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EXCHANGE RATE DETERMINATION, RISK SHARING AND THE ASSET MARKET
VIEW

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ABSTRACT

Recent research in international finance suggests that changes in real exchange rates can be understood and interpreted using only asset returns and agents' intertemporal marginal rates of substitution. This asset market view of exchange rates has been used to gain insights into exchange rate determination, foreign exchange risk premia, and international risk sharing. We show that asset markets alone are not sufficient to understand how real exchange rates are determined, nor are they sufficient to economically interpret time-series variation in real exchange rates. Instead, we argue that it is necessary to make specific assumptions about preferences, frictions in the market for goods and services, the nature of endowments or production, and the assets that agents can trade.

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To fix ideas, we briefly formalize the asset market view of the real exchange rate between two agents. Let Amy and Bob be the representative agents in the United States and the United Kingdom. The real U.S./U.K. exchange rate is defined as the value of a unit of Bob's consumption basket of goods and services, at prices in the U.K., relative to a unit of Amy's consumption basket, valued at prices in the U.S., where both values are expressed in common units. Let e denote the real U.S./U.K. exchange rate today and let e' denote its value next period. Let m be Amy's intertemporal marginal rate of substitution (IMRS) over her consumption basket and let $\tilde{m}$ be Bob's IMRS over his basket. If Amy and Bob can frictionlessly trade assets that span every possible state of the world next period, then

$$\underbrace{\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array}}_{\ln e' \,-\, \ln e} = \underbrace{\begin{array}{c}\text{Bob's log IMRS over}\\ \text{his consumption basket}\end{array}}_{\ln \tilde{m}} - \underbrace{\begin{array}{c}\text{Amy's log IMRS over}\\ \text{her consumption basket}\end{array}}_{\ln m}. \quad (1)$$

We refer to Eq. (1) as the asset market, or stochastic discount factor (SDF), view of exchange rates. It is a workhorse for many recent papers in international finance and has been used to draw inferences about international risk sharing, to characterize risk premia in the foreign exchange market, and to gain insights into exchange rate determination.[1]

We argue that the asset market view, alone, is not sufficient to explain time-series variation in real exchange rates, nor is it sufficient to infer differences in agents' IMRSs from that variation. Our argument follows from the observation that the real exchange can only vary if either the composition of Amy's and Bob's consumption baskets differs, or there are frictions in the market for goods and services, so that Amy and Bob face different prices in that market. Both of these conditions are rooted squarely in the nature of both preferences over goods and the market for goods and services. Yet the asset market view of exchange rates, alone, provides no economic insights on either front. Indeed, it is broadly appealing because it holds whenever the asset market is economically complete, regardless of the specific nature of the market for goods and services.

Agents' preferences over, and frictions in the markets for, goods and services are central to any explanation or interpretation of time-series variation in real exchange rates. To illustrate, suppose that Amy's and Bob's consumption baskets are composed differently. In this case, the real exchange rate represents the relative price of different baskets, which can vary over time even within the same location. Moreover, there is no economic basis for comparing

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[1]Examples include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Smith and Wickens (2002); Brandt, Cochrane, and Santa-Clara (2006); Lustig and Verdelhan (2006); Brennan and Xia (2006); Lustig and Verdelhan (2007); Bakshi, Carr, and Wu (2008); Verdelhan (2010); Colacito and Croce (2011); Lustig, Roussanov, and Verdelhan (2011); and Stathopoulos (2011).

Amy's IMRS over one basket with Bob's IMRS over a *different* basket. In fact, if there is frictionless trade in goods and services, then the growth in the exchange rate is always equal to *any* agent's log IMRS measured over Bob's consumption basket minus that *same* agent's log IMRS measured over Amy's basket. In other words, with frictionless trade in goods and services, different consumption baskets are all that's relevant in Eq. (1), *not* different agents. This standard change of numeraire units always applies, irrespective of asset market completeness.

Suppose, instead, that Amy and Bob choose the same consumption basket. In this case, the real exchange rate measures the relative price of this common basket across the two countries. Regardless of whether the asset market is complete or incomplete, with identical consumption baskets the real exchange rate can only vary if there are frictions in the market for goods and services. Any difference between Amy's and Bob's log IMRSs over a common consumption basket can be economically interpreted as imperfect risk sharing. In general, these differences reflect frictions in the market for goods and services, as well as frictions in the asset market (e.g., incomplete asset markets). Ironically, if there is frictionless trade in an economically complete set of assets, so that the asset market view in Eq. (1) holds, then differences in agents' IMRSs (measured over a common basket) can only reflect frictions in the market for goods and services. Yet virtually all recent papers that draw economic insights from the asset market view of real exchange rates are completely silent on the specific nature of any such frictions.

More generally, Eq. (1) does not necessarily hold when asset markets are incomplete, which is an additional reason why imperfect risk sharing and real exchange rate variation are not necessarily one and the same thing. With incomplete markets, the right hand side of Eq. (1) is often re-interpreted as the difference between the log of an SDF that applies in the U.K. and the log of an SDF that applies in the U.S. Empirical SDFs are sometimes formed by projecting agents' IMRSs in the two countries onto vectors of returns on the *same* assets, measured in the respective real currency units of the two countries. For example, Brandt, Cochrane, and Santa-Clara (2006) and Lustig and Verdelhan (2012), both argue that Eq. (1) holds for these empirical minimum-variance SDFs, even under incomplete markets. In Section 1.2.1 we show that, in fact, Eq. (1) does *not* generally hold for these minimum-variance SDFs. Additionally, in Section 1.3 we make clear that minimum-variance SDFs provide a measure of shared risk, but are silent on the risks that agents do not share due to asset market incompleteness. In the language of finance, agents may have different IMRSs, however, the projection of every agent's IMRS onto the space of asset returns must be identical (e.g., see Hansen and Jagannathan, 1991). It is true that these projections onto asset returns may look different when they are measured in different numeraire units, but

this result is only because they are identical when measured in a common numeraire that agents can frictionlessly trade with each other.

We argue that models and assumptions are necessary to understand and interpret time-series variation in real exchange rates. In Section 1.4 we describe the necessary ingredients in any such model. In Section 2 we provide an example of a simple two-country endowment economy, based closely on Backus and Smith (1993). We assume that agents in these two economies have standard preferences over two goods. One good is frictionlessly traded. We alternately assume that the second good is not traded, or is frictionlessly traded. In this model, real exchange rate fluctuations arise out of variation in countries' endowments of the two goods. But the exact nature of these fluctuations–the mapping from endowments to consumptions and real exchange rates–depends crucially on the precise nature of the assumed preference differences, imperfections in goods markets, and imperfections in asset markets.

Our model also illustrates that the theoretical link between imperfect risk sharing and real exchange rate variation is tenuous. We construct equilibria with lots of real exchange rate variation combined with perfect or no risk sharing. Similarly, we construct equilibria with no real exchange rate variation combined with perfect or no risk sharing. The weak connection between risk sharing and exchange rate behavior is readily understood if we consider a single economy with two agents, in which all goods have the same prices in all markets. The fact that these agents face the same prices, and use the same numeraire to denominate those prices, does not tell us what the economy's overall risk sharing characteristics are. It only tells us that, if risk is not perfectly shared, goods market imperfections play no role. More generally, agents must agree on the relative prices of all goods and assets that they can frictionlessly trade with each other in a competitive equilibrium. Therefore, no model-free economic inferences about *different* agents can be made using only these relative prices.

The literature sometimes uses reduced-form models of the IMRSs of representative agents in different countries, in conjunction with the asset market view, to investigate the period-by-period behavior of exchange rates. In Section 3 we discuss issues with this approach. For example, the reduced-form model of the domestic agent's IMRS is often identified using asset returns denominated in (real or nominal) domestic currency units, while the reduced-form model for the foreign agent's IMRS is identified using the *same* asset returns denominated, instead, in (real or nominal) foreign currency units. The real exchange rate is a necessary input into such exercises, so implications for the real exchange rate cannot be treated as an output as well. In addition, for Eq. (1) to hold, the set of asset returns used in the reduced-form model must completely span the variation in the two agents' IMRSs. However, papers in this literature only use a very small subset of the assets that are available for agents to invest in.

1   Real Exchange Rates

We begin with a formal definition of the real exchange rate between two agents, Amy and Bob. They could be located anywhere in the world, including places that use the same nominal currency to denominate prices (e.g., different countries within the eurozone, or different states in the U.S.). To be concrete, we assume that they are representative agents in the United States and the United Kingdom. We make the standard assumption that there is frictionless trade in nominal currencies so that, without loss of generality, we use U.S. dollars to denominate the prices of *all* goods and assets, regardless of their location. If the price of a good or asset is denominated in a different nominal currency, then it can be converted to the U.S. dollar equivalent at the relevant nominal exchange rate.

Let P be the dollar value today of one unit of Amy's consumption basket of goods and services, at U.S. prices, and let P' be its (uncertain) dollar value next period. Similarly, let $\tilde{P}$ be the U.S. dollar value today of one unit of Bob's consumption basket, at U.K. prices, and let $\tilde{P}'$ be its (uncertain) dollar value next period. The real U.S./U.K. exchange rate, e, is defined as the value of a unit of Bob's consumption basket relative to a unit of Amy's consumption basket,

$$e \equiv \tilde{P}/P \quad \text{and} \quad e' \equiv \tilde{P}'/P' \,. \tag{2}$$

Empirically, P is usually measured as the dollar value of the basket of consumer goods and services that is used to compute the U.S. consumer price index (CPI). Likewise, let $\tilde{P}^*$ denote the U.K. pound value, measured at U.K. prices, of the basket of consumer goods and services that is used to compute the U.K. CPI. If S is the nominal dollar/pound exchange rate (i.e., the U.S. dollar price of one U.K. pound), then $\tilde{P} \equiv S\tilde{P}^*$ is the U.S. dollar value of this U.K. basket. Therefore, the *growth* in the real U.S./U.K. exchange rate is typically measured as

$$\ln(e'/e) \;=\; \ln(\tilde{P}^{*\prime}/\tilde{P}^*) \;+\; \ln(S'/S) \;-\; \ln(P'/P) \,. \tag{3}$$

If the composition of Amy's and Bob's consumption baskets is the same *and* they face identical prices (measured in common units) for the goods and services in their baskets, then the real U.S./U.K. exchange rate is constant. Therefore, the real U.S./U.K. exchange rate can only vary if either (or both):

1. The composition of Amy's and Bob's consumption baskets is different; *and/or*

2. Amy and Bob face different prices, in the U.S. and the U.K., for identical goods in their baskets (where prices are measured in common units such as U.S. dollars).

Hence, to understand *why* real exchange rates vary over time, at a minimum it is necessary

to understand *why* different agents may face different prices for identical goods and services, and/or *why* the composition of their consumption baskets may differ.

From an empirical standpoint, *both* of these necessary conditions for a variable real exchange rate are satisfied for virtually every country pair in the world. Different countries use different baskets of consumer goods and services to compute their respective consumer price indices. Also, identical goods and services frequently have different prices in different countries (i.e., purchasing power parity does not typically hold across countries, or even in different locations within the same country).

1.1 The Asset Market View of Real Exchange Rates

Next we provide a formal development of the asset market view of real exchange rates in Eq. (1). Let λ be Amy's marginal utility today over units of her consumption basket, and let λ' denote the discounted value of her (uncertain) marginal utility next period. Amy's intertemporal marginal rate of substitution (IMRS), or discounted marginal utility growth, over units of her consumption basket is $m \equiv \lambda'/\lambda$. Today, Amy can purchase $1/P$ units of her consumption basket in the U.S. with a dollar. Therefore, λ/P is her indirect marginal utility today over dollars and $mP/P' \equiv (P\lambda')/(P'\lambda)$ is her IMRS over dollars.

Analogous to Amy, let $\tilde{\lambda}$ denote Bob's marginal utility today over a unit of his consumption basket and let $\tilde{\lambda}'$ denote the (uncertain) discounted value of his marginal utility next period. Then $\tilde{m} \equiv \tilde{\lambda}'/\tilde{\lambda}$ is Bob's IMRS over units of his consumption basket, and $\tilde{m}\tilde{P}/\tilde{P}' \equiv (\tilde{P}\tilde{\lambda}')/(\tilde{P}'\tilde{\lambda})$ is his IMRS over dollars.

As its name suggests, the asset market (or SDF) view of exchange rates focuses on the role that asset markets play in understanding exchange rates. Consider a set of k assets that can be located anywhere in the world. The standard assumption in this literature is that trade in assets is frictionless, so that Amy and Bob can both trade the same set of assets at the same dollar-denominated prices. Let $\mathbf{X}$ be the $k \times 1$ vector of uncertain asset payoffs next period, and let $\mathbf{P_X}$ be the $k \times 1$ vector of asset prices today. Again, without loss of generality, we assume that the payoffs and prices of all the assets are measured in U.S. dollars, regardless of the nominal currency where the assets are located. Denote the vector of uncertain dollar-denominated returns on the assets by $\mathbf{R} \equiv \mathbf{X}/\mathbf{P_X}$.

If Amy maximizes her expected discounted utility, then her first order condition for optimality (i.e., her Euler equation) implies that

$$\mathbf{P_X}\lambda/P = \mathbb{E}\left[\mathbf{X}\lambda'/P'\right], \quad \text{or equivalently,} \quad \mathbf{1} = \mathbb{E}\left[\mathbf{R}mP/P'\right]. \tag{4}$$

In Eq. (4), $\mathbf{1}$ denotes a $k \times 1$ vector of 1's and $\mathbb{E}\left[\cdot\right]$ denotes the standard expectations operator.

Likewise, Bob's Euler equation is

$$\mathbf{1} = \mathbb{E}[\mathbf{R}\tilde{m}\tilde{P}/\tilde{P}'] . \tag{5}$$

Together, Eqs. (4) and (5) do *not* imply that Amy and Bob always equate IMRSs over dollars (i.e., in general, $mP/P' \neq \tilde{m}\tilde{P}/\tilde{P}'$). However, Eqs. (4) and (5) *do* imply that the *linear projections* of their IMRSs over dollars, onto the dollar-denominated asset returns $\mathbf{R}$, must agree. In other words,

$$\mathbb{E}\left[\mathbf{R}mP/P'\right] = \mathbf{1} = \mathbb{E}[\mathbf{R}\tilde{m}\tilde{P}/\tilde{P}'] \quad \Rightarrow \quad \text{proj}\left[mP/P' \mid \mathbf{R}\right] = \text{proj}[\tilde{m}\tilde{P}/\tilde{P}' \mid \mathbf{R}] . \tag{6}$$

In the special case that the dollar-denominated asset returns span every possible state of the world next period,[2] then $\text{proj}\left[mP/P' \mid \mathbf{R}\right] = mP/P'$ and $\text{proj}[\tilde{m}\tilde{P}/\tilde{P}' \mid \mathbf{R}] = \tilde{m}\tilde{P}/\tilde{P}'$ and therefore Amy and Bob must *always* equate IMRSs over dollars (i.e., in *every* state of the world next period), so that

$$mP/P' = \tilde{m}\tilde{P}/\tilde{P}' . \tag{7}$$

Eq. (7) can be rearranged and written in logs to produce the asset market view of exchange rates in Eq. (1),

$$\underbrace{\begin{array}{c}\text{growth in the real} \\ \text{U.S./U.K. exchange rate}\end{array}}_{\ln e' \;-\; \ln e} = \underbrace{\begin{array}{c}\text{Bob's log IMRS over} \\ \text{his consumption basket}\end{array}}_{\ln \tilde{m}} - \underbrace{\begin{array}{c}\text{Amy's log IMRS over} \\ \text{her consumption basket}\end{array}}_{\ln m} , \tag{1}$$

where the real exchange rate is defined by Eq. (2). Eq. (1), or equivalently Eq. (7), is a testable implication of any model in which Amy and Bob can frictionlessly trade assets with returns that completely span all possible states of the world next period.

As we noted in the introduction, the asset market view of exchange rates is a workhorse for many recent papers in international finance. For example, Brandt, Cochrane, and Santa-Clara (2006) use it to draw inferences about international risk sharing,[3] Lustig and Verdelhan (2007) and Bakshi, Carr, and Wu (2008) use it to characterize risk premia in the foreign exchange market,[4] and Verdelhan (2010) uses it to gain insights into exchange rate deter-

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[2]In discrete time, the returns on k assets can span, at most, k distinct states of the world (events) next period.

[3]We provide a detailed analysis of Brandt, Cochrane, and Santa-Clara (2006) in Section 1.2.

[4]Section IIIB ("Mechanism: Where Do Consumption Betas of Currencies Come From?") of Lustig and Verdelhan (2007) states that:

> Investing in foreign currency is like betting on the difference between your own and your neighbor's intertemporal marginal rate of substitution (IMRS). These bets are very risky if your IMRS is not correlated with that of your neighbor, but they provide a hedge when her

mination.[5] Numerous other examples of papers that draw economic insights from the asset market view of exchange rates are cited in the first footnote of the introduction.

In the introduction we argued that the *asset* market view, alone, is not sufficient to explain time-series variation in real exchange rates, nor is it sufficient to infer differences in agents' IMRSs from that variation. As we highlighted earlier, the real exchange rate can only vary if either (or both) the composition of Amy's and Bob's consumption baskets differs, or they face different prices for identical goods and services in their baskets. Therefore, agents' preferences over goods and services, and frictions in that market, must be central to any explanation or interpretation of time-series variation in real exchange rates. Yet the asset market view of exchange rates provides no economic insights on either front.

Here, we also show that Amy and Bob are not uniquely identified in Eq. (1) unless there are frictions in the market for goods and services (so that they face different prices in that market). To illustrate, suppose that they face the same dollar prices in the U.S. and the U.K. for identical goods and services in their baskets. In this case, Amy can trade one unit of Bob's basket for $\tilde{P}/P \equiv e$ units of her basket. Therefore, her marginal utility over units of Bob's basket is λe and her IMRS over units of his basket is me'/e. Likewise, Bob's marginal utility over units of Amy's basket is $\tilde{\lambda}/e$ (since he can trade a unit of Amy's basket for $P/\tilde{P} \equiv 1/e$ units of his basket), so his IMRS over units of her basket is $\tilde{m}e/e'$. Thus, if Amy and Bob face the same prices for identical goods and services, then they are *not uniquely identified* in the asset market view of real exchange rates, since an equivalent economic interpretation of Eq. (1) is

$$
\underbrace{\begin{array}{c}\text{growth in the real}\\ \text{U.S./U.K. exchange rate}\end{array}}_{\ln e' \,-\, \ln e} = \underbrace{\begin{array}{c}\text{Amy's log IMRS}\\ \text{over Bob's basket}\end{array}}_{\ln(me'/e)} - \underbrace{\begin{array}{c}\text{Bob's log IMRS}\\ \text{over Amy's basket}\end{array}}_{\ln(\tilde{m}e/e')}. \tag{8}
$$

IMRS is highly correlated and more volatile.

The introduction of Bakshi, Carr, and Wu (2008) states that:

> In particular, because the ratio of the stochastic discount factors in two economies governs the exchange rate between them, the exchange rate market offers a direct information source for assessing the relative risk-taking behavior of investors in international economies.

[5]The introduction of Verdelhan (2010) states that:

> When markets are complete, the real exchange rate, measured in units of domestic goods per foreign good, equals the ratio of foreign to domestic pricing kernels. Exchange rates thus depend on foreign and consumption growth shocks. If the conditional variance of the domestic stochastic discount factor (SDF) is large relative to its foreign counterpart, then domestic consumption growth shocks determine variation in exchange rates.

Therefore, when agents face the same prices for identical goods and services, it is impossible to infer any differences in their IMRSs using only the growth in the real exchange rate.[6]

Note that the math below the braces in Eq. (8) always holds whenever Eq. (7) holds. However, the economic labels of Amy's and Bob's log IMRSs over each others' baskets is only valid if they face the same prices in the U.S. and the U.K. for identical goods and services. It is also worth noting that, if agents face the same prices for identical goods and services, then a version of Eq. (1) holds for any *single* agent. For example, we can write

$$
\underbrace{\begin{array}{c}\text{growth in the real}\\\text{U.S./U.K. exchange rate}\end{array}}_{\ln e' \,-\, \ln e} = \underbrace{\begin{array}{c}\text{Amy's log IMRS}\\\text{over Bob's basket}\end{array}}_{\ln(me'/e)} - \underbrace{\begin{array}{c}\text{Amy's log IMRS}\\\text{over Amy's basket}\end{array}}_{\ln m} . \tag{9}
$$

Again, the math below the braces in Eq. (9) holds trivially, but the economic labels require that agents face the same prices for identical goods and services. This standard change of numeraire units always applies, regardless of whether asset markets are complete or incomplete. In other words, with frictionless trade in goods and services, only the different consumption baskets are relevant in Eq. (1), *not* the different agents.

1.2 Inferences About International Risk Sharing

In this section we provide a more detailed analysis of Brandt, Cochrane, and Santa-Clara (2006). As we mentioned in Section 1.1, they use the asset market view of exchange rates in Eq. (1) to interpret variation in the real exchange rate as evidence of imperfect risk sharing between foreign and domestic investors. To measure the *extent* of imperfect risk sharing, they take the variance of both sides of Eq. (1). From the abstract of Brandt, Cochrane, and Santa-Clara (2006):

> Exchange rates depreciate by the difference between domestic and foreign marginal utility growth or discount factors. Exchange rates vary a lot, as much as 15% per year. However, equity premia imply that marginal utility growth varies much more, by at least 50% per year. Therefore, marginal utility growth must be highly correlated across countries: international risk sharing is better than you think. Conversely, if risks really are not shared internationally, exchange rates should vary more than they do: exchange rates are too smooth.

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[6]Of course, even if one could infer any differences in their IMRSs, there is still the issue that we highlighted in the introduction that economic theory does not provide a basis for comparing *different* agents' IMRSs over *different* baskets.

What exactly constitutes perfect risk sharing, and how is it related to the difference in agents' IMRSs (or discounted marginal utility growths)?

First, consider again the example in Section 1.1 in which Amy and Bob face the same prices for identical goods and services in their baskets. In that example we showed that Amy's marginal utility over units of Bob's basket is λe, while Bob's marginal utility over units of Amy's basket is $\tilde{\lambda}/e$. Therefore, if asset markets are complete and agents face the same prices for identical goods and services in their baskets then, in *every* period,

$$\underbrace{\text{Amy's log IMRS over Amy's basket}}_{\ln m} = \underbrace{\text{Bob's log IMRS over Amy's basket}}_{\ln(\tilde{m}e/e')}, \quad (10)$$

and

$$\underbrace{\text{Amy's log IMRS over Bob's basket}}_{\ln(me'/e)} = \underbrace{\text{Bob's log IMRS over Bob's basket}}_{\ln \tilde{m}}. \quad (11)$$

It is straightforward to extend this analysis to show that, when asset markets are complete, Amy and Bob *always* equate IMRSs over *any* basket of goods and services that they can frictionlessly trade with each other (i.e., for which they face identical prices). In particular, if Amy and Bob face identical prices for *all* goods and services, then with complete asset markets they *always* equate IMRSs over *any* common basket of goods and services (i.e., in *every* possible state of the world next period). In this case, Amy and Bob share risk perfectly. However, note that even if Amy and Bob share risk perfectly, their IMRSs in Eq. (1) can still differ when they are measured over *different* baskets of goods and services. In this case, the real exchange rate can also still vary because it too reflects the relative price of *different* baskets.

With frictionless trade in assets, risk sharing can only be imperfect if asset markets are incomplete *and/or* agents face different prices for identical goods and services. If asset markets are incomplete, then Amy's and Bob's IMRSs can differ across states of the world that are not spanned by the available assets. If they face different prices for identical goods and services, then there must be a friction in that market that prevents these prices from being equal in different locations, and that friction can also prevent perfect risk sharing.

Contrary to the premise of Brandt, Cochrane, and Santa-Clara (2006), these conditions for imperfect risk sharing *do not* completely overlap with the conditions for a variable real exchange rate. For example, as we noted earlier, if the composition of Amy's and Bob's consumption baskets is the same *and* they face identical prices for the goods and services in their baskets, then the real exchange rate is constant. Yet, risk sharing can still be imperfect

if asset markets are incomplete. Conversely, if the asset market is complete and agents face identical prices for the goods and services in their baskets, then risk sharing is perfect (since they *always* equate IMRSs over *any* common basket of goods and services). Yet the exchange rate can still vary if the composition of Amy's and Bob's consumption baskets differs.

	Composition of Consumption Baskets	
Asset Markets	Identical	Different
Complete	Yes	No
Incomplete	No	No

Table 1: Does a variable real exchange rate directly reflect imperfect risk sharing?

Table 1 provides the necessary conditions such that variation in the real exchange rate is a direct reflection of imperfect risk sharing. When asset markets are complete and the composition of Amy's and Bob's consumption baskets is identical, then the exchange rate can only vary if they face different prices for identical goods and services in their baskets. Likewise, if asset markets are complete, then risk sharing can only be imperfect if there are frictions in the market for goods and services that prevent prices from being the same in different locations. In other words, under these specific assumptions, both imperfect risk sharing and variation in the real exchange rate can only occur if there are frictions in the market for goods and services. Unfortunately, as Table 1 indicates, the link between risk sharing and a volatile real exchange rate only holds in this one special case. As previously noted, the difference in composition of Amy's and Bob's consumption baskets can contribute to variation in the real exchange rate, but it need not affect risk sharing. Likewise, incomplete asset markets can contribute to imperfect risk sharing, without affecting the volatility of the real exchange rate.

To emphasize, if we only observe variation in the real exchange rate, and nothing more, then we cannot be sure of the extent to which that variation reflects the relative prices of different baskets of goods and services, versus different prices for identical goods and services in those baskets. Similarly, incomplete asset markets are a source of imperfect risk sharing, but one cannot learn the *extent* of market incompleteness (i.e., the extent to which agents' IMRSs are not spanned by asset returns) from asset returns alone. Therefore, very specific *assumptions* are required to make any inferences about international risk sharing using only observations of asset returns and variation in the real exchange rate. Moreover, any such inferences are not robust to different assumptions.

1.2.1 International Risk Sharing with Incomplete Asset Markets

Brandt, Cochrane, and Santa-Clara (2006) also examine international risk sharing when asset markets do not completely span agents' IMRSs. As the basis for their analysis, they claim that, even when asset markets are incomplete, Eq. (1) continues to hold for the *linear projection*s of agents' IMRSs onto the available asset returns (denominated in units of their respective consumption baskets).[7] Here, we show that Eq. (1) *does not* in fact hold for these linear projections.

For notational convenience, let $\mathbf{r} \equiv \mathbf{R}P/P'$ denote the vector of uncertain asset returns denominated in units of Amy's consumption basket in the U.S. Similarly, let $\tilde{\mathbf{r}} \equiv \mathbf{R}\tilde{P}/\tilde{P}'$ denote the vector of uncertain asset returns denominated in units of Bob's consumption basket in the U.K. Brandt, Cochrane, and Santa-Clara (2006) claim that

$$\ln e' - \ln e \;=\; \ln(\mathrm{proj}[\tilde{m} \mid \tilde{\mathbf{r}}]) \;-\; \ln(\mathrm{proj}[m \mid \mathbf{r}]), \tag{12}$$

or equivalently,

$$\mathrm{proj}\,[m \mid \mathbf{r}] \times e'/e \;=\; \mathrm{proj}[\tilde{m} \mid \tilde{\mathbf{r}}]. \tag{13}$$

Note that the left hand side of Eq. (13) is linear in $\mathbf{r}e'/e$, while the right hand side is instead linear in $\tilde{\mathbf{r}} \equiv \mathbf{r}e/e'$. In general, when there are fewer assets than future possible states of the world next period (i.e., when asset markets are incomplete), $\mathbf{r}e/e'$ is not in the linear span of $\tilde{\mathbf{r}} \equiv \mathbf{r}e'/e$. Therefore, Eqs. (12) and (13) do *not* necessarily hold when asset markets are incomplete.[8]

1.3 Projections onto Asset Returns

In the previous section we showed that Eq. (1) *does not* hold for the *linear projection*s of agents' IMRSs onto the asset returns (denominated in units of their respective consumption baskets). In this section we consider whether it is possible to learn anything about risk sharing using only the projections of agents' IMRSs onto the returns of assets that they can frictionlessly trade with each other.

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[7]On page 675 of Brandt, Cochrane, and Santa-Clara (2006) they state that:

> These discount factors are the projections of any possible domestic and foreign discount factors onto the relevant spaces of asset payoffs, and they are also the minimum-variance discount factors. We show that Eq. (1) continues to hold with this particular choice of discount factors.

In Section 14.2.2 of Lustig and Verdelhan (2012) they also claim that Eq. (1) holds with incomplete markets. Alvarez, Atkeson, and Kehoe (2007) also make this claim.

[8]Eqs. (12) and (13) also do *not* necessarily hold with incomplete asset markets in a continuous-time setting (for instance, with jumps).

For notational convenience, let $M \equiv mP/P'$ and $\tilde{M} \equiv \tilde{m}\tilde{P}/\tilde{P}'$ denote Amy's and Bob's respective IMRSs over U.S. dollars. We can separate their IMRSs over dollars into the portion that is linearly spanned by the available dollar-denominated asset returns, and the portion that is orthogonal to those returns. In other words,

$$M = \text{proj}\,[M \mid \mathbf{R}] + \varepsilon, \quad \text{where} \quad \mathbb{E}\,[\mathbf{R}\varepsilon] = \mathbf{0}, \tag{14}$$

and

$$\tilde{M} = \text{proj}[\tilde{M} \mid \mathbf{R}] + \tilde{\varepsilon}, \quad \text{where} \quad \mathbb{E}\,[\mathbf{R}\tilde{\varepsilon}] = \mathbf{0}. \tag{15}$$

In Section 1.1 we highlighted the well-known result in Eq. (6) that

$$\mathbb{E}\,[\mathbf{R}M] = \mathbf{1} = \mathbb{E}[\mathbf{R}\tilde{M}] \quad \Rightarrow \quad \text{proj}\,[M \mid \mathbf{R}] = \text{proj}[\tilde{M} \mid \mathbf{R}]. \tag{6}$$

Therefore, from these projections we learn that Amy's and Bob's IMRSs over dollars share a common component, but we learn nothing about the components of their IMRSs, ε and $\tilde{\varepsilon}$, that are orthogonal to the available asset returns. These include all of Amy's and Bob's risks that go unshared due to asset market incompleteness, as well as any shared risk that happens to not be spanned by the available asset returns (more generally, shared risk not spanned by the specific asset returns used in the projections). Simply put, in equilibrium Amy and Bob must agree on the prices of assets and goods that they can frictionlessly trade with each other, and therefore it is impossible to make any model-free inferences about differences in their IMRSs using only these relative prices.

1.4   Necessary Ingredients in a Model

What are the necessary ingredients of a model that attempts to explain real exchange rates? As we've already argued, to understand *why* real exchange rates vary over time, at a minimum it is necessary to understand *why* agents may face different prices for identical goods and services, and/or *why* the composition of their consumption baskets may differ (or at least correctly account for any differences). Therefore, any model that is used to explain or interpret time-series variation in real exchange rates must directly specify:

- agents' preferences over goods;

- any frictions in goods (and services) market trade that may restrict prices of identical goods (and services) from being equal in different locations;

- agents' preferences over time and uncertainty;

- the available assets that agents can trade; and

- endowments or production technology.

These ingredients are required to map structural shocks to endowments (or the production technology) into agents' consumptions and, hence, outcomes of the real exchange rate. Any model that is missing one of these key ingredients is silent about how the real exchange rate is determined, and is therefore of limited usefulness for understanding and interpreting time-series variation in real exchange rates.

It is important to note that in open economies, there is a fundamental distinction between models that treat aggregate consumptions in each country as exogenous and those that treat each country's endowments (or production technology shocks) as exogenous. In a closed economy without physical investment or government purchases, aggregate consumption must equal the aggregate endowment (production). However, in an open economy, each country's aggregate consumption can differ from its aggregate output. Take an endowment economy as an example. To explain how real exchange rates are determined (which is necessary to interpret time-series variation in real exchange rates), such a model must map each country's endowments into its aggregate consumption and, hence, the real exchange rate between the two countries (i.e., the relative value of aggregate consumption baskets across countries). Any model that treats aggregate consumption in each country as exogenous is silent about this map and, hence, is silent on the mechanism that determines real exchange rates.

A number of recent papers treat aggregate consumption in the foreign and domestic economies as exogenous and assume a utility function for the representative agent in each economy.[9] For example, Verdelhan (2010) uses representative agents with external habit preferences (see Campbell and Cochrane, 1999), while Bansal and Shaliastovich (2010) use representative agents with Epstein-Zin recursive utility functions (see Epstein and Zin, 1989). As we have highlighted above, any such model that treats aggregate consumption in each country as exogenous has limited potential to provide insights into time-series variation in real exchange rates.[10]

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[9]Colacito and Croce (2011) treat aggregate endowments as exogenous. However, they assume complete home bias so that aggregate consumptions equal aggregate endowments in each country. With complete home bias, the real exchange rate is not uniquely determined. *Any* real exchange rate clears the market since agents are assumed to have no desire to consume goods that they are not endowed with. Nevertheless, Colacito and Croce (2011) use the asset market view of real exchange rates in Eq. (1) to uniquely characterize the growth in the real exchange rate.

[10]Of course there *may* be asset markets, endowments (production technology), goods market frictions, and preferences over goods that *could* generate these aggregate consumptions in each country. Our point is that these elements are required to understand how aggregate consumption and the real exchange rate are *jointly* determined, but they are missing in many (if not most) of the recent papers in this literature.

To be clear, there is no issue with the standard exercise in consumption-based asset pricing that takes shocks to aggregate consumption and asset returns as exogenous, and uses Euler equations of representative agents (e.g., Eqs. (4) and (5) for Amy and Bob) to draw insights into the *average* (or *expected*) returns of various assets. That approach can be fruitful for understanding the *average* (or *expected*) returns to investing in assets in other countries (including investments in nominal currencies) based on how the returns on those assets covary with agents' IMRS. Our point is simply that models that treat shocks to consumptions in each country as exogenous are not particularly useful for understanding or interpreting changes in real exchange rates in *each period* (rather than average, or expected, changes).

Ultimately, there are two simple diagnostics that any model that purports to explain variation in real exchange rates must pass. First, the model cannot rely on complete asset markets to *define* the real exchange rate. The real exchange rate is the relative price of one basket of goods and services to another, and therefore any model must describe how that relative price is determined, regardless of whether asset markets are complete or incomplete. Second, the *level* of the real exchange rate should be determined in the model (up to a constant that reflects the size of the baskets of goods that are being compared), and not simply *growth* (or changes) in the real exchange rate. If a model doesn't help one to understand the relative price of one basket of goods and services to another, then it doesn't help one to understand how real exchange rates are determined. Models of the real exchange rate that rely *solely* on the asset market view of exchange rates in Eq. (1) fail both of these simple diagnostics.

2   An Endowment Economy

In Section 1 we showed that the asset market view of exchange rates in Eq. (1) is a first order condition that must be satisfied in any model, but it is not a substitute for a model. Aggregate consumption growths and intertemporal marginal rates of substitution do not determine the real exchange rate any more than the real exchange rate determines aggregate consumption growths and intertemporal marginal rates of substitution. A full-fledged model with the minimum necessary ingredients described in Section 1.4 is required to understand how aggregate consumptions and the real exchange rate are *jointly* determined in equilibrium. In this section we provide an example of such a model. The model is a generalization of Backus and Smith (1993) along three dimensions. First, we allow for preference differences across countries. Second, we allow for incomplete markets. Third, like Backus and Smith (1993) we have two goods, but we explicitly compare the case where the second good is

non-traded to the case in which it is frictionlessly traded. We don't view this model as a solution to existing exchange rate puzzles, rather it is illustrative of our point about the joint determination of consumption and the real exchange rate.[11]

We describe an endowment economy with two countries ("home" and "foreign") and representative households within each country. Utility is defined over two goods, A and B. All goods are perishable and households live for two periods.

The representative household in the home economy has an instantaneous utility function

$$U(c_A, c_B) = u[c(c_A, c_B)], \tag{16}$$

where c_A and c_B denote, respectively, the consumption of goods A and B by the home household, $c(\cdot)$ is a homogeneous of degree 1 quasi-concave function of its arguments, and u is a monotonic function with standard properties. Similarly, the representative household in the foreign economy has the instantaneous utility function

$$\tilde{U}(\tilde{c}_A, \tilde{c}_B) = u[\tilde{c}(\tilde{c}_A, \tilde{c}_B)], \tag{17}$$

where $\tilde{c}_A$ and $\tilde{c}_B$ denote, respectively, the consumption of goods A and B by the foreign household, and $\tilde{c}(\cdot)$ is a homogeneous of degree 1 quasi-concave function of its arguments.

Both economies are cashless and use good A as the numeraire. Our model would have the same implications for the real exchange rate if we chose different numeraires. Goods markets meet sequentially. Good A is frictionlessly tradable. We alternately assume that good B is frictionlessly tradable or non-tradable. We let P_B and P'_B denote the prices of good B in the home economy in the first and second periods. Similarly, we let $\tilde{P}_B$ and $\tilde{P}'_B$ denote the prices of good B in the foreign economy in the first and second periods. When good B is frictionlessly tradable, its price must be the same in both countries,

$$P_B = \tilde{P}_B \quad \text{and} \quad P'_B = \tilde{P}'_B. \tag{18}$$

The natural definition of the consumer price index (CPI) in the home country is a variable P such that $c_A + P_B c_B = P\, c(c_A, c_B)$. Since $c(\cdot)$ and $\tilde{c}(\cdot)$ are homogeneous of degree one functions, it can be shown that there are homogeneous of degree one functions $H(\cdot)$ and

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[11]Dating back, at least, to the contributions of Stockman (1980) and Lucas (1982), a large literature has developed explicit equilibrium models of the exchange rate. To name but a few, Cole and Obstfeld (1991), Backus, Kehoe, and Kydland (1992), Dumas (1992), Backus and Smith (1993), Baxter and Crucini (1995), Obstfeld and Rogoff (1995), Sercu, Uppal, and Van Hulle (1995), Stockman and Tesar (1995), Betts and Devereux (1996), Chari, Kehoe, and McGrattan (2002), Apte, Sercu, and Uppal (2004), Bacchetta and Wincoop (2006), Kocherlakota and Pistaferri (2007), Pavlova and Rigobon (2007), Bodenstein (2008), Benigno and Thoenissen (2008), and Corsetti, Dedola, and Leduc (2008).

$\tilde{H}(\cdot)$ whose form depends on $c(\cdot)$ and $\tilde{c}(\cdot)$, such that the home and foreign CPIs are:[12]

$$P = H(1, P_B) \quad \text{and} \quad \tilde{P} = \tilde{H}(1, \tilde{P}_B). \tag{19}$$

Similarly the CPIs in period two are

$$P = H(1, P'_B) \quad \text{and} \quad \tilde{P}' = \tilde{H}(1, \tilde{P}'_B). \tag{20}$$

Identical to Eq. (2) in Section 1, the real exchange rates in periods one and two are

$$e \equiv \tilde{P}/P \quad \text{and} \quad e' \equiv \tilde{P}'/P'. \tag{21}$$

In the special case where preferences are identical in the two countries, we have $H(\cdot) = \tilde{H}(\cdot)$. If, additionally, both goods are traded, $e = 1 = e'$, regardless of the asset market structure. If preferences differ across countries and both goods are traded, variation in the real exchange rate *can* arise even though $\tilde{P}_B = P_B$. All that is needed is variation in P_B. We can make these statements even though we've said nothing about asset markets. This is one concrete sense in which the link between exchange rates and asset markets is tenuous.

As was the case in Section 1, we assume that there are k assets with $k \times 1$ random payoff vector $\mathbf{X}$. The $k \times 1$ price vector today for these assets is $\mathbf{P_X}$. The payoffs and prices of the assets are measured in units of good A. The asset payoffs, and all variables in period two, depend on the state of the world in period two. For notational simplicity, however, we suppress the dependence of period two variables on the state of the world.

The household in the home country chooses c_A, c_B, c'_A, c'_B, and the $k \times 1$ vector $\mathbf{a}$, to maximize

$$u[c(c_A, c_B)] + \beta \, \mathbb{E}\left\{ u[c(c'_A, c'_B)] \right\}, \tag{22}$$

subject to

$$c_A + P_B c_B + \mathbf{P_X} \cdot \mathbf{a} = y_A + P_B y_B \quad \text{and} \quad c'_A + P'_B c'_B = y'_A + P'_B y'_B + \mathbf{X} \cdot \mathbf{a}. \tag{23}$$

Here $0 < \beta < 1$, c_A and c_B are the household's current consumption of the two goods, c'_A and c'_B are the household's plans for future consumption of the two goods (in every possible state of the world), the jth element of $\mathbf{a}$ is the household's net purchases of asset j, and y_A, y_B, y'_A and y'_B are the household's current and future endowments of the two goods.

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[12]For details, see the section on price aggregation in the appendix.

Similarly, the foreign household chooses $\tilde{c}_A$, $\tilde{c}_B$, $\tilde{c}'_A$, $\tilde{c}'_B$, and $\tilde{\mathbf{a}}$ to maximize

$$u[\tilde{c}(\tilde{c}_A, \tilde{c}_B)] + \beta\,\mathbb{E}\left\{u[\tilde{c}(\tilde{c}'_A, \tilde{c}'_B)]\right\}, \tag{24}$$

subject to

$$\tilde{c}_A + \tilde{P}_B\tilde{c}_B + \mathbf{P_X} \cdot \tilde{\mathbf{a}} = \tilde{y}_A + \tilde{P}_B\tilde{y}_B \quad\text{and}\quad \tilde{c}'_A + \tilde{P}'_B\tilde{c}'_B = \tilde{y}'_A + \tilde{P}'_B\tilde{y}'_B + \mathbf{X}\cdot\tilde{\mathbf{a}}. \tag{25}$$

Here $\tilde{c}_A$ and $\tilde{c}_B$ are the household's current consumption of the two goods, $\tilde{c}'_A$ and $\tilde{c}'_B$ are the household's plans for future consumption of the two goods (in every possible state of the world), $\tilde{\mathbf{a}}$ is a $k \times 1$ vector whose jth element is the household's net purchases of asset j, and $\tilde{y}_A$, $\tilde{y}_B$, $\tilde{y}'_A$ and $\tilde{y}'_B$ are the household's current and future endowments of the two goods.

The market clearing conditions for good A are

$$c_A + \tilde{c}_A = y_A + \tilde{y}_A \quad\text{and}\quad c'_A + \tilde{c}'_A = y'_A + \tilde{y}'_A. \tag{26}$$

When good B is tradable we have the following market clearing conditions

$$c_B + \tilde{c}_B = y_B + \tilde{y}_B \quad\text{and}\quad c'_B + \tilde{c}'_B = y'_B + \tilde{y}'_B. \tag{27}$$

When it is non-tradable, instead, we have

$$c_B = y_B, \quad \tilde{c}_B = \tilde{y}_B, \quad c'_B = y'_B \quad\text{and}\quad \tilde{c}'_B = \tilde{y}'_B. \tag{28}$$

The market clearing condition in asset markets is

$$\mathbf{a} + \tilde{\mathbf{a}} = \mathbf{0}. \tag{29}$$

Definition. A competitive equilibrium is values of the quantities c_A, c_B, c'_A, c'_B, $\mathbf{a}$, $\tilde{c}_A$, $\tilde{c}_B$, $\tilde{c}'_A$, $\tilde{c}'_B$, $\tilde{\mathbf{a}}$ and prices, P_B, P'_B, $\tilde{P}_B$, $\tilde{P}'_B$, and $\mathbf{P_X}$ such that the quantities solve the home and foreign country optimization problems (taking the prices as given), and such that the market clearing conditions are satisfied. When good B is frictionlessly traded, Eq. (18) must also be satisfied.

2.1   Risk Sharing and IMRSs

Our model always implies a simple relationship between the two countries' discounted marginal utility growths, or IMRSs, defined over aggregate consumption. We define these

IMRSs as

$$m \equiv \beta u_c(c')/u_c(c) \quad \text{and} \quad \tilde{m} = \beta u_{\tilde{c}}(\tilde{c}')/u_{\tilde{c}}(\tilde{c}) \,. \tag{30}$$

Similarly, we can define the two countries' IMRSs over goods A and B:

$$m_A \equiv \beta u_{c_A}(c')/u_{c_A}(c) \,, \quad \tilde{m}_A \equiv \beta u_{\tilde{c}_A}(\tilde{c}')/u_{\tilde{c}_A}(\tilde{c}) \,, \tag{31}$$

$$m_B \equiv \beta u_{c_B}(c')/u_{c_B}(c) \,, \quad \tilde{m}_B \equiv \beta u_{\tilde{c}_B}(\tilde{c}')/u_{\tilde{c}_B}(\tilde{c}) \,. \tag{32}$$

Definition.   Perfect risk sharing describes any competitive equilibrium in which $\tilde{m}_A = m_A$ and $\tilde{m}_B = m_B$ in every possible state of the world next period.

Our definition of perfect risk sharing is the same as the one in Section 1. For any individual good, the IMRSs are equated across agents. For any identical basket of goods, suitably defined, the same is true.

As we show in the Appendix, equilibrium in the goods market always produces the intuitive result that

$$\frac{u_c(c)}{u_{c_A}(c)} = P \,, \quad \frac{u_c(c')}{u_{c_A}(c')} = P' \,, \quad \frac{u_{\tilde{c}}(\tilde{c})}{u_{\tilde{c}_A}(\tilde{c})} = \tilde{P} \,, \quad \text{and} \quad \frac{u_{\tilde{c}}(\tilde{c}')}{u_{\tilde{c}_A}(\tilde{c}')} = \tilde{P}' \,. \tag{33}$$

Combining Eq. (33) with the definitions of IMRSs in Eqs. (30) and (31), produces

$$\frac{m}{m_A} = \frac{P'}{P} \quad \text{and} \quad \frac{\tilde{m}}{\tilde{m}_A} = \frac{\tilde{P}'}{\tilde{P}} \,, \tag{34}$$

so that

$$\frac{\tilde{m}}{m} = \frac{e'}{e}\Xi \,, \quad \text{with} \quad \Xi \equiv \frac{\tilde{m}_A}{m_A} \,. \tag{35}$$

In Eq. (35), $\Xi = 1$ whenever risk sharing is perfect in frictionlessly traded goods, and $\Xi \neq 1$ when risk sharing in those goods is imperfect. For example, when asset markets are complete, agents equate IMRSs across frictionlessly traded goods, and so $m_A = \tilde{m}_A$ and therefore $\Xi = 1$. But in *any* incomplete markets setting, in general, $m_A \neq \tilde{m}_A$ and thus $\Xi \neq 1$. Also, note that Ξ is the same for any frictionlessly traded good (or basket of goods). In particular, $\tilde{m}_B/m_B = \tilde{m}_A/m_A \equiv \Xi$ whenever good B is frictionlessly traded.

2.2   Four Specific Examples

This section discusses four specific examples of our model. The four special cases we consider combine different assumptions about financial markets (complete markets vs. financial autarky) and goods market frictions (good B is frictionlessly traded vs. good B is non-

traded). By explicitly solving for the equilibrium in these four cases, we demonstrate that real exchange rates and agents' IMRSs are jointly determined by the laws of motion of the endowments, together with our assumptions about preferences, goods market frictions, and asset markets. We also illustrate a point we made in Section 1: The conditions under which risk sharing is imperfect, and those under which the real exchange rate varies, are different.

We adopt the assumption that $u(c) = \ln c$, and the consumption aggregates in the two countries are $c = c_A^\theta c_B^{1-\theta}$, and $\tilde{c} = \tilde{c}_A^{\tilde\theta} c_B^{1-\tilde\theta}$. These assumptions are useful because equilibrium prices and quantities can be worked out with pencil and paper. They imply that the CPIs in the two countries, measured in units of good A, are

$$P = \rho P_B^{1-\theta}, \qquad \text{and} \qquad \tilde{P} = \tilde\rho \tilde{P}_B^{1-\tilde\theta}, \tag{36}$$

with $\rho = \theta^{-\theta}(1-\theta)^{\theta-1}$, and $\tilde\rho = \tilde\theta^{-\tilde\theta}(1-\tilde\theta)^{\tilde\theta-1}$. The real exchange rates in periods one and two are

$$e = (\tilde\rho/\rho)\tilde{P}_B^{1-\tilde\theta}/P_B^{1-\theta} \quad \text{and} \quad e' = (\tilde\rho/\rho)\tilde{P}_B'^{1-\tilde\theta}/P_B'^{1-\theta}. \tag{37}$$

We derive all of the solutions in detail in the Appendix. We use some notation in what follows. The global endowment of good A in period one is $Y_A = y_A + \tilde{y}_A$, while in period two it is $Y_A' = y_A' + \tilde{y}_A'$. Analogously, for good B we have $Y_B = y_B + \tilde{y}_B$, and $Y_B' = y_B' + \tilde{y}_B'$. The growth rates of the global endowments are $G_A = Y_A'/Y_A$ and $G_B = Y_B'/Y_B$. We also define $g_A = y_A'/y_A$, $g_B = y_B'/y_B$, $\tilde{g}_A = \tilde{y}_A'/\tilde{y}_A$ and $\tilde{g}_B = \tilde{y}_B'/\tilde{y}_B$. The home country's shares of the global endowment of good A are $s_A = y_A/Y_A$ and $s_A' = y_A'/Y_A'$, in periods one and two, respectively. Similarly, $s_B = y_B/Y_B$ and $s_B' = y_B'/Y_B'$. We let $\bar{s}_A' = \mathbb{E}[s_A']$ and $\bar{s}_B' = \mathbb{E}[s_B']$ denote the home country's average shares of the global endowments in period two.

2.2.1 Complete Markets, No Goods Market Frictions

When asset markets are complete internationally and there are no goods market frictions (i.e., good B is frictionlessly traded), then $P_B = \tilde{P}_B$ and $P_B' = \tilde{P}_B'$, and IMRSs in the individual goods are always equated across countries. As we show in the Appendix, in good A the IMRS is β/G_A. In good B the IMRS is β/G_B. Risk is shared perfectly, regardless of preferences.

In the case where preferences are identical, $e = 1$ and $e' = 1$. When preferences differ across countries the expressions in Eq. (37) simplify to $e = (\tilde\rho/\rho)P_B^{\theta-\tilde\theta}$ and $e' = (\tilde\rho/\rho)P_B'^{\theta-\tilde\theta}$, where $P_B = \kappa Y_A/Y_B$, $P_B' = \kappa Y_A'/Y_B'$ and

$$\kappa = \frac{(1-\tilde\theta)(1+\beta) + (\tilde\theta-\theta)(s_A + \beta\bar{s}_A')}{\tilde\theta(1+\beta) + (\theta-\tilde\theta)(s_B + \beta\bar{s}_B')}. \tag{38}$$

Hence,
$$\ln(e'/e) = (\theta - \tilde{\theta}) \ln(P'_B/P_B) = (\theta - \tilde{\theta}) \ln(G_A/G_B). \qquad (39)$$

Real exchange rate fluctuations are driven by differences in the global growth rates of the endowments of goods A and B. We see that if the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If the foreign country's preferences put more weight on good B than home country preferences (i.e.,$\tilde{\theta} < \theta$), then the foreign basket becomes relatively more expensive (the foreign country's real exchange rate appreciates).

2.2.2   Complete Markets, Good B is Non-traded

Now consider the case where asset markets are complete internationally, but good B is non-traded. In this case, in general, $P_B \neq \tilde{P}_B$. IMRSs in good A are always equated across countries: $m_A = \tilde{m}_A = \beta/G_A$. IMRSs in good B are, respectively, $m_B = \beta/g_B$ and $\tilde{m}_B = \beta/\tilde{g}_B$, so risk is not shared perfectly unless $g_B = \tilde{g}_B$ in every possible state of the world next period.

When preferences differ across countries the real exchange rates are given by Eq. (37), with prices given by

$$P_B = \kappa \frac{Y_A}{y_B}, \quad \tilde{P}_B = \tilde{\kappa} \frac{Y_A}{\tilde{y}_B}, \quad P'_B = \kappa \frac{Y'_A}{y'_B}, \quad \tilde{P}'_B = \tilde{\kappa} \frac{Y'_A}{\tilde{y}'_B}, \qquad (40)$$

and

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_A + \beta \bar{s}'_A), \quad \tilde{\kappa} = \frac{1-\tilde{\theta}}{(1+\beta)\tilde{\theta}} [1 - s_A + \beta(1 - \bar{s}'_A)]. \qquad (41)$$

This implies that

$$\ln(e'/e) = (1-\theta) \ln g_B - (1-\tilde{\theta}) \ln \tilde{g}_B + (\theta - \tilde{\theta}) \ln G_A. \qquad (42)$$

Here, the real exchange rate depends on the relative growth rates of the endowment of good B in the two countries, but the two growth rates matter to different extents due to preference differences. Additionally, as was the case when good B was traded, if the foreign country's preferences put more weight on good B than home country preferences ($\tilde{\theta} < \theta$) then, other things being equal, the foreign county's real exchange rate appreciates when the global endowment of good A grows.

If preferences are identical, then the real exchange rate in Eq. (37) simplifies to $e = (\tilde{P}_B/P_B)^{1-\theta}$ and $e' = (\tilde{P}'_B/P'_B)^{1-\theta}$ with prices still given by Eq. (40), but Eq. (41) simplified

to

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_A + \beta \bar{s}'_A), \quad \tilde{\kappa} = \frac{1-\theta}{(1+\beta)\theta}[1 - s_A + \beta(1 - \bar{s}'_A)]. \tag{43}$$

This means that

$$\ln(e'/e) = (1 - \theta)\ln(g_B/\tilde{g}_B). \tag{44}$$

Here, the real exchange rate depends entirely on the relative growth rates of the endowment of good B in the two countries. If the endowment grows more slowly in the foreign country, its basket becomes relatively more expensive and its real exchange rate appreciates.

2.2.3   Financial Autarky, No Goods Market Frictions

The third case we consider is where no assets are traded internationally, but goods markets are frictionless. In this case, $P_B = \tilde{P}_B$ and $P'_B = \tilde{P}'_B$ in every possible state of the world next period. Risk sharing, in general, is imperfect. As we show in the Appendix, the ratio of IMRSs in the two countries is the same in goods A and B. That is

$$\frac{\tilde{m}_A}{m_A} = \frac{\tilde{m}_B}{m_B} = \Xi = \frac{\theta(1 - s_A) + (1 - \theta)(1 - s_B)}{\tilde{\theta}s_A + (1 - \tilde{\theta})s_B} \times \frac{\tilde{\theta}s'_A + (1 - \tilde{\theta})s'_B}{\theta(1 - s'_A) + (1 - \theta)(1 - s'_B)}. \tag{45}$$

This expression is the same when preferences are identical, except that $\theta = \tilde{\theta}$.

In the case where preferences are identical, $e = 1$ and $e' = 1$ in every possible state of the world next period. Risk sharing, on the other hand, can be good or bad. Suppose, for example, that the home country's shares of the global endowments vary and comove positively. In this case, Ξ deviates from one a lot, implying that risk sharing is limited. On the other hand, suppose that business cycles are strongly correlated across countries, so that the home country's shares of the global endowments do not change very much across different states of the world next period. In this case, Ξ will be close to one in all states, implying a high degree of risk sharing.

When preferences differ across countries then $e = (\tilde{\rho}/\rho)P_B^{\theta-\tilde{\theta}}$ and $e' = (\tilde{\rho}/\rho)P_B'^{\theta-\tilde{\theta}}$, where $P_B = \kappa Y_A/Y_B$, $P'_B = \kappa' Y'_A/Y'_B$, and

$$\kappa = \frac{1 - \tilde{\theta} + (\tilde{\theta} - \theta)s_A}{\tilde{\theta} + (\theta - \tilde{\theta})s_B}, \qquad \kappa' = \frac{1 - \tilde{\theta} + (\tilde{\theta} - \theta)s'_A}{\tilde{\theta} + (\theta - \tilde{\theta})s'_B}. \tag{46}$$

Hence,

$$\ln(e'/e) = (\theta - \tilde{\theta})\left[\ln(G_A/G_B) + \ln(\kappa'/\kappa)\right]. \tag{47}$$

As in the case of complete markets, real exchange rate fluctuations are driven by differences in

the growth rates of the two endowments. If the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If the foreign country's preferences put more weight on good B than home country preferences ($\tilde{\theta} < \theta$) then the foreign basket becomes relatively more expensive (the foreign country's real exchange rate appreciates). But the way in which the countries' shares of the global endowments fluctuate also matters for the real exchange rate. In the example we just described, the real exchange rate rises more in states of the world where $\kappa' > \kappa$. This could reflect, for example, a rise in the foreign country's share of the global endowment of good A (a drop of s'_A) at the same time as the global endowment of A rises relative to the global endowment of B.

2.2.4 Financial Autarky, Good B is Non-traded

The final case we consider combines financial autarky with the assumption that good B is non-traded. In this case, each country simply consumes its own endowments. IMRSs in the individual goods are determined by the country-specific endowment growth rates. For good A they are $m_A = \beta/g_A$ and $\tilde{m}_A = \beta/\tilde{g}_A$. In good B they are $m_B = \beta/g_B$ and $m_B = \beta/\tilde{g}_B$. Risk is not shared unless growth rates happen to coincide. The real exchange rates in the two periods are given by Eq. (37), with

$$P_B = \frac{(1-\theta)}{\theta}\frac{y_A}{y_B}, \quad \tilde{P}_B = \frac{(1-\tilde{\theta})}{\tilde{\theta}}\frac{\tilde{y}_A}{\tilde{y}_B}, \quad P'_B = \frac{(1-\theta)}{\theta}\frac{y'_A}{y'_B}, \quad \text{and} \quad \tilde{P}'_B = \frac{(1-\tilde{\theta})}{\tilde{\theta}}\frac{\tilde{y}'_A}{\tilde{y}'_B}. \quad (48)$$

Hence

$$\ln(e'/e) = (1-\tilde{\theta})\ln(\tilde{g}_A/\tilde{g}_B) - (1-\theta)\ln(g_A/g_B). \quad (49)$$

Suppose endowment growth rates are identical across goods; i.e., $g_A = g_B$ and $\tilde{g}_A = \tilde{g}_B$. Notice that this implies $e = e'$. There is no variation in the real exchange rate. The extent of risk sharing, in contrast, depends only on whether $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. It could be good or bad. Suppose, on the other hand, that risk sharing is perfect; i.e., $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. We only get the result that $e = e'$ if $\theta = \tilde{\theta}$.

2.2.5 Discussion

Consider Table 1 from Section 1. It states that under complete markets, the observation that real exchange rates are variable only implies imperfect risk sharing when the two countries have the same consumption basket. In our model, the countries have identically-composed consumption baskets if and only if $\theta = \tilde{\theta}$, because θ and $\tilde{\theta}$ are the constant expenditure shares of good A in the two countries.

So suppose that $\theta = \tilde{\theta}$. Under complete markets, we saw that $\ln(e'/e) = 0$ and risk sharing is perfect if trade in both goods is frictionless. On the other hand, $\ln(e'/e) = (1-\theta)\ln(g_B/\tilde{g}_B)$ and $\tilde{m}_B/m_B = g_B/\tilde{g}_B$ if good B is non-traded. If one is willing to *assume* that markets are complete, *and* that countries have identical preferences, risk sharing and exchange rate changes are intimately linked in our model.

Under incomplete markets, however, there is no link, in general, between risk sharing and exchange rates, even when $\theta = \tilde{\theta}$. When $\theta = \tilde{\theta}$, and trade in both goods is frictionless, $\ln(e'/e) = 0$ yet Ξ can depart arbitrarily from one, and therefore risk sharing can be arbitrarily imperfect. When $\theta = \tilde{\theta}$, and good B is non-traded, $\ln(e'/e) = 0$ when risk sharing happens to be perfect (i.e., when $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$ in every possible state of the world next period), but we also have $\ln(e'/e) = 0$ when risk sharing is "poor" and $g_A = g_B \neq \tilde{g}_A = \tilde{g}_B$
.

More generally, our model illustrates that there is no direct link between the degree of risk sharing and real exchange rate variability.

3   Reduced-Form Models

Finally, in this section we discuss the large number of papers in the recent international asset pricing literature that provide reduced-form models of the IMRSs of domestic and foreign representative agents.[13]

We begin by formalizing the notion of a stochastic discount factor (SDF). Consider again the setup in Section 1. Let $\mathbf{R}$ denote the $k \times 1$ random vector of asset returns denominated in nominal U.S. dollars. An SDF for these dollar-denominated asset returns is a strictly positive random variable $M > 0$ such that

$$1 = \mathbb{E}\left[\mathbf{R}M\right]. \tag{50}$$

From Eqs. (4) and (5) in Section 1.1, Amy's and Bob's IMRSs over nominal U.S. dollars are both examples of SDFs for the asset returns denominated in dollars.

If there are no arbitrage opportunities in the asset returns then, under some mild regularity conditions, it is well known that there always exists an SDF that can be constructed using only the distribution of those returns. For example, define

$$\hat{M} \equiv \mathbf{R}^\top \left(\mathbb{E}\left[\mathbf{R}\mathbf{R}^\top\right]\right)^{-1}\mathbf{1}. \tag{51}$$

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[13]A few examples of these papers include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Brennan and Xia (2006); Brandt, Cochrane, and Santa-Clara (2006); Bakshi, Carr, and Wu (2008); and Lustig, Roussanov, and Verdelhan (2011).

If $\hat{M} > 0$ then this linear function of the asset returns is a valid SDF that satisfies Eq. (50), since

$$\mathbb{E}[\mathbf{R}\hat{M}] \equiv \mathbb{E}\left[\mathbf{R}\mathbf{R}^\top \left(\mathbb{E}\left[\mathbf{R}\mathbf{R}^\top\right]\right)^{-1}\mathbf{1}\right] = \mathbf{1}. \tag{52}$$

Call an SDF for the asset returns a *reduced-form* SDF if it can be constructed using only the distribution of those returns, without reference to any other variables in the economy such as quantities. More formally, M is a reduced-form SDF for the dollar-denominated asset returns, $\mathbf{R}$, if it is measurable with respect to the σ-algebra, $\sigma(\mathbf{R})$, generated by those returns.

Papers in this literature directly model Amy's IMRS over nominal U.S. dollars as a reduced-form SDF, M, for the dollar-denominated asset returns, $\mathbf{R}$. Likewise, they model Bob's IMRS over nominal U.K. pounds as a reduced-form SDF, $\tilde{M}^*$, for the same asset returns, $\mathbf{R}S/S'$, denominated instead in U.K. pounds (recall that S denotes the nominal dollar/pound exchange rate). Finally, these papers almost always assume that Amy and Bob equate IMRSs over *any* basket of goods and assets that they can frictionlessly trade with each other.[14] In particular, if Amy and Bob are assumed to always equate IMRSs over nominal U.S. dollars (which they can frictionlessly trade with each other), then

$$M = \tilde{M}^* S/S', \quad \text{or equivalently,} \quad S'/S = \tilde{M}^*/M. \tag{53}$$

Some papers in this literature prefer to work with real rather than nominal returns. These papers instead directly model Amy's IMRS over her consumption basket as a reduced-form SDF, m, for the asset returns, $\mathbf{r} \equiv \mathbf{R}P/P'$, denominated in units of her consumption basket. Likewise, they model Bob's IMRS over his consumption basket as a reduced-form SDF, $\tilde{m}$, for the same asset returns, $\tilde{\mathbf{r}} \equiv \mathbf{R}\tilde{P}/\tilde{P}'$, denominated instead in units of his consumption basket. Again, if Amy and Bob always equate IMRSs over nominal U.S. dollars (or U.K. pounds) then Eq. (53) can be written as

$$mP/P' = \tilde{m}\tilde{P}/\tilde{P}', \quad \text{or equivalently,} \quad e'/e = \tilde{m}/m. \tag{54}$$

(Once more, $e \equiv \tilde{P}/P$ and $e' \equiv \tilde{P}'/\tilde{P}$, are the real U.S./U.K. exchange rates today and next period.) There is a one-to-one mapping between models of reduced-form SDFs for real returns and those for nominal returns, $M \equiv mP/P'$ and $\tilde{M}^* \equiv (\tilde{m}\tilde{P}S')/(\tilde{P}'S)$, so their economic implications are identical.

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[14]Note that in discrete-time, if the k asset returns vary over $n > k$ states of the world, then Eq. (50) represents k equations with $n > k$ unknowns and therefore, in general, there are an infinite number of reduced-form SDFs. Thus, with $n > k$ states of the world, Amy and Bob do not *necessarily* equate IMRSs over any basket that they can frictionlessly trade with each other, but they *may*.

Again, akin to Eq. (1), it is tempting to look at Eq. (53) or Eq. (54) and conclude that changes in the real (or nominal) U.S./U.K. exchange rate can be explained by the ratio of Bob's IMRS to Amy's IMRS. Likewise, it's tempting to conclude that one can learn about properties of the ratio of Bob's IMRS to Amy's IMRS by observing the growth in the (real or nominal) U.S/U.K. exchange rate. However, as we argued in Section 1, the real exchange rate can only vary if either (or both) the composition of Amy's and Bob's consumption baskets differs, or they face different prices for identical goods and services in their baskets. A reduced-form model of their IMRSs provides no economic insights on either front. In short, a reduced-form model cannot be used to explain or understand period-by-period changes in the real exchange rate because it does not contain the necessary ingredients that we provided in Section 1.4. Moreover, the reduced-form SDFs, m and $\tilde{m}$ (or M and $\tilde{M}^*$), are functions of the asset returns themselves, which include the period-by-period variation in the real (or nominal) exchange rate. Therefore, exchange rate growth cannot be *both* an input *and* an output of these reduced-form models.[15]

Furthermore, there are at least a couple of reasons to question whether reduced-form models of m and $\tilde{m}$ can reliably be interpreted as Amy's and Bob's IMRSs, respectively. First, it is impossible to empirically test, using only asset returns, whether a reduced-form SDF is equal to an agent's IMRS. Of course, one *can* test whether SDFs, m and $\tilde{m}$, price the asset returns,

$$1 = \mathbb{E}[\mathbf{r}m] \quad \text{and} \quad 1 = \mathbb{E}[\tilde{\mathbf{r}}\tilde{m}], \tag{55}$$

(Again, $\mathbf{r} \equiv \mathbf{R}P/P'$ and $\tilde{\mathbf{r}} \equiv \mathbf{R}\tilde{P}/\tilde{P}'$ are the asset returns denominated in units of Amy's and Bob's consumption baskets respectively.) However, while Eq. (54) implies Eq. (55), the reverse implication does not hold in general. The second reason to give pause is that the set of asset returns that are used to construct the reduced-form SDFs, m and $\tilde{m}$, must completely span agents' IMRSs. Instead, papers in this literature construct their reduced-form models using only a very small subset of the assets that are available to agents. For example, the returns that are used to construct the reduced-form SDFs in these papers usually include nominal currency investments, as well as zero-coupon bonds in two (occasionally more) countries or options on foreign currencies. Even if one accepts the assumption that asset markets span agents' IMRSs, one might reasonably expect that the set of required assets includes the major equity markets, other government bond markets, corporate bond markets,

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[15]Brennan and Xia (2006) provide a notable exception. In the empirical section of the paper, they assume that m can be identified using only domestic government bonds and $\tilde{m}$ can be identified using only foreign government bonds. The growth in the exchange rate, e'/e, is not an input to either exercise. However, it's not clear what one can reliably conclude from this exercise. Even if asset markets are complete, so that Eq. (54) holds for some m and $\tilde{m}$, there is no general economic restriction that m is only a function of the returns on domestic assets and $\tilde{m}$ is only a function of the returns on foreign assets.

other foreign currencies, and perhaps even commodity and real estate markets.

3.1 Comparison to Standard Approach with a Single SDF

Papers in the broader asset pricing literature model a *single* reduced-form SDF, rather than the *multiple* reduced-form SDFs that are modeled in the recent international asset pricing literature. In this section we compare these two approaches. To do so, it is important to first understand how a *single* SDF is affected by a change in the numeraire that is used to denominate the *same* asset returns.

An SDF for the asset returns denominated in nominal U.S. dollars effectively assigns a strictly positive dollar value to each state of the world next period. To illustrate, suppose that the returns on the k assets vary over $n \geq k$ states of the world next period, indexed by $z = 1, 2, \ldots, n$. Let $\pi(z)$ be the probability that state z occurs next period, so that Eq. (50) can be written more explicitly as

$$\mathbf{1} = \mathbb{E}\left[\mathbf{R}M\right] \equiv \sum_{z=1}^{n} \pi\left(z\right) \mathbf{R}(z) M\left(z\right) . \tag{56}$$

In Eq. (56), $\pi\left(z\right) M\left(z\right)$ is commonly interpreted as the dollar price today of a claim that pays one dollar next period when state z occurs (i.e., the price of an Arrow-Debreu state contingent claim), whether or not such claims are available in asset markets.

What price today does a given SDF, M, for asset returns denominated in nominal U.S. dollars, assign to a claim that pays one unit of a different numeraire (i.e., something other than nominal U.S. dollars) in state z next period? To address this question, let Y be the nominal U.S. dollar value today of the new numeraire, and let $Y'\left(z\right)$ denote its dollar value in state z next period. A claim that pays one unit of the new numeraire in state z next period is equivalent to $Y'\left(z\right)$ units of a claim that pays one U.S. dollar in state z next period. From Eq. (56), for a given SDF, M, one unit of this U.S. dollar state contingent claim is worth $\pi\left(z\right) M\left(z\right)$ dollars today. Therefore, $Y'\left(z\right)$ units of it are worth $\pi\left(z\right) M\left(z\right) Y'\left(z\right)$ dollars today, or equivalently, $\pi\left(z\right) M\left(z\right) Y'\left(z\right)/Y$ units of the new numeraire (since one dollar today is worth $1/Y$ units of the new numeraire). Thus, if M is an SDF for the asset returns denominated in U.S. dollars then MY'/Y is that *same* SDF for the *same* asset returns denominated in the new numeraire.[16]

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[16]It is straightforward to verify this change of numeraire units for an SDF. The *same* asset returns denominated in this new numeraire are $\mathbf{R}Y/Y'$, and therefore,

$$\mathbf{1} = \mathbb{E}\left[\mathbf{R}M\right] \equiv \mathbb{E}\left[\mathbf{R}\frac{Y}{Y'}M\frac{Y'}{Y}\right], \quad \text{since} \quad \mathbf{R}\frac{Y}{Y'}M\frac{Y'}{Y} \equiv \mathbf{R}M .$$

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For example, if M is a SDF for the asset returns denominated in nominal U.S. dollars, then MS'/S is that *same* SDF when the *same* asset returns, $\mathbf{R}S/S'$, are instead denominated in nominal U.K. pounds. If the *same* asset returns, $\mathbf{R}P/P'$, are instead denominated in units of Amy's consumption basket in the U.S., then MP'/P is that *same* SDF. If the same asset returns, $\mathbf{R}\tilde{P}/\tilde{P}'$, are denominated in units of Bob's consumption basket in the U.K., then $M\tilde{P}'/\tilde{P}$ is that *same* SDF. And so on.

Given the nominal U.S./U.K. exchange rate, S'/S, and a reduced-form SDF, M, one can always *define* $\tilde{M}^* \equiv MS'/S$ and Eq. (53) is, *by definition*, guaranteed to hold. Therefore, any model of two reduced-form SDFs, say M and $\tilde{M}^*$, that are assumed to satisfy Eq. (53), is mathematically *identical* to a reduced-form model of the growth in the nominal U.S./U.K. exchange rate, S'/S, and a *single* reduced-form SDF (either M or $\tilde{M}^*$). Likewise, a model of two reduced-form SDFs, m and $\tilde{m}$, that are assumed to satisfy Eq. (54) is mathematically *identical* to a reduced-form model of the growth in the real U.S./U.K. exchange rate, e'/e, and a *single* reduced-form SDF (either m or $\tilde{m}$).

Therefore, the approach of modeling two reduced-form SDFs and assuming that Eq. (53) or Eq. (54) holds, is mathematically *indistinguishable* from the standard approach of modeling a single reduced-form SDF and the growth in the (nominal or real) U.S./U.K. exchange rate. The standard approach of modeling a single reduced-form SDF always applies, regardless of whether asset markets are complete or incomplete. Moreover, a single reduced-form SDF is not typically associated with any particular agent, since it applies equally well to any agent who can trade the assets that are used to construct that reduced-form SDF.

3.2   Incomplete Markets

In Section 3.1 we argued that any model of two reduced-form SDFs, say M and $\tilde{M}^*$, that are assumed to satisfy Eq. (53), is mathematically *identical* to a reduced-form model of the growth in the nominal U.S./U.K. exchange rate, S'/S, and a *single* reduced-form SDF (either M or $\tilde{M}^*$). Brandt and Santa-Clara (2002) provide reduced-form models of M and $\tilde{M}^*$, but they do *not* impose that Eq. (53) holds. In other words, they do not assume that asset markets are complete so that Amy and Bob always equate IMRSs over *any* basket of goods and assets that they can frictionlessly trade with each other. Instead, they assume that

$$MS'/S = \tilde{M}^* O\,, \quad \text{or equivalently,} \quad \frac{S'}{S} = \frac{\tilde{M}^*}{M} O\,, \tag{57}$$

where $\mathbb{E}[O] = 1$ and O is independent of M, $\tilde{M}^*$, and all assets.[17] They state that "the key insight of our model is that when markets are incomplete, the volatility of the exchange rate is not uniquely determined by the domestic and foreign stochastic discount factors."

There are two problems with the reduced-form approach in Brandt and Santa-Clara (2002). The most fundamental issue is that they effectively specify two *different* reduced-form SDFs, M and $\tilde{M}^*S/S'$, for the same asset returns, $\mathbf{R}$, denominated in U.S. dollars.[18] In general, there *is* more than one reduced-form SDF that prices the asset returns. However, Amy and Bob can frictionlessly trade the assets and U.S. dollars, and therefore any reduced-form SDF (constructed from the distribution of the dollar-denominated asset returns) applies equally well to both Amy *and* Bob. In other words, there is never a benefit to modeling more than one reduced-form SDF (even when more than one exists) since the association of a specific agent with a specific reduced-form SDF is completely arbitrary. For example, Brandt and Santa-Clara (2002) associate Amy with the reduced-form SDF, M, and Bob with the reduced-form SDF, $\tilde{M}^*S/S'$, but there are no economic restrictions that prevent one from switching these associations. Equally as important, the standard approach of modeling a single reduced-form SDF places no additional restrictions on the dynamics of the exchange rate.

The second related issue in Brandt and Santa-Clara (2002) is that their particular model assigns two different prices to the same zero-coupon bond, which implies that the two SDFs that they specify in their model cannot both simultaneously satisfy Eq. (50). They use their reduced-form model of M and $\tilde{M}^*$ to price foreign and domestic zero-coupon bonds. For example, if a zero-coupon bond that pays one U.K. pound next period costs B^* pounds today, then

$$B^* = \mathbb{E}[1MS'/S] \quad \text{and} \quad B^* = \mathbb{E}[1\tilde{M}^*]. \tag{58}$$

Brandt and Santa-Clara (2002) note that Eq. (58) can be satisfied if the growth in the exchange rate, S'/S, is given by Eq. (57). However, the assumption in Eq. (57) that O is independent of M, $\tilde{M}^*$, and all assets, introduces an arbitrage opportunity into their model. In particular, both M and $\tilde{M}^*$ must also price a one-period U.S. dollar zero-coupon bond that costs B today, so that

$$B = \mathbb{E}[1M] \quad \text{and} \quad B = \mathbb{E}[1\tilde{M}^*S/S']. \tag{59}$$

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[17]See equation (24) on page 176 in Brandt and Santa-Clara (2002).

[18]Recall that if $\tilde{M}^*$ is an SDF for those asset returns, $\mathbf{R}S/S'$, denominated in pounds, then

$$\mathbf{1} = \mathbb{E}[(\mathbf{R}S/S')\,\tilde{M}^*].$$

Therefore, $\tilde{M}^*S/S'$ is an SDF for the dollar-denominated asset returns, $\mathbf{R}$.

This additional economic restriction in Eq. (59) reveals a contradiction (or internal inconsistency) in their model, since (by Jensen's inequality)

$$B = \mathbb{E}[1\tilde{M}^*S/S'] \equiv \mathbb{E}\left[1M/O\right] = \mathbb{E}\left[1M\right]\mathbb{E}\left[1/O\right] > \mathbb{E}\left[1M\right]/\mathbb{E}\left[O\right] = \mathbb{E}\left[1M\right] = B\,. \quad (60)$$

That is, the model in Brandt and Santa-Clara (2002) is not free of arbitrage opportunities, since it assigns two different prices to the same zero-coupon bond.[19]

3.3   Long's Numeraire Portfolio

In Section 3 we defined a reduced-form SDF as one that can be constructed using only the distribution of the asset returns. For example, from Eq. (51),

$$\hat{M} \equiv \mathbf{R}^\top \left(\mathbb{E}\left[\mathbf{RR}^\top\right]\right)^{-1}\mathbf{1} \quad (61)$$

is the linear reduced-form (minimum variance) SDF for the asset returns denominated in U.S. dollars (when $\hat{M}$ is strictly positive). In Section 3.1 we showed that if M is an SDF for the asset returns denominated in U.S. dollars then MY'/Y is that *same* SDF for the *same* asset returns denominated in a new numeraire with dollar price Y. However, as we showed in Section 1.2.1, $\hat{M}Y'/Y$ is *not* the linear reduced-form (minimum variance) SDF for the asset returns denominated in this different numeraire, since (in general)

$$\hat{M}Y'/Y \neq \mathbf{R}_Y^\top \left(\mathbb{E}\left[\mathbf{R}_Y\mathbf{R}_Y^\top\right]\right)^{-1}\mathbf{1}\,, \quad (62)$$

where, for notational convenience, $\mathbf{R}_Y \equiv \mathbf{R}Y/Y'$.[20]

Is there a reduced-form SDF that does not depend on the numeraire that is used to denominate the returns? Yes.

Let $\hat{\boldsymbol{\theta}}$ denote the the optimal portfolio weights for an investor who maximizes expected log wealth next period denominated in U.S. dollars,

$$\hat{\boldsymbol{\theta}} \equiv \arg\max_{\mathbf{1}^\top\boldsymbol{\theta}=1}\mathbb{E}\left[\ln(\mathbf{R}^\top\boldsymbol{\theta})\right]\,. \quad (63)$$

The first order condition for optimality in Eq. (63) is

$$\mathbf{0} = \mathbb{E}[\mathbf{R}/(\mathbf{R}^\top\hat{\boldsymbol{\theta}})] - \hat{\lambda}\mathbf{1}\,,$$

[19]Similarly, Anderson et al. (2010) show that, in the special case of an affine setting, the assumptions in Brandt and Santa-Clara (2002) are infeasible.

[20]The simplest proof of the inequality in Eq. (62) is to note that the left hand side is linear in $\mathbf{R}Y'/Y$ while the right hand side is instead linear in $\mathbf{R}_Y \equiv \mathbf{R}Y/Y'$.

where $\hat{\lambda}$ denotes the Lagrange multiplier on the constraint that the optimal portfolio weights must sum to one (i.e., $\mathbf{1}^\top \hat{\boldsymbol{\theta}} = 1$). If we multiply this first order condition through by $\hat{\boldsymbol{\theta}}$ then

$$\mathbf{0}^\top \hat{\boldsymbol{\theta}} = \mathbb{E}[(\mathbf{R}^\top \hat{\boldsymbol{\theta}})/(\mathbf{R}^\top \hat{\boldsymbol{\theta}})] - \hat{\lambda}\mathbf{1}^\top \hat{\boldsymbol{\theta}} \quad \Rightarrow \quad 1 = \mathbb{E}[\mathbf{R}/(\mathbf{R}^\top \hat{\boldsymbol{\theta}})]. \tag{64}$$

To our knowledge, Long (1990) was the first to recognize from Eq. (64) that $(\mathbf{R}^\top \hat{\boldsymbol{\theta}})^{-1}$ is an SDF that satisfies Eq. (50) for the asset returns denominated in U.S. dollars. He referred to $\mathbf{R}^\top \hat{\boldsymbol{\theta}}$ as the *numeraire portfolio*.

Interestingly, the optimal portfolio weights, $\hat{\boldsymbol{\theta}}$, in Eqs. (63) and (64) do not depend on the numeraire that is used to denominated the asset returns. Again, let Y and Y' be the U.S. dollar value today and next period of a different numeraire, so that $\mathbf{R}_Y \equiv \mathbf{R}Y/Y'$ are the returns on the *same* assets when they are instead denominated in this numeraire. Let $\hat{\boldsymbol{\theta}}_Y$ denote the optimal portfolio weights for an investor who maximizes expected log wealth next period, denominated in this different numeraire, so that

$$\hat{\boldsymbol{\theta}}_Y \equiv \arg \max_{\mathbf{1}^\top \boldsymbol{\theta}=1} \mathbb{E}\left[\ln(\mathbf{R}_Y^\top \boldsymbol{\theta})\right] \equiv \arg \max_{\mathbf{1}^\top \boldsymbol{\theta}=1} \left\{\mathbb{E}\left[\ln(\mathbf{R}^\top \boldsymbol{\theta})\right] - \mathbb{E}\left[\ln\left(Y'/Y\right)\right]\right\}. \tag{65}$$

There are two equivalent ways to see that the optimal portfolio weights do not depend on the choice of numeraire that is used to denominate the asset returns (i.e., $\hat{\boldsymbol{\theta}}_Y = \hat{\boldsymbol{\theta}}$). First, the optimization problems in Eqs. (63) and (65) are identical since

$$\arg \max_{\mathbf{1}^\top \boldsymbol{\theta}=1} \left\{\mathbb{E}\left[\ln(\mathbf{R}^\top \boldsymbol{\theta})\right] - \mathbb{E}\left[\ln\left(Y'/Y\right)\right]\right\} = \arg \max_{\mathbf{1}^\top \boldsymbol{\theta}=1} \mathbb{E}\left[\ln(\mathbf{R}^\top \boldsymbol{\theta})\right]. \tag{66a}$$

Second, and obviously related, $\hat{\boldsymbol{\theta}}$ satisfies the first order condition in Eq. (64), if and only if it also satisfies the first order condition for the optimization problem in Eq. (65), since

$$\frac{\mathbf{R}_Y}{\mathbf{R}_Y^\top \hat{\boldsymbol{\theta}}} \equiv \frac{\mathbf{R}Y/Y'}{(\mathbf{R}Y/Y')^\top \hat{\boldsymbol{\theta}}} \equiv \frac{\mathbf{R}}{\mathbf{R}^\top \hat{\boldsymbol{\theta}}}. \tag{66b}$$

Thus, given the optimal portfolio weights, $\hat{\boldsymbol{\theta}}$, in Eq. (63), $(\mathbf{r}^\top \hat{\boldsymbol{\theta}})^{-1}$ is an SDF for the asset returns denominated in units of Amy's consumption basket (where, again, $\mathbf{r} \equiv \mathbf{R}P/P'$), and $(\tilde{\mathbf{r}}^\top \hat{\boldsymbol{\theta}})^{-1}$ is an SDF for the asset returns denominated in units of Bob's consumption basket (where $\tilde{\mathbf{r}} \equiv \mathbf{R}\tilde{P}/\tilde{P}'$). Trivially, the asset market view of exchange rates holds for these SDFs constructed from asset returns, since

$$\ln[1/(\tilde{\mathbf{r}}^\top \hat{\boldsymbol{\theta}})] - \ln[1/(\mathbf{r}^\top \hat{\boldsymbol{\theta}})] \equiv \ln(\tilde{P}'/\tilde{P}) - \ln(P'/P) \equiv \ln e' - \ln e. \tag{67}$$

However, Eq. (67) provides no economic insights or restrictions, because the real U.S./U.K.

exchange rate could literally be anything and Eq. (67) would still hold.

4 Conclusion

The recent literature in international finance has used the asset market view of real exchange rates to explain and interpret a number of empirical properties of both real and nominal exchange rates. In this paper we showed that, alone, the asset market view is not useful for understanding, or giving an economic interpretation to, changes in exchange rates. Instead, we argue that in order to explain how real exchange rates are determined, it is necessary to make specific assumptions about preferences, frictions in the market for goods and services, the assets agents can trade, and the nature of endowments or production.

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A Appendix

In this appendix we provide detailed solutions for the model in Section 2. It is only for completeness and is not intended for publication.

A.1 Aggregate Prices

The overall consumption aggregate for the domestic household is $c(c_A, c_B)$. Given a particular set of prices (in an arbitrary numeraire) for the individual goods, we can solve the household's static expenditure minimization problem

$$\min_{c_A, c_B} P_A c_A + P_B c_B \quad \text{subject to} \quad c = c(c_A, c_B). \tag{68}$$

Because $c(\cdot)$ is a homogenous of degree one function, minimized expenditure is equal to Pc where $P = H(P_A, P_B)$. The function $H(\cdot)$ is also homogenous of degree one in its arguments, and is related to the function $c(\cdot)$ [see Varian (1984)]. The aggregate price index has the interpretation of being the Lagrange multiplier on the constraint at the optimum. To see this, notice that the first order conditions for the expenditure minimization problem are

$$P_A = \theta c_{c_A}(c_A, c_B) \quad P_B = \theta c_{c_B}(c_A, c_B). \tag{69}$$

Multiplying these through these conditions by c_A and c_B and adding up you get $P_A c_A + P_B c_B = \theta c$ hence $P = \theta$. We also have

$$H(P_A, P_B) = H[Pc_{c_A}(\cdot), Pc_{c_B}(\cdot)] = PH[c_{c_A}(\cdot), c_{c_B}(\cdot)],$$

establishing that at the optimum, $H[c_{c_A}(\cdot), c_{c_B}(\cdot)] = 1$.

Of course, a similar approach may be used for the foreign household.

A.2 Overall Marginal Utility

The asset payoffs, and all variables in period two, depend on the state of the world in period two. For concreteness, in this appendix we assume that the state of the world is indexed by $z \in \mathcal{Z} = \{1, 2, \ldots, n\}$, with n finite. The assumption that the number possible states of the world in period two is finite, or even countable, is not important and is only for ease of exposition.

By nesting the expenditure minimization problem described in Section A.1 within the domestic household's problem, we can rewrite the latter as follows. The household in the

home country chooses c, $c'(z)$, and $\mathbf{a}$ to maximize

$$u(c) + \beta \sum_{z=1}^{n} u\left[c'(z)\right] \pi(z) \tag{70}$$

subject to

$$Pc + \mathbf{P_X} \cdot \mathbf{a} = y_A + P_B y_B \,, \tag{71}$$

$$P'(z)c'(z) = y'_A(z) + P'_B(z)y'_B(z) + \mathbf{X}(z) \cdot \mathbf{a} \,, \quad z = 1, \dots, n. \tag{72}$$

The first order conditions for c, $c'(z)$, and $\mathbf{a}$ are

$$u_c(c) = P\lambda \,, \tag{73}$$

$$\beta u_c[c'(z)]\pi(z) = P'(z)\mu(z) \,, \quad z = 1, \dots, n, \tag{74}$$

$$\mathbf{P_X}\lambda = \sum_{z=1}^{n} \mu(z)\mathbf{X}(z) \,. \tag{75}$$

Here λ is the Lagrange multiplier on the constraint (73), and $\mu(z)$ is the Lagrange multiplier on the constraint (74). So, combining (73) and (74), we get the following expression for the home household's discounted marginal utility growth, or intertemporal marginal rate of substitution, defined over its basket:

$$m(z) = \frac{\beta u_c[c'(z)]}{u_c(c)} = \frac{P'(z)}{P}\frac{\mu(z)}{\lambda\pi(z)} \,. \tag{76}$$

The household in the foreign country chooses $\tilde{c}$, $\{\tilde{c}'(z)\}_{z=1}^{n}$, and $\tilde{\mathbf{a}}$ to maximize

$$u(\tilde{c}) + \beta \sum_{z=1}^{n} u\left[\tilde{c}'(z)\right] \pi(z) \tag{77}$$

subject to

$$\tilde{P}\tilde{c} + \mathbf{P_X} \cdot \tilde{\mathbf{a}} = \tilde{y}_A + \tilde{P}_B \tilde{y}_B \,, \tag{78}$$

$$\tilde{P}'(z)\tilde{c}'(z) = \tilde{y}'_A(z) + \tilde{P}'_B(z)\tilde{y}'_B(z) + \mathbf{X}(z) \cdot \tilde{\mathbf{a}} \,, \quad z = 1, \dots, n. \tag{79}$$

The first order conditions for $\tilde{c}$, $\{\tilde{c}'(z)\}_{z=1}^{n}$, and $\tilde{\mathbf{a}}$ are

$$u_c(\tilde{c}) = \tilde{P}\tilde{\lambda} \,, \tag{80}$$

$$\beta u_c[\tilde{c}'(z)]\pi(z) = \tilde{P}'(z)\tilde{\mu}(z) \,, \quad z = 1, \dots, n, \tag{81}$$

$$\mathbf{P_X}\tilde{\lambda} = \sum_{z=1}^{n} \tilde{\mu}(z)\mathbf{X}(z) \,. \tag{82}$$

Here $\tilde{\lambda}$ is the Lagrange multiplier on the constraint (80), and $\tilde{\mu}(z)$ is the Lagrange multiplier on the constraint (81). So, combining (80) and (81), we get the following expression for the home household's discounted marginal utility growth, or intertemporal marginal rate of substitution, defined over its basket:

$$\tilde{m}(z) = \frac{\beta u_c[\tilde{c}'(z)]}{u_c(\tilde{c})} = \frac{\tilde{P}'(z)}{\tilde{P}} \frac{\tilde{\mu}(z)}{\tilde{\lambda}\pi(z)} \,. \tag{83}$$

Notice that m is an SDF for payoffs and prices measured in home country basket units. This is because Eqs. (75) and (76) combined imply

$$\frac{\mathbf{P_X}}{P} = \sum_{z=1}^{n} m(z)\frac{\mathbf{X}(z)}{P'(z)}\pi(z) \,. \tag{84}$$

Similarly, $\tilde{m}$ is an SDF for payoffs and prices measured in foreign country basket units. This is because Eqs. (82) and (83) combined imply

$$\frac{\mathbf{P_X}}{\tilde{P}} = \sum_{z=1}^{n} \tilde{m}(z)\frac{\mathbf{X}(z)}{\tilde{P}'(z)}\pi(z) \,. \tag{85}$$

From (76) and (83), the ratio of $\tilde{m}$ to m is

$$\frac{\tilde{m}(z)}{m(z)} = \left[\frac{\tilde{P}'(z)}{\tilde{P}}\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \bigg/ \left[\frac{P'(z)}{P}\frac{\mu(z)}{\lambda}\right] = \left[\frac{e'(z)}{e}\right] \cdot \left[\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \bigg/ \left[\frac{\mu(z)}{\lambda}\right] \,. \tag{86}$$

We define
$$\Xi(z) = \left[\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \bigg/ \left[\frac{\mu(z)}{\lambda}\right] \,.$$

Notice that since good A is the numeraire, the first order conditions for c_A and $\tilde{c}_A$, given in Eq. (69), along with Eqs. (73) and (80) imply that the time one marginal utilities of good A in the two countries are

$$u_c(c)c_{c_A}(c_A, c_B) = \lambda \quad u_c(\tilde{c})\tilde{c}_{c_A}(\tilde{c}_A, c_B) = \tilde{\lambda} \,. \tag{87}$$

Similarly, when these first order conditions are combined with Eqs. (74) and (81), we get

expressions for the time two discounted marginal utilities of good A in the two countries:

$$\beta u_c[c'(z)]c_{c_A}[c'_A(z), c'_B(z)] = \mu(z)/\pi(z), \quad z = 1, \ldots, n, \tag{88}$$

$$\beta u_c[\tilde{c}'(z)]\tilde{c}_{c_A}[\tilde{c}'_A(z), \tilde{c}'_B(z)] = \tilde{\mu}(z)/\pi(z), \quad z = 1, \ldots, n. \tag{89}$$

Thus, $m_A(z) = \mu(z)/[\lambda\pi(z)]$ and $\tilde{m}_A(z) = \tilde{\mu}(z)/[\tilde{\lambda}\pi(z)]$ are the discounted marginal utility growths of good A in the two countries. Consequently,

$$\frac{\tilde{m}(z)}{m(z)} = \left[\frac{e'(z)}{e}\right] \cdot \Xi(z), \tag{90}$$

with $\Xi(z) = \tilde{m}_A(z)/m_A(z)$ being a measure of risk sharing in the frictionlessly traded good (good A).

The first order conditions for c_B and $\tilde{c}_B$, given in Eq. (69), along with Eqs. (73) and (80) imply that the time one marginal utilities of good B in the two countries are

$$u_c(c)c_{c_B}(c_A, c_B) = \lambda P_B \quad u_c(\tilde{c})\tilde{c}_{c_B}(\tilde{c}_A, c_B) = \tilde{\lambda}\tilde{P}_B. \tag{91}$$

Similarly, when these first order conditions are combined with Eqs. (74) and (81), we get expressions for the time two discounted marginal utilities of good B in the two countries:

$$\beta u_c[c'(z)]c_{c_B}[c'_A(z), c'_B(z)] = \mu(z)P'_B(z)/\pi(z), \quad z = 1, \ldots, n, \tag{92}$$

$$\beta u_c[\tilde{c}'(z)]\tilde{c}_{c_B}[\tilde{c}'_A(z), \tilde{c}'_B(z)] = \tilde{\mu}(z)\tilde{P}'_B(z)/\pi(z), \quad z = 1, \ldots, n. \tag{93}$$

Thus, $m_B(z) = m_A(z)P'_B(z)/P_B$ and $\tilde{m}_B(z) = \tilde{m}_A(z)\tilde{P}'_B(z)/\tilde{P}_B$ are the discounted marginal utility growths of good B in the two countries. Consequently, $\Xi(z)[\tilde{P}'_B(z)/\tilde{P}_B]/[P'_B(z)/P_B]$ is a measure of how well risk is shared in good B. If good B is frictionlessly traded the price terms in this expression cancel out and the measure of risk sharing in good B is also $\Xi(z)$.

When the securities span variation in households' marginal utilities (i.e., if financial markets are complete) the first order conditions for a and a become equivalent to

$$\boldsymbol{\psi}\lambda = \boldsymbol{\mu}, \qquad \boldsymbol{\psi}\tilde{\lambda} = \tilde{\boldsymbol{\mu}}, \tag{94}$$

where $\boldsymbol{\mu}$ is an $n \times 1$ vector whose zth element is $\mu(z)$, $\tilde{\boldsymbol{\mu}}$ is an $n \times 1$ vector whose zth element is $\tilde{\mu}(z)$ and $\boldsymbol{\psi}$ is an $n \times 1$ vector whose zth element is $\psi(z)$, the price of a claim that pays one unit of good A in state z. Notice that when financial markets are complete, this implies $m_A(z) = \tilde{m}_A(z) = \psi(z)/\pi(z)$ and $\Xi(z) = 1$.

A.3 Equilibrium in the Special Cases

To solve the model in the special cases we assume from the start that there is a complete set of state contingent securities indexed by z. Security z pays one unit of good A in state z and zero otherwise. It's price is $\psi(z)$ in the home country and $\tilde{\psi}(z)$ in the foreign country. If there is international trade in these assets (the complete markets case), we have $\psi(z) = \tilde{\psi}(z)$. Under financial autarky, the prices can be different.

The first order conditions for the individual consumption goods and holdings of the securities are

$$\theta c_A^{-1} = \lambda, \tag{95}$$

$$(1 - \theta)c_B^{-1} = P_B\lambda, \tag{96}$$

$$\beta\theta c_A'(z)^{-1}\pi(z) = \mu(z), \quad z = 1, \ldots, n, \tag{97}$$

$$\beta(1 - \theta)c_B'(z)^{-1}\pi(z) = P_B'(z)\mu(z), \quad z = 1, \ldots, n, \tag{98}$$

$$\psi(z)\lambda = \mu(z), \quad z = 1, \ldots, n. \tag{99}$$

$$\tilde{\theta}\tilde{c}_A^{-1} = \tilde{\lambda}, \tag{100}$$

$$(1 - \tilde{\theta})\tilde{c}_B^{-1} = \tilde{P}_B\tilde{\lambda}, \tag{101}$$

$$\beta\tilde{\theta}\tilde{c}_A'(z)^{-1}\pi(z) = \tilde{\mu}(z), \quad z = 1, \ldots, n, \tag{102}$$

$$\beta(1 - \tilde{\theta})\tilde{c}_B'(z)^{-1}\pi(z) = \tilde{P}_B'(z)\tilde{\mu}(z), \quad z = 1, \ldots, n, \tag{103}$$

$$\tilde{\psi}(z)\tilde{\lambda} = \tilde{\mu}(z), \quad z = 1, \ldots, n. \tag{104}$$

We can rewrite the first order conditions for the consumptions, using the first order conditions for the securities, as:

$$\theta = \lambda c_A \tag{105}$$

$$1 - \theta = \lambda c_B P_B \tag{106}$$

$$\beta\theta = \frac{\psi(z)\lambda}{\pi(z)}c_A'(z) \tag{107}$$

$$\beta(1 - \theta) = \frac{\psi(z)\lambda}{\pi(z)}c_B'(z)P_B'(z) \tag{108}$$

$$\tilde{\theta} = \tilde{\lambda}\tilde{c}_A \tag{109}$$

$$1 - \tilde{\theta} = \tilde{\lambda}\tilde{P}_B\tilde{c}_B \tag{110}$$

$$\beta\tilde{\theta} = \frac{\tilde{\psi}(z)\tilde{\lambda}}{\pi(z)}\tilde{c}'_A(z) \tag{111}$$

$$\beta(1-\tilde{\theta}) = \frac{\tilde{\psi}(z)\tilde{\lambda}}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{112}$$

Here, we have dropped the $z = 1, \ldots, n$, from the equations for convenience.

In what follows we will use the notation $L = \lambda^{-1}$, $\tilde{L} = \tilde{\lambda}^{-1}$. From Eqs. (105), (105), (105) and (105), we see that L and $\tilde{L}$ are the households' respective total expenditures on goods in period one. We also define the global endowments: $Y_A = y_A + \tilde{y}_A$, $Y_B = y_B + \tilde{y}_B$, $Y'_A(z) = y'_A(z) + \tilde{y}'_A(z)$, $Y'_B(z) = y'_B(z) + \tilde{y}'_B(z)$. Additionally we define $G_A(z) = Y'_A(z)/Y_A$, $G_B(z) = Y'_B(z)/Y_B$, $g_A(z) = y'_A(z)/y_A$, $g_B(z) = y'_B(z)/y_B$, $\tilde{g}_A(z) = \tilde{y}'_A(z)/\tilde{y}_A$, $\tilde{g}_B(z) = \tilde{y}'_B(z)/\tilde{y}_B$, We also use the following notation

$$s_A = y_A/Y_A \quad s_B = y_B/Y_B \quad s'_A(z) = y'_A(z)/Y'_A(z) \quad s'_B(z) \equiv y'_B(z)/Y'_B(z)$$

$$\bar{s}'_A = \sum_{z=1}^{n} s'_A(z)\pi(z) \quad \bar{s}'_B = \sum_{z=1}^{n} s'_B(z)\pi(z)$$

A.3.1 When International Asset Markets are Complete

Here we have $\psi(z) = \tilde{\psi}(z)$, which allows us to rewrite the first order conditions for the consumptions as

$$\theta L = c_A \tag{113}$$

$$(1-\theta)L = c_B P_B \tag{114}$$

$$\beta\theta L = \frac{\psi(z)}{\pi(z)}c'_A(z) \tag{115}$$

$$\beta(1-\theta)L = \frac{\psi(z)}{\pi(z)}c'_B(z)P'_B(z) \tag{116}$$

$$\tilde{\theta}\tilde{L} = \tilde{c}_A \tag{117}$$

$$(1-\tilde{\theta})\tilde{L} = \tilde{P}_B\tilde{c}_B \tag{118}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\psi(z)}{\pi(z)}\tilde{c}'_A(z) \tag{119}$$

$$\beta(1-\tilde{\theta})\tilde{L} = \frac{\psi(z)}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{120}$$

The home country household's lifetime budget constraint is

$$c_A + P_B c_B + \sum_{z=1}^{n} \psi(z) \left[c'_A(z) + P'_B(z) c'_B(z) \right] = y_A + P_B y_B + \sum_{z=1}^{n} \psi(z) \left[y'_A(z) + P'_B(z) y'_B(z) \right]$$

(121)

From Eqs. (113), (115), (113), and (115) we see that discounted marginal utility growth in good A in the two countries are equated:

$$m_A(z) = \beta \frac{c_A}{c'_A(z)} = \frac{\psi(z)}{\pi(z)} \qquad \tilde{m}_A(z) = \beta \frac{\tilde{c}_A}{\tilde{c}'_A(z)} = \frac{\psi(z)}{\pi(z)}$$

(122)

From Eqs. (114), (116), (114), and (116), discounted marginal utility growths in good B are

$$m_B(z) = \beta \frac{c_B}{c'_B(z)} = \frac{\psi(z)}{\pi(z)} \frac{P'_B(z)}{P_B} \qquad \beta \frac{\tilde{c}_B}{\tilde{c}'_B(z)} = \frac{\psi(z)}{\pi(z)} \frac{\tilde{P}'_B(z)}{\tilde{P}_B}$$

(123)

A.3.2 When Good B is Traded

The market clearing conditions for good A are

$$c_A + \tilde{c}_A = Y_A$$

(124)

$$c'_A(z) + \tilde{c}'_A(z) = Y'_A(z)$$

(125)

$$c_B + \tilde{c}_B = Y_B$$

(126)

$$c'_B(z) + \tilde{c}'_B(z) = Y'_B(z)$$

(127)

These market clearing conditions, together with the first order conditions, (113)–(120), imply

$$\theta L + \tilde{\theta}\tilde{L} = Y_A$$

(128)

$$\beta(\theta L + \tilde{\theta}\tilde{L}) = \frac{\psi(z)}{\pi(z)} Y'_A(z)$$

(129)

$$(1-\theta)L + (1-\tilde{\theta})\tilde{L} = P_B Y_B$$

(130)

$$\beta[(1-\theta)L + (1-\tilde{\theta})\tilde{L}] = \frac{\psi(z)}{\pi(z)} P'_B(z) Y'_B(z)$$

(131)

Given a value of L we can solve the Eqs. (128) and (130) for L and P_B:

$$\tilde{L} = \frac{Y_A}{\tilde{\theta}} - \frac{\theta}{\tilde{\theta}} L$$

(132)

$$P_B = \frac{\frac{\tilde{\theta}-\theta}{\tilde{\theta}}L + (\frac{1-\tilde{\theta}}{\tilde{\theta}})Y_A}{Y_B} \tag{133}$$

If you combine Eqs. (128) and (129) you get

$$m_A(z) = \beta G_A(z)^{-1} = \frac{\psi(z)}{\pi(z)} \tag{134}$$

If you combine Eqs. (130) and (131) and previous results you get

$$P_B'(z)/P_B = G_A(z)/G_B(z) \tag{135}$$

Marginal utility growth in good B is

$$m_B(z) = m_A(z)P_B'(z)/P_B = \beta G_B(z)^{-1} \tag{136}$$

Marginal utility growths are across countries in both goods (but not across goods) are equated: $m_A(z) = \tilde{m}_A(z)$ and $m_B(z) = \tilde{m}_B(z)$. This is true regardless of preferences.

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that Eqs. (133) becomes

$$P_B = \frac{1-\theta}{\theta}\frac{Y_A}{Y_B} \tag{137}$$

and Eq. (135) implies

$$P_B'(z) = \frac{1-\theta}{\theta}\frac{Y_A'(z)}{Y_B'(z)} \tag{138}$$

Since trade is frictionless and preferences are identical $e = e'(z) = 1$.

We can solve for allocations by solving for L. To do this we consider the lifetime budget constraint, (121), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_A + \beta \bar{s}_A' + (\frac{1-\theta}{\theta})(s_B + \beta \bar{s}_B') \right] Y_A \tag{139}$$

This implies

$$L = \frac{\theta(s_A + \beta \bar{s}_A') + (1-\theta)(s_B + \beta \bar{s}_B')}{\theta(1+\beta)}Y_A \tag{140}$$

Eq. (132) then implies that

$$\tilde{L} = \frac{\theta[(1-s_A) + \beta(1-\bar{s}_A')] + (1-\theta)[(1-s_B) + \beta(1-\bar{s}_B')]}{\theta(1+\beta)}Y_A \tag{141}$$

Different Preferences With different preferences we need to solve for L. To do this we consider the lifetime budget constraint, (121), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_A + \beta \bar{s}'_A + (\frac{1-\tilde{\theta}}{\tilde{\theta}})(s_B + \beta \bar{s}'_B) \right] Y_A + \frac{\tilde{\theta}-\theta}{\tilde{\theta}}(s_B + \beta \bar{s}'_B)L \qquad (142)$$

This implies

$$L = \frac{\tilde{\theta}(s_A + \beta \bar{s}'_A) + (1-\tilde{\theta})(s_B + \beta \bar{s}'_B)}{\tilde{\theta}(1+\beta) + (\theta - \tilde{\theta})(s_B + \beta \bar{s}'_B)} Y_A \qquad (143)$$

Eq. (132) then implies that

$$\tilde{L} = \frac{\theta[(1-s_A) + \beta(1-\bar{s}'_A)] + (1-\theta)[(1-s_B) + \beta(1-\bar{s}'_B)]}{\tilde{\theta}(1+\beta) + (\theta - \tilde{\theta})(s_B + \beta \bar{s}'_B)} Y_A \qquad (144)$$

and (133) implies that

$$P_B = \frac{(1-\tilde{\theta})(1+\beta) + (\tilde{\theta}-\theta)(s_A + \beta \bar{s}'_A)}{\tilde{\theta}(1+\beta) + (\theta - \tilde{\theta})(s_B + \beta \bar{s}'_B)} \frac{Y_A}{Y_B}, \qquad (145)$$

Given Eq. (135) we have

$$P'_B(z) = \frac{G_A(z)}{G_B(z)} P_B = \frac{(1-\tilde{\theta})(1+\beta) + (\tilde{\theta}-\theta)(s_A + \beta \bar{s}'_A)}{\tilde{\theta}(1+\beta) + (\theta - \tilde{\theta})(s_B + \beta \bar{s}'_B)} \frac{Y'_A(z)}{Y'_B(z)}. \qquad (146)$$

Since good B is traded, $\tilde{P}_B = P_B$ and $\tilde{P}'_B(z) = P'_B(z)$ so

$$e = (\tilde{\rho}/\rho) P_B^{\theta-\tilde{\theta}} \qquad e'(z) = (\tilde{\rho}/\rho) P'_B(z)^{\theta-\tilde{\theta}}$$

But this means

$$\ln\left[e'(z)/e\right] = (\theta - \tilde{\theta}) \ln\left[P'_B(z)/P_B\right] = (\theta - \tilde{\theta}) \ln\left[G_A(z)/G_B(z)\right]$$

A.3.3 When Good B is Non-traded

The market clearing conditions for good A are (124) and (125). For good B they are

$$c_B = y_B, \quad \tilde{c}_B = \tilde{y}_B \qquad (147)$$

$$c'_B(z) = y'_B(z), \quad \tilde{c}'_B(z) = \tilde{y}'_B(z) \tag{148}$$

The market clearing conditions and the first order conditions together imply that Eqs. (128) and (129) hold along with

$$(1 - \theta)L = P_B y_B \qquad (1 - \tilde{\theta})\tilde{L} = \tilde{P}_B \tilde{y}_B \tag{149}$$

$$\beta(1 - \theta)L = \frac{\psi(z)}{\pi(z)} P'_B(z) y'_B(z), \quad \beta(1 - \tilde{\theta})\tilde{L} = \frac{\psi(z)}{\pi(z)} \tilde{P}'_B(z) \tilde{y}'_B(z) \tag{150}$$

Given the results so far, the lifetime budget constraint of the home household, (121), becomes:

$$(1 + \beta)\theta L = \kappa Y_A,$$

where $\kappa = s_A + \beta \bar{s}'_A$, and implies

$$L = \frac{\kappa}{\theta(1 + \beta)} Y_A. \tag{151}$$

If we combine Eqs. (128) and (151) we have

$$\tilde{L} = \frac{\tilde{\kappa}}{\tilde{\theta}(1 + \beta)} Y_A \tag{152}$$

where $\tilde{\kappa} = 1 - s_A + \beta(1 - \bar{s}'_A)$.

Combining Eqs. (128) and (129) you get

$$m_A(z) = \beta G_A(z)^{-1} = \frac{\psi(z)}{\pi(z)} \tag{153}$$

Combining Eqs. (149), (151) and (152) we have

$$P_B = \frac{(1 - \theta)\kappa}{\theta(1 + \beta)} \frac{Y_A}{y_B} \qquad \tilde{P}_B = \frac{(1 - \tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1 + \beta)} \frac{Y_A}{\tilde{y}_B}. \tag{154}$$

If we combine (149) and (150), and make use of (153) we get

$$\frac{P'_B(z)}{P_B} = \frac{G_A(z)}{g_B(z)} \qquad \frac{\tilde{P}'_B(z)}{\tilde{P}_B} = \frac{G_A(z)}{\tilde{g}_B(z)}. \tag{155}$$

Therefore, we can write

$$P'_B(z) = \frac{(1 - \theta)\kappa}{\theta(1 + \beta)} \frac{Y'_A(z)}{y'_B(z)} \qquad \tilde{P}'_B(z) = \frac{(1 - \tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1 + \beta)} \frac{Y'_A(z)}{\tilde{y}'_B(z)}. \tag{156}$$

Discounted marginal utility growth in good B is

$$m_B(z) = \frac{\psi(z)}{\pi(z)} \frac{P'_B(z)}{P_B} = \beta/g'_B(z) \qquad \tilde{m}_B(z) = \frac{\psi(z)}{\pi(z)} \frac{\tilde{P}'_B(z)}{\tilde{P}_B} = \beta/\tilde{g}_B(z) \qquad (157)$$

Identical Preferences We have

$$e = \left(\tilde{P}_B/P_B\right)^{1-\theta} = \left(\frac{\tilde{\kappa}}{\kappa}\frac{y_B}{\tilde{y}_B}\right)^{1-\theta}.$$

$$e'(z) = \left(\tilde{P}'_B(z)/P'_B(z)\right)^{1-\theta} = \left[\frac{\tilde{\kappa}}{\kappa}\frac{y'_B(z)}{\tilde{y}'_B(z)}\right]^{1-\theta}.$$

And this means

$$\ln[e'(z)/e] = (1-\theta)\ln\left[g'_B(z)/\tilde{g}'_B(z)\right]$$

Different Preferences

$$e = (\frac{\tilde{\rho}}{\rho})\frac{\tilde{P}_B^{1-\tilde{\theta}}}{P_B^{1-\theta}} = (\frac{\tilde{\rho}}{\rho})\frac{\left[\frac{(1-\tilde{\theta})}{(1+\beta)\tilde{\theta}}\tilde{\kappa}\frac{Y_A}{\tilde{y}_B}\right]^{1-\tilde{\theta}}}{\left[\frac{(1-\theta)}{(1+\beta)\theta}\kappa\frac{Y_A}{y_B}\right]^{1-\theta}} = (\frac{\theta}{\tilde{\theta}})\frac{\left(\tilde{\kappa}\tilde{y}_B^{-1}\right)^{1-\tilde{\theta}}}{\left(\kappa y_B^{-1}\right)^{1-\theta}}\left(\frac{Y_A}{1+\beta}\right)^{\theta-\tilde{\theta}}.$$

$$e'(z) = (\frac{\tilde{\rho}}{\rho})\frac{\tilde{P}'_B(z)^{1-\tilde{\theta}}}{P'_B(z)^{1-\theta}} = (\frac{\theta}{\tilde{\theta}})\frac{[\tilde{\kappa}\tilde{y}'_B(z)^{-1}]^{1-\tilde{\theta}}}{[\kappa y'_B(z)^{-1}]^{1-\theta}}\left[\frac{Y'_A(z)}{1+\beta}\right]^{\theta-\tilde{\theta}}$$

So

$$\ln[e'(z)/e] = (1-\theta)\ln g'_B(z) - (1-\tilde{\theta})\ln\tilde{g}'_B(z) + (\theta-\tilde{\theta})\ln G'_A(z)$$

A.3.4   Financial Autarky

Since the countries are in financial autarky, we no longer have $\psi(z) = \tilde{\psi}(z)$, so the rearranged first order conditions for the consumptions are

$$\theta L = c_A \qquad (158)$$

$$(1-\theta)L = c_B P_B \qquad (159)$$

$$\beta\theta L = \frac{\psi(z)}{\pi(z)}c'_A(z) \qquad (160)$$

$$\beta(1-\theta)L = \frac{\psi(z)}{\pi(z)}c'_B(z)P'_B(z) \qquad (161)$$

45

$$\tilde{\theta}\tilde{L} = \tilde{c}_A \tag{162}$$

$$(1 - \tilde{\theta})\tilde{L} = \tilde{P}_B\tilde{c}_B \tag{163}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\tilde{\psi}(z)}{\pi(z)}\tilde{c}'_A(z) \tag{164}$$

$$\beta(1 - \tilde{\theta})\tilde{L} = \frac{\tilde{\psi}(z)}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{165}$$

The home country's flow budget constraints must be satisfied with no asset holdings so we have

$$c_A + P_B c_B = y_A + P_B y_B \tag{166}$$

$$c'_A(z) + P'_B(z)c'_B(z) = y'_A(z) + P'_B(z)y'_B(z). \tag{167}$$

Using the first order conditions for the consumptions we get expressions for discounted marginal utility growth:

$$m_A(z) = \beta\frac{c_A}{c'_A(z)} = \frac{\psi(z)}{\pi(z)} \qquad \tilde{m}_A(z) = \beta\frac{\tilde{c}_A}{\tilde{c}'_A(z)} = \frac{\tilde{\psi}(z)}{\pi(z)} \tag{168}$$

Discounted marginal utility growth in good B is

$$m_B(z) = \beta\frac{c_B}{c'_B(z)} = \frac{\psi(z)}{\pi(z)}\frac{P'_B(z)}{P_B} \qquad \tilde{m}_B(z) = \beta\frac{\tilde{c}_B}{\tilde{c}'_B(z)} = \frac{\tilde{\psi}(z)}{\pi(z)}\frac{\tilde{P}'_B(z)}{\tilde{P}_B} \tag{169}$$

A.3.5 When Good B is Traded

The market clearing conditions for goods are Eqs. (124)–(127). The market clearing conditions and the first order conditions together imply

$$\theta L + \tilde{\theta}\tilde{L} = Y_A \tag{170}$$

$$\frac{\theta}{\psi(z)}L + \frac{\tilde{\theta}}{\tilde{\psi}(z)}\tilde{L} = \frac{1}{\beta\pi(z)}Y'_A(z) \tag{171}$$

$$(1 - \theta)L + (1 - \tilde{\theta})\tilde{L} = P_B Y_B \tag{172}$$

$$(1 - \theta)\frac{L}{\psi(z)} + (1 - \tilde{\theta})\frac{\tilde{L}}{\tilde{\psi}(z)} = \frac{1}{\beta\pi(z)}P'_B(z)Y'_B(z), \quad z = 1, \dots, n, \tag{173}$$

We can rearrange Eqs. (170) and (172) to get:

$$\tilde{L} = \frac{1}{\tilde{\theta}}(Y_A - \theta L) \tag{174}$$

$$P_B = \frac{1}{\tilde{\theta}}\frac{(\tilde{\theta} - \theta)L + (1 - \tilde{\theta})Y_A}{Y_B}, \tag{175}$$

We can rearrange Eqs. (171) and (173) to get:

$$\frac{\beta\pi(z)}{\tilde{\psi}(z)}\tilde{L} = \frac{1}{\tilde{\theta}}\left[Y_A'(z) - \theta\frac{\beta\pi(z)}{\psi(z)}L\right]. \tag{176}$$

$$P_B'(z) = \frac{1}{\tilde{\theta}}\frac{(\tilde{\theta} - \theta)\frac{\beta\pi(z)}{\psi(z)}L + (1 - \tilde{\theta})Y_A'(z)}{Y_B'(z)} \tag{177}$$

The flow budget constraint, (166), and Eq. (175) imply that

$$L = y_A + \frac{\frac{\tilde{\theta}-\theta}{\tilde{\theta}}L + (1 - \tilde{\theta})\frac{Y_A}{\tilde{\theta}}}{Y_B}y_B$$

or

$$L = \frac{\tilde{\theta}s_A + (1 - \tilde{\theta})s_B}{\tilde{\theta} + (\theta - \tilde{\theta})s_B}Y_A \tag{178}$$

The flow budget constraint, (167), and Eq. (177) imply that

$$\frac{\beta\pi(z)}{\psi(z)}L = \frac{\tilde{\theta}s_A'(z) + (1 - \tilde{\theta})s_B'(z)}{\tilde{\theta} + (\theta - \tilde{\theta})s_B'(z)}Y_A'(z). \tag{179}$$

Using (178) we then have

$$m_A(z) = \frac{\psi(z)}{\pi(z)} = \beta\frac{\xi_A(z)}{G_A(z)} \quad \text{with} \quad \xi_A(z) = \frac{\frac{\tilde{\theta}s_A + (1-\tilde{\theta})s_B}{\tilde{\theta} + (\theta - \tilde{\theta})s_B}}{\frac{\tilde{\theta}s_A'(z) + (1-\tilde{\theta})s_B'(z)}{\tilde{\theta} + (\theta - \tilde{\theta})s_B'(z)}} \tag{180}$$

Substituting (178) into (174) we get

$$\tilde{L} = \frac{\theta(1 - s_A) + (1 - \theta)(1 - s_B)}{\theta + (\tilde{\theta} - \theta)(1 - s_B)}Y_A \tag{181}$$

Substituting (179) into (176) we get

$$\frac{\beta\pi(z)}{\tilde{\psi}(z)}\tilde{L} = \frac{\theta[1 - s_A'(z)] + (1 - \theta)[1 - s_B'(z)]}{\theta + (\tilde{\theta} - \theta)[1 - s_B'(z)]}Y_A'(z) \tag{182}$$

Given these results, discounted marginal utility growth in good A in the foreign country is

$$\tilde{m}_A(z) = \frac{\tilde{\psi}(z)}{\pi(z)} = \beta \frac{\tilde{\xi}_A(z)}{G_A(z)} \quad \text{with} \quad \tilde{\xi}_A(z) = \frac{\frac{\theta(1-s_A)+(1-\theta)(1-s_B)}{\theta+(\tilde{\theta}-\theta)(1-s_B)}}{\frac{\theta[1-s'_A(z)]+(1-\theta)[1-s'_B(z)]}{\theta+(\tilde{\theta}-\theta)[1-s'_B(z)]}} \tag{183}$$

Substituting (178) into (175)

$$P_B = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B} \frac{Y_A}{Y_B} \tag{184}$$

Substituting (179) into (177)

$$P'_B(z) = \left[\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}\right] \frac{Y'_A(z)}{Y'_B(z)} \tag{185}$$

Discounted marginal utility growth in good B in the two countries is

$$m_B(z) = \beta \frac{\xi_A(z)}{G_A(z)} \frac{P'_B(z)}{P_B} = \beta \frac{\xi_A(z)}{G_B(z)} \xi_B(z) \quad \text{with} \quad \xi_B(z) = \frac{\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}}{\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B}} \tag{186}$$

$$\tilde{m}_B(z) = \beta \frac{\tilde{\xi}_A(z)}{G_A(z)} \frac{P'_B(z)}{P_B} = \beta \frac{\tilde{\xi}_A(z)}{G_B(z)} \xi_B(z) \tag{187}$$

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that Eqs. (184) and (185) simplify to

$$P_B = \frac{1-\theta}{\theta} \frac{Y_A}{Y_B} \tag{188}$$

$$P'_B(z) = \frac{1-\theta}{\theta} \frac{Y'_A(z)}{Y'_B(z)} \tag{189}$$

Since both goods are frictionlessly traded and preferences are identical $e = e'(z) = 1$.

The expressions for ξ_A and $\tilde{\xi}_A$ in Eqs. (180) and (183) simplify to

$$\xi_A(z) = \frac{\theta s_A + (1-\theta)s_B}{\theta s'_A(z) + (1-\theta)s'_B(z)} \tag{190}$$

$$\tilde{\xi}_A(z) = \frac{\theta(1-s_A)+(1-\theta)(1-s_B)}{\theta[1-s'_A(z)]+(1-\theta)[1-s'_B(z)]} \tag{191}$$

The expression for ξ_B in Eq. (186) simplifies to $\xi_B(z) = 1$, implying that

$$m_B(z) = \beta \frac{\xi_A(z)}{G_B(z)} \quad \tilde{m}_B(z) = \beta \frac{\tilde{\xi}_A(z)}{G_B(z)} \tag{192}$$

The wedge between marginal utility growths in good A, good B, and in terms of aggregate consumption is

$$\tilde{m}_A(z)/m_A(z) = \tilde{m}_B(z)/m_B(z) = \tilde{m}(z)/m(z) = \tilde{\xi}_A(z)/\xi_A(z).$$

Different Preferences   Given the expressions for prices, above,

$$e = (\tilde{\rho}/\rho) P_B^{\theta - \tilde{\theta}} = (\tilde{\rho}/\rho) \left(\frac{1 - \tilde{\theta} + \left(\tilde{\theta} - \theta \right) s_A}{\tilde{\theta} + \left(\theta - \tilde{\theta} \right) s_B} \frac{Y_A}{Y_B} \right)^{\theta - \tilde{\theta}}$$

and

$$e'(z) = (\tilde{\rho}/\rho) P_B'(z)^{\theta - \tilde{\theta}} = (\tilde{\rho}/\rho) \left(\frac{1 - \tilde{\theta} + (\tilde{\theta} - \theta) s_A'(z)}{\tilde{\theta} + (\theta - \tilde{\theta}) s_B'(z)} \frac{Y_A'(z)}{Y_B'(z)} \right)^{\theta - \tilde{\theta}}$$

A.3.6   When Good B is Non-traded

Because the B good cannot be traded the goods market clearing conditions and the home household budget constraints together imply,

$$c_A = y_A, \quad \tilde{c}_A = \tilde{y}_A \tag{193}$$

$$c_A'(z) = y_A'(z), \quad \tilde{c}_A'(z) = \tilde{y}_A'(z) \tag{194}$$

$$c_B = y_B, \quad \tilde{c}_B = \tilde{y}_B, \tag{195}$$

$$c_B'(z) = y_B'(z), \quad \tilde{c}_B'(z) = \tilde{y}_B'(z), \tag{196}$$

So

$$L = y_A/\theta, \tag{197}$$

$$P_B = \frac{1 - \theta}{\theta} \frac{y_A}{y_B} \tag{198}$$

$$\frac{\psi(z)}{\pi(z)} = \beta/g_A(z) \tag{199}$$

$$P'_B(z) = \frac{1-\theta}{\theta} \frac{y'_A(z)}{y'_B(z)} \tag{200}$$

$$\tilde{L} = \tilde{y}_A / \tilde{\theta} \tag{201}$$

$$\tilde{P}_B = \frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}_A}{\tilde{y}_B}, \tag{202}$$

$$\frac{\tilde{\psi}(z)}{\pi(z)} = \beta / \tilde{g}_A(z) \tag{203}$$

$$\tilde{P}'_B(z) = \frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}'_A(z)}{\tilde{y}'_B(z)} \tag{204}$$

Discounted marginal utility growths in goods A and B are

$$m_A(z) = \beta / g_A(z) \qquad \tilde{m}_A(z) = \beta / \tilde{g}_A(z)$$

$$m_B(z) = \beta / g_B(z) \qquad \tilde{m}_B(z) = \beta / \tilde{g}_B(z)$$

Identical Preferences   If preferences are identical we have $\theta = \tilde{\theta}$ so that

$$e = \left(\frac{\tilde{y}_A / \tilde{y}_B}{y_A / y_B} \right)^{1-\theta} = \left(\frac{(1-s_A)/s_A}{(1-s_B)/s_B} \right)^{1-\theta}.$$

$$e'(z) = \left(\frac{\tilde{y}'_A(z) / \tilde{y}'_B(z)}{y_A(z) / y_B(z)} \right)^{1-\theta} = \left(\frac{[1-s'_A(z)]/s'_A(z)}{[1-s'_B(z)]/s'_B(z)} \right)^{1-\theta}.$$

Different Preferences

$$e = (\tilde{\rho}/\rho) [\frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}_A}{\tilde{y}_B}]^{1-\tilde{\theta}} / [\frac{1-\theta}{\theta} \frac{y_A}{y_B}]^{1-\theta}.$$

$$e'(z) = (\tilde{\rho}/\rho) [\frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}'_A(z)}{\tilde{y}'_B(z)}]^{1-\tilde{\theta}} / [\frac{1-\theta}{\theta} \frac{y'_A(z)}{y'_B(z)}]^{1-\theta}.$$