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ENDOGENOUS LIQUIDITY AND DEFAULTABLE BONDS

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ABSTRACT

This paper studies the interaction between fundamental and liquidity for defaultable corporate bonds that are traded in an over-the-counter secondary market with search frictions. Bargaining with dealers determines a bond's endogenous liquidity, which depends on both the firm fundamental and the time-to-maturity of the bond. Corporate default decisions interact with the endogenous secondary market liquidity via the rollover channel. A default-liquidity loop arises: Earlier endogenous default worsens a bond's secondary market liquidity, which amplifies equity holders' rollover losses, which in turn leads to earlier endogenous default. Besides characterizing in closed form the full inter-dependence between liquidity premium and default premium for credit spreads, we also study the optimal maturity implied by the model based on the tradeoff between liquidity provision and inefficient default.

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1 Introduction

The recent 2007-2008 financial crisis and the ongoing sovereign crisis have vividly demonstrated the intricate interaction between asset fundamental and asset liquidity in financial markets. Liquidity tends to dry up for assets with deteriorating fundamentals when solvency becomes a concern, reflected by soaring liquidity premia and/or prohibitive transaction costs in trading. In the mean time, asset fundamentals worsen further due to endogenous reactions (say, default) of market participants in response to worsening liquidity in financial markets.

The fundamental-liquidity spiral is at the center of the academic policy research on financial crises, and this paper aims to deliver such a feedback loop in the context of corporate bond markets.[1] It has been well documented that secondary corporate bond markets – which are mainly over-the-counter (OTC) markets – are much less liquid than equity markets.[2] On the one hand, Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011) document a strong empirical pattern that the liquidity for corporate bonds (measured as the transaction cost) deteriorates dramatically for bonds with lower fundamental, i.e., bonds that are issued by firms closer to default (reflected by higher credit derivative swaps, CDS). On the other hand, the recent financial crisis of 2007-2008 illustrates that the deterioration of secondary market liquidity, through adversely affecting the refinancing operations of firms, can exacerbate the incentives of equity holders to default (He and Xiong (2012b), hereafter HX12). Taken together, these two observations imply a positive feedback loop between the secondary market liquidity and asset fundamentals for corporate bonds.

To deliver such an default-liquidity spiral effect, we adopt two standard ingredients from the existing literature. First, we model the endogenous liquidity in the secondary corporate bond market as a search-based over-the-counter (OTC) market á la Duffie, Garlenau, and Pedersen

[1]Corporate bond markets, for both financial and non-financial firms, make up a large part of the U.S. financial system. According to flow of funds, the values of corporate bonds reaches about 4.7 trillion in the first quarter of 2010, which consists of about one third of total liabilities of U.S. corporate businesses.

[2]For instance, Edwards, Harris, and Piwowar (2007) study the U.S. OTC secondary trades in corporate bonds and estimate the transaction cost to be around 150 bps, and Bao, Pan, and Wang (2011) find an even larger number. The fact that equity markets–while being presumably subject to more asymmetric information problems–are more liquid imply the importance of search friction in corporate bond markets. Other empirical papers that investigate secondary bond market liquidity are Hong and Warga (2000), Schultz (2001), Green, Hollifield, and Schurhoff (2007a,b); Harris and Piwowar (2006).

(2005). Bond investors who are hit by liquidity shocks prefer early payments, and with a certain matching technology they meet and trade with an intermediary dealer at an endogenous bid-ask spread. A novel feature is that the endogenous liquidity for the secondary bond market depends on both the firm's distance-to-default and the bond's time-to-maturity.

The second important ingredient for the feedback between fundamental and liquidity is the endogenous default decision by equity holders. This mechanism is borrowed from the standard Leland-type corporate finance structural models, i.e., Leland (1994) and Leland and Toft (1996) (hereafter LT96). More specifically, a firm rolls over (refinances) maturing bonds by issuing new bonds of the same face value. When firm fundamentals deteriorate, equity holders will face heavier rollover losses due to falling prices of newly issued bonds. Equity holders default optimally when absorbing further losses is unprofitable, at which point bond investors with defaulted claims step in to recover part of the firm value subject to dead-weight bankruptcy cost.

The secondary market liquidity of *defaulted* bonds, i.e., bonds of firms that have defaulted, is important in deriving the endogenous bond liquidity before the firm defaults. We model the (il)liquidity of defaulted bonds based on the fact that bankruptcy leads to a delay in the payout of any cash due to lengthy court proceedings, as for example in the Lehman Brothers bankruptcy (see footnote 15). This serves as one of the boundary conditions needed to solve the system of partial differential equations (PDEs) that describes the bond valuations.[3] We solve for debt and equity valuation, the endogenous default boundary, and the endogenous liquidity in closed form in Section 3.

Consistent with empirical findings in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011), we show in Section 4.2 that the endogenous bid-ask spread is decreasing with the firm's distance-to-default, holding the time-to-maturity constant; and decreasing in the bond's time-to-maturity, holding the distance-to-default constant. Moreover, our model produces a novel testable empirical prediction that the slope with respect to time-to-maturity of the bid-ask spread

[3]This arises because bond valuations depend on firm fundamental, the bond's time-to-maturity, and the liquidity state of bond holders.

will be greater for bonds with higher distance-to-default. Intuitively, as the stated maturity of corporate bonds plays no role in bankruptcy procedures, the difference between bonds with different time-to-maturities vanishes when firms are close to default.

The derived endogenous liquidity allows us to study the positive feedback loop between liquidity and default. Imagine an exogenous negative cash flow shock that pushes the firm closer to default, lowering the bond's fundamental value. More importantly, because bonds of defaulted firms suffer greater illiquidity, the outside option of bondholders when bargaining with the dealer declines. This worsens the secondary market liquidity and lowers the bond prices even further. The wider refinancing gap between the newly issued bond prices and promised principals gives rise to heavier rollover losses, which causes equity holders to default earlier and thus pushes the firm even closer to default. As a result, lower distance-to-default reduces the fundamental value of the corporate bonds even further, and so forth. The outcome of these spirals is a unique fixed point bankruptcy threshold at which equity holders default.

The feedback loop between fundamental and liquidity is important in understanding the endogenous link between liquidity and solvency in modern financial markets. More specifically, our paper characterizes a full inter-dependence between liquidity and default components in the credit spread for corporate bonds. This contrasts with the widely-used reduced-form approach in the empirical research, where it is common to decompose firms' credit spreads into independent liquidity-premium and default-premium components (e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010)). Our fully solved structural model calls for more structural approaches in future studies on the impact of liquidity factors upon the credit spread of corporate bonds; indeed, our ongoing project suggests that our positive spiral may amplify small liquidity frictions into quantitatively significant liquidity and default premia.

Our paper belongs to the literature on the role of secondary market trading frictions in structural models of corporate finance (Black and Cox (1976), Leland (1994) and LT96). Ericsson and Renault (2006) analyze the interaction between secondary liquidity and the bankruptcy-renegotiation in a LT96 framework. HX12 take the simplified secondary market friction introduced in the classic article of Amihud and Mendelson (1986), i.e., bond investors hit by liquidity shocks are forced sell their holdings immediately at an exogenous and constant proportional transaction cost. Because in HX12 the bond market liquidity is modeled in an exogenous way, that paper can only speak to the one-way economic channel from exogenous liquidity to default. In contrast, our paper endogenizes the secondary market liquidity by micro-founding the bond trading in a search-based secondary market, and derives the equilibrium liquidity *jointly* with equilibrium asset prices.[4]

We investigate the model performance for BB rated corporate bonds, and show that the positive feedback mechanism between fundamental and liquidity can be quantitatively important for structural models with credit risk. In our model with the full positive spiral between liquidity and default, the credit spread is calibrated to 320 bps to roughly match the observed credit spread of a BB rated bond. As one benchmark, in LT96 model where the secondary corporate bond markets are perfectly liquid, the implied credit spread is only about 181 bps. This implies that incorporating illiquidity of corporate bonds can be important in explaining about a half of the observed credit spread. As more stringent benchmark, the HX12 model with exogenous and constant bond illiquidity produces a credit spread of about 288 bps. Therefore, our calibration suggests that the positive spiral between liquidity and default (which only exists in our model, not in HX12) can help understand about $(320 - 288)/320 = 10\%$ of the observed credit spread.

Our paper also makes a contribution to the search based asset-pricing literature, as represented by Duffie, Garlenau, and Pedersen (2005, 2007); Weill (2007); Lagos and Rocheteau (2007, 2009); Biais and Weill (2009); Feldhutter (2011). To our knowledge, this literature with concentration on OTC markets has thus far focused on the determinants of contact intensities and behavior of intermediaries, while eschewing time-varying asset fundamentals and asset maturities. Undoubt-

[4]Another possibility to micro-found secondary market liquidity is to assume some adverse selection with regard to the bankruptcy recovery value, a path we do not pursue in this model due to the difficulties inherent in tracking persistent private information. Well-known endogenous market illiquidity models based on private information are Kyle (1985), Glosten and Milgrom (1985), and Back and Baruch (2004). Besides the advantage of being able to be integrated seamlessly into the dynamic firm setting in LT96, the search based framework is suitable for the secondary market for corporate bonds, especially considering the fact that equity markets have much higher liquidity while being subject to more severe asymmetric information problems. For adverse selection in search markets, see Lauermann and Wolinsky (2011) and Guerrieri, Shimer, and Wright (2010) (in directed search, rather than random as we assumed here).

edly, asset-specific dynamics are important for the corporate bonds market, and we fill this gap by incorporating the firm's distance-to-default and the bond's time-to-maturity in deriving the asset (bond) valuations.[5] Moreover, our paper demonstrates that, via the rollover channel, the endogenous search-based secondary market liquidity can have a significant impact on the firms' behavior on the real side.

Positive feedback is an active research topic in different areas. For instance, strategic complementarity naturally gives rise to positive feedback effect in the global games literature (Morris and Shin 2009), and a similar effect emerges in He and Xiong (2012a) who study dynamic coordinations among creditors whose debt contracts mature at different times. Our paper is more related to the literature that emphasizes the interaction between firms and financial markets. For example, Goldstein, Ozdenoren, and Yuan (2011) show that market prices can feedback to firm's investment decisions through the information channel; Brunnermeier and Pedersen (2009) illustrate the positive feedback loop between funding liquidity and market liquidity; Cheng and Milbradt (2012) show how managerial risk-shifting feeds back on bondholders decision to run, which in turn feeds back on managerial incentives; and Manso (2011) points out that credit ratings affect a firm's default decision, which feeds back into the rating decision.

Our paper is also related to the literature of debt maturity structure (Diamond, 1993, Leland, 1998, etc). For the use of short-term debt with a higher rollover frequency, there exists a trade-off between better liquidity provision and earlier inefficient default. Regarding the liquidity provision of short-term debt, bond investors hit by liquidity shocks can either sell to dealers or sit out shocks by waiting to receive the face value when the bond matures. Shorter maturity improves upon the waiting option, resulting in a lower rent extracted by dealers and thus a greater secondary market liquidity. On the other hand, equity holders are absorbing rollover gains/losses ex post. As

[5] The existing literature often assumes infinite maturity and constant asset payoffs; for instance, focusing on a very different market, Vayanos and Weill (2008) use a search framework to explain the difference between off-the-run and on-the-run treasury yields. As far as we know, the only paper with deterministic time dynamics in a search framework is the contemporaneous Afonso and Lagos (2011), which introduces deterministic time dynamics via an end-of-day trading close in the federal funds market. More importantly, endogenous default with stochastic fundamental is one key building block for our paper. Because corporate bond payoffs are highly nonlinear in firm fundamentals, our closed-form solution with stochastic fundamentals is nontrivial.

shown in LT96 and emphasized in HX12, shorter-term debt with a higher rollover frequency leads to heavier rollover losses in bad times, which pushes equity holders to default earlier and thus to incur greater dead-weight bankruptcy costs. This tradeoff allows us to endogenize the firm's initial choice of debt maturity, and unlike traditional capital structure models an optimal finite maturity structure arises.

The paper is organized as follows. Section 2 lays out the model, and Section 3 solves the model in closed-form. Section 4 illustrates the positive feedback loop between fundamental and liquidity, and Section 5 provides extensions and discussions. Section 6 concludes. All proofs can be found in the Appendix.

2 The Model

2.1 Firm Cash Flows and Debt Maturity Structure

We consider a continuous-time model where a firm has assets-in-place that generate (after-tax) cash flows at a rate of $\delta_t > 0$, where $\{\delta_t : 0 \leq t < \infty\}$ follows a geometric Brownian motion under the risk-neutral probability measure:

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dZ_t, \tag{1}$$

where μ is the constant growth rate of cash flow rate, σ is the constant asset volatility, and $\{Z_t : 0 \leq t < \infty\}$ is a standard Brownian motion, representing random shocks to the firm fundamental. We assume the risk-free rate r to be constant in this economy.

We follow LT96 in assuming that the firm maintains a stationary debt structure. At each moment in time, the firm has a continuum of bonds outstanding with an aggregate principal of p and an aggregate coupon payment of c, where p and c are constants that we take as exogenously given. We normalize the measure of bonds to 1, so that each bond has a principal face value of p and a coupon flow payment of c. All bonds have an initial maturity T but differ in their current time-to-maturity $\tau \in [0, T]$. Expirations of the bonds are uniformly spread out across time;[6] that is, during a time interval $(t, t + dt)$, a fraction $\frac{1}{T}dt$ of the bonds matures and needs to be rolled over. Thus, $\frac{1}{T}$ is the firm's rollover frequency on its debt. Denote by $D(\delta, \tau)$ the value of one unit of bond, which depends on firm fundamental δ and its time-to-maturity τ.

Following the LT96 framework, we assume that the firm commits to a stationary debt structure denoted (c, p, T). In other words, when a bond matures, the firm will replace it by issuing a new bond with identical (initial) maturity T, principal value p, and coupon rate c, in the primary market (to be modeled shortly). This simple stationary debt maturity structure gives us a convenient dynamic setting to analyze the interaction between liquidity and default, and we believe the general economic mechanism identified in this paper is robust to this assumption.

In the main analysis we take the firm's debt maturity T as given; Section 5.1 discusses the optimal ex-ante choice of debt maturity T^* that maximizes firm value.[7]

2.2 Secondary Bond Market and Search-Based Liquidity

As in Duffie, Garlenau, and Pedersen (2005), individual bond investors are subject to idiosyncratic liquidity shocks, and once hit by shocks they need to search for market-makers/dealers to trade. More specifically, at any time with intensity ξ an individual bond holder is hit by an idiosyncratic liquidity shock. We model this sudden need for liquidity as an upward jump in the discount rate from the common interest rate r to a higher level $\overline{r} > r$.[8] For simplicity, this higher discount rate persists until either the agent manages to sell his debt-holdings, or the face value p is paid out when the debt matures. After either event, the investor exits the market forever.[9] It is important to note that this individual liquidity shock is uninsurable and thus results in an incomplete market and type dependent valuations as explained below.

[6] This staggered debt maturity structure is consistent with recent empirical findings (Choi, Hackbarth, and Zechner, 2012).

[7] One could also endogenize the firm initial leverage (c, p) based on the trade-off between tax benefit and bankruptcy cost by following LT96. We leave this exercise for future research.

[8] A constant holding cost per unit of time once in the liquidity shocked state (as in Duffie, Garlenau, and Pedersen (2005)) will deliver similar results.

[9] This assumption is for easier exposition, and can easily be relaxed as shown in the Appendix.

We further assume an infinite mass of non-liquidity-shocked H type buyers (i.e., high valuation agent) on the sidelines to simplify the calculations. Lastly, we simply assume that each bond investor only holds one unit of bond, and indicate the investor who has been hit by a liquidity shock by L (i.e., liquidity state or low valuation agent).

In practice, secondary corporate bond markets are less liquid than equity or primary debt markets. Thus, we assume that the secondary debt markets are subject to the following trading friction. An L bond investor who wants to sell his debt-holdings has to wait an exponential time with intensity λ to meet a dealer. When they meet, bargaining occurs over the economic surplus generated. We follow Duffie, Garlenau, and Pedersen (2007) and assume Nash-bargaining weights β of the investor and $1-\beta$ of the dealer to model this bargaining.

The illiquidity of secondary bond markets give rise to wedges in bond valuations for different investor types. Define $D_H(\delta,\tau)$ and $D_L(\delta,\tau)$ to be the valuations of debt of the high (or normal) type and the low (or liquidity) type, respectively. Suppose that a contact between a type L investor and a dealer occurs. We assume that the dealer faces a competitive inter-dealer market with a continuum of dealers, and at any time they can (collectively) contact H type investors who are competitive as well. Thus, the particular dealer in question can turn around and instantaneously sell directly (or through another dealer) to H type investors at a price of $D_H(\delta,\tau)$, which implies that the surplus from trade is

$$S(\delta,\tau) \equiv D_H(\delta,\tau) - D_L(\delta,\tau).$$

The transaction price at which L types sell to the dealer, $X(\delta,\tau)$, thus implements the following splits of the surplus according to the bargaining weights,

$$\begin{aligned} D_H(\delta,\tau) - X(\delta,\tau) &= (1-\beta) S(\delta,\tau) \\ X(\delta,\tau) - D_L(\delta,\tau) &= \beta \cdot S(\delta,\tau), \end{aligned} \tag{2}$$

so that

$$X(\delta,\tau) = \underbrace{\beta \cdot S(\delta,\tau)}_{appropriated\,surplus} + \underbrace{\beta D_L(\delta,\tau)}_{outside\,option}. \tag{3}$$

Relating to the micro-structure literature, in our model, the *ask* price at which dealers sell to H type investors is simply their valuation D_H, while the *bid* price at which L type investors sell their bond holdings to dealers is X. This implies that $D_H - X = (1-\beta)(D_H - D_L)$ is also the (dollar) bid-ask spread. Thus, H type investors are indifferent between buying and not buying the bond, whereas L type investors strictly prefer selling the bond when they have the opportunity for any $\beta > 0$.

The endogenous transaction cost $(1-\beta)(D_H - D_L)$ captures the liquidity of the secondary market for corporate bonds. Later we will calculate the percentage bid-ask spread as the dollar spread divided by the mid point of transaction prices (bid price X and ask price D_H). In a preview of the solution, by the dynamic nature of the model, the difference at issuance for a say AAA bond, $D_H - D_L$, will be determined by the probability that the firm defaults before the bond matures interacted with the wedge that prevails at bankruptcy, and the probability that the bond matures before the firm defaults interacted with the necessarily zero wedge at maturity between D_H and D_L.

2.3 Primary Bond Market and Debt Rollover

As mentioned, at any time the firm replaces the maturing bonds with newly issued ones in the so-called primary market, where the firm hires a dealer who can place the new debt to H type investors. As dealers are competitive in the primary market, the firm receives the full bond value of the high type D_H.

As a crucial part of our feedback mechanism, the H type incorporates in his bond valuation D_H the possibility that he will be hit by a liquidity shock in the future and thus has to use the illiquid secondary market to sell the bond. In other words, due to either fluctuating firm fundamental or changing secondary market illiquidity, the newly issued bond price D_H might be higher or lower than the required principal repayments to the maturing bonds. Equity holders are the residual claimants of any rollover gains/losses. Again, following LT96, we assume that any gain will be immediately paid out to equity holders and any loss will be funded by issuing more equity at the market price. Thus, over a short time interval $(t, t + dt)$, the net cash flow to equity holders (omitting dt) is

$$NC_t = \underbrace{\delta_t}_{CF} - \underbrace{(1 - \pi)\, c}_{Coupon} + \underbrace{\frac{1}{T}\left[D_H\left(\delta_t, T\right) - p\right]}_{Rollover}. \tag{4}$$

The first term is the firm's cash flow. The second term is the after-tax coupon payment to bond investors, where π denotes the marginal tax benefit rate of debt.[10] The third term captures the firm's rollover gains/losses by issuing new bonds to replace maturing bonds. This term can be understood as *repricing* the bonds at a rate of $\frac{1}{T}$. In this transaction, there is a $\frac{1}{T}dt$ fraction of bonds maturing, which requires a principal payment of $\frac{1}{T}pdt$; while the primary market value of the newly issued bonds is $\frac{1}{T}D_H(\delta_t, T)dt$. When the newly issued bond price $D_H(\delta_t, T)$ drops so that $D_H\left(\delta_t, T\right) < p$ (i.e., a discount bond), equity holders have to absorb the negative cash-flow stemming from rollover $\frac{1}{T}[D_H(\delta_t, T) - p]dt$. Thus, the rollover frequency $\frac{1}{T}$ (or the inverse of debt maturity) affects the extent of rollover losses/gains.

2.4 Bankruptcy

When the firm issues additional equity to fund these rollover losses, the equity issuance dilutes the value of existing shares.[11] Equity holders are willing to buy more shares and bail out the maturing debt holders as long as the equity value is still positive (i.e. the option value of keeping the firm alive justifies absorbing the rollover losses). When the firm defaults, its equity value drops to zero. The default threshold δ_B is endogenously determined by equity holders, which is an important

[10]For each dollar received by bond investors, the government is subsidizing π dollars so that equity holders only have to pay $1 - \pi$ dollars. The tax advantage of debt π affects the equity holders' endogenous default decision.

[11]A simple example works as follows. Suppose a firm has 1 billion shares of equity outstanding, and each share is initially valued at \$10. The firm has \$10 billion of debt maturing now, but the firm's new bonds with the same face value can only be sold for \$9 billion. To cover the shortfall, the firm needs to issue more equity. As the proceeds from the share offering accrue to the maturing debt holders, the new shares dilute the existing shares and thus reduce the market value of each share. If the firm only needs to roll over its debt once, then the firm needs to issue 1/9 billion shares and each share is valued at \$9. The \$1 price drop reflects the rollover loss borne by each share.

ingredient for the feedback loop between firm fundamentals and secondary market liquidity.[12]

When the firm declares bankruptcy, we simply assume that creditors can only recover a fraction α of the firm's unlevered value from liquidation.[13] As usual, the bankruptcy cost is ex post borne by debt holders but represents a deadweight loss to equity holders ex ante. Since the stated maturity for bonds per se does not matter in bankruptcy, for simplicity we assume equal seniority of all creditors.

Because one driving force of our model is that agents value receiving cash early, our bankruptcy treatment has to be careful in this regard. If bankruptcy leads investors to receive the proceeds immediately, L type investors who are trying to sell their bonds could view default as a beneficial outcome.[14] In other words, bankruptcy confers a benefit similar to maturity that may outweigh the deadweight loss stemming from the bankruptcy cost $1-\alpha$. This "liquidity by default" runs counter to the fact that in practice bankruptcy leads to a much more illiquid secondary market, the freezing of assets within the company, and a delay in the payout of any cash depending on court proceeding.[15]

Motivated by these facts, we make the following assumption for defaulted bonds. Suppose that after bankruptcy recovery is based on the unlevered firm value, $\frac{\delta_B}{r-\mu}$. To capture the uncertain timing of the court decision, we introduce a court delay so that the payout of cash $\alpha\frac{\delta_B}{r-\mu}$ occurs at a Poisson arrival time with intensity θ. We focus on situations where $\alpha\frac{\delta_B}{r-\mu} < p$ (which holds for all our examples) so that the recovery rate to bond holders is below 100%. Also, the secondary market for defaulted bonds is illiquid with contact intensity λ_B. Additionally, we assume that there can be a different discount rate $\overline{r}_B > r$ for liquidity shocked agents. We interpret a prohibitively high $\overline{r}_B$

[12]To focus on the liquidity effect originating from the debt market, we ignore any additional frictions in the equity market such as transaction costs and asymmetric information. It is important to note that while we allow the firm to freely issue more equity, the equity value can be severely affected by the firm's debt rollover losses. This feedback effect allows the model to capture difficulties faced by many firms in raising equity during a financial-market meltdown even in the absence of any friction in the equity market.

[13]The bankruptcy cost is standard in the trade-off literature, and can be interpreted in different ways, such as loss of customers or or legal fees. However, as we will introduce inefficient delay in court rulings shortly, our analysis goes through even if there is no bankruptcy cost, i.e., $\alpha = 1$.

[14]This would be the case for example for a CDS contract written on the firm which features immediate payouts at the time of a bankruptcy/credit event.

[15]The Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty, payouts to the debt holders only started trickling out after about three and a half years.

as possible regulatory or charter restrictions that amount to the agent not being allowed to hold defaulted assets. Then, the defaulted bond values D_H^B and D_L^B satisfy

$$\begin{aligned} rD_H^B &= \theta\left(\alpha\frac{\delta_B}{r-\mu} - D_H^B\right) + \xi\left(D_L^B - D_H^B\right), \\ \overline{r}_B D_L^B &= \theta\left(\alpha\frac{\delta_B}{r-\mu} - D_L^B\right) + \lambda_B\left(X^B - D_L^B\right), \end{aligned}$$

where as before $X^B = \beta D_L^B + (1-\beta)\, D_H^B$ is the transaction price received by L type investors. Plugging X^B into the above equations, we can solve for $D_i^B = \alpha_i \frac{\delta_B}{r-\mu}$ for $i \in \{H, L\}$ where

$$\begin{aligned} \alpha_H &= \tfrac{\theta\alpha(\overline{r}_B+\theta+\lambda_B\beta+\xi)}{\overline{r}_B(\xi+\theta)+r(\overline{r}_B+\theta+\lambda_B\beta)+\theta(\xi+\theta+\lambda_B\beta)}, \\ \alpha_L &= \tfrac{\theta\alpha(r+\theta+\lambda_B\beta+\xi)}{\overline{r}_B(\xi+\theta)+r(\overline{r}_B+\theta+\lambda_B\beta)+\theta(\xi+\theta+\lambda_B\beta)}. \end{aligned} \tag{5}$$

Note that this establishes the boundary conditions at the bankruptcy boundary δ_B, $D_i = \alpha_i \frac{\delta_B}{r-\mu}$ for $i \in \{H, L\}$. One can easily see that $\alpha_H > \alpha_L$ as $\overline{r}_B > r$. We denote the (bold face) vector $\boldsymbol{\alpha} \equiv [\alpha_H, \alpha_L]^\top$ as the *effective* bankruptcy cost factors from the perspective of different bond holders. Clearly, the wedge $\alpha_H - \alpha_L$ characterizes the illiquidity of the defaulted bonds when the firm (i.e. equity holders) declares bankruptcy. Throughout the paper we focus on the situation where the illiquidity in the default state is sufficiently high, in order to conform our model to the regular empirical pattern that bonds closer to default are more illiquid (e.g., Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011)).

2.5 Summary of Setup

The model setup is summarized in a schematic representation given in Figure 1, and for exposition purposes we omit including the bankruptcy decision that are driven by the stochastic process in δ.

Primary market. Let us start with the firm. It (re)issues debt to H-types via the primary market, who value the debt at D_H, as represented via the "Reissue" arrow. After the H-types buy the debt, there is a chance the bond matures before either a bankruptcy occurs or a liquidity shock

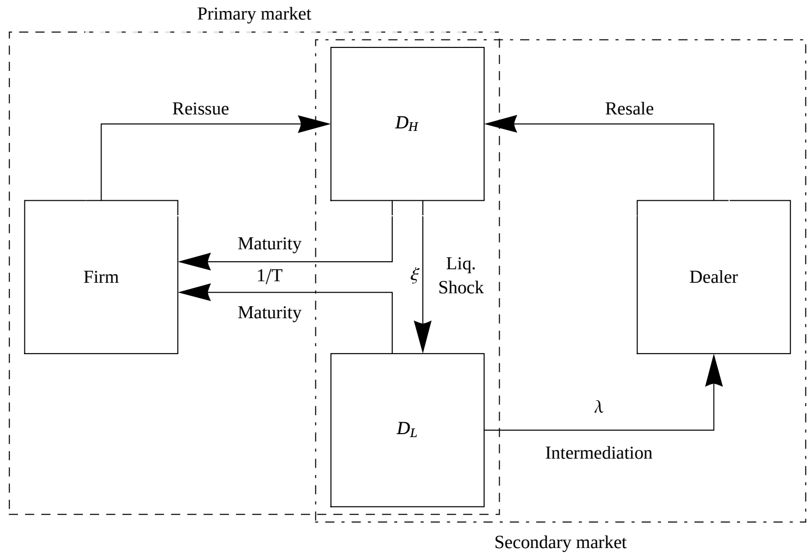

Figure 1: Schematic representation of model

hits. In this case, the bond goes back to the firm, which pays back the principal to the agent. This event is summarized in the "Maturity" arrow. This subpart of the graph represents the LT96 model. With liquidity shocks, an H-type transitions to an L-type with intensity ξ who values the bond at D_L, as represented by the "Liq. shock" arrow. Absent bankruptcy and retrading opportunities, the bond matures and the L-type will be paid back the face-value of the bond, again summarized by the "Maturity" arrow. $1/T$ indicates the flow of bonds that mature.

Secondary market. Once we introduce a secondary market, L-types can now sell the bond to H-types via the help of dealers. To do so, they try to contact dealers with an intensity λ, as indicated by the "Intermediation" arrow. They sell their bond to the dealer for $X(\delta, \tau)$. The dealer turns around and immediately (re)sells the bond to H-types, as indicated by the "Resale" arrow, for a price $D_H(\delta, \tau)$. It is important that H-types are indifferent between staying out of the market, buying bonds of maturity T at reissue via the primary market, or buying bonds of maturities $\tau \in (0, T)$ on the secondary market, as this relieves us from having to track the value functions of agents not holding the bond.

3 Model Solutions

3.1 Debt Valuations and Credit Spread

We first derive bond valuations by taking the firm's default boundary δ_B as given. Recall that $D_H(\delta,\tau)$ and $D_L(\delta,\tau)$ are the value of one unit of bond with time-to-maturity $\tau \le T$, an annual coupon payment of c, and a principal value of p to a type H and L investor, respectively. We have the following system of PDEs for the values of D_H and D_L, where we omit the two-dimensional argument (δ,τ) for both debt value functions:

$$
\begin{aligned}
rD_H &= c - \frac{\partial D_H}{\partial \tau} + \mu\delta \cdot \frac{\partial D_H}{\partial \delta} + \frac{\sigma^2\delta^2}{2}\frac{\partial^2 D_H}{\partial \delta^2} + \underbrace{\xi\left[D_L - D_H\right]}_{Liqidity\, shock}, \\
\overline{r}D_L &= c - \frac{\partial D_L}{\partial \tau} + \mu\delta \cdot \frac{\partial D_L}{\partial \delta} + \frac{\sigma^2\delta^2}{2}\frac{\partial^2 D_L}{\partial \delta^2} + \underbrace{\lambda\left[X - D_L\right]}_{Secondary\, market} .
\end{aligned}
\tag{6}
$$

The boundary conditions are $D_H = D_L = p$ at $\tau = 0$ because of the principal payment at maturity, and $D_i = \alpha_i \frac{\delta_B}{r-\mu}$ at $\delta = \delta_B$ where $i \in \{H, L\}$ as discussed in Section 2.4.[16]

The first equation in (6) is the type H bond valuation. The left-hand side rD_H is the required (dollar) return from holding the bond for type H investors. There are four terms on the right-hand side, capturing expected returns from holding the bond. The first term is the coupon payment. The next three terms capture the expected value change due to change in time-to-maturity τ (the second term) and fluctuation in the firm's fundamental δ_t (the third and fourth terms). The last term is a loss $D_L - D_H$ caused by the liquidity shock that transforms H investors into L investors, multiplied by the intensity of the liquidity shock.

The second equation in (6), the type L bond valuation, follows a similar explanation to the one above. The two differences are that the left hand side now has a higher required return $\overline{r} > r$, and there is the value impact of the secondary market reflected in the last term of the right hand side. A type L investor meets a dealer with an intensity of λ and is then able to sell his bond (with a

[16] And, given any time-to-maturity τ, when $\delta \to \infty$, D_H and D_L converge to the values of default-free bonds (but still subject to liquidity shocks and search frictions). Their expressions are given in footnote 29.

private value D_L) at a price of $X = (1-\beta) D_L + \beta D_H$. Plugging in equation (2) into equation (6), we have $\lambda [X - D_L] = \lambda\beta [D_H - D_L]$. One can interpret $\lambda\beta$ as the bargaining weighted intensity of "transitioning" (via a sale) back from the L state to the H state.[17] It is easy to show that when $\lambda \to \infty$, debt values converge to the LT96 case with perfectly liquid secondary markets. The surplus from intermediating trades vanishes because the outside option of meeting another dealer becomes very large.

We can now define the matrix $\mathbf{A}$ that incorporates the discount factors and the effective transition intensities ξ and $\lambda\beta$ of the states. Then, the following decomposition holds:

$$\mathbf{A} \equiv \begin{bmatrix} r+\xi & -\xi \\ -\lambda\beta & \overline{r}+\lambda\beta \end{bmatrix} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}.$$

We let $\hat{\mathbf{D}} \equiv Diag \begin{bmatrix} \hat{r}_1 & \hat{r}_2 \end{bmatrix}$, where $\hat{r}_i = \frac{r+\xi+\overline{r}+\lambda\beta \pm \sqrt{[(r+\xi)-(\overline{r}+\lambda\beta)]^2+4\xi\lambda\beta}}{2}$ satisfying $\hat{r}_1 > \overline{r} > \hat{r}_2 > r$, be the matrix of eigenvectors of $\mathbf{A}$, and denote by $\mathbf{P}$ be the matrix of stacked eigenvalues. For a given δ_B, we derive the closed-form solution for the bond values in the next proposition.[18]

Proposition 1 *The debt values are given by*

$$\begin{bmatrix} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{bmatrix} = \mathbf{P} \begin{bmatrix} A_1 + B_1 e^{-\hat{r}_1\tau}[1-F(\delta,\tau)] + C_1 G_1(\delta,\tau) \\ A_2 + B_2 e^{-\hat{r}_2\tau}[1-F(\delta,\tau)] + C_2 G_2(\delta,\tau) \end{bmatrix}. \tag{7}$$

Here, by defining $a \equiv \frac{\mu}{\sigma^2} - 0.5$, $\gamma_1 \equiv 0$, $\gamma_2 \equiv -2a$, $\eta_{j1,2} \equiv -a \pm \sqrt{a^2 + \frac{2}{\sigma^2}\hat{r}_j}$, *and* $q(\delta,\chi,t) \equiv \frac{\log(\delta_B)-\log(\delta)-(\chi+a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$, *the constants in (7) are given by:*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \equiv c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \equiv p\mathbf{P}^{-1}\mathbf{1} - c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1}, \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \equiv \frac{\delta_B}{r-\mu}\mathbf{P}^{-1}\boldsymbol{\alpha} - c\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{1},$$

[17] Although the debt values are functions of only the product $\beta\lambda$, the bid-ask spread is given by $(1-\beta)(D_H - D_L)$, and thus has β entering on its own independent of the product $\beta\lambda$. This will be important when calibrating our model, as it allows a separation of the identification of λ and β.

[18] All derivations, because of the linear decomposition, would go through even if creditors would be subject to possibly different shock states, $\overline{r}_1, \overline{r}_2, \ldots$ and if the capital structure of the firm consisted of different issues of debt differing in T or the like.

and the functions are given by

$$F(\delta,\tau) \equiv \sum_{i=1}^{2} \left(\frac{\delta}{\delta_B}\right)^{\gamma_i} N\left[q\left(\delta,\gamma_i,\tau\right)\right], \ G_j(\delta,\tau) \equiv \sum_{i=1}^{2} \left(\frac{\delta}{\delta_B}\right)^{\eta_{ji}} N\left[q\left(\delta,\eta_{ji},\tau\right)\right],$$

where $N(x)$ is the cumulative distribution function for a standard normal distribution.

A closer inspection of the solution reveals a linear combination (via the matrix $\mathbf{P}$) of two sub-solutions each closely related to the original LT96 solution: the first term A_i gives the value of a risk-free consol bond, the term multiplied by B_i encapsulates the probability that the bond will mature before default, and the term multiplied by C_i encapsulates the probability that the bond will default before maturity. Relative to LT96, each of these independent sub-solutions $i = \{1, 2\}$ has a distorted discount rate $\hat{r}_i > r$, a distorted coupon rate $\hat{c}_i \equiv \left(c\mathbf{P}^{-1}\mathbf{1}\right)_i$ and a distorted recovery rate $\hat{\alpha}_i \equiv \left(\mathbf{P^{-1}}\boldsymbol{\alpha}\right)_i$.[19]

Credit Spreads. Recall that the bond credit spread is the spread between the corporate bond yield and the risk-free rate r. Given a bond of value $D(\delta,\tau)$, the bond yield y is defined as the solution to the following equation:

$$D(\delta,\tau) = \frac{c}{y}(1 - e^{-y\tau}) + pe^{-y\tau}, \qquad (8)$$

so that the right-hand side is the present value of a bond (discounted by y) with a constant coupon payment c and a principal payment p, conditional on it being held to maturity without default or re-trading. For the remainder of the paper, we simply use the ask price $D_H(\delta,\tau)$ in Proposition 1 as our bond price for the left-hand side of equation (8).

3.2 Equity Valuation

The next key step is the equity holders' decision to default, given that they receive the net cash flow in (4) every instant. Because equity is naturally an infinite maturity security and we are

[19]Given a matrix $\mathbf{M}$, $(\mathbf{M})_i$ selects the i-th row and $(\mathbf{M})_{ij}$ selects the i-th row and j-th column.

investigating a stationary (debt maturity structure) setting, the equity value $E\left(\delta;\delta_B\right)$ satisfies the following ordinary differential equation:

$$rE = \delta - (1-\pi)\,c + \underbrace{\frac{1}{T}\left[D_H\left(\delta,T\right) - p\right]}_{Rollover} + \mu\delta E' + \frac{\sigma^2\delta^2}{2}E'', \tag{9}$$

where the left hand side is the required rate of return of equity holders. On the right hand side, the first three terms are the equity holders net cash flows, and the next two terms are capturing the instantaneous change of the firm fundamental. As mentioned earlier, the term involving square brackets is the cash-flow term that arises from rolling over debt (while keeping coupon, principal, and maturity stationary), with $\frac{1}{T}$ being the rollover frequency.

It is worthwhile to point out that equity value in our model is no longer the difference between the levered firm value and debt value adjusted for tax benefits and bankruptcy costs, a common calculation performed in Leland-type models. This is because part of the firm value goes to the dealers in the secondary bond market, and part vanishes because of inefficient holdings of debt by L types. Instead, we need to solve for $E\left(\delta\right)$ directly via (9), which is non-trivial due to the highly-nonlinear form of $D_H\left(\delta,T\right)$ given in (7). The next proposition gives the equity value.

Proposition 2 *Given a default boundary δ_B, the equity value is given by*

$$E\left(\delta;\delta_B\right) = K\left(\frac{\delta}{\delta_B}\right)^{\kappa_2} + \frac{\delta}{r-\mu} + K_0 - \frac{g_F\left(\delta\right)}{T}\sum_{j=1}^{2}P_{1j}B_j e^{-\hat{r}_j T} + \frac{1}{T}\sum_{j=1}^{2}P_{1j}C_j g_{G_j}\left(\delta\right), \tag{10}$$

where P_{ij} gives the element of $\mathbf{P}$ in row i and column j, $\kappa_{1,2} \equiv -a \pm \frac{\sqrt{a^2\sigma^4+2\sigma^2 r}}{\sigma^2}$, $\Delta\kappa \equiv \kappa_1 - \kappa_2$, *and*

$$
\begin{aligned}
K_0 &\equiv \frac{1}{r}\left\{-(1-\pi)c+\frac{1}{T}\left[\sum_{j=1}^{2}P_{1j}A_j+\sum_{j=1}^{2}P_{1j}B_je^{-\hat{r}_jT}-p\right]\right\},\\
K &\equiv -\left[\delta_B+K_0-\frac{1}{T}g_F(\delta_B)\sum_{j=1}^{2}P_{1j}B_je^{-\hat{r}_jT}+\frac{1}{T}\sum_{j=1}^{2}P_{1j}C_jg_{G_j}(\delta_B)\right],\\
g_F(x) &\equiv \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{\frac{x^{\kappa_2}}{\delta_B^{\gamma_i}}H(x,\gamma_i,\kappa_2,T)-\frac{x^{\kappa_1}}{\delta_B^{\gamma_i}}H(x,\gamma_i,\kappa_1,T)\right\},\\
g_{G_j}(x) &\equiv \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{\frac{x^{\kappa_2}}{\delta_B^{\eta_{ij}}}H(x,\eta_{ij},\kappa_2,T)-\frac{x^{\kappa_1}}{\delta_B^{\eta_{ij}}}H(x,\eta_{ij},\kappa_1,T)\right\},\\
H(\delta,\chi,\kappa,T) &\equiv \frac{1}{\kappa-\chi}\left\{\delta^{\chi-\kappa}N\left[q(\delta,\chi,T)\right]-\delta_B^{\chi-\kappa}e^{\frac{1}{2}\left[(\kappa+a)^2-(\chi+a)^2\right]\sigma^2T}N\left[q(\delta,\kappa,T)\right]\right\},
\end{aligned}
$$

where $q(\cdot,\cdot,\cdot)$ *is given in Proposition 1.*

3.3 Endogenous Default Boundary

So far we have taken the default boundary δ_B as given. We now use the standard smooth pasting condition $E_\delta(\delta_B^*;\delta_B^*)=0$ to determine the optimal δ_B^* chosen by equity holders in closed form.

Proposition 3 *The endogenous default boundary* δ_B^* *is given by*

$$
\delta_B^*(T) = (r-\mu)\left[\kappa_2-1+\frac{1}{T}\sum_{j=1}^{2}P_{1j}\hat{\alpha}_jh_{G_j}\right]^{-1}\left(-\kappa_2K_0+\frac{h_F}{T}\sum_{j=1}^{2}P_{1j}B_je^{-\hat{r}_jT}+\frac{1}{T}\sum_{j=1}^{2}P_{1j}A_jh_{G_j}\right),
$$

where $\hat{\boldsymbol{\alpha}}\equiv\mathbf{P}^{-1}\boldsymbol{\alpha}$, P_{ij} *gives the element of* $\mathbf{P}$ *in row* i *and column* j, *and*

$$
\begin{aligned}
h_F &\equiv -\frac{2}{\sigma^2}\sum_{i=1}^{2}\frac{1}{\kappa_1-\gamma_i}\left\{N\left[-(\gamma_i+a)\sigma\sqrt{T}\right]-e^{rT}N\left[-(\kappa_1+a)\sigma\sqrt{T}\right]\right\},\\
h_{G_j} &\equiv -\frac{2}{\sigma^2}\sum_{i=1}^{2}\frac{1}{\kappa_1-\eta_{ij}}\left\{N\left[-(\eta_{ij}+a)\sigma\sqrt{T}\right]-e^{(r-\hat{r}_j)T}N\left[-(\kappa_1+a)\sigma\sqrt{T}\right]\right\}.
\end{aligned}
$$

Relating to existing literature, in the absence of debt rollover, secondary market frictions cannot affect the equity holders' default decision once debt is in place. Infinite debt maturity features no rollover and thus no feedback between liquidity and default, and thus the model converges to the bankruptcy boundary derived in Leland (1994).

3.4 Firm value

Following LT96, we assume that at time 0 the firm is issuing new bonds to H type investors only with a uniform distribution of maturities on $[0, T]$. Given the results established above, the levered initial firm value $TV_0(\delta_0, T; \delta_B)$ is[20]

$$\begin{aligned} TV_0(\delta_0, T; \delta_B) &= E(\delta_0; \delta_B) + \frac{1}{T}\int_0^T D_H(\delta_0, \tau; \delta_B)\, d\tau \\ &= E(\delta_0; \delta_B) + [1, 0] \cdot \mathbf{P}\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1\left(\frac{1-e^{-\hat{r}_1 T}}{\hat{r}_1 T} - I_1(\delta_0, T)\right) \\ B_2\left(\frac{1-e^{-\hat{r}_2 T}}{\hat{r}_2 T} - I_2(\delta_0, T)\right) \end{bmatrix} + \begin{bmatrix} C_1 J_1(\delta_0, T) \\ C_2 J_2(\delta_0, T) \end{bmatrix}\right) \end{aligned} \tag{11}$$

where

$$\begin{aligned} I_j(\delta, T) &= \frac{1}{\hat{r}_j T}\left[G_j(\delta, T) - e^{-\hat{r}_j T} F(\delta, T)\right], \\ J_j(\delta, T) &= \frac{1}{(\eta_{1j} + a)\,\sigma\sqrt{T}} \sum_{i=1}^{2} (-1)^i \left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}} N\left[q(\delta, \eta_{ij}, T)\right] q(\delta, \eta_{ij}, T). \end{aligned}$$

We will use this measure in section 5.1 to study the optimal maturity structure decision by the firm at time 0.

4 Endogenous Liquidity, Feedback Effects and Credit Spreads

We discuss the model's implications in this section. We first explain the parameter choices in Section 4.1. Section 4.2 analyzes the endogenous liquidity that depends on both firm fundamental and time-to-maturity. Based on endogenous liquidity, Section 4.3 illustrates the positive feedback effect between fundamental and liquidity for corporate bonds. Section 4.4 discusses the model's implications on the observed credit spreads.

[20] The reader should note the difference that we have one unit measure of bonds, whereas LT96 expand the measure of bonds according to maturity. We could also define firm value as the steady state sum of the individual valuations of equity holders and the two types of debt holders, which would result in

$$TV_{ss}(\delta_0, T; \delta_B) = E(\delta_0; \delta_B) + \frac{1}{T}\int_0^T \left[p_H(\tau) D_H(\delta_0, \tau; \delta_B) + p_L(\tau) D_L(\delta_0, \tau; \delta_B)\right] d\tau$$

where $p_H(\tau), p_L(\tau)$ are steady-state proportions given in the Appendix A.5.

4.1 Parameters

We present the baseline parameters in Table 1 which are broadly consistent with those used in the literature to calibrate standard structural credit risk models. Although a thorough calibration is beyond the scope of this paper, we choose our parameters to roughly match the empirical characteristic of BB rated corporate bonds.

We set the risk-free rate $r = 8\%$ which is also used by Huang and Huang (2003). We use a debt tax benefit rate $\pi = 27\%$, [21] and set a debt maturity of $T = 10$ years for illustration. Without loss of generality, we normalize the initial cash flow level δ_0 to 1.

We set the L type discount rate $\overline{r}$ to 10%. Relative to the H type's normal discount rate $r = 8\%$, the implied high-low valuation wedge for default-free bonds (and if investors are holding them forever) is 20%. This wedge due to inefficient holding is consistent with the estimation result in Feldhutter (2011).[22] The inefficiency wedge is also reflected in the divergence of default recovery rates $\alpha_H - \alpha_L$ across H type and L type investors. Because the recovery rate in most structural credit risk models (Chen, 2010; Bhamra, Kuehn, and Strebulaev, 2010) is mainly based on bond trading prices right after default, mapping to our model it is close to but above the L type recovery rate, which we set at $\alpha_L = 55\%$.[23] For the H type investors' recovery rate, we set $\alpha_H = 67\%$, which corresponds to a high-low valuation wedge of $12\%/55\% \approx 22\%$, a conservative implementation of the parameters emerging from the empirical study of defaulted bonds in Altman and Eberhart (1994).[24]

[21] While tax rate of bond income is 32%, many institutions holding corporate bonds enjoy tax exemption. Thus, we use an effective bond income tax rate of 25%. Then, the formula given by Miller (1977) implies a debt tax benefit of $1 - [(1 - 32\%)\,(1 - 15\%)\,/\,(1 - 25\%)] = 26.5\%$ where 32% is the marginal rate of corporate tax and 15% is the marginal rate of capital gain tax.

[22] In Feldhutter (2011), L type is assumed to carry an extra holding cost as in Duffie, Garlenau, and Pedersen (2005), as opposed to a higher discount rate in our model. Based on the estimates of Feldhutter (2011), for a ten-year default-free bond, the effective discount of a bond (that is held by L type investors forever) is about 17%, which is close to our assumption.

[23] Chen (2010) finds that across 9 different aggregate states, bonds have default recovery rates around 60%. These recovery rates are typically estimated from two standard sources. The first is the Moody's recovery data which is based on bond price around 30 days after default. The other is an NYU recovery database which is based on prices as close as possible to default.

[24] Altman and Eberhart (1994) document that for senior and secured bonds that went through bankruptcy, the bid price for the bond at default (which proxies for D_L) is about 52, and the average bond emergence payoff is about 84 which is paid out about 1.92 years after default (the payoffs are weighted average of senior and secured bonds). The data covers 1980 to 1992, with an average annual interest rate of about 9.3% (FRB release, H.15). Using this rate to

We rely on implied bond illiquidity to determine parameters on search frictions. More specifically, we choose these parameters to match the implied bid-ask spread (evaluated at the initial cash flow level δ_0) to 100 bps, which corresponds to the transaction cost of BB rated bonds documented in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011). We set $\beta = 5\%$ (i.e., dealers get 95% of the trading surplus) based on the estimation result in Feldhutter (2011), and choose the liquidity shock intensity ξ and dealer-meeting intensity λ to match both the (initial) percentage bid-ask spread of 100 bps and the empirical holding time (turnover) of 1 year.[25] This leads us to set $\xi = 1.07$ and $\lambda = 15.5$. That is, investors are hit by liquidity shocks about every 11.2 months, and then it takes about three weeks for them to sell their holdings completely.

The cash flow drift $\mu = 2\%$ and volatility $\sigma = 20\%$ are standard in the literature. For bonds that are priced at par, we pin down the coupon $c = 1.31$ and the principal $p = 11.67$ to produce an initial credit spread of 320 bps, which corresponds to the observed credit spread of the BB rated corporate bonds (Huang and Huang, 2003).

Finally, for illustration we may later use LT96 and HX12 models as alternative benchmarks, which take the same parameters as in Table 1.[26] For the HX12 model, the transaction cost is assumed to be constant at k (and L type investors are forced to immediately sell with this transaction cost). We set $k = 99.4bps \approx 1\%$ for the HX12 benchmark for fair comparison as to also have a bid-ask spread of $100bps$.[27]

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discount the emergence payoff, we obtain the proxy for D_H of 71, which implies a high-low valuation wedge around 32%.

[25] According to Bao, Pan, and Wang (2011), the average turnover in their sample is about 1 year. In our model, the average time that an investor is holding the bond (including the time that the investor remains at H type and that he is L type but searching) is $\frac{1}{\xi} + \frac{1}{\lambda}$. Thus we require that $\frac{1}{\xi} + \frac{1}{\lambda} = 1$. And, we require that the effective percentage bid-ask spread $\frac{(1-\beta)[D_H(\delta_0,T) - D_L(\delta_0,T)]}{\frac{1}{2}X(\delta_0,T) + \frac{1}{2}D_H(\delta_0,T)} = 100bps$ at δ_0.

[26] More specifically, both models take the common discount rate $r = 8\%$ and recovery value $\alpha_H = 80\%$ as in Table 1, along with other parameters. For HX12, we set liquidity shock intensity $\xi = 1$ (which is slightly lower than $\xi = 1.07$ given in Table 1) so that the average holding time is kept at 1 year; this choice makes a negligible quantitative difference.

[27] There is a small adjustment to k, because the proportional bid-ask spread (over the midpoint) is $bidask = \frac{k}{1-k/2}$.Thus, $k = \frac{bidask}{1+\frac{1}{2}bidask} < bidask$, i.e., k will be slightly lower than the targeted proportional bid-ask spread $bidask$.

Firm Characteristics			Illiquid Secondary Market		
Parameter	Interpretation	Value	Parameter	Interpretation	Value
δ_0	Initial cash flow level	1	r	Discount rate	8%
σ	Volatility	20%	$\overline{r}$	Liq. shock discount rate	10%
μ	Drift	2%	ξ	Intensity of liquidity shock	1.07
π	Tax shield	27%	λ	Intensity to meet dealers	15.5
p	Principal	10.67	β	Bargaining power of investors	5%
c	Coupon	1.31	α_H	Recovery value H type	67%
T	Bond maturity	10	α_L	Recovery value L type	55%

Table 1: Model parameters that are calibrated to BB rated bonds.

4.2 Endogenous Liquidity

4.2.1 Endogenous bid-ask spread

As mentioned, the (dollar) bid-ask spread is simply the difference between the bid price $X(\delta,\tau)$ and the ask price $D_H(\delta,\tau)$:

$$(1-\beta)\,S(\delta,\tau) = D_H(\delta,\tau) - X(\delta,\tau)\,, \tag{12}$$

which is just a constant positive fraction of the surplus S. In the following proofs, for better analytical properties we concentrate on the behavior of S.

Time-to-maturity. First, let us study the effect of time-to-maturity by fixing firm fundamental. Formally, we have the following proposition.

Proposition 4 *Under the following sufficient conditions*

$$c - p\hat{r}_2 \quad \geq \quad 0, \;\; \textit{and}\; \frac{\delta_B}{r-\mu}\left[\alpha_L\left(\hat{r}_2-\overline{r}\right)+\alpha_H\left(r-\hat{r}_2\right)\right]+p\left(\overline{r}-r\right)\geq 0,$$

we have $S_\tau(\delta,\tau) > 0$, i.e. the bid-ask spread is larger for bonds with longer time-to-maturity.

The intuition for this result is simple. Because a shorter time-to-maturity delivers the full principal back to L type investors sooner, this enhances L type's outside option in the bargaining and reduces the rent extracted by dealers, thereby resulting in a smaller bid-ask spread. In fact, by the boundary conditions the surplus vanishes as time-to-maturity goes towards 0, i.e.,

$$\lim_{\tau \to 0} S(\delta, \tau) = 0.$$

If the bond is almost immediately demandable from the firm, L type investors gain little value from trade with dealers, and as a result the bid-ask spread vanishes.[28] This indicates that short-term debt provides liquidity for bond investors, and we will discuss the role of liquidity provision in more detail in Section 5.1.

Distance-to-default. Second, let us fix the time-to-maturity $\tau > 0$ and investigate the bid-ask spread by varying the distance-to-default (i.e., $\delta - \delta_B$). Formally, we have the following proposition.

Proposition 5 *Under sufficient conditions provided in the Appendix A.4.2, we have $S_\delta(\delta, \tau) < 0$, i.e. the bid-ask spread is smaller for bonds with higher firm fundamental.*

The sufficient conditions in Proposition 5 can be understood as follows. Recall that, in Section 2.4, motivated by the empirical facts, we assume that bond investors need to wait quite a long time before they receive the cash pay-out. It is easy to show that as the firm fundamental converges towards δ_B, for any bonds that still have time-to-maturity left, i.e. $\tau > 0$, we have

$$\lim_{\delta \to \delta_B} S(\delta, \tau) = (\alpha_H - \alpha_L) \frac{\delta_B}{r - \mu} > 0, \tag{13}$$

We focus on the situation where the post-default illiquidity $\alpha_H - \alpha_L$ derived in equation (5) is sufficiently high, especially relative to the bid-ask spread for default-free bonds.[29] As a a result,

[28] This implies that S goes down for τ close to zero. Unfortunately, for global monotonicity established in Proposition 4, we need some extra sufficient conditions due to the complex nature of the functions involved.

[29] The intuition is straightforward: When $\delta = \infty$, so that bonds are risk-free, we have

$$\begin{aligned} \begin{bmatrix} D_H(\infty, \tau) \\ D_L(\infty, \tau) \end{bmatrix} &= \mathbf{A}^{-1}\mathbf{c} + \exp(-\mathbf{A}\tau)\left(\mathbf{p} - \mathbf{A}^{-1}\mathbf{c}\right) \\ &= \frac{c}{(r+\xi)(\overline{r}+\lambda\beta) - \xi\lambda\beta} \begin{bmatrix} \overline{r}+\xi+\lambda\beta \\ r+\xi+\lambda\beta \end{bmatrix} + \exp(-\mathbf{A}\tau) \begin{bmatrix} p - \frac{c(\overline{r}+\xi+\lambda\beta)}{(r+\xi)(\overline{r}+\lambda\beta)-\xi\lambda\beta} \\ p - \frac{c(r+\xi+\lambda\beta)}{(r+\xi)(\overline{r}+\lambda\beta)-\xi\lambda\beta} \end{bmatrix} \end{aligned}$$

the endogenous illiquidity rises when the cash flow rate δ deteriorates and the firm is closer to bankruptcy.

4.2.2 Proportional bid-ask spread and empirical implications

So far for analytical tractability we have focused on dollar bid-ask spread $S(\delta, \tau)$. However, the effective percentage bid-ask spread $\Delta(\delta, \tau)$ is commonly used as an illiquidity measure, and we here define it as the dollar bid-ask spread $S(\delta, \tau)$ divided by the mid point of transaction prices (bid price X and ask price D_H):

$$\Delta(\delta, \tau) = \frac{(1-\beta)\left[D_H(\delta, \tau) - D_L(\delta, \tau)\right]}{\frac{1}{2}X(\delta, \tau) + \frac{1}{2}D_H(\delta, \tau)} = \frac{2(1-\beta) S(\delta, \tau)}{(1+\beta) D_H(\delta, \tau) + (1-\beta) D_L(\delta, \tau)}. \tag{14}$$

The percentage illiquidity $\Delta(\delta, \tau)$ shares the same qualitative properties as $S(\delta, \tau)$. In addition, as the firm fundamental deteriorates the bond value decreases and thus amplifies $\Delta(\delta, \tau)$, this negative force naturally strengthens our result of increasing illiquidity for bonds closer to default. To the extent that the percentage illiquidity is more empirically relevant, the sufficient conditions in Proposition 5 are much stronger than necessary, and our theoretical results should be more general than they appear.

We plot the bid-ask spread in Figure 2 as a function of both time-to-maturity (that is time dynamics) in the left-hand panel and distance-to-default (that is state dynamics) in the right hand panel. The highest time-to-maturity is just the maturity for newly issued bonds, which in the figure is $T = 10$. The distance-to-default is captured by the difference between the current firm fundamental δ and the endogenous bankruptcy boundary $\delta_B^* = 0.56$.

The left hand panel of Figure 2 shows that the endogenous proportional bid-ask spread is lower for shorter time-to-maturities (recall Proposition 4), and the right hand panel of Figure 2 shows that the bid-ask spread rises when the firm fundamental deteriorates towards the bankruptcy boundary

Together with $S_\tau(\delta, \tau) < 0$, we know that S reaches a maximum when $\tau = T$. The most important part of the proof is that $S(\delta_B, \tau) - \lim_{\delta \to \infty} S(\delta, \tau) < 0$. That is, a necessary condition is that the bid-ask spread of the default-free bond is below that of the defaulted bond. Unfortunately we are unable to show the sufficiency of this condition due to the complex nature of the functions involved, and in the proof of Proposition 5 we impose stronger sufficient conditions.

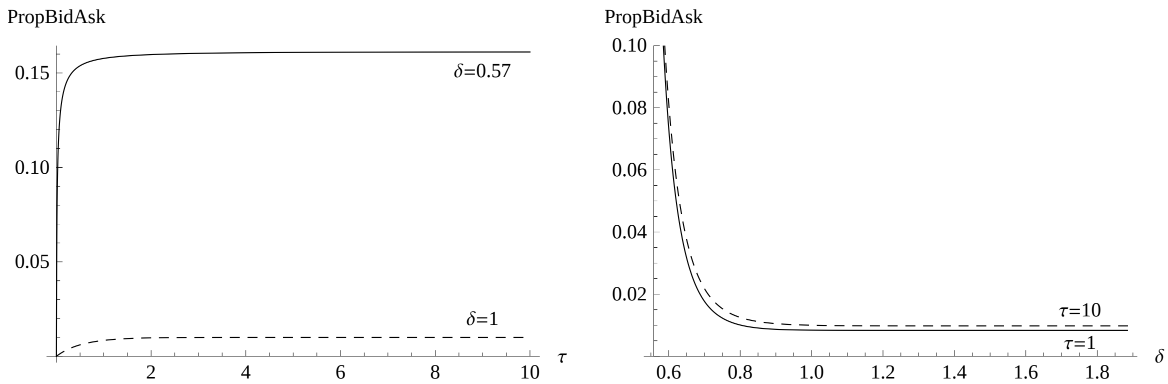

Figure 2: Left panel: **Proportional bid-ask spread Δ w.r.t.** τ, i.e. $D_H - X$, for $\delta = 1$ (dashed) and $\delta = .57$ (solid). Note that $\delta_B = .56$. Right panel: **Proportional bid-ask spread Δ w.r.t.** δ, i.e. $D_H - X$, for $\tau = 10$ (dashed) and $\tau = 1$ (solid).

δ_B (recall Proposition 5). Both of them are consistent with the empirical regularity in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011).

Interaction between time-to-maturity and distance-to-default. We now investigate the impact of the interaction between time-to-maturity and distance-to-default on the endogenous bid-ask spread. As our goal is to provide some novel empirical predictions, in this subsection we focus on the percentage bid-ask spread $\Delta(\delta, \tau)$ in (14) which is commonly used in the empirical literature.

Similar to $S(\delta, \tau)$, under our parametrization we find that $\Delta(\delta, \tau)$ is increasing with τ for $\delta > \delta_B$ as shorter maturity provides better liquidity. However, we also know from (13) that, as we approach the bankruptcy boundary δ_B, $\Delta(\delta, \tau)$ becomes independent of $\tau > 0$, i.e., we have the same liquidity across all maturities. Thus, when the firm edges closer and closer to default, the slope of $\Delta(\delta, \tau)$ with respect to time-to-maturity τ becomes flatter and flatter. In other words, for financially healthy firms, the difference between the bid-ask spreads of long-term bond and short-term bond is greater than that of firms in imminent danger of bankruptcy. Formally, we have the following proposition.

Proposition 6 *Given any time-to-maturity $\tau > 0$, when the firm gets close to default, we have $lim_{\delta \to \delta_B} \frac{\partial \Delta(\delta, \tau)}{\partial \tau} = 0$, i.e., the bond illiquidity is independent of time-to-maturity of the bond.*

Proof. When $\delta \to \delta_B$ while $\tau > 0$, the boundary condition is $\lim_{\delta\to\delta_B} \Delta(\delta,\tau) = \frac{2(1-\beta)(\alpha_H-\alpha_L)}{(1+\beta)\alpha_H-(1-\beta)\alpha_L}$ which is independent of τ. ■

This property is intuitive. Default, by forcing firms to enter lengthy bankruptcy proceeding that puts all debt holders of equal seniority on equal footing, eliminates difference due to maturities. For financially healthy firms, default is remote, and therefore the time-to-maturity has a positive and significant impact on the bid-ask spread. However, when default is imminent, although the bid-ask spreads for both long-term and short-term bonds soar, their difference diminishes as it is more likely that the stated time-to-maturity eventually becomes irrelevant. This intuition is quite general, as it only relies on the fact that maturity plays no role in bankruptcy.

The above discussion suggest the following regression specification:

$$\Delta_{i,t} = b_0 + \underset{(+)}{b_{Maturity}} \cdot Maturity_{i,t} + \underset{(+)}{b_{CDS}} \cdot CDS_{i,t} + \underset{(-)}{b_{Maturity*CDS}} \cdot Maturity_{i,t} \times CDS_{i,t}. \quad (15)$$

As shown, our model predicts a positive $b_{Maturity}$, i.e., bonds with longer time-to-maturity should have a higher bid-ask spread. Further, the model predicts a positive b_{CDS}, i.e., the bond that is closer to default should have a higher bid-ask spread as well. These two predictions conform with the empirical findings in Edwards, Harris, and Piwowar (2007), and Bao, Pan, and Wang (2011). Finally, Proposition 6 implies that $b_{Maturity*CDS} < 0$, i.e., the difference between the bid-ask spreads of long-term and short-term bonds in financially healthy firms is greater than that of financially distressed firms. As just explained, this new testable prediction is intuitive and awaiting future empirical research.

4.3 Feedback Loop between Fundamental and Liquidity

By linking the secondary market liquidity endogenously to firm fundamental, we now demonstrate the positive default-liquidity spiral in which the deterioration of firm fundamental, via worsening liquidity of the secondary bond market, edges the firm even closer to default, which in turn leads to further deterioration in secondary market liquidity.

In this section we aim to provide a full account of the positive feedback mechanism between fundamental and liquidity. Although we will benchmark our results to that of LT96 and HX12 for illustrative purposes, our contribution is well beyond just endogenizing liquidity in credit risk models. Besides being theoretically challenging, characterizing the full inter-dependence between liquidity and default represents an economically significant leap toward understanding the role of liquidity in determining credit spreads for corporate bonds. More broadly, it established an the endogenous link between liquidity risk and solvency risk in financial markets.

4.3.1 Rollover losses, endogenous liquidity, and endogenous default

The endogenous pro-cyclical secondary market liquidity and the endogenous default decision taken by equity holders are the two building blocks for the positive feedback loop between fundamental and liquidity. To understand the mechanism, consider the rollover losses borne by equity holders as a function of the firm cash flow rate δ. The dashed line in the left panel of Figure 3 graphs the benchmark rollover losses implied by the LT96 model where the secondary bond market is perfectly liquidity. There, the (absolute value) of rollover losses $\frac{1}{T}\left[D\left(\delta,T\right)-p\right]$ rises when the firm fundamental δ deteriorates, simply because forward looking bond investors adjust the market price of newly issued bonds downward when the firm is closer to default. As shown in the dashed line, the similar pattern holds for the HX12 model with the constant (proportional) transaction cost $k\approx 1\%$. Because of higher refinancing costs for liquidity compensation, compared to the LT96 model equity holders in HX12 suffer greater rollover losses.

The new force in our model is that the endogenous secondary market liquidity further amplifies the rollover losses. The right panel of Figure 3 graphs the percentage bid-ask spread $\Delta\left(\delta,T=10\right)$ defined in (14). Recall that our calibration requires $\Delta\left(\delta_0=1,T=10\right)=1\%$ for better comparison to HX12, and the bond illiquidity remains constant in HX12 ($k\approx 1\%$) or LT96 ($k=0$). The countercyclical pattern of bid-ask spread $\Delta\left(\delta,T=10\right)$, i.e., more severe illiquidity for lower δ's, implies the same countercyclical pattern of the implied endogenous transaction cost $k\left(\delta\right)$ in (**??**). This suggests a worsening secondary market liquidity for firms with lower fundamentals in our

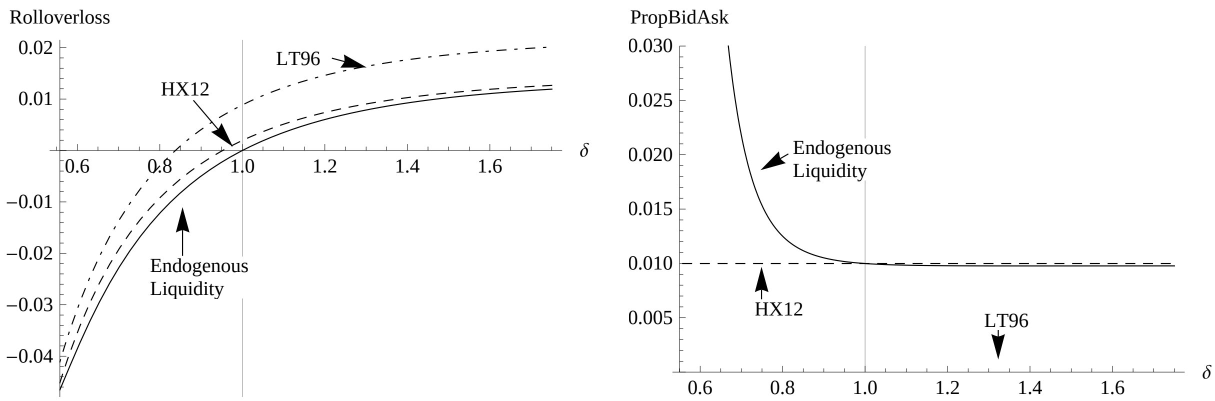

Figure 3: Left panel: **Rollover loss** $\frac{1}{T}\left[D_H(\delta, T) - p\right]$ as a function of fundamental value δ of our model (solid line), the LT96 model (dash-dotted line) with perfectly liquidity bond market, and the HX12 model (dashed line) with exogenous transaction cost $k \approx 1\%$. The rollover losses in our model are zero at issuance (i.e., when $\delta = 1$) because we assume that bonds are issued at par. Right panel: **Proportional bid-ask spread** $\Delta(\delta, \tau)$ defined in (14) for our model (solid line), relative to constant proportional bid-ask spread of 100*bps* in HX12 (dashed line) and no bid-ask spread in LT96.

model.

Relative to models with constant secondary market liquidity, the endogenous search market depresses the bond market price $D_H(\delta, T)$ further for low fundamental states as the implied transaction costs rise. This explains the left panel in Figure 3 where rollover losses in our model (the solid line) are more sensitive to the firm cash flow state δ relative to the HX12 model (the dashed line). This pro-cyclical secondary market liquidity is empirically relevant, because it significantly reduces the equity holders' option value of servicing the debt especially in bad times, and hence the firm defaults earlier.

4.3.2 Positive feedback between fundamental and liquidity

The above discussion implies an important positive feedback loop between firm fundamental and secondary market liquidity for corporate bonds, which is illustrated in Figure 4. For the purpose of illustration, the following discussion takes place in the counterfactual world of constant transaction costs (say HX12 with positive constant $k > 0$ or LT96 with $k = 0$) to explain how the fixed endogenous default threshold δ_B^* arises.

For investors of corporate bonds, the bond fundamental can be measured as the firm's distance

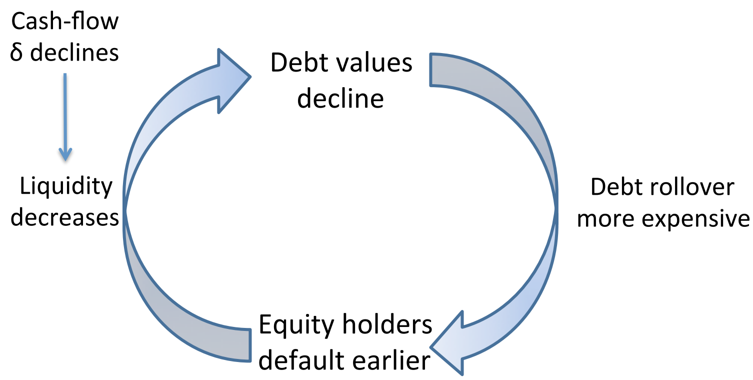

Figure 4: **Feedback loop** between secondary market liquidity and equity holders' default decision.

to default, i.e., $\delta - \delta_B$. Imagine a *negative* shock to firm cash flow rate δ. Since this negative shock brings the firm closer to default, this constitutes a pure-fundamental driven negative shock to bond investors and lowers the holding values of D_H and D_L. This force is present in LT96 and HX12.

The novelty of our model is that, a negative δ shock not only lowers debt values, but also worsens the secondary market liquidity. The lower distance to default worsens the L types' outside option when bargaining with a dealer, as default leads to protracted bankruptcy court decisions. Consequently, bonds in the secondary market becomes more illiquid, as indicated by the left large arrow with "declining liquidity" in Figure 4. This is the pro-cyclicality of liquidity we already discussed above.

Rational H type bond investors will thus value bonds less, i.e., a lower D_H, because they expect to face a less liquid secondary market once hit by liquidity shocks. As shown in Figure 4, the worsening liquidity in the secondary market gives rise to a lower primary market bond issuing price D_H relative to an environment with constant market liquidity.

The lower bond prices now feed back to the equity holders' default decision via the rollover channel, indicated by the arrow on the right of Figure 4. This is because equity holders are absorbing heavier rollover losses (i.e. net cash flow NC_t in (4) goes down), as suggested by the left panel of Figure 3. Equity holders hence default earlier at a higher threshold δ_B, relative to an environment with a constant market liquidity.

The higher default threshold now translates into a shorter distance to default $\delta - \delta_B$. But just as discussed before, the search-based secondary market kicks in again: as shown on the left-hand side in Figure 4, the shorter distance to default *further* worsens market liquidity via the declining outside option of the L type investors. The loop repeats as the lower liquidity now again lowers effective bond prices, and finally stops at the fixed point δ_B given in Proposition 3.

4.4 Credit spreads

The positive liquidity-default spiral illustrated in the previous subsection can have significant quantitative effect on observed corporate bond spreads, or equivalently primary market and secondary market ask prices $D_H(\delta, \tau)$.

Recall the definition of the bond yield y in (8); since our focus is the credit spread of newly issued bonds, we study the H type debt value D_H (the ask price) with $\tau = T$. In Figure 5 we plot the credit spread $y - r$ as a function of δ. As the first benchmark, the dash-dotted line plots the credit spread in the LT96 model with a perfectly liquid secondary corporate bond market. The more stringent benchmark is the credit spread implied by the HX12 model (the dashed line), which takes into account the fact that the higher financing cost due to bond illiquidity pushes equity to default earlier than in LT96. The solid line in Figure 5 gives the credit spread under our model, which incorporates the full liquidity-default spiral discussed in Section 4.3.2.

Because both our model and HX12 account for the illiquidity of corporate bonds, their implied credit spreads are higher than the LT96 benchmark without the liquidity factor. The difference in illiquidity between our model (solid line) and HX12 (dashed line), which surges especially when the firm is not doing well as shown in Figure 5, is due to the positive liquidity-default spiral effect.

Because in our model the bond becomes more illiquid once the firm edges closer to default (and thereby receiving a much lower credit rating, say C), while HX12 assume a constant illiquidity throughout the entire bond life, it is not surprising to observe a significant divergence in implied credit spreads across two models in these bad states. To isolate this issue, we compare the credit spreads implied by different models conditional on the initial cash flow $\delta_0 = 1$ (thus conditional on the BB rating), at which we have controlled for the bond liquidity (bid-ask spread) of 100 bps.

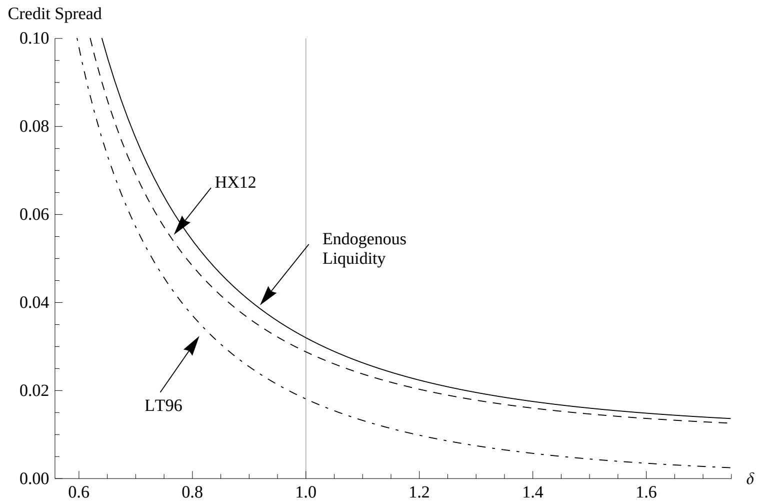

Figure 5: **Credit spread** $y - r$ of bonds at issuance (i.e., $T = 10$) as a function of fundamental cash-flow δ for our model (solid line), the LT96 model (dash-dotted line), and the HX12 model with $k = 1\%$ (dashed line)

Recall that we have calibrated our model for BB rated bonds to generate an initial credit spread of 320 bps; this is about twice of that implied by the LT96 model (about 181 bps at $\delta_0 = 1$ in the dash-dotted line). For HX12 without endogenous pro-cyclical liquidity, the implied initial credit spread at $\delta_0 = 1$ is only about 288 bps. Since we have controlled for bond liquidity given the bond's initial rating, the difference of $320 - 288 = 32$ bps (or 10% of the credit spreads of BB rated bonds) comes entirely from our novel positive spiral between liquidity and default. Intuitively, equity holders default earlier in our model with pro-cyclical endogenous secondary market liquidity ($\delta_B^* = 0.56$), compared to the HX12 model with exogenous constant liquidity ($\delta_{B,HX12}^* = 0.55$).

4.5 Liquidity Premium and Default Premium

It has been widely recognized that the credit spread of corporate bonds not only reflects a default premium determined by the firm's credit risk, but also a liquidity premium due to the illiquidity of the secondary debt market, e.g., Longstaff, Mithal, and Neis (2005), and Chen, Lesmond, and Wei (2007). However, both academics and policy makers tend to treat the default premium and liquidity premium as independent, and thus ignore interactions between them. For instance, it is common practice to decompose firms' credit spreads into independent liquidity-premium and default-premium components and then assessing their quantitative contributions, e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010).

This treatment of independence between liquidity and default is contrary to the data, which suggests that these components exhibit strong positive correlation. We have seen in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011) that liquidity deteriorates for bonds that are issued by firms with high CDS spreads. In an report issued by Barclay Capital, Dastidar and Phelps (2009) study the quote-based bond liquidity measure directly, and document the same robust empirical regularity in not only cross-section (investment grade vs. speculative grate) but also time-series (2005-06 before crisis vs 2008-09 during crisis).

In our model, the endogenous inter-dependence between the liquidity and default premia for corporate bonds captures this important empirical regularity. By endogenizing the secondary market liquidity, our model points out that the origin of shock to liquidity premia can be traced back to the deterioration of firm fundamental itself. Thus, both default premium and liquidity premium are inter-dependent, and the positive feedback loop further amplifies and reinforces both premia in a nontrivial way. More importantly, this positive spiral effect may be quantitatively significant in explaining the observed credit spreads (about $32/320 = 10\%$), as illustrated in Section 4.4.

Another important implication of our model is related to the "credit spread puzzle" in the structural credit risk literature. Structural credit models have difficulty in producing the quantitatively significant AAA credit spread observed in the data, once calibrated to historic default probabilities and asset prices (e.g., Huang and Huang (2003)). In our model, in Figure 5 there remains a non-negligible credit spread even for large δ (hence default-free bonds, see footnote 29 for expressions). This is because in our setting the liquidity risk is uninsurable on the agent level and thus does not affect the translation of the physical probabilities to the risk-neutral probabilities, which is consistent with the idea that the AAA spread can be explained by liquidity reasons. Perhaps more interestingly, the positive liquidity-default spiral emphasized in this paper has the potential to amplify the relatively small liquidity shocks to quantitatively significant liquidity and default premia. We are performing a thorough calibration exercise on this topic in an ongoing project.

5 Extensions and Discussions

5.1 Optimal Debt Maturity

Beyond the feedback loop between fundamental and liquidity, the debt maturity features a natural trade-off between liquidity provision and earlier inefficient default. This natural trade-off allows us to derive the optimal debt maturity (given the stationary maturity structure). Segura and Suarez (2011) present a related trade-off in a banking model without secondary markets but with periodic disruptions of the primary market for debt funding. Although the probability of these disruptions is *exogenous*, the severity of the disruptions is determined by how short the bank's maturity structure is. This is traded off against short-term debt being cheaper outside crisis states. In contrast, our model features an *endogenous* probability of default that is driven by the maturity structure and we also trade this off against cheaper short-term debt away from the bankruptcy boundary.

5.1.1 Liquidity provision: the bright side of short maturity

Section 4.2 has shown that bonds with shorter maturity have a more liquid secondary market, suggesting the role of liquidity provision for short-term debt. The efficiency gain due to short-term maturity arises from two channels.

First, debt holders hit by liquidity shocks become inefficient holders of bonds, and due to trading frictions the inefficient holding lasts for a while. As detailed in Appendix A.5, the steady-state proportion of L types as the firm is able to issue to only H types is

$$\mu_L(T) = \frac{\xi}{\lambda+\xi} - \underbrace{\frac{\xi\left[1-e^{-T(\lambda+\xi)}\right]}{T(\lambda+\xi)^2}}_{Allocative\,efficiency}, \tag{16}$$

with $\mu_L'(T) > 0$, $\lim_{T\to\infty}\mu_L(T) = \frac{\xi}{\lambda+\xi}$ and $\lim_{T\to 0}\mu_L(T) = 0$. Hence, the second term in (16) is the *allocative efficiency gain* of shortening the bond maturity T. Intuitively, shortening maturity alleviates this inefficiency because of the firm's superior primary market liquidity: whenever debt matures, the firm moves debt from inefficient L investors to efficient H investors via new bond issuance.[30]

Second, a shorter maturity reduces the rent extracted by dealers in the secondary market, thus leading to a *bargaining efficiency gain*. Intuitively, a shorter maturity, by allowing L investors to receive principal payment earlier, raises their outside option of waiting and in turn lowers the dealer's rent.

5.1.2 Earlier default: the dark side of short maturity

On the other hand, as first shown in LT96 (and formally proven in HX12 and Diamond and He, 2012[31]), shorter debt maturity in an LT96 style model leads to earlier default and thus greater dead-weight bankruptcy cost. In other words, the optimal maturity in LT96 and HX12 is $T^* = \infty$, so that debt should always take the form of an infinitely lived consol bond. As discussed, the equity holders' rollover losses are $\frac{1}{T}\left[D_H(\delta, T) - P\right]$. In bad times (low fundamental δ), notwithstanding the fact that short-term debt has a greater market price $D_H(\delta, T)$, the effect of a higher rollover frequency $\frac{1}{T}$ dominates, leading to heavier rollover losses. As a result, equity holders default earlier if the firm is using shorter maturity debt.

5.1.3 Optimal Interior Debt Maturity

Relative to LT96 model where the debt maturity affects the equity holders' default decision, in our model the firm—being short of intermediating the market for its debt itself—uses the inefficient tool of the maturity structure to provide liquidity services to bondholders. This extra force naturally leads to an endogenous optimal maturity structure. In Figure 6 we plot in the left panel the ex ante levered firm value $TV(\delta_0)$ given in equation (11) for our model (solid line) and the LT96

[30] The firm could, instead of providing liquidity via maturity, allow bondholders with liquidity shocks to put back their bonds at the face value p. There are two important drawbacks. First, if the firm cannot distinguish who was hit by a liquidity shock, whenever $D_H < p$ everyone will put back their debt at the same time. In fact, the put provision is akin to making bonds demand deposits and we are at traditional models of bank runs. Second, even if the liquidity shock is observable, there will be an additional flow term $\xi\left[D_H - p\right] dt$ as L investors are putting back their bonds to the firm every instant. This additional refinancing losses may influence the bankruptcy boundary in an adverse way and destroy the liquidity thus provided. The full implications of expanded bond contract terms (beyond the choice of initial maturity T covered in this paper) is left for future work.

[31] HX12 prove this claim for given (c, p) in the LT96 framework, while Diamond and He, 2012 prove this claim controlling for leverage (adjusting (c, p) to maintain the same debt value as shifts in the bankruptcy boundary caused by maturity shortening move the value of debt) in the random maturity framework of Leland (1998).

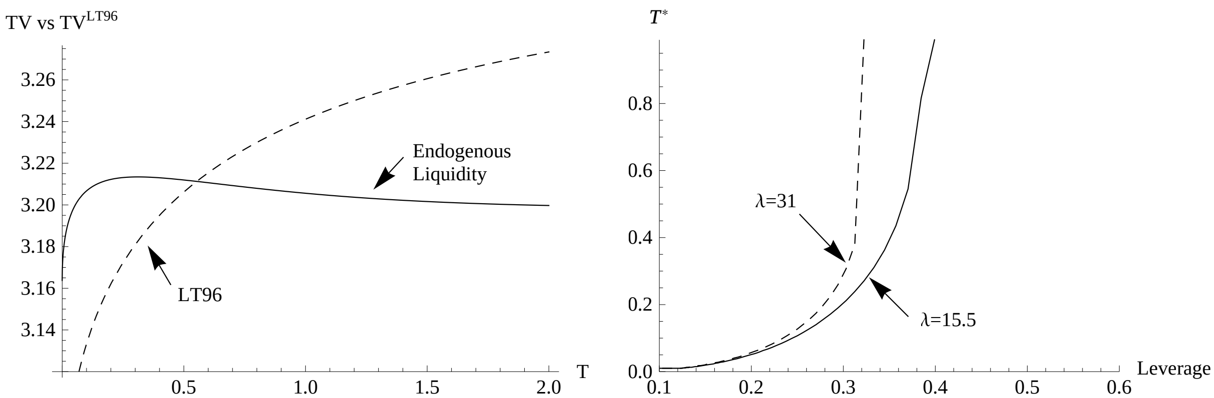

Figure 6: Left panel: **Total firm value** in the main model (solid line) and without frictions in the LT 96 model (dashed line). Right panel: **Optimal maturity** T^* in the main model as a function of initial book leverage for different levels of search frictions, $\lambda = 15.5$ (solid line) and $\lambda = 31$ (dashed line). Both lines at some point jump to $T^* = \infty$ for high enough finite leverage.

benchmark model (dashed line) as a function of the debt maturity T and initial book leverage $(r-\mu)\,p/\delta_0 = 1/3$.[32] The hump shape of firm value suggests the existence of an interior solution for the optimal maturity structure in our model. In contrast, without the benefit of liquidity provision, the total firm value in the LT96 model (dashed line) is monotonically increasing in debt maturity T.

In the right panel of Figure 6 we draw the optimal maturity T^* as a function of initial leverage. The solid line depicts the optimal maturity for a secondary market with baseline intermediation, i.e., $\lambda = 15.5$, whereas the dashed line depicts the optimal maturity for a secondary market with high (thus more efficient) intermediation, i.e., $\lambda = 31$. For low (high) initial leverage, bankruptcy becomes more (less) remote, and the effect of liquidity provision (bankruptcy cost) dominates, resulting in a shorter (longer) optimal debt maturity. Additionally, for our baseline intermediated markets with $\lambda = 15.5$, the firm provides liquidity to its debt holders through shorter maturity. In contrast, for a better intermediated market with $\lambda = 31$, the optimal maturity shifts out uniformly, and jumps to infinity for firms with relatively low initial leverage. In other words, a better functioning secondary market reduces the need to provide liquidity via shorter maturity and thus alleviates the bankruptcy pressure generated by the short debt structure.

[32] That is, the ratio of the aggregate face value p over the unlevered firm value $\frac{\delta_0}{r-\mu}$.

5.2 Discussion of Asymmetric Information

In our model, the important driving force behind the spiking illiquidity near default is that there is a significant valuation wedge between H and L type investors for defaulted bonds, as summarized by the individual recovery values α_H and α_L. In the literature as well as in practice, an equally compelling explanation for the deteriorating liquidity of corporate bonds near default is a possibly worsening adverse selection problem due to information asymmetry. More specifically, one can imagine that some bond investors have private information regarding the bond's recovery value in default. As the firm edges closer to default, the informed agent's information becomes more valuable and he is more likely to attempt to sell his bonds. Thus, to guard against such adversely selected investors, a market maker in the Glosten and Milgrom (1985) tradition would raise the bid-ask spread.

Modeling such persistent adverse selection with long-lived bond investors, however, requires a lot more technical apparatus and thus awaits future research. To the extent that an adverse-selection-based model could conceivably lead to a similar qualitative result if asymmetric information is concentrated in the bond's recovery value,[33] then on the quantitative front our model has the advantage of incorporating standard structural bond valuation models in a simpler setting but still delivering the first-order empirical patterns.

6 Conclusion

We investigate the liquidity-fundamental spiral in the corporate bond market, by studying the endogenous liquidity of defaultable bonds in a search-based OTC markets together with the endogenous default decision by equity holders from the firm side.

By solving a system of PDEs, we derive the endogenous secondary market liquidity jointly with the debt valuations, equity valuations, and endogenous default policy, in closed-form. The

[33] If, instead, the adverse selection might not necessarily worsen when the firm goes closer to the default boundary, we would not expect, absent the bargaining frictions presented in this article, a monotonically increasing pattern of illiquidity towards default.

fundamentals of corporate bonds, which is mainly driven by the firm's distance-to-default, affects the endogenous liquidity of corporate bonds. And, through the rollover channel in which equity holders are absorbing refinancing losses in bad times, worsening liquidity of corporate bonds at the same time significantly hurts the equity holders' option value of keeping the firm alive. As a result, illiquidity of secondary corporate bond market feeds back to the fundamental of corporate bonds by edging the firm closer to bankruptcy. We hope our fully solved structural model can pave the way of bringing more structural approach in the empirical study of the impact of liquidity on corporate bonds.

In earlier versions of working papers, we further incorporate endogenous firm investment and show that this mechanism, i.e., a feedback loop between the firm fundamental and the firm's (debt) financing liquidity, should encompass a broader set of firm level decisions beyond default.

References

Afonso, G., and R. Lagos (2011): "Trade Dynamics in the Market for Federal Funds," *Working Paper*.

Altman, E. I., and A. C. Eberhart (1994): "Do seniority provisions protect bondholders' investments?," *Journal of Portfolio Management*, 20(4), 67.

Amihud, Y., and H. Mendelson (1986): "Asset pricing and the bid-ask spread," *Journal of Financial Economics*, 17, 223–249.

Back, K., and S. Baruch (2004): "Information in securities markets: Kyle meets Glosten and Milgrom," *Econometrica*, 72(2), 433–465.

Bao, J., J. Pan, and J. Wang (2011): "The Illiquidity of Corporate Bonds," *Journal of Finance*, 66(3), 911–946.

Beber, A., M. Brandt, and K. Kavajecz (2009): "Flight-to-quality or flight-to-liquidity? Evidence from the Euro-area bond market," *Review of Financial Studies*, 22, 925–957.

Bhamra, H. S., L.-A. Kuehn, and I. A. Strebulaev (2010): "The Levered Equity Risk Premium and Credit Spreads: A Unified Framework," *Review of Financial Studies*, 23(2), 645–703.

Biais, B., and P.-O. Weill (2009): "Liquidity shocks and order book dynamics," *Working Paper*.

Black, F., and J. C. Cox (1976): "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance*, 31(2), 351–367.

Brunnermeier, M. K., and L. H. Pedersen (2009): "Market liquidity and funding liquidity," *Review of Financial Studies*, 22(6), 2201–2238.

Chen, H. (2010): "Macroeconomic conditions and the puzzles of credit spreads and capital structures," *Journal of Finance*.

Chen, L., D. Lesmond, and J. Wei (2007): "Corporate yield spreads and bond liquidity," *Journal of Finance*, 62, 119–149.

Cheng, I.-H., and K. Milbradt (2012): "The hazards of debt: Rollover Freezes, Incentives, and Bailouts," *Review of Financial Studies*, 25(4), 1070–1110.

Choi, J., D. Hackbarth, and J. Zechner (2012): "Granularity of Corporate Debt : Theory and Tests," .

Dastidar, S., and B. Phelps (2009): "Introducing LCS: Liquidity Cost Scores for US Credit Bonds," *Barclay Capital Report*.

Diamond, D. W. (1993): "Seniority and maturity of debt contracts," *Journal of Financial Economics*, 33, 341–368.

Diamond, D. W., and Z. He (2012): "A theory of debt maturity: the long and short of debt overhang," *Working Paper*.

Duffie, D., N. Garlenau, and L. H. Pedersen (2005): "Over-the-Counter Markets," *Econometrica*, 73(6), 1815–1847.

——— (2007): "Valuation in Over-the-Counter Markets," *Review of Financial Studies*, 20(6), 1865–1900.

Edwards, A., L. Harris, and M. Piwowar (2007): "Corporate Bond Market Transaction Costs and Transparency," *Journal of Finance*, 62, 1421–1451.

Ericsson, J., and O. Renault (2006): "Liquidity and credit risk," *Journal of Finance*, p. 2219.

Feldhutter, P. (2011): "The same bond at different prices : identifying search frictions and selling pressures," *Review of Financial Studies*.

Glosten, L. R., and P. R. Milgrom (1985): "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders," *Journal of Financial Economics*, 14(71-100).

Goldstein, I., E. Ozdenoren, and K. Yuan (2011): "Trading Frenzies and Their Impact on Real Investment," *Working Paper*.

Green, R. C., B. Hollifield, and N. Schurhoff (2007a): "Dealer intermediation and price behavior in the aftermarket for new bond issues," *Journal of Financial Economics*, 86, 643–682.

——— (2007b): "Financial intermediation and the costs of trading in an opaque market," *Review of Financial Studies*, 20(2), 275–314.

Guerrieri, V., R. Shimer, and R. Wright (2010): "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 78(6), 1823–1862.

Harris, L., and M. Piwowar (2006): "Secondary Trading Costs in the Municipal Bond Market," *Journal of Finance*, 61, 1361–1397.

He, Z., and W. Xiong (2012a): "Dynamic debt runs," *Review of Financial Studies*, (Forthcoming).

——— (2012b): "Rollover Risk and Credit Risk," *Journal of Finance*, 67, 391–429.

Hong, G., and A. Warga (2000): "An Empirical Study of Bond Market Transactions," *Financial Analyst Journal*, 56, 32–46.

Huang, J.-Z., and M. Huang (2003): "How much of the corporate-treasury spread is due to credit risk?," *Working Paper*.

Kyle, A. S. (1985): "Continuous auctions and insider trading," *Econometrica*, 53(6), 1315–1335.

Lagos, R., and G. Rocheteau (2007): "Search in asset markets: Market structure, liquidity, and welfare," *American Economic Review*, 97(2), 198–202.

Lagos, R., and G. Rocheteau (2009): "Liquidity in asset markets with search frictions," *Econometrica*, 77(2), 403–426.

Lauermann, S., and A. Wolinsky (2011): "Search with Adverse Selection," *Working Paper*.

Leland, H. (1994): "Corporate debt value, bond covenants, and optimal capital structure," *Journal of Finance*, 49(4), 1213–1252.

——— (1998): "Agency costs, risk management, and capital structure," *Journal of Finance*, 53(4), 1213–1243.

Leland, H., and K. B. Toft (1996): "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads," *Journal of Finance*, 51(3), 987–1019.

Longstaff, F. A., S. Mithal, and E. Neis (2005): "Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Swap Market," *Journal of Finance*, 60, 2213–2253.

Manso, G. (2011): "Feedback Effects of Credit Ratings," *Working Paper*.

Miller, M. H. (1977): "Debt and taxes," *Journal of Finance*, 32, 261–275.

Morris, S., and H. S. Shin (2009): "Illiquidity component of credit risk," *Working Paper*.

Schultz, P. (2001): "Corporate Bond Trading Costs: A Peek Behind the Curtain," *Journal of Finance*, 56, 677–698.

Schwarz, K. (2010): "Mind the Gap: Disentangling Credit and Liquidity in Risk Spreads," *Working Paper*.

Segura, A., and J. Suarez (2011): "Dynamic maturity transformation," *Working Paper*.

Vayanos, D., and P.-O. Weill (2008): "A Search-Based Theory of the On-the-Run Phenomenon," *Journal of Finance*, 53(3), 1361–1398.

Weill, P.-O. (2007): "Leaning against the wind," *Review of Economic Studies*, 74, 1329–1354.

A Appendix

A.1 Notation

First, let us call $r_H \equiv r$, $r_L \equiv \overline{r}$, $\xi_H \equiv \xi$ and $\xi_L \equiv \lambda\beta$, and $\tilde{\mu} = \mu - \frac{\sigma^2}{2}$. Second, define the log-transform $\tilde{\delta} = \log(\delta)$ so that $d\tilde{\delta} = \tilde{\mu}dt + \sigma dZ$. Third, for brevity we use the notation $D' \equiv \frac{\partial D}{\partial \delta}$ and $\dot{D} \equiv \frac{\partial D}{\partial \tau}$. We will, with abuse of notation, write $q\left(\tilde{\delta}, ...\right)$ to mean $\frac{\tilde{\delta}_B - \tilde{\delta} + ...}{...}$. Let $N(x)$ be the cumulative normal function.

A.2 2x2 matrix formulas

As the 2x2 specification is frequently used in the text, we present the results here in compact form. Suppose

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $\mathbf{A} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}$ where

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \mathbf{P} &= \begin{bmatrix} 1 & \frac{b}{\hat{r}_2 - a} \\ \frac{c}{\hat{r}_1 - d} & 1 \end{bmatrix} \\ \hat{\mathbf{D}} &= \begin{bmatrix} \hat{r}_1 & 0 \\ 0 & \hat{r}_2 \end{bmatrix}, \end{aligned}$$

where of course alternative versions of $\mathbf{P}$ can be chosen. However, to show convergence to frictionless markets we chose this form of $\mathbf{P}$ as it allows convergence to an upper triangular form. The roots

$$\begin{aligned} \hat{r}_{1/2} &= \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \\ &= \frac{a+d \pm \sqrt{(a-d)^2 + 4bc}}{2} \end{aligned}$$

solve $\det[\mathbf{A} - \rho\mathbf{I}] = 0$, i.e. $\hat{r}_{1/2}$ are the roots of the characteristic polynomial

$$g(\hat{r}) = (a-\hat{r})(d-\hat{r}) - bc = \hat{r}^2 - (a+d)\hat{r} + (ad-bc).$$

If $a > 0$ and $d > 0$ and $b < 0$ and $c < 0$ as well as $(ad-bc) > 0$, then both roots $\hat{r}_{1/2} > 0$.

Identifying $a = r_H + \xi_H, b = -\xi_H, c = -\xi_L, d = r_L + \xi_L$, we have

$$\hat{r}_i = \frac{r_H + r_L + \xi_H + \xi_L - (-1)^i \sqrt{\left[(r_H + \xi_H) - (r_L + \xi_L)\right]^2 + 4\xi_H \xi_L}}{2}.$$

We can also derive bounds on $\hat{r}_i$ by noting the following results:

$$\begin{aligned} g(r_H) &= \xi_H (r_L - r_H) > 0 \\ g(r_L) &= -\xi_L (r_L - r_H) < 0 \\ g(r_H + \xi_H) &= -\xi_H \xi_L < 0 \\ g(r_L + \xi_L) &= -\xi_H \xi_L < 0 \\ g(r_H + \xi_H + \xi_L) &= -\xi_L (r_L - r_H) < 0 \\ g(r_L + \xi_H + \xi_L) &= \xi_H (r_L - r_H) > 0 \end{aligned}$$

so that we know that

$$\begin{aligned} r_H &< \hat{r}_1 < \min\{r + \xi_H, r_L\} \\ \max\{r_H + \xi_H + \xi_L, r_L + \xi_L\} &< \hat{r}_2 < r_L + \xi_H + \xi_L. \end{aligned}$$

It is easy to show that as $\xi_H \to 0$, $\hat{r}_1 = r_L + \xi_L$ and $\hat{r}_2 = r_H$, and $\lim_{b \to 0} \mathbf{P} = \begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix}$, so that D_H converges towards the LT96 solution.

Next, consider $\lambda \to \infty$ such that $\xi_L \to \infty$, that is, what happens when the market becomes very liquid. Note that we can rewrite the characteristic polynomial as

$$g(\hat{r}) = \xi_L \left[(r_H + \xi_H - \hat{r}) \left(\frac{r_L}{\xi_L} + 1 - \frac{\hat{r}}{\xi_L} \right) - \xi_H \right]$$

Suppose now that $\hat{r}$ is finite. Then we know that the square bracket, as $\xi_L \to \infty$, becomes

$$(r_H + \xi_H - \hat{r}) - \xi_H = 0$$

so that $\hat{r}_2 = r_H > 0$. Thus, as both roots are positive, we must have that the second root $\hat{r}_1 \to \infty$. The diagonal decomposition becomes unstable, in that $\lim_{\lambda \to \infty} \mathbf{P} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Finally, for $r = r_H = r_L$ we can show that $\mathbf{P}^{-1}\mathbf{1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that $\hat{\mathbf{c}} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, and for $\alpha = \alpha_H = \alpha_L$ we have $\hat{\boldsymbol{\alpha}} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$.

A.3 Proofs of Section 3

A.3.1 Debt

Proof of Proposition 1.

Applying the log transform $\tilde{\delta} = \log(\delta)$ to the system of PDEs we are left with a linear system of PDEs:

$$\begin{aligned} \begin{bmatrix} r_H + \xi_H & -\xi_H \\ -\xi_L & r_L + \xi_L \end{bmatrix} \begin{bmatrix} d_H \\ d_L \end{bmatrix} &= \begin{bmatrix} c \\ \rho c \end{bmatrix} + \tilde{\mu} \begin{bmatrix} d_H \\ d_L \end{bmatrix}' + \frac{\sigma^2}{2} \begin{bmatrix} d_H \\ d_L \end{bmatrix}'' - \begin{bmatrix} \dot{d}_H \\ d_L \end{bmatrix} \\ \iff \mathbf{A} \times \mathbf{d} &= \mathbf{c} + \tilde{\mu}\mathbf{d}' + \frac{\sigma^2}{2}\mathbf{d}'' - \dot{\mathbf{d}} \end{aligned}$$

Here we allow for general changes to the coupon payment c by premultiplying by a parameter $\rho \leq 1$ to acknowledge that there might be linear holding costs above and beyond the higher discount rate. In the paper, we have $\rho = 1$. Let us decompose $\mathbf{A} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}$ where $\hat{\mathbf{D}}$ is a diagonal matrix with its diagonal elements the eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is a matrix of the respective stacked eigenvectors. The resulting eigenvalues are defined

$$g(\hat{r}) = (r_H + \xi_H - \hat{r})(r_L + \xi_L - \hat{r}) - \xi_L \xi_H = 0$$

and $g(r_H) = \xi_H (r_L - r_H) > 0$ and $g(r_L) = -\xi_L (r_L - r_H) < 0$. We thus have $\hat{r}_i = \frac{r+\xi+\overline{r}+\lambda\beta \pm \sqrt{[(r+\xi)-(\overline{r}+\lambda\beta)]^2 + 4\xi\lambda\beta}}{2}$.

Premultiplying the system by $\mathbf{P}^{-1}$ and noting that $\mathbf{P}^{-1}\mathbf{A} = \hat{\mathbf{D}}\mathbf{P}^{-1}$ we have a delinked system PDEs with a common bankruptcy boundary $\tilde{\delta}_B \equiv \log(\delta_B)$ and payout boundary $t = 0$

$$\begin{aligned} \hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{d} &= \mathbf{P}^{-1}\mathbf{c} + \tilde{\mu}\mathbf{P}^{-1}\mathbf{d}' + \frac{\sigma^2}{2}\mathbf{P}^{-1}\mathbf{d}'' - \mathbf{P}^{-1}\dot{\mathbf{d}} \\ \iff \hat{\mathbf{D}}\mathbf{y} &= \widehat{\mathbf{c}} + \tilde{\mu}\mathbf{y}' + \frac{\sigma^2}{2}\mathbf{y}'' - \dot{\mathbf{y}} \end{aligned}$$

where $\mathbf{y} = \mathbf{P}^{-1}\mathbf{d}$ and $\hat{\mathbf{c}} = \mathbf{P}^{-1}\mathbf{c}$. The rows of the system are now delinked, and we are left with two PDEs of the form

$$\hat{r}_i y_i = \widehat{c}_i + \tilde{\mu} y_i' + \frac{\sigma^2}{2} y_i'' - \dot{y}_i$$

with given boundary conditions at $t = 0$ and $\tilde{\delta} = \tilde{\delta}_B$, whose solutions are known from LT96. The decomposition works because the boundaries are the same across rows. The solution takes the form

$$\begin{aligned} y_i &= A_i + B_i e^{-\hat{r}_i t} (1 - F_i) + C_i G_i \\ F_j\left(\tilde{\delta}, t\right) &= \sum_{i=1}^{2} e^{\left(\tilde{\delta} - \tilde{\delta}_B\right)\gamma_{ij}} N\left[q\left(\tilde{\delta}, \gamma_{ij}, t\right)\right] \\ G_j\left(\tilde{\delta}, t\right) &= \sum_{i=1}^{2} e^{\left(\tilde{\delta} - \tilde{\delta}_B\right)\eta_{ij}} N\left[q\left(\tilde{\delta}, \eta_{ij}, t\right)\right] \end{aligned}$$

where

$$q\left(\tilde{\delta},\chi,t\right)=\frac{\tilde{\delta}_B-\tilde{\delta}-(\chi+a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$$

and constants

$$\begin{aligned}A_i &= \frac{\hat{c}_i}{\hat{r}_i}\\ B_i &= \left(\hat{p}_i-\frac{\hat{c}_i}{\hat{r}_i}\right)\\ C_i &= \left(\hat{\alpha}_i\frac{e^{\tilde{\delta}_B}}{r-\mu}-\frac{\hat{c}_i}{\hat{r}_i}\right)\end{aligned}$$

and some yet to be determined parameters γ_{ij},η_{ij}. Note that $\lim_{t\to 0} q\left(\tilde{\delta},\chi,t\right)=\lim_{t\to 0}\frac{\tilde{\delta}_B-\tilde{\delta}}{\sigma\sqrt{t}}=-\infty$ as $\tilde{\delta}_B<\tilde{\delta}$, so $N\left[q\left(\tilde{\delta},\chi,0\right)\right]=0$ for all i and $\tilde{\delta}>\tilde{\delta}_B$. Further note that $\lim_{\tilde{\delta}\to\infty} q\left(\tilde{\delta},\chi,t\right)=-\infty$, so $\lim_{\tilde{\delta}\to\infty} N\left[q\left(\tilde{\delta},\chi,t\right)\right]=0$. Substituting the candidate solution y_i into the PDE with $A_i=\widehat{\frac{c_i}{\hat{r}_i}}, B_i=\hat{p}_i-\frac{\hat{c}_i}{\hat{r}_i}, C_i=\hat{\alpha}_i\frac{\exp\left(\tilde{\delta}_B\right)}{r-\mu}-\frac{\hat{c}_i}{\hat{r}_i}$, we see that

$$\begin{aligned}b_i e^{-\hat{r}_i t}\left[\hat{r}_i\left(1-F_i\right)+\tilde{\mu}F_i'+\frac{\sigma^2}{2}F''-\left[\hat{r}_i\left(1-F_i\right)+\dot{F}_i\right]\right]&\\ +c_i\left[\hat{r}_i G_i-\tilde{\mu}G_i'-\frac{\sigma^2}{2}G_i''+\dot{G}_i\right] &= 0\\ \iff b_i e^{-\hat{r}_i t}\left[\tilde{\mu}F'+\frac{\sigma^2}{2}F''-\dot{F}\right]&\\ +c_i\left[\hat{r}_i G_i-\tilde{\mu}G_i'-\frac{\sigma^2}{2}G_i''+\dot{G}_i\right] &= 0\end{aligned}$$

We see that both $\dot{F}_i$ and $\dot{G}_i$ have no term $N(\cdot)$. As q is linear in $\tilde{\delta}$, we have $q''=0$ (where $q'=q_{\tilde{\delta}}$ and $\dot{q}=q_t$). We thus have, for F,

$$\begin{aligned}N\left[q\left(\tilde{\delta},\gamma,t\right)\right]\left[\tilde{\mu}\gamma+\frac{\sigma^2}{2}\gamma^2\right]&\\ +\phi\left[q\left(v,\gamma,t\right)\right]\left[\tilde{\mu}q'+\frac{\sigma^2}{2}\left[2\gamma q'-q\left(q'\right)^2\right]-\dot{q}\right] &= 0\end{aligned}$$

So the roots for F_i are $\gamma_1=0=-a+a$ and $\gamma_2=-\frac{2\tilde{\mu}}{\sigma^2}=-a-a$ where $a\equiv\frac{\tilde{\mu}}{\sigma^2}$. We see that this is independent of i, that is, it is independent of what row of $\mathbf{y}$ we picked, as $\hat{r}_i$ is cancelled out. Further, for G, we have

$$\begin{aligned}N\left[q\left(v,\eta,t\right)\right]\left[\tilde{\mu}\eta+\frac{\sigma^2}{2}\eta^2-\hat{r}_i\right]&\\ +\phi\left[q\left(v,\eta,t\right)\right]\left[\tilde{\mu}q'+\frac{\sigma^2}{2}\left[2\eta q'-q\left(q'\right)^2\right]-\dot{q}\right] &= 0\end{aligned}$$

so the roots for G_i are $\eta_{i1}=\frac{-\tilde{\mu}+\sqrt{\tilde{\mu}^2+2\sigma^2\hat{r}_i}}{\sigma^2}=-a+\frac{\sqrt{\tilde{\mu}^2+2\sigma^2\hat{r}_i}}{\sigma^2}$ and $\eta_{i2}=-a-\frac{\sqrt{\tilde{\mu}^2+2\sigma^2\hat{r}_i}}{\sigma^2}$. Simply plugging in the functional form of q results in the term in square brackets in the second row to vanish.

For the boundary condition, we have

$$\begin{aligned}\mathbf{y}\left(\tilde{\delta},0\right) &= \mathbf{P}^{-1}\mathbf{1}\cdot p=\hat{\mathbf{p}}\\ \mathbf{y}\left(\tilde{\delta}_B,t\right) &= \mathbf{P}^{-1}\boldsymbol{\alpha}\frac{\exp\left(\tilde{\delta}_B\right)}{r-\mu}=\hat{\boldsymbol{\alpha}}\frac{\exp\left(\tilde{\delta}_B\right)}{r-\mu}\end{aligned}$$

which defines the remaining parameters of the solution.

As a last step, we retranslate the system back into the original debt functions by premultiplying by $\mathbf{P}$ and noting that $F(v,t)=F_i(v,t)=F_{-i}(v,t)$ by the symmetry of the γ's, and by rewriting it in terms of $\delta=\exp\left(\tilde{\delta}\right)$. ■

A.3.2 Equity

Proof of Proposition 2.

Equity has the following ODE where for notational ease we define $m = \frac{1}{T}$

$$rE = \exp\left(\tilde{\delta}\right) - (1-\pi)\, c + \tilde{\mu} E' + \frac{\sigma^2}{2} E'' + m\left[D_H\left(\tilde{\delta}, T\right) - p\right]$$

The term in square brackets is the cash-flow term that arises out of rollover of debt (while keeping coupon, principal and maturity stationary), a term first pointed out by LT96. We will establish the (closed-form) solution in several steps.

First, the homogenous solutions to the ODE are $M\left(\tilde{\delta}\right) = e^{\kappa_1 \tilde{\delta}}$ and $U\left(\tilde{\delta}\right) = e^{\kappa_2 \tilde{\delta}}$ where

$$\frac{\sigma^2}{2}\kappa^2 + \tilde{\mu}\kappa - r = 0$$

so that

$$\kappa_{1/2} = \frac{-\tilde{\mu} \pm \sqrt{\tilde{\mu}^2 + 2\sigma^2 r}}{\sigma^2} = -a \pm \frac{\sqrt{\tilde{\mu}^2 + 2\sigma^2 r}}{\sigma^2}$$

and $\kappa_1 > 1 > 0 > \kappa_2$.

Next, let us establish the Wronskian

$$\begin{aligned} Wr(s) &= M(s)\, U'(s) - M'(s)\, U(s) \\ &= -(\kappa_1 - \kappa_2)\exp\left\{(\kappa_1 + \kappa_2)\, s\right\} \\ &= -\Delta\kappa \cdot M(s)\, U(s) \end{aligned}$$

Then, by the variation of coefficient solutions to linear ODEs, a technique described in most textbooks on differential equations, we have for an ODE

$$rg = \tilde{\mu} g' + \frac{\sigma^2}{2} g'' + part(s)$$

the following particular solution g_p

$$\begin{aligned} g_p(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\, \frac{M(s)\, U(x) - M(x)\, U(s)}{Wr(s)} ds \\ &= \frac{2}{\sigma^2}\int_x^l part(s)\, \frac{e^{-\kappa_2 s} e^{\kappa_2 x} - e^{\kappa_1 x} e^{-\kappa_1 s}}{-\Delta\kappa} ds \\ g_p'(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\, \frac{M(s)\, U'(x) - M'(x)\, U(s)}{Wr(s)} ds \\ &= \frac{2}{\sigma^2}\int_x^l part(s)\, \frac{\kappa_2 M(s)\, U(x) - \kappa_1 M(x)\, U(s)}{Wr(s)} ds \\ g_p''(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\, \frac{\kappa_2^2 M(s)\, U(x) - \kappa_1^2 M(x)\, U(s)}{Wr(s)} ds - \frac{2}{\sigma^2} part(x) \end{aligned}$$

for an arbitrary limit $l \in (v_B, \infty)$.

Second, as the debt term D_H is bounded, to impose the condition that equity does not grow orders of magnitude faster than the unlevered value of the firm $V\left(\tilde{\delta}\right) = \frac{e^{\tilde{\delta}}}{r-\mu}$ we need $\lim_{\tilde{\delta}\to\infty}\left|\frac{E\left(\tilde{\delta}\right)}{V\left(\tilde{\delta}\right)}\right| < \infty$. Let us write the solution as

$$E\left(\tilde{\delta}\right) = K_U U\left(\tilde{\delta}\right) + K_M M\left(\tilde{\delta}\right) + V\left(\tilde{\delta}\right) + K_0 + \int_{\tilde{\delta}}^l \frac{2}{\sigma^2} part(s)\, \frac{M(s)\, U\left(\tilde{\delta}\right) - M\left(\tilde{\delta}\right) U(s)}{Wr(s)} ds$$

where we incorporated all constant terms of the ODE into the definition of K_0 and $part(s)$ is thus just composed of cumulative normal functions of the form $N\left[-aa \cdot \tilde{\delta} + bb\right]$ where $aa > 0$. Let us gather terms of $U\left(\tilde{\delta}\right)$ and $M\left(\tilde{\delta}\right)$ to get

$$E\left(\tilde{\delta}\right) = U\left(\tilde{\delta}\right)\left[K_U + \int_{\tilde{\delta}}^l \frac{2}{\sigma^2} part(s)\, \frac{M(s)}{Wr(s)} ds\right] + M\left(\tilde{\delta}\right)\left[K_M - \int_{\tilde{\delta}}^l \frac{2}{\sigma^2} part(s)\, \frac{U(s)}{Wr(s)} ds\right] + \frac{e^{\tilde{\delta}}}{r - \tilde{\mu}} + K_0$$

First, let us note that the integrals all converge, as $N\left[-aa \cdot \tilde{\delta} + bb\right]$ converges faster than any function $e^{cst \cdot \tilde{\delta}}$ for any constant cst. Second, to impose the boundary condition of $\lim_{\tilde{\delta}\to\infty}\left|\frac{E(\tilde{\delta})}{V(\tilde{\delta})}\right|<\infty$, we note that $\lim_{\tilde{\delta}\to\infty}U\left(\tilde{\delta}\right)=0$ so the first term in the above equation converges for any choice of K_U. However, the second term contains $M\left(\tilde{\delta}\right)$ which explodes to infinity faster than $e^{\tilde{\delta}}$ as $\kappa_1>1$. We thus need to pick

$$K_M(l)=-\int_l^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds$$

as a necessary condition to have the term stay bounded. Next, plugging it in, we see that the term in question becomes

$$M\left(\tilde{\delta}\right)\left[K_M(l)-\int_{\tilde{\delta}}^l \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds\right]=-M\left(\tilde{\delta}\right)\int_{\tilde{\delta}}^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds$$

and we now show that this term converges to 0 as $\tilde{\delta}\to\infty$. Let us rewrite to get

$$\begin{aligned}\lim_{\tilde{\delta}\to\infty}-M\left(\tilde{\delta}\right)\int_{\tilde{\delta}}^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds &= \lim_{\tilde{\delta}\to\infty}\frac{-\int_{\tilde{\delta}}^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds}{\frac{1}{M(\tilde{\delta})}}=\frac{\text{"0"}}{\text{"0"}}\\ &\overset{\{L'Hopital\}}{=} \lim_{\tilde{\delta}\to\infty}\frac{\frac{2}{\sigma^2}part\left(\tilde{\delta}\right)\frac{U(\tilde{\delta})}{Wr(\tilde{\delta})}}{\frac{M'(\tilde{\delta})}{\left[M(\tilde{\delta})\right]^2}}=0\end{aligned}$$

and again, we see that since $U\left(\tilde{\delta}\right),Wr\left(\tilde{\delta}\right),M\left(\tilde{\delta}\right),M'\left(\tilde{\delta}\right)$ are all of exponential form and $part\left(\tilde{\delta}\right)$ is of cumulative normal form this term converges to zero rapidly, and the solution to $E\left(\tilde{\delta}\right)$ is verified. Let us take the arbitrary limit $l\to\infty$ and define $g_p(x)\equiv g_p(x|\infty)$. We note that the complement of the integrals (i.e. $\int_l^\infty \cdot ds$) vanishes, so that $\lim_{l\to\infty}K_M(l)=0$. We see that $g_p(x)$ and $g_p'(x)$ (and so forth) consists of a finite sum of integrals of the form $\int_x^\infty e^{cst\cdot s}N\left[q(s,\chi,T)\right]ds$ where cst is a constant.

Third, let us briefly establish two auxiliary results. First, let us note that for $aa>0$ we have

$$aa\int_x^\infty \phi(-aa\cdot s+bb)\,ds=\int_{-\infty}^{-aa\cdot x+bb}\phi(y)\,dy=N\left[-aa\cdot x+bb\right]$$

by simple change of variables. Second, note that

$$\begin{aligned}e^{cst\cdot x}\phi(-aa\cdot x+bb) &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[(-aa\cdot x+bb)^2-2cst\cdot x\right]\right\}\\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\left(-aa\cdot x+bb+\frac{cst}{aa}\right)^2+bb^2-\left(bb+\frac{cst}{aa}\right)^2\right]\right\}\\ &= \phi\left(-aa\cdot x+bb+\frac{cst}{aa}\right)e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)}\end{aligned}$$

by a simple completion of the square. Now, we can solve the integral in question via integration by parts:

$$\begin{aligned}&\int_x^\infty e^{cst\cdot s}N\left[-aa\cdot s+bb\right]ds\\ =\ & \frac{e^{cst\cdot s}}{cst}N\left[-aa\cdot s+bb\right]\Big|_{s=x}^{\infty}+\frac{1}{cst}\left[aa\cdot\int_x^\infty e^{cst\cdot s}\phi(-aa\cdot s+bb)\,ds\right]\\ =\ & -\frac{e^{cst\cdot x}}{cst}N\left[-aa\cdot x+bb\right]+\frac{1}{cst}\left[a\int_x^\infty \phi\left(-aa\cdot s+bb+\frac{cst}{aa}\right)ds\right]e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)}\\ =\ & -\frac{e^{cst\cdot x}}{cst}N\left[-aa\cdot x+bb\right]+\frac{1}{cst}N\left[-aa\cdot x+bb+\frac{cst}{aa}\right]e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)}\end{aligned}$$

where we again used the fact that the cumulative normal vanishes faster than any exponential function explodes.

Next, note that $D_i\left(\tilde{\delta},t\right)=...+...e^{\left(\tilde{\delta}-\tilde{\delta}_B\right)\chi}N\left[q\left(\tilde{\delta},\chi,t\right)\right]+...$ for some χ, so that we are essentially facing integrals

$$\begin{aligned}
& \frac{2}{\sigma}\int_x^\infty e^{\left(s-\tilde{\delta}_B\right)\chi}N\left[q\left(s,\chi,t\right)\right]\frac{M\left(s\right)U\left(x\right)}{Wr\left(s\right)}ds \\
= & \frac{2}{\sigma}\frac{1}{-\Delta\kappa}e^{\kappa_2 x}e^{-\tilde{\delta}_B\chi}\int_x^\infty e^{\left(\chi-\kappa_2\right)s}N\left[q\left(s,\chi,t\right)\right]ds \\
= & \frac{2}{\sigma}\frac{1}{-\Delta\kappa}e^{\kappa_2 x}e^{-\tilde{\delta}_B\chi}\frac{1}{\chi-\kappa_2} \\
& \times\left[-e^{\left(\chi-\kappa_2\right)}N\left[q\left(x,\chi,t\right)\right]+N\left[q\left(x,\kappa_2,t\right)\right]e^{\left(\chi-\kappa_2\right)\left\{\tilde{\delta}_B-\frac{1}{2}\left[\left(\kappa+a\right)^2-\left(\chi+a\right)^2\right]\sigma^2T\right\}}\right]
\end{aligned}$$

Here, we used $cst=\left(\chi-\kappa_2\right)$, $aa=\frac{1}{\sigma\sqrt{T}}$, $b=\frac{\tilde{\delta}_B-\left(\chi+a\right)\sigma^2T}{\sigma\sqrt{T}}$, $q\left(x,\chi,t\right)+\left(\chi-\kappa\right)\sigma\sqrt{t}=q\left(x,\kappa,t\right)$ and the fact that

$$\begin{aligned}
\left(\chi-\kappa\right)\left(-\right)\left[\chi+a-\frac{1}{2}\left(\chi-\kappa\right)\right] &= \left(\chi-\kappa\right)\left(-\right)\left[\frac{1}{2}\chi+\frac{1}{2}a+\frac{1}{2}\kappa+\frac{1}{2}a\right] \\
&= \frac{1}{2}\left[\left(\kappa+a\right)^2-\left(\chi+a\right)^2\right]
\end{aligned}$$

where we note that the last term is independent of if we pick the larger or smaller root, as both κ and all possible χ are centered around $-a$. Lastly, we note that $\frac{2}{\sigma}\int_x^\infty e^{\left(s-\tilde{\delta}_B\right)\chi}N\left[q\left(s,\chi,t\right)\right]\frac{M(x)U(s)}{Wr(s)}ds$ has the same form of solution only with κ_1 replacing κ_2. Define

$$\begin{aligned}
H\left(x,\chi,\kappa,T\right) &\equiv \int_x^\infty e^{\left(\chi-\kappa\right)\cdot s}N\left[q\left(s,\chi,T\right)\right]ds \\
&= -\frac{1}{cst}\left\{e^{cst\cdot x}N\left[q\left(x,\chi,T\right)\right]-e^{cst\cdot\tilde{\delta}_B}\exp\left\{-cst\left(\chi+a-\frac{1}{2}cst\right)\sigma^2T\right\}N\left[q\left(x,\chi,T\right)+cst\cdot\sigma\sqrt{T}\right]\right\} \\
&= \frac{1}{\kappa-\chi}\left\{e^{\left(\chi-\kappa\right)x}N\left[q\left(x,\chi,T\right)\right]-e^{\left(\chi-\kappa\right)\tilde{\delta}_B}e^{\frac{1}{2}\left[\left(\kappa+a\right)^2-\left(\chi+a\right)^2\right]\sigma^2T}N\left[q\left(x,\kappa,T\right)\right]\right\}
\end{aligned}$$

The solution to the particular part for F then is

$$\begin{aligned}
g_F\left(x\right) &\equiv \frac{2}{\sigma^2}\int_x^\infty F\left(s\right)\frac{M\left(s\right)U\left(x\right)-M\left(x\right)U\left(s\right)}{Wr\left(s\right)}ds \\
&= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^2\left\{e^{\kappa_2 x}e^{-\gamma_i\tilde{\delta}_B}H\left(x,\gamma_i,\kappa_2,T\right)-e^{\kappa_1 x}e^{-\gamma_i\tilde{\delta}_B}H\left(x,\gamma_i,\kappa_1,T\right)\right\} \\
g_F'\left(x\right) &\equiv \frac{2}{\sigma^2}\int_x^\infty F\left(s\right)\frac{\kappa_2 M\left(s\right)U\left(x\right)-\kappa_1 M\left(x\right)U\left(s\right)}{Wr\left(s\right)}ds \\
&= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^2\left\{\kappa_2 e^{\kappa_2 x}e^{-\gamma_i\tilde{\delta}_B}H\left(x,\gamma_i,\kappa_2,T\right)-\kappa_1 e^{\kappa_1 x}e^{-\gamma_i\tilde{\delta}_B}H\left(x,\gamma_i,\kappa_1,T\right)\right\}
\end{aligned}$$

and the solution to the particular part for G_j is

$$\begin{aligned}
g_{G_j}\left(x\right) &\equiv \frac{2}{\sigma^2}\int_x^\infty G_j\left(s\right)\frac{M\left(s\right)U\left(x\right)-M\left(x\right)U\left(s\right)}{Wr\left(s\right)}ds \\
&= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^2\left\{e^{\kappa_2 x}e^{-\eta_{ji}\tilde{\delta}_B}H\left(x,\eta_{ji},\kappa_2,T\right)-e^{\kappa_1 x}e^{-\eta_{ji}\tilde{\delta}_B}H\left(x,\eta_{ji},\kappa_1,T\right)\right\} \\
g_{G_j}'\left(x\right) &\equiv \frac{2}{\sigma^2}\int_x^\infty G_j\left(s\right)\frac{\kappa_2 M\left(s\right)U\left(x\right)-\kappa_1 M\left(x\right)U\left(s\right)}{Wr\left(s\right)}ds \\
&= \frac{1}{-\Delta\kappa}\frac{2}{\sigma^2}\sum_{i=1}^2\left\{\kappa_2 e^{\kappa_2 x}e^{-\eta_{ji}\tilde{\delta}_B}H\left(x,\eta_{ji},\kappa_2,T\right)-\kappa_1 e^{\kappa_1 x}e^{-\eta_{ji}\tilde{\delta}_B}H\left(x,\eta_{ji},\kappa_1,T\right)\right\}
\end{aligned}$$

Plugging in $x=\tilde{\delta}_B$, and noting that $q\left(\tilde{\delta}_B,\chi,t\right)=-\left(\chi+a\right)\sigma\sqrt{t}$, we make the important observation that

$$e^{\kappa\tilde{\delta}_B}e^{-\chi\tilde{\delta}_B}H\left(\tilde{\delta}_B,\chi,\kappa,T\right)=\frac{1}{\kappa-\chi}\left\{N\left[-\left(\chi+a\right)\sigma\sqrt{T}\right]-e^{\frac{1}{2}\left[\left(\kappa+a\right)^2-\left(\chi+a\right)^2\right]\sigma^2T}N\left[-\left(\kappa+a\right)\sigma\sqrt{T}\right]\right\}$$

is independent of $\tilde{\delta}_B$. We thus conclude that for any particular part $g_p\left(\tilde{\delta}_B\right)$, of the form given above, and its derivative

$g_p'\left(\tilde{\delta}_B\right)$ are **independent** of $\tilde{\delta}_B$ besides $C\left(\tilde{\delta}_B\right)$ containing $e^{\tilde{\delta}_B}$. Also note that for $\chi = \{\gamma_1, \gamma_2\}$ we have

$$e^{\frac{1}{2}\left[(\kappa+a)^2-(\gamma+a)^2\right]\sigma^2 T} = e^{rT}$$

and for $\chi = \{\eta_{i1}, \eta_{i2}\}$ we have

$$e^{\frac{1}{2}\left[(\kappa+a)^2-\left(\eta_{ij}+a\right)^2\right]\sigma^2 T} = e^{(r-\hat{r}_i)T}$$

Total equity is now easily written out to be

$$\begin{aligned} E\left(\tilde{\delta}\right) &= Ke^{\kappa_2\left(\tilde{\delta}-\tilde{\delta}_B\right)} + \frac{e^{\tilde{\delta}}}{r-\mu} + K_0 + g_p\left(\tilde{\delta}\right) \\ &= Ke^{\kappa_2\left(\tilde{\delta}-\tilde{\delta}_B\right)} + \frac{e^{\tilde{\delta}}}{r-\mu} + K_0 - m\left(P_{11}B_1e^{-\hat{r}_1 T} + P_{12}B_2e^{-\hat{r}_2 T}\right)g_F\left(\tilde{\delta}\right) + P_{11}mC_1\left(\tilde{\delta}_B\right)g_{G_1}\left(\tilde{\delta}\right) + P_{12}mC_2\left(\tilde{\delta}_B\right)g_{G_2}\left(\tilde{\delta}\right) \end{aligned}$$

where we scaled K by $e^{-\kappa_2\tilde{\delta}_B}$. The constant term K_0 is

$$K_0 = \frac{1}{r}\left\{-(1-\pi)c + m\left[A_1 + A_2 + \sum_j P_{1j}B_i e^{-\hat{r}_j T} - p\right]\right\}$$

The constant K is derived by setting

$$\begin{aligned} 0 = E\left(\tilde{\delta}_B\right) &= K + \frac{e^{\tilde{\delta}_B}}{r-\mu} + K_0 - m\left(\sum_j P_{1j}B_i e^{-\hat{r}_j T}\right)g_F\left(\tilde{\delta}_B\right) \\ &\quad + m\sum_{j=1}^{2} C_j\left(\tilde{\delta}_B\right)g_{G_j}\left(\tilde{\delta}_B\right) \\ \Longleftrightarrow K\left(\tilde{\delta}_B\right) &= -\left[\frac{e^{\tilde{\delta}_B}}{r-\mu} + K_0 - m\left(\sum_j P_{1j}B_i e^{-\hat{r}_j T}\right)g_F\left(\tilde{\delta}_B\right) + m\sum_{j=1}^{2} C_j\left(\tilde{\delta}_B\right)g_{G_j}\left(\tilde{\delta}_B\right)\right] \end{aligned}$$

The term in brackets only features linear combinations of constants independent of $\tilde{\delta}_B$. ■

Proof of Proposition 3.

The optimal $\delta_B = e^{\tilde{\delta}_B}$ is now easily derived. Plugging in $K\left(\tilde{\delta}_B\right)$ into the smooth pasting condition $E'\left(\tilde{\delta}_B\right) = 0$, we can derive $\delta_B = e^{\tilde{\delta}_B}$ in closed form:

$$\begin{aligned} 0 &= E'\left(\tilde{\delta}_B\right) \\ &= K\left(\tilde{\delta}_B\right)\kappa_2 + \frac{e^{\tilde{\delta}_B}}{r-\mu} - m\left(B_1e^{-\hat{r}_1 T} + P_{12}B_2e^{-\hat{r}_2 T}\right)g_F'\left(\tilde{\delta}_B\right) + m\sum_{j=1}^{2} P_{1j}C_j\left(\tilde{\delta}_B\right)g_{G_j}'\left(\tilde{\delta}_B\right) \\ &= \kappa_2\left[-\frac{e^{\tilde{\delta}_B}}{r-\mu} - K_0 + m\left(B_1e^{-\hat{r}_1 T} + P_{12}B_2e^{-\hat{r}_2 T}\right)g_F\left(\tilde{\delta}_B\right) - m\sum_{j=1}^{2} P_{1j}\left(\hat{\alpha}_j\frac{e^{\tilde{\delta}_B}}{r-\mu} - A_j\right)g_{G_j}\left(\tilde{\delta}_B\right)\right] \\ &\quad + \frac{e^{\tilde{\delta}_B}}{r-\mu} - m\left(B_1e^{-\hat{r}_1 T} + P_{12}B_2e^{-\hat{r}_2 T}\right)g_F'\left(\tilde{\delta}_B\right) + m\sum_{j=1}^{2} P_{1j}\left(\hat{\alpha}_j\frac{e^{\tilde{\delta}_B}}{r-\mu} - A_j\right)g_{G_j}'\left(\tilde{\delta}_B\right) \\ &= -\frac{e^{\tilde{\delta}_B}}{r-\mu}\left[\kappa_2 - 1 + m\sum_{j=1}^{2} P_{1j}\hat{\alpha}_j\left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g_{G_j}'\left(\tilde{\delta}_B\right)\right\}\right] \\ &\quad - \kappa_2 K_0 + m\left(B_1e^{-\hat{r}_1 T} + P_{12}B_2e^{-\hat{r}_2 T}\right)\left\{\kappa_2 g_F\left(\tilde{\delta}_B\right) - g_F'\left(\tilde{\delta}_B\right)\right\} + m\sum_{j=1}^{2} P_{1j}A_j\left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g_{G_1}'\left(\tilde{\delta}_B\right)\right\} \end{aligned}$$

which yields

$$\delta_B = e^{\tilde{\delta}_B} = (r-\mu) \times \left[\kappa_2 - 1 + m\sum_{j=1}^{2} P_{1j}\hat{\alpha}_j \left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g'_{G_j}\left(\tilde{\delta}_B\right)\right\}\right]^{-1}$$
$$\times \left[\begin{array}{c} -\kappa_2 K_0 + m\left(B_1 e^{-\hat{r}_1 T} + P_{12} B_2 e^{-\hat{r}_2 T}\right)\left\{\kappa_2 g_F\left(\tilde{\delta}_B\right) - g'_F\left(\tilde{\delta}_B\right)\right\} \\ +m\sum_{j=1}^{2} P_{1j} A_j \left\{\kappa_2 g_{G_j}\left(\tilde{\delta}_B\right) - g'_{G_j}\left(\tilde{\delta}_B\right)\right\} \end{array}\right]$$

where we note that the right hand side is independent of $\tilde{\delta}_B$ by previous results. We can simplify further by noting that each of the terms in curly brackets can be written as

$$\begin{aligned}
& \kappa_2 g_F\left(\tilde{\delta}_B\right) - g'_F\left(\tilde{\delta}_B\right) \\
= & \kappa_2 \frac{2}{\sigma^2}\int_{\tilde{\delta}_B}^{\infty} F(s) \frac{M(s) U\left(\tilde{\delta}_B\right) - M\left(\tilde{\delta}_B\right) U(s)}{Wr\left(\tilde{\delta}_B\right)} ds - \frac{2}{\sigma^2}\int_{\tilde{\delta}_B}^{\infty} F(s) \frac{\kappa_2 M(s) U\left(\tilde{\delta}_B\right) - \kappa_1 M\left(\tilde{\delta}_B\right) U(s)}{Wr\left(\tilde{\delta}_B\right)} ds \\
= & \frac{2}{\sigma^2}\int_{\tilde{\delta}_B}^{\infty} F(s) \frac{(\kappa_1 - \kappa_2) M\left(\tilde{\delta}_B\right) U(s)}{Wr\left(\tilde{\delta}_B\right)} ds \\
= & -\frac{2}{\sigma^2}\sum_{i=1}^{2} e^{(\kappa_1 - \gamma_i)\tilde{\delta}_B} H\left(\tilde{\delta}_B, \gamma_i, \kappa_1, T\right) \\
= & -\frac{2}{\sigma^2}\sum_{i=1}^{2} \frac{1}{\kappa_1 - \gamma_i}\left\{N\left[-(\gamma_i + a)\sigma\sqrt{T}\right] - e^{\frac{1}{2}\left[(\kappa_1+a)^2 - (\gamma_i+a)^2\right]\sigma^2 T} N\left[-(\kappa_1 + a)\sigma\sqrt{T}\right]\right\}
\end{aligned}$$

We thus established a closed form, albeit quite complex, for the optimal $\tilde{\delta}_B$.

The limit $\lim_{T\to\infty}\delta_B$ can be easily derived by noting that the normal distributions either converge to 0 or 1, so the only difficulty remaining is the term $e^{\frac{1}{2}\left[(\kappa_1+a)^2 - (\gamma_i+a)^2\right]\sigma^2 T}$. Let us establish a series of results:

First, we note that in addition to $e^{\frac{1}{2}\left[(\kappa_1+a)^2 - (\gamma_i+a)^2\right]\sigma^2 T} = e^{r_H T}$, we have

$$e^{\frac{1}{2}\left[(\kappa_1+a)^2 - \left(\eta_{ji}+a\right)^2\right]\sigma^2 T} = e^{\left(r_H - \hat{r}_j\right)T}$$

and since we established that $\hat{r}_j > r_H$ we note that this term is converging to zero.

Second, we note that

$$\begin{aligned}
\lim_{T\to\infty}\frac{N\left[-(\kappa_1+a)\sigma\sqrt{T}\right]}{e^{-r_H T}} &= \frac{\text{"0"}}{\text{"0"}} = \lim_{T\to\infty}\frac{\left(N\left[-(\kappa_1+a)\sigma\sqrt{T}\right]\right)'}{\left(e^{-r_H T}\right)'} \\
&= \lim_{T\to\infty}\frac{(\kappa_1+a)\sigma}{2r_H\sqrt{T}}\exp\left\{-\frac{1}{2}(\kappa_1+a)^2\sigma^2 T + r_H T\right\} \\
&= \lim_{T\to\infty}\frac{(\kappa_1+a)\sigma}{2r_H\sqrt{T}}\exp\left\{-T\left[\frac{\tilde{\mu}^2}{2\sigma^2} + r_H - r_H\right]\right\} \\
&= \lim_{T\to\infty}\frac{(\kappa_1+a)\sigma}{2r_H\sqrt{T}}\exp\left\{-\frac{\tilde{\mu}^2}{2\sigma^2}T\right\} = 0
\end{aligned}$$

where we used the fact that $(\kappa_1+a)^2 = \frac{\tilde{\mu}^2 + 2\sigma^2 r_H}{\sigma^4}$. Thus, all terms involving functions g vanish and no complication arises from premultiplying by $m = \frac{1}{T}$, and we are left with

$$\lim_{T\to\infty}\frac{\delta_B}{r-\mu} = \lim_{T\to\infty} V_B = \lim_{T\to\infty}\frac{-\kappa_2 K_0(T)}{\kappa_2 - 1} = \frac{\kappa_2(1-\pi)c}{\kappa_2 - 1}$$

where $V_B = \frac{\delta_B}{r-\mu}$ which is the same result as in Leland (1994) once we identify (in Leland's notation) $x = -\kappa_2$, so that $\lim_{T\to\infty} V_B = \frac{(1-\pi)\frac{c}{r}x}{x+1}$. In the infinite maturity limit, the equity holders care about the illiquidity they impose on bondholders via the valuation spread between H and L only at the beginning when issuing bonds, but since there is no rollover their default decision is not affected by bond market illiquidity for a given level of aggregate face value and coupon.

Next, let us investigate $T \to 0$, which essentially renders the secondary bond market completely liquid. But of course there is a large effect of $T \to 0$ on the bankruptcy decision of the equity holders. Using L'Hopital's rule, we need to investigate

$$\lim_{T\to 0}\frac{1}{T}\left[\kappa_2 g_F\left(v_B\right)-g_F'\left(v_B\right)\right]$$

We see that two terms that exactly give $\kappa_i-\chi$ explode at the rate $\frac{1}{\sqrt{T}}$, so that in the limit we have

$$\lim_{T\to\infty}\frac{\delta_B^*\left(T\right)}{r-\mu}=\frac{\sum_{j=1}^{2}P_{1j}\left(B_j+A_j\right)}{\sum_{j=1}^{2}P_{1j}\hat{\alpha}_j}=\frac{p\left[\begin{array}{cc}P_{11} & P_{12}\end{array}\right]\mathbf{P^{-1}1}}{\left[\begin{array}{cc}P_{11} & P_{12}\end{array}\right]\mathbf{P}^{-1}\boldsymbol{\alpha}}$$

If $\alpha=\alpha_H=\alpha_L$, we are back to the L96 solution of $V_B=\frac{p}{\alpha}$. ■

A.4 Proofs of Section 4

Recall that debt values are given by

$$\begin{aligned}
\left[\begin{array}{c}D_H\left(\delta,\tau\right)\\ D_L\left(\delta,\tau\right)\end{array}\right] &= \mathbf{P}\left[\begin{array}{c}A_1+B_1e^{-\hat{r}_1\tau}\left[1-F\left(\delta,\tau\right)\right]+C_1G_1\left(\delta,\tau\right)\\ A_2+B_2e^{-\hat{r}_2\tau}\left[1-F\left(\delta,\tau\right)\right]+C_2G_2\left(\delta,\tau\right)\end{array}\right]\\
&= \mathbf{P}\left[\begin{array}{c}A_1\\ A_2\end{array}\right]+\left[1-F\left(\delta,\tau\right)\right]\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}B_1\\ B_2\end{array}\right]+\mathbf{P}\left[\begin{array}{cc}G_1\left(\delta,\tau\right) & 0\\ 0 & G_2\left(\delta,\tau\right)\end{array}\right]\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right]\\
&= \mathbf{P}\left[\begin{array}{c}A_1\\ A_2\end{array}\right]+\left[1-F\left(\delta,\tau\right)\right]\exp\left(-\mathbf{A}\tau\right)\mathbf{P}\left[\begin{array}{c}B_1\\ B_2\end{array}\right]+\mathbf{P}\left[\begin{array}{cc}G_1\left(\delta,\tau\right) & 0\\ 0 & G_2\left(\delta,\tau\right)\end{array}\right]\mathbf{P}^{-1}\mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right]
\end{aligned}$$

Here, by defining $a\equiv\frac{\mu-\frac{\sigma^2}{2}}{\sigma^2}$, $\gamma_1\equiv 0$, $\gamma_2\equiv -2a$, $\eta_{i1,2}\equiv -a\pm\frac{\sqrt{a^2\sigma^4+2\sigma^2\hat{r}_i}}{\sigma^2}$, and $q\left(\delta,\chi,t\right)\equiv\frac{\log(\delta_B)-\log(\delta)-(\chi+a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$, the constants in (7) are given by:

$$\begin{array}{lll}
\left[\begin{array}{c}A_1\\ A_2\end{array}\right]\equiv c\hat{\mathbf{D}}^{-1}\mathbf{P^{-1}1}, & \left[\begin{array}{c}B_1\\ B_2\end{array}\right]\equiv p\mathbf{P^{-1}1}-c\hat{\mathbf{D}}^{-1}\mathbf{P^{-1}1}, & \left[\begin{array}{c}C_1\\ C_2\end{array}\right]\equiv\frac{\delta_B}{r-\mu}\mathbf{P}^{-1}\boldsymbol{\alpha}-c\hat{\mathbf{D}}^{-1}\mathbf{P^{-1}1}\\
\mathbf{P}\left[\begin{array}{c}A_1\\ A_2\end{array}\right]\equiv c\mathbf{A}^{-1}\mathbf{1}, & \mathbf{P}\left[\begin{array}{c}B_1\\ B_2\end{array}\right]\equiv p\mathbf{1}-c\mathbf{A}^{-1}\mathbf{1}, & \mathbf{P}\left[\begin{array}{c}C_1\\ C_2\end{array}\right]\equiv\frac{\delta_B}{r-\mu}\boldsymbol{\alpha}-c\mathbf{A}^{-1}\mathbf{1}
\end{array}$$

and the functions F and G are given by

$$F\left(\delta,\tau\right)\equiv\sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\gamma_i}N\left[q\left(\delta,\gamma_i,\tau\right)\right],\ G_j\left(\delta,\tau\right)\equiv\sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}}N\left[q\left(\delta,\eta_{ij},\tau\right)\right],$$

where $N\left(x\right)$ is the cumulative distribution function for a standard normal distribution.

Define $\boldsymbol{\omega}\equiv\left[1,-1\right]\mathbf{A}=\left[\begin{array}{c}\left(r_H+\xi_H+\xi_L\right)\\ -\left(r_L+\xi_H+\xi_L\right)\end{array}\right]^{\top}$ and $S\equiv D_H-D_L=\left[1,-1\right]\left[\begin{array}{c}D_H\\ D_L\end{array}\right]$. We will also write the shorthand $\sqrt{\cdot}$ for $\sqrt{\left[\left(r+\xi\right)-\left(\overline{r}+\lambda\beta\right)\right]^2+4\xi\lambda\beta}$ and note that $\hat{r}_1-\sqrt{\cdot}=\hat{r}_2>0$.

A.4.1 Time-to-maturity τ derivative

Proof of Proposition 4.

First, we know that at $\tau=0$, the derivative with respect to τ is

$$\dot{S}\left(\delta,0\right)\quad=\left[1-F\left(\delta,0\right)\right]p\left(r_L-r_H\right)+\lim_{\tau\to 0}\dot{F}\left(\delta,0\right)\frac{\delta_B}{r-\mu}\left(\alpha_H-\alpha_L\right)=p\left(r_L-r_H\right)>0$$

and hence our result always holds in the vicinity of $\tau=0$.

Now we prove the general results under sufficient conditions listed in Proposition 4. First, we note that $q_\tau\left(\delta,\chi,\tau\right)=\frac{\log(\delta)-\log(\delta_B)-(\chi+a)\sigma^2\tau}{\sigma\sqrt{\tau}}\frac{1}{2\tau}$, so δ and δ_B have reversed signs. Then, we have

$$\left[\begin{array}{c}\dot{D_H}\left(\delta,\tau\right)\\ D_L\left(\delta,\tau\right)\end{array}\right]=\mathbf{P}\left[\begin{array}{c}-\hat{r}_1B_1e^{-\hat{r}_1\tau}\left[1-F\left(\delta,\tau\right)\right]-B_1e^{-\hat{r}_1\tau}\dot{F}\left(\delta,\tau\right)+C_1\dot{G_1}\left(\delta,\tau\right)\\ -\hat{r}_2B_2e^{-\hat{r}_2\tau}\left[1-F\left(\delta,\tau\right)\right]-B_2e^{-\hat{r}_2\tau}\dot{F}\left(\delta,\tau\right)+C_2\dot{G_2}\left(\delta,\tau\right)\end{array}\right]$$

and the derivatives of the auxiliary functions are

$$
\begin{aligned}
\dot{F}(\delta,\tau) &= \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\gamma_i}\phi\left[q(\delta,\gamma_i,\tau)\right]q_\tau(\delta,\gamma_i,\tau)\\
&= \phi\left[q(\delta,0,\tau)\right]\sum_{i=1}^{2}q_\tau(\delta,\gamma_i,\tau)\\
&= \phi\left[q(\delta,0,\tau)\right]\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}}>0\\
\dot{G}_j(\delta,\tau) &= \sum_{i=1}^{2}\left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}}\phi\left[q(\delta,\eta_{ij},\tau)\right]q_\tau(\delta,\eta_{ij},\tau)\\
&= \phi\left[q(\delta,0,\tau)\right]e^{-\hat{r}_j\tau}\sum_{i=1}^{2}q_\tau(\delta,\eta_{ij},\tau)\\
&= \phi\left[q(\delta,0,\tau)\right]e^{-\hat{r}_j\tau}\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}}\\
&= e^{-\hat{r}_j\tau}\dot{F}(\delta,\tau)>0
\end{aligned}
$$

where we used

$$
\begin{aligned}
\left(\frac{\delta}{\delta_B}\right)^{\gamma_i}\phi\left[q(\delta,\gamma_i,\tau)\right] &= \phi\left[q(\delta,0,\tau)\right]\\
\left(\frac{\delta}{\delta_B}\right)^{\eta_{ij}}\phi\left[q(\delta,\eta_{ij},\tau)\right] &= \phi\left[q(\delta,0,\tau)\right]e^{-\hat{r}_j\tau}
\end{aligned}
$$

This is easily derived:

$$
\begin{aligned}
\left(\frac{\delta}{\delta_B}\right)^{\gamma_i}\phi\left[q(\delta,\gamma_i,\tau)\right] &= e^{-\gamma_i\left(\tilde{\delta}_B-\tilde{\delta}\right)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\tilde{\delta}_B-\tilde{\delta}-(\gamma_i+a)\sigma^2t}{\sigma\sqrt{t}}\right]^2}\\
&= \exp\left\{-\gamma_i\left(\tilde{\delta}_B-\tilde{\delta}\right)\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\left[\frac{\left(\tilde{\delta}_B-\tilde{\delta}\right)^2}{2\sigma^2t}-2\frac{(\gamma_i+a)\sigma^2t\left(\tilde{\delta}_B-\tilde{\delta}\right)}{2\sigma^2t}+\frac{\left[(\gamma_i+a)\sigma^2t\right]^2}{2\sigma^2t}\right]\right\}\\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\left[\frac{\left(\tilde{\delta}_B-\tilde{\delta}\right)^2}{2\sigma^2t}-2\frac{a\left(\tilde{\delta}_B-\tilde{\delta}\right)\sigma^2t}{2\sigma^2t}+\frac{(\gamma_i+a)^2\left(\sigma^2t\right)^2}{2\sigma^2t}\right]\right\}\\
&= \frac{1}{\sqrt{2\pi}}\exp\left\{-\left[\frac{\left(\tilde{\delta}_B-\tilde{\delta}\right)^2}{2\sigma^2t}-2\frac{a\left(\tilde{\delta}_B-\tilde{\delta}\right)\sigma^2t}{2\sigma^2t}+\frac{a^2\left(\sigma^2t\right)^2}{2\sigma^2t}+\frac{\left(2\gamma_ia+\gamma_i^2\right)\left(\sigma^2t\right)^2}{2\sigma^2t}\right]\right\}\\
&= \phi\left[q(\delta,0,\tau)\right]\exp\left\{-\frac{\left(2\gamma_ia+\gamma_i^2\right)\sigma^2t}{2}\right\}
\end{aligned}
$$

and we finally note that $\tilde{\mu}\gamma+\frac{\sigma^2}{2}\gamma^2=0\iff\frac{2\tilde{\mu}}{\sigma^2}\gamma+\gamma^2=0\iff 2\gamma a+\gamma^2=0$ which gives the result in conjunction with the fact that $(\gamma_i+a)+(\gamma_{-i}+a)=0$ as they are complementary roots centered around $-a$. Plugging in, we have

$$\begin{aligned}
\left[\begin{array}{c} \dot{D_H(\delta,\tau)} \\ D_L(\delta,\tau) \end{array}\right] &= \mathbf{P}\left[\begin{array}{c} -\hat{r}_1 B_1 e^{-\hat{r}_1\tau}\left[1-F(\delta,\tau)\right]+(C_1-B_1)e^{-\hat{r}_1\tau}\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2 e^{-\hat{r}_2\tau}\left[1-F(\delta,\tau)\right]+(C_2-B_2)e^{-\hat{r}_2\tau}\dot{F}(\delta,\tau) \end{array}\right] \\
&= \mathbf{P}\left[\begin{array}{cc} e^{-\hat{r}_1\tau} & 0 \\ 0 & e^{-\hat{r}_2\tau} \end{array}\right]\left[\begin{array}{c} -\hat{r}_1 B_1\left[1-F(\delta,\tau)\right]+(C_1-B_1)\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2\left[1-F(\delta,\tau)\right]+(C_2-B_2)\dot{F}(\delta,\tau) \end{array}\right] \\
&= \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\left[\begin{array}{c} -\hat{r}_1 B_1\left[1-F(\delta,\tau)\right]+(C_1-B_1)\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2\left[1-F(\delta,\tau)\right]+(C_2-B_2)\dot{F}(\delta,\tau) \end{array}\right] \\
&= \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\left(-\left[1-F(\delta,\tau)\right]\hat{\mathbf{D}}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right]+\dot{F}(\delta,\tau)\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right) \\
&= \exp(-\mathbf{A}\tau)\left(-\left[1-F(\delta,\tau)\right]\mathbf{AP}\left[\begin{array}{c} B_1 \\ B_2 \end{array}\right]+\dot{F}(\delta,\tau)\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)
\end{aligned}$$

where we used the fact that $\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)=\exp(-\mathbf{A}\tau)\mathbf{P}$ and $\mathbf{P}\hat{\mathbf{D}}=\mathbf{AP}$. Premultiplying by the difference vector $[1,-1]$ and plugging in the definitions of $\mathbf{A}, B_i, C_i$, we have

$$\begin{aligned}
\dot{S}(\delta,\tau) &= [1,-1]\left[\begin{array}{c} \dot{D_H(\delta,\tau)} \\ D_L(\delta,\tau) \end{array}\right] \\
&= [1,-1]\exp(-\mathbf{A}\tau)\left\{\left[1-F(\delta,\tau)\right]\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]+\dot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_B}{r-\mu}\alpha_H-p \\ \frac{\delta_B}{r-\mu}\alpha_L-p \end{array}\right]\right\}
\end{aligned}$$

Let us derive a formula for a general vector $\left[\begin{array}{c} x \\ y \end{array}\right]$:

$$\begin{aligned}
[1,-1]\exp(-\mathbf{A}\tau)\left[\begin{array}{c} x \\ y \end{array}\right] &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left[x\left(r_L-r_H-\xi_H-\xi_L\right)-y\left(r_H-r_L-\xi_L-\xi_H\right)\right]+\sqrt{\cdot}\left(1+e^{\tau\sqrt{\cdot}}\right)[x-y]\right\} \\
&= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left([r_L,-r_H]\left[\begin{array}{c} x \\ y \end{array}\right]-\boldsymbol{\omega}\left[\begin{array}{c} x \\ y \end{array}\right]\right)+\sqrt{\cdot}\left(1+e^{\tau\sqrt{\cdot}}\right)[1,-1]\left[\begin{array}{c} x \\ y \end{array}\right]\right\} \\
&= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left([r_L,-r_H]-\boldsymbol{\omega}+\sqrt{\cdot}[1,-1]\right)\left[\begin{array}{c} x \\ y \end{array}\right]+2\sqrt{\cdot}[1,-1]\left[\begin{array}{c} x \\ y \end{array}\right]\right\}
\end{aligned}$$

When $x>y$, it is clear that for $\tau=0$, we have $[1,-1]\exp(-\mathbf{A}\cdot 0)\left[\begin{array}{c} x \\ y \end{array}\right]=(x-y)>0$. Further, if it is to hold for any τ, we need

$$\left(e^{\tau\sqrt{\cdot}}-1\right)\left([r_L,-r_H]\left[\begin{array}{c} x \\ y \end{array}\right]-\boldsymbol{\omega}\left[\begin{array}{c} x \\ y \end{array}\right]+\sqrt{\cdot}[1,-1]\left[\begin{array}{c} x \\ y \end{array}\right]\right)\geq 0$$

Our derivation of $\dot{S}$ has two terms of this form, multiplied by $[1-F]>0$ and $\dot{F}>0$. To ensure positivity, this implies conditions on $p, c, r_H, r_L, \alpha_H, \alpha_L, \delta_B$ once we identify $\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]$ and $\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} \frac{\delta_B}{r-\mu}\alpha_H-p \\ \frac{\delta_B}{r-\mu}\alpha_L-p \end{array}\right]$.

Define$V_B\equiv\frac{\delta_B}{r-\mu}$; thus, we have the following two conditions for these two cases, i.e., for $\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]$,

$$-(r_L-r_H)\left[p\left(r_H+r_L+\xi_H+\xi_L-\sqrt{\cdot}\right)-2c\right] \quad > \quad \Longleftrightarrow w_1\equiv c-p\hat{r}_2>0 \Longleftrightarrow (r_L-r_H)\,2\left[c-p\hat{r}_2\right]>0$$

and for $\left[\begin{array}{c} V_B\alpha_H-p \\ V_B\alpha_L-p \end{array}\right]$,

$$\begin{aligned}
V_B\left[\alpha_L\left(r_L-r_H+\xi_H+\xi_L-\sqrt{\cdot}\right)-\alpha_H\left(r_H-r_L+\xi_H+\xi_L-\sqrt{\cdot}\right)\right]+2p(r_H-r_L) &> 0 \\
\Longleftrightarrow V_B\left[\alpha_L(-2r_H+2\hat{r}_2)-\alpha_H(-2r_L+2\hat{r}_2)\right]+2p(r_H-r_L) &> 0 \\
\Longleftrightarrow w_2\equiv V_B\left[\alpha_L(\hat{r}_2-r_H)+\alpha_H(r_L-\hat{r}_2)\right]-p(r_L-r_H) &> 0
\end{aligned}$$

Note that $r_H<\hat{r}_2<r_L$. So we need sufficiently high $c>p\hat{r}_2$ and also sufficiently high α_L, α_H in the face of a large discount differential r_L-r_H. We thus have proved the following proposition. Thus, under the sufficient conditions listed in Proposition 4, we have

$$\begin{aligned} w_1 \equiv c - p\hat{r}_2 &\geq 0 \\ w_2 \equiv V_B\left[\alpha_L\left(\hat{r}_2 - r_H\right) + \alpha_H\left(r_L - \hat{r}_2\right)\right] - p\left(r_L - r_H\right) &\geq 0, \end{aligned}$$

which implies that $S_\tau(\delta,\tau) > 0$, i.e. the bid-ask spread $(1-\beta)S(\delta,\tau)$ is larger for bonds with longer time-to-maturity.
■

A.4.2 Proof of $S' < 0$ via the system of PDEs and LHS

Proof of Proposition 5.
We aim to prove $S' < 0$ under the following sufficient conditions:

$$\frac{\delta_B}{r-\mu}\left(\alpha_H - \alpha_L\right) - p\left(r_L - r_H\right) > 0 \quad \text{(A.1)}$$

$$-\alpha_L\left(r_L - r_H\right) + \left(\alpha_H - \alpha_L\right)\frac{r_H + \xi_H + \xi_L}{2} > 0 \quad \text{(A.2)}$$

$$\frac{\delta_B}{r-\mu}\left(\alpha\left(r_L - r_H\right)\hat{r}_1 + \left(\alpha_H - \alpha_L\right)\frac{\left[\left(r_H + \xi_H\right)\left(r_L - r_H - \xi_H\right) - \xi_L\left(r_L + \xi_L + 2\xi_H\right)\right]}{2\sqrt{\cdot}}\right) - c\frac{\left(r_L - r_H\right)}{\sqrt{\cdot}} > 0 \quad \text{(A.3)}$$

$$\left(\alpha_H - \alpha_L\right)\frac{\delta_B}{r-\mu} - \left(r_L - r_H\right)\frac{c}{\left[\left(r_H + \xi_H\right)\left(r_L + \xi_L\right) - \xi_H\xi_L\right]} > 0 \quad \text{(A.4)}$$

First, note that when we subtract the second line from the first line of the differential equation we have

$$\begin{aligned} [1,-1]\begin{bmatrix} r_H + \xi_H & -\xi_H \\ -\xi_L & r_L + \xi_L \end{bmatrix}\begin{bmatrix} D_H \\ D_L \end{bmatrix} &= [1,-1]\left(\begin{bmatrix} c \\ c \end{bmatrix} + \tilde{\mu}\delta\begin{bmatrix} D_H \\ D_L \end{bmatrix}' + \frac{\sigma^2}{2}\delta^2\begin{bmatrix} D_H \\ D_L \end{bmatrix}'' - \begin{bmatrix} \dot{D}_H \\ D_L \end{bmatrix}\right) \\ \iff \boldsymbol{\omega}\begin{bmatrix} D_H \\ D_L \end{bmatrix} + \dot{S} &= \tilde{\mu}S' + \frac{\sigma^2}{2}S'' \\ \iff LHS &= \tilde{\mu}S' + \frac{\sigma^2}{2}S'' \end{aligned}$$

where

$$\boldsymbol{\omega} \equiv \left[r_H + \xi_H + \xi_L, -\left(r_L + \xi_L + \xi_H\right)\right].$$

Let us first establish a limit of $LHS(\delta,\tau)$:

$$\lim_{\tau\to 0} LHS(\delta,\tau) = \boldsymbol{\omega}\begin{bmatrix} D_H(\delta,0) \\ D_L(\delta,0) \end{bmatrix} + \lim_{\tau\to 0}\dot{S}(\delta,\tau) = -p\left(r_L - r_H\right) + p\left(r_L - r_H\right) = 0.$$

Outline of the proof:

1. Show that $L\dot{H}S$ as a function of τ only changes sign once.
2. Show, when τ is small, that LHS increases, that is

$$L\dot{H}S(\delta,\tau) > 0.$$

3. Show that $LHS(\delta,\infty) \geq 0$.
4. Show that

$$S(\delta_B,\tau) - \lim_{\delta\to\infty} S(\delta,\tau) > 0$$

Then we are done: (1.) implies that the can at most be one local extrema. By (2.), we know that there is a local maximum in LHS in terms of τ, i.e., LHS has to go up and then down again to approach from above the value in (3.), which is zero or something positive. Finally, (4.) gives us a contradiction if ever $S' > 0$. First, by continuity of the expectation, we have that $S' < 0$ for some part of the state space (δ_B,∞), as otherwise the surplus couldn't be less at ∞ than at 0. Suppose now that there is an interval on which $S' < 0$. This means that there exist a local maximum with $S' = 0 > S''$. But this would imply $LHS = \tilde{\mu}S' + \frac{\sigma^2}{2}S'' < 0$, a contradiction. Thus, $S' > 0$ everywhere.

Step 1: Recall that

$$\begin{aligned}\begin{bmatrix} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau)\end{bmatrix} &= \exp(-\mathbf{A}\tau)\,\mathbf{P}\begin{bmatrix} -\hat{r}_1 B_1\left[1-F(\delta,\tau)\right]+(C_1-B_1)\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2\left[1-F(\delta,\tau)\right]+(C_2-B_2)\dot{F}(\delta,\tau)\end{bmatrix} \\ &= \exp(-\mathbf{A}\tau)\,\mathbf{P}\left(-\left[1-F(\delta,\tau)\right]\hat{\mathbf{D}}\begin{bmatrix} B_1 \\ B_2\end{bmatrix}+\dot{F}(\delta,\tau)\begin{bmatrix} C_1-B_1 \\ C_2-B_2\end{bmatrix}\right)\end{aligned}$$

Thus, we have

$$\begin{aligned}\begin{bmatrix} \ddot{D_H}(\delta,\tau) \\ D_L(\delta,\tau)\end{bmatrix} &= \exp(-\mathbf{A}\tau)(-\mathbf{A})\,\mathbf{P}\left(-\left[1-F(\delta,\tau)\right]\hat{\mathbf{D}}\begin{bmatrix} B_1 \\ B_2\end{bmatrix}+\dot{F}(\delta,\tau)\begin{bmatrix} C_1-B_1 \\ C_2-B_2\end{bmatrix}\right) \\ &\quad+\exp(-\mathbf{A}\tau)\,\mathbf{P}\left(\dot{F}(\delta,\tau)\,\hat{\mathbf{D}}\begin{bmatrix} B_1 \\ B_2\end{bmatrix}+\ddot{F}(\delta,\tau)\begin{bmatrix} C_1-B_1 \\ C_2-B_2\end{bmatrix}\right) \\ &= \exp(-\mathbf{A}\tau)\left(\left[1-F(\delta,\tau)\right]\mathbf{A}^2\mathbf{P}\begin{bmatrix} B_1 \\ B_2\end{bmatrix}-\dot{F}(\delta,\tau)\,\mathbf{AP}\begin{bmatrix} C_1-B_1 \\ C_2-B_2\end{bmatrix}\right) \\ &\quad+\exp(-\mathbf{A}\tau)\left(\dot{F}(\delta,\tau)\,\mathbf{AP}\begin{bmatrix} B_1 \\ B_2\end{bmatrix}+\dot{F}(\delta,\tau)(\ldots)\,\mathbf{P}\begin{bmatrix} C_1-B_1 \\ C_2-B_2\end{bmatrix}\right) \\ &= \exp(-\mathbf{A}\tau)\left(\left[1-F(\delta,\tau)\right]\mathbf{A}^2\mathbf{P}\begin{bmatrix} B_1 \\ B_2\end{bmatrix}\right) \\ &\quad+\exp(-\mathbf{A}\tau)\,\dot{F}(\delta,\tau)\left(\mathbf{AP}\begin{bmatrix} 2B_1-C_1 \\ 2B_2-C_2\end{bmatrix}+(\ldots)\,\mathbf{IP}\begin{bmatrix} C_1-B_1 \\ C_2-B_2\end{bmatrix}\right)\end{aligned}$$

where we used the fact that $\mathbf{AP}=\mathbf{P}\hat{\mathbf{D}}$ and $\mathbf{A}\exp(-\mathbf{A}\tau)=\mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\hat{\mathbf{D}}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\hat{\mathbf{D}}\mathbf{P}^{-1}=\exp(-\mathbf{A}\tau)\,\mathbf{A}$ as diagonal matrices of the same order commute.

Thus, if we can show that $L\dot{H}S>0$ for any $\delta>\delta_B$ we are done. Note that $\frac{\partial^2 S(\delta,\tau)}{\partial\tau^2}=\ddot{S}$ equals to

$$\begin{aligned}\ddot{S} &= [1,-1](-\mathbf{A})\begin{bmatrix} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau)\end{bmatrix}+[1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\begin{bmatrix} c-pr_H \\ c-pr_L\end{bmatrix}+\ddot{F}(\delta,\tau)\begin{bmatrix} \frac{\delta_B}{r-\mu}\alpha_H-p \\ \frac{\delta_B}{r-\mu}\alpha_L-p\end{bmatrix}\right\} \\ &= [1,-1]\exp(-\mathbf{A}\tau)\left\{-\mathbf{A}\left(\left[1-F(\delta,\tau)\right]\begin{bmatrix} c-pr_H \\ c-pr_L\end{bmatrix}+\dot{F}(\delta,\tau)\begin{bmatrix} \frac{\delta_B\alpha_H}{r-\mu}-p \\ \frac{\delta_B\alpha_L}{r-\mu}-p\end{bmatrix}\right)-\dot{F}(\delta,\tau)\begin{bmatrix} c-pr_H \\ c-pr_L\end{bmatrix}+\ddot{F}(\delta,\tau)\begin{bmatrix} \frac{\delta_B\alpha_H}{r-\mu}-p \\ \frac{\delta_B\alpha_L}{r-\mu}-p\end{bmatrix}\right\} \\ &= -\boldsymbol{\omega}\begin{bmatrix} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau)\end{bmatrix}+[1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\begin{bmatrix} c-pr_H \\ c-pr_L\end{bmatrix}+\ddot{F}(\delta,\tau)\begin{bmatrix} \frac{\delta_B}{r-\mu}\alpha_H-p \\ \frac{\delta_B}{r-\mu}\alpha_L-p\end{bmatrix}\right\}\end{aligned}$$

where we used the fat that
$\mathbf{A}\exp(-\mathbf{A}\tau)=\mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\hat{\mathbf{D}}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\hat{\mathbf{D}}\mathbf{P}^{-1}=\exp(-\mathbf{A}\tau)\,\mathbf{A}$ as diagonal matrices of the same order commute.

We realize that the $\boldsymbol{\omega}\begin{bmatrix} \dot{D_H}(\delta,\tau) \\ D_L(\delta,\tau)\end{bmatrix}$ parts cancel out in $L\dot{H}S$, and we are left with

$$L\dot{H}S(\delta,\tau)=[1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\begin{bmatrix} c-pr_H \\ c-pr_L\end{bmatrix}+\ddot{F}(\delta,\tau)\begin{bmatrix} \frac{\delta_B}{r-\mu}\alpha_H-p \\ \frac{\delta_B}{r-\mu}\alpha_L-p\end{bmatrix}\right\}$$

Further note that with $\dot{F}(\delta,\tau)=\phi\left[q(\delta,0,\tau)\right]\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}}$, $q_\tau(\delta,0,\tau)=\frac{\log\left(\frac{\delta}{\delta_B}\right)-a\sigma^2\tau}{2\sigma\tau^{3/2}}$, and $\phi'(x)=-x\phi(x)$, we have

$$\begin{aligned}
\ddot{F}(\delta,\tau) &= \phi'\left[q(\delta,0,\tau)\right]q_\tau(\delta,0,\tau)\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}}+\phi\left[q(\delta,0,\tau)\right]\frac{\log\left(\frac{\delta}{\delta_B}\right)}{\sigma\tau^{3/2}}\left(-\frac{3}{2\tau}\right)\\
&= \dot{F}(\delta,\tau)\left[-q(\delta,0,\tau)q_\tau(\delta,0,\tau)-\frac{3}{2\tau}\right]\\
&= \dot{F}(\delta,\tau)\left[-\frac{-\log\left(\frac{\delta}{\delta_B}\right)-a\sigma^2\tau}{\sigma\sqrt{\tau}}\cdot\frac{\log\left(\frac{\delta}{\delta_B}\right)-a\sigma^2\tau}{2\sigma\tau^{3/2}}-\frac{3}{2\tau}\right]\\
&= \dot{F}(\delta,\tau)\left[\frac{\log\left(\frac{\delta}{\delta_B}\right)^2-a^2\left(\sigma^2\right)^2\tau^2}{2\sigma^2\tau^2}-\frac{3}{2\tau}\right]\\
&= \dot{F}(\delta,\tau)\left[\frac{\log\left(\frac{\delta}{\delta_B}\right)^2}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right]
\end{aligned}$$

so that

$$L\dot{H}S(\delta,\tau)=\dot{F}(\delta,\tau)\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\left\{\left(\frac{\log\left(\frac{\delta}{\delta_B}\right)^2}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\begin{bmatrix}\frac{\delta_B}{r-\mu}\alpha_H-p\\ \frac{\delta_B}{r-\mu}\alpha_L-p\end{bmatrix}-\begin{bmatrix}c-pr_H\\ c-pr_L\end{bmatrix}\right\}$$

Let us now write out this term in more detail. First, note that

$$\begin{aligned}
\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\begin{bmatrix}V_B\alpha_H-p\\ V_B\alpha_L-p\end{bmatrix} &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}\\
\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\begin{bmatrix}c-pr_H\\ c-pr_L\end{bmatrix} &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}
\end{aligned}$$

Then, let $x\equiv\log\left(\frac{\delta}{\delta_B}\right)^2\in(0,\infty)$, to simplify to

$$L\dot{H}S=\dot{F}\times\frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}-\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}\right]$$

As $\dot{F}\times\frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}>0$, we know that the term $[\cdot]$ determines the sign of $L\dot{H}S$. Writing it out, we have

$$\begin{aligned}
&\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}-\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}\right]\\
=&\left(e^{\tau\sqrt{\cdot}}-1\right)\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)w_2-w_1\right]+2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]
\end{aligned}$$

We note that $\lim_{\tau\to0}\frac{e^{\tau\sqrt{\cdot}}-1}{\tau}=\frac{"0"}{"0"}=\sqrt{\cdot}>0$, so that $\lim_{\tau\to\infty}\frac{e^{\tau\sqrt{\cdot}}-1}{\tau^2}=\infty$. Thus, at τ in the vicinity of 0, the sign of the term is determined by w_2. Next, when $\tau\to\infty$, we have the sign being determined by $-\frac{a^2\sigma^2}{2}w_2-w_1<0$.

Multiplying out $w_2\left(e^{\tau\sqrt{\cdot}}-1\right)>0$, and defining $Q_1(x,\tau)=\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)$, we have

$$\begin{aligned}
Q(x,\tau) &= Q_1(x,\tau)-\frac{w_1}{w_2}+\frac{2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\
&= Q_1(x,\tau)-\frac{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\
&= Q_1(x,\tau)-\frac{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-w_3}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\
&= Q_2(x,\tau)-Q_2(\tau)
\end{aligned}$$

where from (A.1) we know that

$$w_3 \equiv 2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]>0.$$

Note that $Q_1(x,\tau)$ changes sign only once. Then, we know that

$$\dot{Q}_2(\tau) = \frac{\sqrt{\cdot}e^{\tau\sqrt{\cdot}}w_1\left(e^{\tau\sqrt{\cdot}}-1\right)w_2-\left[\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-w_3\right]\sqrt{\cdot}e^{\tau\sqrt{\cdot}}w_2}{(\cdot)^2}=\frac{w_2w_3\sqrt{\cdot}e^{\tau\sqrt{\cdot}}}{(\cdot)^2}$$

Thus, if $w_2w_3>0$, then $\dot{Q}_2(\tau)>0$ and we know that $Q(x,\tau)$ is composed of a part that crosses from positive to negative as τ increase $(Q_1(x,\tau))$ and of a part that is monotonically decreasing as τ increases $(-Q_2(\tau))$.

Step 2: From the derivation above, we know that for τ in the vicinity of 0, the sign of the $L\dot{H}S$ is determined by w_2. Next, when $\tau\to\infty$, we have the sign being determined by $-\frac{a^2\sigma^2}{2}w_2-w_1<0$.

Step 3: Note that

$$LHS(\delta,\infty)=\boldsymbol{\omega}\mathbf{P}\begin{bmatrix}\left(\frac{\delta}{\delta_B}\right)^{\eta_{12}} & 0\\ 0 & \left(\frac{\delta}{\delta_B}\right)^{\eta_{22}}\end{bmatrix}\mathbf{P}^{-1}\mathbf{P}\begin{bmatrix}C_1\\ C_2\end{bmatrix}$$

with $\eta_{12}<\eta_{22}<0$, so that $0<X_1=\left(\frac{\delta}{\delta_B}\right)^{\eta_{12}}<\left(\frac{\delta}{\delta_B}\right)^{\eta_{22}}=X_2$. Note that for $\delta\to\delta_B$, the LHS is positive under (A.2)

$$\lim_{\delta\to\delta_B}LHS(\delta,\infty)=-\alpha_L\left(r_L-r_H\right)+\left(\alpha_H-\alpha_L\right)\frac{r_H+\xi_H+\xi_L}{2}>0.$$

First, let us note the following results:

$$\begin{aligned}
\mathbf{P}\begin{bmatrix}C_1\\ C_2\end{bmatrix} &= \frac{\delta_B}{r-\mu}\boldsymbol{\alpha}-c\mathbf{A}^{-1}\mathbf{1}\\
\boldsymbol{\omega}\mathbf{P}\begin{bmatrix}X_1 & 0\\ 0 & X_2\end{bmatrix}\mathbf{P}^{-1}\boldsymbol{\alpha} &= \boldsymbol{\omega}\mathbf{P}\begin{bmatrix}X_1 & 0\\ 0 & X_2\end{bmatrix}\mathbf{P}^{-1}\begin{bmatrix}\alpha_H\\ \alpha_L\end{bmatrix}\\
&= \frac{\alpha_L\left(r_L-r_H\right)\left[\hat{r}_1\left(X_2-X_1\right)-X_2\sqrt{\cdot}\right]}{\sqrt{\cdot}}\\
&\quad+\left(\alpha_H-\alpha_L\right)\left(X_2-X_1\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\\
&\quad+\left(\alpha_H-\alpha_L\right)\left(X_1+X_2\right)\frac{r_H+\xi_H+\xi_L}{2}\\
\boldsymbol{\omega}\mathbf{P}\begin{bmatrix}X_1 & 0\\ 0 & X_2\end{bmatrix}\mathbf{P}^{-1}\mathbf{A}^{-1}\mathbf{1} &= \frac{\left(r_L-r_H\right)\left(X_2-X_1\right)}{\sqrt{\cdot}}>0
\end{aligned}$$

Combining these results, we have that

$$\begin{aligned}
&LHS(\delta,\infty)\\
=\;&\frac{\delta_B}{r-\mu}\times\left\{\frac{\alpha_L\left(r_L-r_H\right)\left[\hat{r}_1\left(X_2-X_1\right)-X_2\sqrt{\cdot}\right]}{\sqrt{\cdot}}+\left(\alpha_H-\alpha_L\right)\left(X_2-X_1\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\right\}\\
&+\frac{\delta_B}{r-\mu}\times\left\{\left(\alpha_H-\alpha_L\right)\left(X_1+X_2\right)\frac{r_H+\xi_H+\xi_L}{2}\right\}-c\frac{\left(r_L-r_H\right)\left(X_2-X_1\right)}{\sqrt{\cdot}}\\
=\;&\left(X_2-X_1\right)\left[\frac{\delta_B}{r-\mu}\left(\alpha\left(r_L-r_H\right)\hat{r}_1+\left(\alpha_H-\alpha_L\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\right)-c\frac{\left(r_L-r_H\right)}{\sqrt{\cdot}}\right]\\
&+X_2\frac{\delta_B}{r-\mu}\left(\frac{r_H+\xi_H+\xi_L}{2}\left(\alpha_H-\alpha_L\right)-\alpha_L\left(r_L-r_H\right)\right)+X_1\frac{\delta_B}{r-\mu}\frac{r_H+\xi_H+\xi_L}{2}\left(\alpha_H-\alpha_L\right)
\end{aligned}$$

Because $0<X_1<X_2$, the sufficient conditions for $LHS(\delta,\infty)>0$ are

$$\begin{aligned}
\frac{\delta_B}{r-\mu}\left(\alpha\left(r_L-r_H\right)\hat{r}_1+\left(\alpha_H-\alpha_L\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\right)-c\frac{\left(r_L-r_H\right)}{\sqrt{\cdot}} &> 0,\\
\frac{r_H+\xi_H+\xi_L}{2}\left(\alpha_H-\alpha_L\right)-\alpha_L\left(r_L-r_H\right) &> 0.
\end{aligned}$$

Here, the first condition is from (A.3) and the second is from (A.2).

Step 4: We have

$$\begin{aligned} S(\delta_B,\tau) &= \frac{\delta_B}{r-\mu}(\alpha_H-\alpha_L) \\ \lim_{\delta\to\infty} S(\delta,\tau) &= [1,-1]\left[c\mathbf{A}^{-1}\mathbf{1}+\exp(-\mathbf{A}\tau)\left(p\mathbf{1}-c\mathbf{A}^{-1}\mathbf{1}\right)\right] \end{aligned}$$

Under our assumption that $S_\tau(\delta,\tau)>0$, we know that the highest $S(\delta,\tau)$ is at $\tau=\infty$. Noting

$$\begin{aligned} [1,-1]\,\mathbf{A}^{-1}\mathbf{1} &= \frac{r_L-r_H}{(r_H+\xi_H)(r_L+\xi_L)-\xi_H\xi_L} \\ [1,-1]\exp(-\mathbf{A}\tau)\,\mathbf{1} &= \frac{(r_L-r_H)\,e^{-\hat{r}_1\tau}\left(e^{\sqrt{\cdot}\tau}-1\right)}{\sqrt{\cdot}} \\ [1,-1]\exp(-\mathbf{A}\tau)\,\mathbf{A}^{-1}\mathbf{1} &= -\frac{(r_L-r_H)\,e^{-\hat{r}_1\tau}\left[\hat{r}_1\left(e^{\sqrt{\cdot}\tau}-1\right)+\sqrt{\cdot}\right]}{\sqrt{\cdot}\,[(r_H+\xi_H)(r_L+\xi_L)-\xi_H\xi_L]} \end{aligned}$$

we have from (A.4)

$$\begin{aligned} S(\delta_B,\tau)-\lim_{\delta\to\infty}S(\delta,\tau) &> \lim_{\tau\to\infty}\left\{S(\delta_B,\tau)-\lim_{\delta\to\infty}S(\delta,\tau)\right\} \\ &= (\alpha_H-\alpha_L)\frac{\delta_B}{r-\mu}-(r_L-r_H)\frac{c}{[(r_H+\xi_H)(r_L+\xi_L)-\xi_H\xi_L]}>0. \end{aligned}$$

Taken together, we established parameter restrictions that result in $S_\delta(\delta,\tau)<0$. ■

Looser sufficiency conditions can be established for $S_\delta(\delta,\tau)$ in the vicinity of $\tau=0$ or $\delta=\delta_B$. We omit these proofs for brevity.

A.5 The steady-state distribution of types, trading volume

We now derive the cross-sectional (w.r.t. τ) steady-state distribution of L types. Let $p_H(t,\tau)$ be the proportion at time t of H types of maturity τ. Then we have

$$\frac{\partial p_H(t,\tau)}{\partial t}-\frac{\partial p_H(t,\tau)}{\partial\tau}=\lambda p_L(t,\tau)-\xi p_H(t,\tau)$$

as when time advances, maturity shrinks. To impose a steady-state, we note that $\frac{\partial p_H(t,\tau)}{\partial t}=0$ and that $p_H(t,T)=1$, i.e., at any time t, due to the firm being able to issue to only H types, the proportion of H types with the longest maturity T is always 1. Further note that $p_H+p_L=1,\forall\tau$, so that in the end we have

$$\begin{aligned} -\frac{\partial p_H(\tau)}{\partial\tau} &= \lambda p_L(t,\tau)-\xi p_H(t,\tau) \\ p_H(\tau) &= \frac{\lambda+\xi e^{(\tau-T)(\lambda+\xi)}}{\lambda+\xi} \\ p_L(\tau) &= \frac{\xi}{\lambda+\xi}\left[1-e^{(\tau-T)(\lambda+\xi)}\right] \end{aligned}$$

We of course have to adjust by the density of bonds $\frac{1}{T}$ when looking at the steady state mass of H and L types, μ_H and μ_L. We solve to get

$$\begin{aligned} \mu_H(T) &= \frac{1}{T}\int_0^T p_H(\tau)\,d\tau \\ &= \frac{\lambda}{\lambda+\xi}+\frac{\xi\left(1-e^{-T(\lambda+\xi)}\right)}{T(\lambda+\xi)^2} \\ \mu_L(T) &= \frac{\xi}{\lambda+\xi}-\frac{\xi\left(1-e^{-T(\lambda+\xi)}\right)}{T(\lambda+\xi)^2} \end{aligned}$$

and we note that $\mu_L'(T) > 0 > \mu_H'(T)$ (note that $\frac{\partial p_i(\tau)}{\partial T} \neq 0$), $\lim_{T\to 0}\mu_H(T) = 1$ and $\lim_{T\to 0}\mu_L(T) = 0$, as well as $\lim_{T\to\infty}\mu_H(T) = \frac{\lambda}{\lambda+\xi}$ and $\lim_{T\to\infty}\mu_L(T) = \frac{\xi}{\lambda+\xi}$.

Trade volume is now easily derived. It is simply the mass of agents that are in state (L, τ) times the intensity with which they meet a market maker and execute trades, λ. Thus, trade volume (scaled by total bonds outstanding) for maturity τ will be

$$Volume(\tau) = \frac{\lambda}{T} p_L(\tau) = \frac{1}{T}\frac{\lambda\xi}{\lambda+\xi}\left[1 - e^{(\tau-T)(\lambda+\xi)}\right]$$

A.6 Steady state in the search market

So far, we have assumed that the intermediation intensity λ is exogenously determined by the dealers. This assumption hinges on the fact that we assume an infinite mass of H type buyers waiting on the sideline who do not hold the asset, but in order to buy have to go through a dealer. We can relax these assumptions in several ways without substantially changing the model.

However, the two assumptions we will not relax are that (i) orders are 'batched' in the sense that there is no difference in intermediation intensities between different maturities $\tau \in [0, T]$, and (ii) that dealers extract all the surplus when bargaining with H type buyers. Relaxing (i) would destroy our closed form solution without providing much more insight.

The problem with relaxing (ii) is more subtle: by removing dealers and allowing direct negotiation between H and L types we essentially (except in extreme circumstances) leave some surplus beyond their own valuation to the H type buyers. If we assume that we have an infinite mass of H type buyers on the sideline, this would not change their behavior as every single one of the buyers expects never to have the opportunity to be able to buy and make a surplus. However, if there is only a finite mass of H type buyers waiting on the sideline (as we will allow later on), then (a) the firm's pricing at issuance will be affected as it will have to entice H type buyers to participate instead of waiting and buying for a discount in the secondary market, (b) we now have to track the value function of the H and L types who do not hold the asset, and (c) the expected surplus for an H type not holding the asset is an integral over all possible maturities he might encounter. This sufficiently complicates the equations to render them not solvable by the methods we employed in this paper, while not adding more insights. In a follow up project, we can easily incorporate a dealer-less market by giving up the deterministic maturity dimension.

First, we can easily relax the model to allow for random transitioning back from the L to the H state for bondholders, say with intensity ζ. Then we can simply use our current valuation formulas, but with $\lambda\beta + \zeta$ taking the place of $\lambda\beta$. Also, the impact on trade-volume can be easily handled but for brevity is not shown here.

Second, with this switching back intensity, we can also close the model to have 4 different finite population measures — H types with and without the bond, and L types with and without the bond under the assumption that a dealer will only intermediate a trade once he found a buying party (H type) and leave no surplus to the H type buyer. This then allows us, in the tradition of most search models, to define the meeting intensity λ as some function of the steady-state masses of these populations, especially of the mass H types without the bond trying to buy and the mass of L types with the bond trying to sell. This would, in steady-state, result in a meeting intensity $\lambda(T)$ that is a function of the maturity structure, which however for a given T would be constant. The valuation equations would simply include $\lambda(T)$ as a constant and thus would not change. The only thing that would change is the optimal maturity calculations we analyzed in Section 5.1 — here, the firm will take into account the impact of its maturity choice T on the liquidity of the secondary market, and thus indirectly on the valuation of its bonds.