-1}\right)$$

and

$$p_t = R_t^L + 1 - \delta = \alpha k_t^{L\frac{\alpha}{1-\theta}-1} \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{-\frac{\nu\theta}{1+\nu-\theta}} + 1 - \delta$$

Using these last 2 equations in

$$q_{t-1} = \beta(1 + \pi\mu_t)$$

$$q_{t-1}p_t = 1$$

we obtain the following system of equations:

$$\alpha q_{t-1} \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{-\frac{\nu\theta}{1+\nu-\theta}} \left(A^{\frac{1}{1-\theta}} k_t^{H\frac{\alpha}{1-\theta}-1} - k_t^{L\frac{\alpha}{1-\theta}-1}\right) = \frac{(q_{t-1} - \beta)(1 - \xi q_{t-1})}{\beta\pi} \tag{36}$$

$$\alpha q_{t-1} \left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}} u_t^{-\frac{\nu\theta}{1+\nu-\theta}} k_t^{L\frac{\alpha}{1-\theta}-1} = 1 - (1 - \delta)q_{t-1} \tag{37}$$

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Taking the ratio of these two equations, we obtain

$$k_t^H = \left(\frac{\beta \pi A^{\frac{1}{1-\theta}} (1 - (1-\delta)q_{t-1})}{\beta \pi (1 - (1-\delta)q_{t-1}) + (q_{t-1} - \beta)(1 - \xi q_{t-1})} \right)^{\frac{1-\theta}{1-\alpha-\theta}} k_t^L = \Phi(q_{t-1}) k_t^L$$

Using it in the definition of u_t, we get

$$u_t = (1 + \pi A^{\frac{1}{1-\theta}} \Phi(q_{t-1})^{\frac{\alpha}{1-\theta}}) k_t^{L \frac{\alpha}{1-\theta}}$$

Plugging this result in (37), we get

$$\alpha q_{t-1} \left(\frac{\theta(1-\tau_t)}{\vartheta} \right)^{\frac{\theta}{1+\nu-\theta}} (1 + \pi A^{\frac{1}{1-\theta}} \Phi(q_{t-1})^{\frac{\alpha}{1-\theta}})^{-\frac{\nu\theta}{1+\nu-\theta}} k_t^{L -\frac{(1-\alpha)(1+\nu)-\theta}{1+\nu-\theta}} = 1 - (1-\delta)q_{t-1}$$

such that

$$k_t^L = \left(\frac{\alpha q_{t-1}}{1 - (1-\delta)q_{t-1}} \right)^{\frac{1+\nu-\theta}{(1+\nu)(1-\alpha)-\theta}} \left(\frac{\theta(1-\tau_t)}{\vartheta} \right)^{\frac{\theta}{(1+\nu)(1-\alpha)-\theta}} (1 + \pi A^{\frac{1}{1-\theta}} \Phi(q_{t-1})^{\frac{\alpha}{1-\theta}})^{-\frac{\nu\theta}{(1+\nu)(1-\alpha)-\theta}}$$

or identically

$$k_t^L = \Psi_s(q_{t-1})$$

It then follows that

$$k_t^H = \Phi(q_{t-1}) \Psi_s(q_{t-1}) = \Psi_e(q_{t-1})$$

and

$$u_t = (1 + \pi A^{\frac{1}{1-\theta}} \Phi(q_{t-1})^{\frac{\alpha}{1-\theta}}) \Psi_s(q_{t-1})^{\frac{\alpha}{1-\theta}} = \Psi_x(q_{t-1})$$

Aggregate capital is then given by

$$\kappa_t = \pi \Psi_e(q_{t-1}) + (1-\pi) \Psi_s(q_{t-1}) = \Psi_k(q_{t-1})$$

while tax revenues take the form

$$\tau_t w_t h_t = \theta \tau_t \left(\frac{\theta(1-\tau_t)}{\vartheta} \right)^{\frac{\theta}{1+\nu-\theta}} \Psi_x(q_{t-1})^{\frac{(1+\nu)(1-\theta)}{1+\nu-\theta}}$$

such that

$$b_t = z_t - g_t + \theta \tau_t \left(\frac{\theta(1-\tau_t)}{\vartheta} \right)^{\frac{\theta}{1+\nu-\theta}} \Psi_x(q_{t-1})^{\frac{(1+\nu)(1-\theta)}{1+\nu-\theta}}$$

Plugging these results in the borrowing constraint, we obtain an equation for q_{t-1} as a function of z_t and τ_t

$$\Psi_e(q_{t-1})(1 - \xi q_{t-1}) = \Psi_k(q_{t-1}) + q_{t-1} \left(z_t - g_t + \theta \tau_t \left(\frac{\theta(1-\tau_t)}{\vartheta} \right)^{\frac{\theta}{1+\nu-\theta}} \Psi_x(q_{t-1})^{\frac{(1+\nu)(1-\theta)}{1+\nu-\theta}} \right)$$

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Once we have $q_{t-1} = \mathcal{Q}(z_t, \tau_t)$, we have all the other quantities of the model in the constrained regime. As in the unconstrained case, we can express the constrained regime in terms of (z_{t-1}, z_t) by using the additional equation

$$q_{t-1}\left(z_t - g_t + \theta\tau_t\left(\frac{\theta(1-\tau_t)}{\vartheta}\right)^{\frac{\theta}{1+\nu-\theta}}\Psi_x(q_{t-1})^{\frac{(1+\nu)(1-\theta)}{1+\nu-\theta}}\right) = z_{t-1}$$

All variables entering the utility function are therefore expressed as a function of the tax rate and z_t. This completes the proofs of Lemmas 3-4, and Proposition 3. ∎

Algorithm. The algorithm we use to numerically solve the Ramsey problem in Proposition 3 is based on a standard value-function iteration.

First, we setup a grid of 5000 nodes for the pairs (τ_t, z_t) and obtain all quantities of the model for each pair in the grid. We therefore obtain the instantaneous utility as a function of (τ_t, z_t). Next, we setup a grid for z_{t-1}, and for each point in this grid and solve

$$\mathcal{Q}(z_t, \tau_t)\mathcal{B}(z_t, \tau_t) = z_{t-1}$$

for τ_t as a function of the pair (z_{t-1}, z_t). (At this point, we make sure that the so-obtainted τ_t lies on the upward sloping part of the Laffer curve.) Using this result, we can express the utility as a function U of the pairs (z_{t-1}, z_t). We interpolate using splines. We finally numerically solve for the value function by iterating on the following contraction mapping:

$$\mathcal{T}V(z) = \max_{z'}\{U(z, z') + \beta V(z')\}$$

Appendix B: The Bewley-like Variant

This appendix considers the variant studied in Section 6. Lemma 5 follows from the discussion in the main text. Here we provide a detailed derivation of the results in Lemma 6. Proposition 4 then follows directly from Lemma 6.

As explained in the main text (see Lemma 5), an equilibrium of the Bewley-like variant of the model satisfies equations (24)–(27). From (25) and (27), we have

$$u'(x_{t+1}^s)\left(\frac{\overline{e} - \pi x_{t+1}^s}{1 - \pi}\right) = \phi + b_{t+1}$$

from which we obtain

$$x_{t+1}^s = \Gamma^s(b_{t+1})$$

using (27) we obtain

$$x_{t+1}^b = \frac{\overline{e} - \pi\Gamma^s(b_{t+1})}{1 - \pi} = \Gamma^b(b_{t+1})$$

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Using (24) and (26), we then get

$$q_t = \beta \left(1 - \pi + \pi \frac{\theta u'(\Gamma^b(b_{t+1}))}{u'(\Gamma^s(b_{t+1}))} \right) = Q(b_{t+1})$$

We then obtain immediately

$$\beta \pi u(x_{t+1}^s) + \beta(1-\pi)u(x_{t+1}^b) = \beta \pi u(\Gamma^s(b_{t+1})) + \beta(1-\pi)u(\Gamma^b(b_{t+1})) = V(b_{t+1})$$

Note that when the financial constraint does not bite, we have $\mu_{t+1} = 0$, such that, as can be seen directly from (26), $q_t = \beta$. Furthermore, (24) reduces to

$$u'(x_{t+1}^s) = \theta u'(x_{t+1}^b) \iff x_{t+1}^s = \widetilde{\theta} x_{t+1}^b$$

where $\widetilde{\theta} = u'^{-1}(\theta)$. Plugging it in the morning resource constraint (27), we immediately get

$$x_{t+1}^b = x^{b\star} = \frac{\overline{e}}{\pi \widetilde{\theta} + 1 - \pi}$$

$$x_{t+1}^s = x^{s\star} = \frac{\overline{e} \widetilde{\theta}}{\pi \widetilde{\theta} + 1 - \pi}$$

such that $V(b_{t+1}) = V^\star$ with

$$V^\star = \beta \pi u \left(\frac{\overline{e}}{\pi \widetilde{\theta} + 1 - \pi} \right) + \beta(1-\pi)u \left(\frac{\overline{e} \widetilde{\theta}}{\pi \widetilde{\theta} + 1 - \pi} \right)$$

such a situation emerges when

$$b_{t+1} > b^\star$$

where $b^\star = u'(x^{s\star})(x^{b\star} - \delta) - \phi$ is the level of debt below which the financial constraint binds. This completes the proof of Lemma 6. Proposition 4 is then immediate.

Finally, the algorithm for the numerical solution of the planner's problem is similar to that in the baseline model.