NBER WORKING PAPER SERIES

# REFINING LINEAR RATIONAL EXPECTATIONS MODELS AND EQUILIBRIA

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Working Paper 18348
http://www.nber.org/papers/w18348

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
August 2012

Seonghoon Cho acknowledges that this work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2010-327-B00089). The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

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NBER Working Paper No. 18348
August 2012
JEL No. C62,D84

**ABSTRACT**

This paper proposes forward convergence as a model refinement scheme for linear rational expectations (LRE) models and an associated no-bubble condition as a solution selection criterion. We relate these two concepts to determinacy and characterize the complete set of economically relevant rational expectations solutions to the LRE models under determinacy and indeterminacy. Our results show (1) why a determinate solution is economically meaningful in most, but not all, cases, and (2) that those models that are not forward-convergent have no economically relevant solutions.

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## 1. Introduction

Recent macroeconomics literature has utilized the concept of determinacy as a primary criterion for characterizing the economic properties of a given rational expectations (RE) model and its solutions. While some researchers argue that determinacy is necessary and sufficient for a model to be economically relevant, others argue that multiple rational expectations equilibria (REEs) can be admissible as well in some cases of indeterminacy. Thus models and their REEs are not dismissed as implausible simply because they are indeterminate. Furthermore, that determinacy alone does not automatically guarantee the economic plausibility of a given model and its determinate equilibrium has been argued by Bullard and Mitra (2002), Cho and McCallum (2009), Honkapohja and Mitra (2004), and others. Several solution selection criteria have been proposed to narrow down the set of relevant equilibria in indeterminate models including, for example, the MSV criterion of McCallum (1983, 2007), the E-stability criterion of Evans and Honkapohja (2001), and a fairly recent proposal by Driskill (2006).

This extent of disagreement over the role of determinacy may have been a consequence of the absence of any step for refining "models" to begin with.[1] Here we

[1] Here the word "determinacy" is being used in the sense that is standard in monetary economics and frequently utilized elsewhere, namely, to designate a model specification in which there exists only one RE solution that is dynamically stable — literally a "single stable solution" (SSS). McCallum (2011) has argued that this terminology is highly inappropriate, however, since the traditional meaning of the word "determinate" is that the model at hand clearly points to a single relevant solution. Thus the usage in question proceeds as if the SSS requirement was equivalent to the desired condition—i.e., that the model at hand provides a unique prediction as to the behavior of the (model) economy. A unique prediction is what "determinacy" is supposed to *mean*, however, so it is unsatisfactory for this word to be used as a synonym

propose a model refinement scheme for a general class of linear models, together with a solution selection criterion for the REEs, and characterize the complete set of economically relevant equilibria under determinacy and indeterminacy. In particular, we adopt the forward convergence and no-bubble conditions proposed by Cho and Moreno (2011), and relate these refinement schemes to determinacy by means of characterization results developed by McCallum (2007).

In their study of hyperinflation and monetary reform, Flood and Garber (1980) introduce the notion of "process consistency" and suggest that it is an essential characteristic of any model variable that pretends to serve as money. Specifically, in the context of a Cagan-type monetary model, they argue that any "process inconsistent" money supply will be rejected by the public because it does not provide a finite solution for the price level. In this setting, process consistency simply amounts to the case that a RE model can be solved forward and they argue that any reasonable model should possess this minimal economic characteristic. Thus their analysis pertains to a prototype model that can be solved forward recursively since it has no lagged variables. This method of solving RE models forward had not, however, been developed and applied to general models with lagged variables until Cho and Moreno (2011) developed the forward method for such models. Their forward convergence property, which we propose as a model refinement, amounts to a generalization of process consistency and here we argue that, indeed, any model that fails to satisfy the forward convergence condition has no cogent RE solution. Moreover, whenever a model is forward-convergent, the Cho and

for the SSS condition, especially since there are examples in which it is clearly inappropriate. In the present paper we retain that usage, nevertheless, in order to facilitate communication.

Moreno methodology provides a well-known forward (forward-looking) solution in the sense of Blanchard (1979).

When a model is solved forward, there remains a term involving the expectation of future endogenous variables, often called "bubbles". Seeking fundamental solutions, researchers often assume that this expectational term will disappear as the iterative process goes to the limit. Blanchard and Kahn (1980) in fact impose a restriction to a given RE model that expectations of the endogenous and predetermined and non-predetermined variables do not explode , which *rules out "bubbles" of the sort considered by Flood and Garber (1980)*. While their "bubbles" are not exactly the same as the expectational term, their no-bubble restriction can be interpreted as a requirement that any fundamental solution should satisfy. Cho and Moreno (2011) have shown that this no-bubble condition holds only for the forward solution and that all other fundamental – often called "bubble-free" – solutions fail to satisfy it. Accordingly, we contend that the no-bubble condition constitutes a relevant solution refinement scheme.

Cho and Moreno (2011), however, do not relate their refinement schemes to determinacy. Accordingly, it is our task here to derive this relationship, drawing on results of McCallum (2007), and to characterize all the REEs in relation to determinate and indeterminate models. In the process, we will also extend one of McCallum's results so as to apply to non-fundamental, as well as fundamental, solutions.

The paper is organized as follows. Section 2 presents a general class of linear

Rational Expectations (RE) models and characterizes the set of REEs. In Section 3, necessary and sufficient conditions for determinacy are stated. In section 4, we formally define the concept of forward convergence and study the relation between determinacy and forward convergence. Section 5 classifies RE models with these two properties and characterizes the full set of REEs. In section 6, we apply our methodology to an example based on a standard New-Keynesian model. Section 7 concludes.

## 2. Linear Rational Expectations Models and Rational Expectations Equilibria

### 2.1 The Model

We consider general linear rational expectations models of the form:

(1) $$x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t\,, \qquad z_t = Rz_{t-1} + e_t\,,$$

where $x_t$ is an $n \times 1$ vector of endogenous variables and $z_t$ is an $l \times 1$ vector of exogenous variables, with $R$ a stable $m \times m$ matrix, and $e_t$ is an $l \times 1$ vector of *i.i.d* and mean zero shock processes. Also, $E_t(\cdot)$ is mathematical expectation operator conditional on the information set available at time $t$.[2] The linear model (1) that we have in mind is a local linear approximation around the steady state of the underlying dynamic stochastic general equilibrium model. Hence, we assume that the steady state is well-defined and known to all economic agents, and we present the model in terms of the deviations of the endogenous variables from their steady states.

[2] McCallum (2007) shows that any model of the class studied by King and Watson (1998) and Klein (2000) – which admits any finite number of lags, expectational leads, and lags of expectational leads – can be written in the form (1).

## 2.2 Classes of Solutions

Any process $x_t$ that is consistent with the model (1) is a solution to the model. We decompose it into two components:

(2) $x_t = x_t^{FUN} + w_t$,

where $x_t^{FUN}$ is a fundamental solution and $w_t$ is a non-fundamental component.[3] Note that for any given $x_t^{FUN}$, there is the corresponding class of $w_t$. That is, the process of $w_t$ is restricted by a particular $x_t^{FUN}$. We discuss the two classes in detail.

### A. Fundamental Solutions

Any fundamental solution has the following form.

(3) $x_t = \Omega x_{t-1} + \Gamma z_t$,

where $(\Omega, \Gamma)$ must satisfy the following conditions:

(4a) $\Omega = (I - A\Omega)^{-1} B$,

(4b) $\Gamma = (I - A\Omega)^{-1} C + F\Gamma R$,

where $F$ is given by:

(5) $F = (I - A\Omega)^{-1} A$.

There are in general multiple solutions for $\Omega$ (thus $\Gamma$ as well).[4] But the number of $\Omega$ satisfying (4a), and thus the number of fundamental solutions, is at most ${}_{2n}C_n$, hence

[3] A fundamental solution includes no extraneous state variables. This concept differs from McCallum's (1983) minimum state variable (MSV) solution, however, in that the MSV solution is in all cases uniquely defined whereas there may be multiple fundamental solutions.

[4] When there are no predetermined variables ($B = O$), then $\Omega = O$ and $F = A$, implying that the fundamental solution is unique if it exists.

finite.

**B. Non-fundamental Solutions**

The class of non-fundamental solutions has the following form:

(6) $$x_t = \Omega x_{t-1} + \Gamma z_t + w_t,$$

where $w_t$ is an arbitrary process satisfying

(7) $$w_t = F E_t w_{t+1}.$$[5]

It is important to note that $F$ is restricted in (5) by a particular $\Omega$.

For each $F$, the following proposition characterizes the complete set of solutions to equation (7), thereby extending the result of McCallum (2007), which refers only to fundamental solutions.

**Proposition 1** *Let $m$ be the rank of the matrix $F$. Any real-valued solution $w_t$ to equation (7) can be written as:*

(8) $$w_t = \Lambda w_{t-1} + V\xi_t,$$

*where $\Lambda$ is an $n \times n$ matrix such that $\Lambda = V\Phi V'$, $V$ is an $n \times k$ matrix with $0 \le k \le m \le n$ such that its columns form a subset of an orthonormal basis associated with the inverses of non-zero eigenvalues of $F$, $\Phi$ is a $k \times k$ block-diagonal matrix such that $V = FV\Phi$, and $\xi_t$ is an arbitrary $n \times 1$ stochastic vector such that $E_t \xi_{t+1} = 0$.*

**Proof**: *See Appendix.*

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[5] To see this, forward equation (6) one period ahead and plug it into (1) as follows:

$$x_t = A[\Omega x_t + \Gamma R z_t] + AE_t w_{t+1} + Bx_{t-1} + Cz_t.$$

Rearranging this equation yields:

$$x_t = (I - A\Omega)^{-1} Bx_{t-1} + [(I - A\Omega)^{-1} C + (I - A\Omega)^{-1} A\Gamma R] z_t + (I - A\Omega)^{-1} AE_t w_{t+1}$$
$$= \Omega x_{t-1} + \Gamma z_t + FE_t w_{t+1}.$$

where $\Omega$, $\Gamma$ and $F$ are given by equations (4) and (5).

While the class of non-fundamental components is large, $\Lambda$ has a simple structure: non-zero eigenvalues of $\Lambda$ are the inverses of some or all of the non-zero eigenvalues of $F$. This implies that we have only to study $F$ in order to deduce stability of $w_t$ and determinacy of the model without solving (7) directly. For a compact exposition of stability and determinacy, we define the *spectral radius operator*.

**Definition (Spectral Radius)**: $r(X) = \max_{\{1 \le i \le n\}} |\xi_i| : R^{n \times n} \to R^{+}$ *where* $\xi_i$ *is the i-th eigenvalue of an* $n \times n$ *matrix* $X$ *.*

Now we can state that a fundamental solution (3) is dynamically stable if and only if $r(\Omega) < 1$. In addition, there exist stationary processes $w_t$ if $r(F) > 1$ because $\Lambda$ in equation (8) can always be constructed such that it contains an eigenvalue equal to $1/r(F) < 1$. Therefore, $r(F) \le 1$ is the condition under which there is no stationary stochastic process of $w_t$.

## 3. Determinacy

A model is said to be determinate if the model has a unique stable RE solution. Different researchers use different representations of the underlying model, but many use essentially the same matrix decomposition theorem to derive the conditions for determinacy, i.e., the main theorem of Blanchard and Kahn (1980). But this method is available only when $A$ is non-singular. Thus, the procedure of researchers often is to reformulate a model into a canonical form of Blanchard and Kahn (1980) if it has a singular $A$, following the steps proposed by King and Watson (1998), for instance. Instead, however, one can use a simpler way to identify determinacy without transformation of (1) following Klein (2000) or McCallum (1998, 2007), who utilize the generalized Schur decomposition theorem. To do so, define $\tilde{B} = \begin{bmatrix} I & -B \\ I & O \end{bmatrix}$ and $\tilde{A} = \begin{bmatrix} A & \mathrm{O} \\ \mathrm{O} & I \end{bmatrix}$. Solving for real-valued $\Omega$ amounts to choosing n roots out of the $2n$ generalized eigenvalues of the matrix pencil $[\tilde{B} - \lambda \tilde{A}]$, namely, $\lambda(\tilde{B}, \tilde{A})$ = $\{\lambda_1, \lambda_2, ..., \lambda_{2n}\}$, where $|\lambda_1| \leq ... \leq |\lambda_{2n}|$.[6]

Following McCallum (2007), determinacy can be stated in terms of the matrices $\Omega$ and $F$ in equations (3) and (5), which govern the stability of fundamental and non-fundamental components of the REEs, respectively. His definition enables us to relate determinacy and the property of forward convergence.

To proceed, we introduce an important property regarding the eigenvalues of $\Omega$ and $F$: for any $\Omega$ associated with $n$ eigenvalues in $\lambda(\tilde{B}, \tilde{A})$, the eigenvalues of the corresponding $F$ are the inverses of the *remaining* eigenvalues in $\lambda(\tilde{B}, \tilde{A})$. For instance, let $\Omega^{MOD}$ denote the solution associated with $n$ smallest eigenvalues, $(\lambda_1, \lambda_2, ..., \lambda_n)$. Then the eigenvalues of $F^{MOD}$ are the set $(1/\lambda_{n+1}, 1/\lambda_{n+2}, ..., 1/\lambda_{2n})$. This implies that if a model is determinate, the determinate solution must be the MOD solution. Following this idea of McCallum (2007), determinacy (indeterminacy) conditions can be stated in the

[6] If a complex root is included, its conjugate member must also be included for $\Omega$ to be real-valued. Modulus equality holds when two eigenvalues form a complex conjugate.

following way.

**Proposition 2** *Linear RE models of the form (1) can be classified as follows.*

*1. The model (1) is determinate if and only if* $r(\Omega^{MOD}) < 1$ *and* $r(F^{MOD}) \leq 1$.

*2. The model (1) is indeterminate if and only if* $r(\Omega^{MOD}) < 1$ *and* $r(F^{MOD}) > 1$.

*3. The model (1) has no stable REEs if and only if* $r(\Omega^{MOD}) \geq 1$.

**Proof**: *See McCallum (2007).* [7]

## 4. The Forward Convergence and No Bubble Conditions

Nothing in the foregoing determinacy/indeterminacy conditions serves to establish economic relevance for the determinate or indeterminate solutions on any basis other than their dynamic stability. The forward method of Cho and Moreno (2011), however, provides model and solution refinement schemes pertaining to another dimension —an additional characteristic. Following their method, we solve the model (1) into the future so as to derive its forward representation as follows:

(9) $$x_t = M_k E_t x_{t+k} + \Omega_k x_{t-1} + \Gamma_k z_t ,$$

where $(M_k, \Omega_k, \Gamma_k)$ is given by: $M_1 = A$, $\Omega_1 = B$, $\Gamma_1 = C$, and for $k > 1$,

(10) $$M_k = F_{k-1} M_{k-1} ,$$

[7] A subtle issue may arise in the case of $r(F) = 1$. For instance, consider a univariate model. When $r(F) = 1$, $w_t = a$ solves (7) for any arbitrary constant $a$. There seems to be no general agreement as to whether this case is considered as indeterminate or determinate. However, since such a solution is non-stochastic, we include $r(F) = 1$ as determinacy. If one alternatively treats this case as indeterminate, then one may define determinacy that excludes $r(F) = 1$ in Assertion 1 and includes it in Assertion 2 of Proposition 2.

(11a) $\Omega_k = (I - A\Omega_{k-1})^{-1} B$,

(11b) $\Gamma_k = (I - A\Omega_{k-1})^{-1} C + F_{k-1}\Gamma_{k-1} R$,

where $F_{k-1}$ is given by:

(12) $F_k = (I - A\Omega_k)^{-1} A$,

provided that the regularity condition $\det(I - A\Omega_k) \neq 0$ is satisfied for all $k = 1, 2, 3, \ldots$.

**Definition (Forward Convergence Condition, FCC)** *The model (1) is said to satisfy the forward convergence condition if* $(\Omega_k, \Gamma_k)$ *defined in (11) converge as* $k \to \infty$.

Note that if $(\Omega_k, \Gamma_k)$ converges to $(\Omega^*, \Gamma^*)$, equations (11) fulfill the conditions (4). Note also that $F_k$ in (11) converges to $F^*$ if and only if $\Omega_k$ converges. Hence, under the FCC, the matrix $F^*$ defined by equation (12) fulfills the condition (5) as well. Therefore, the following forward solution,

(13) $x_t = \Omega^* x_{t-1} + \Gamma^* z_t$

is a fundamental solution and it exists if and only if the model satisfies the FCC. Hence the FCC and the existence of the forward solution are equivalent. Therefore, $(\Omega_k, \Gamma_k)$ is unique and implied by the model. Accordingly, the forward solution is a model-implied relation and, consequently, it is economically sensible by itself, i.e., inherently. Evidently, forward convergence is exactly the same concept for general LRE models as the process consistency of Flood and Garber (1980), a paper that accordingly seems to have been undervalued in the literature.

A key implication of the forward method is that the expectational term $M_k E_t x_{t+k}$ depends on the particular solution with which expectations are formed. In principle, the limiting behavior of this "bubble" term should be verified for each REE, instead of assuming its behavior. Formally we define the no-bubble condition:

**Definition (No bubble condition, NBC)** *A solution to the model (1) is said to satisfy NBC if* $\lim_{k\to\infty} M_k E_t x_{t+k} = O_{n\times 1}$ *in (9) when expectations are formed with that solution.*

The following proposition provides a central result of the forward method.

**Proposition 3** *The forward solution is the unique REE that satisfies the NBC.*

**Proof**: *See Appendix.*

Thus the NBC is the unique feature that differentiates the forward solution from all other fundamental solutions. The NBC removes two kinds of equilibria. First, it refines away all the solutions for those models that fail to satisfy the FCC. Suppose that a stable fundamental solution $x_t = \Omega x_{t-1} + \Gamma z_t$ exists for a model where one or both of $(\Omega_k, \Gamma_k)$ explodes as $k$ tends to infinity. Since this solution must satisfy the forward representation (9) for all $k$, this implies that the expectational term evaluated with this solution must explode as well. This would be the major consideration underlying the restriction in Blanchard and Kahn (1980). In this sense, the FCC can be interpreted as a model refinement scheme and the NBC refines away all those fundamental solutions. The role of the NBC is, however, not confined to models that are not forward-convergent. In addition, the NBC also refines away all the stable fundamental solutions, different from the forward solution for the models that are convergent, that would arise in the case of indeterminacy. Fundamental solutions are often conceived of and referred to as bubble-free solutions. Thus, the notion of bubble-free solutions that violate our no-bubble condition would be basically incoherent. This refinement is a phenomenon that arises when we consider models with predetermined variables.

In both cases, since any REE can be written as sum of a fundamental solution and the associated bubble term, then if any fundamental solutions violate the NBC, any member of the set of REEs associated with those solutions also makes no economic sense --i.e., is not analytically coherent.

## 5. Complete Characterization of the REEs under FCC

Now we bridge the relation between the FCC and determinacy, which is absent from Cho and Moreno (2011). Under the FCC, $F^* = \lim_{k\to\infty} F_k$ exists from equation (12). The following proposition shows that under FCC, determinacy corresponds to $r(F^*) \le 1$ [and indeterminacy to $r(F^*) > 1$] and thus characterizes the full set of REEs.

**Proposition 4** *Suppose that the RE model (1) satisfies the forward convergence condition.*

*1. If* $r(\Omega^*) < 1$ and $r(F^*) \le 1$, *then model (1) is determinate and the unique stationary solution is given by the forward solution:*

(14) $$x_t = \Omega^* x_{t-1} + \Gamma^* z_t .$$

*2. If* $r(\Omega^*) < 1$ and $r(F^*) > 1$, *then (1) is indeterminate and the set of stable REEs for which the fundamental solutions satisfy the NBC is given by*

(15) $x_t = \Omega^* x_{t-1} + \Gamma^* z_t + w_t$,

*where* $w_t$ *is an arbitrary stationary process such that* $w_t = F^* E_t w_{t+1}$ *and the whole set of* $w_t$ *associated with* $F^*$ *can be constructed from Proposition 1.*

**Proof:** *See Appendix.*

The conditions in Assertion 1 of Proposition 4 are sufficient for determinacy, but not necessary. This implies that there can be cases in which a model is determinate but the determinate solution is not the forward solution, i.e., $\Omega^* \neq \Omega^{MOD}$ from Proposition 2, and therefore, the determinate solution violates the NBC. An example of this kind is given by Cho and McCallum (2009). Assertion 1 provides a general way to rule out such models on the grounds of economic implausibility. Nevertheless, Assertion 1indicates why a determinate equilibrium is usually economically sensible: it is because $\Omega^* = \Omega^{MOD}$ for almost all well-formulated models.

Now we consider the indeterminate case in which the model satisfies the FCC, but $r(F^*) > 1$. In this case, there exists a continuum of non-fundamental REEs associated with the forward solution. But there may well exist fundamental solutions associated with $\tilde{\Omega}$ different from $\Omega^*$ and the corresponding $\tilde{F}$ as long as $r(\tilde{\Omega}) < 1$ and $r(\tilde{F}) > 1$.[8] Such

[8] There cannot be a fundamental solution with $\tilde{\Omega}$ different from $\Omega^*$ such that $r(\tilde{\Omega}) < 1$ and $r(\tilde{F}) \leq 1$ because this would imply that the model is determinate.

a solution must violate the NBC from Proposition 3. Therefore, the set of REEs that are consistent with the FCC and NBC are the ones associated with the forward solution.

Proposition 4 excludes the following cases as they have no relevant REEs. First, if $r(\Omega^*) \geq 1$, then the model has no stationary solution of which the fundamental component satisfies the NBC. Second, those models that are not forward convergent have no fundamental solutions satisfying the NBC. This latter case is the one for which the NBC is the most important as a solution refinement: without verifying the FCC, one may find solutions to a model that fails to satisfy the FCC. We present an example of this kind in the following section.

Our results show the importance of examining the FCC of RE models. Moreover, our methodology is sufficient to identify determinate and indeterminate cases and provides a complete set of REEs to any LRE models satisfying the FCC. Another important feature of our methodology is that the solution method and the solution refinements are obtained using only the rationality assumption and the recursive structure of the underlying macroeconomic model, without solving for all the mathematical solutions using matrix decomposition techniques.

## 6. Example

In this section, we present a New-Keynesian model similar to the one considered by Cho and Moreno (2011), but detect determinacy and indeterminacy using Proposition 4 and further investigate the reasons why the fundamental solutions to the model do not make sense when the model is not forward convergent. The model is given by the three equations.

(16a) $\pi_t = \beta E_t \pi_{t+1} + \kappa y_t$,

(16b) $y_t = \mu E_t y_{t+1} + (1-\mu) y_{t-1} - \theta(i_t - E_t \pi_{t+1}) + z_t$,

(16c) $i_t = \phi_\pi E_t \pi_{t+1} + \phi_y y_t$,

where $\pi_t, y_t, i_t$ are respectively inflation, the output gap and the nominal interest rate. $z_t$ is an aggregate demand shock that follows an AR(1) process: $z_t = \rho z_{t-1} + \varepsilon_t$ where $0 \le \rho < 1$ and $\varepsilon_t$ is an *i.i.d.* shock. By substituting out $i_t$, the model (16) can be cast into a bivariate system as $x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t$ where $x_t = [\pi_t \ \ y_t]'$ and $A = B_1^{-1} A_1$ $B = B_1^{-1} B_2$, $C = B_1^{-1} C_1$ with

$$B_1 = \begin{bmatrix} 1 & -\kappa \\ 0 & 1+\theta\phi_y \end{bmatrix},\ A_1 = \begin{bmatrix} \beta & 0 \\ -\theta(\phi_\pi - 1) & \mu \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1-\mu \end{bmatrix},\ C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We set the parameter values as $\beta = 0.99,\ \kappa = 0.3,\ \mu = 0.55,\ \theta = 1,\ \phi_y = 0.1,\ \rho = 0.8$.

**Case 1**. When $\phi_\pi = 1.5$, the FCC holds and $r(\Omega^*) = 0.46$ with $r(F^*) = 0.75$. Therefore, the model is determinate and the determinate forward solution is given by:

$$x_t = \begin{bmatrix} 0 & 0.26 \\ 0 & 0.46 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1.66 \\ 0.62 \end{bmatrix} z_t .$$

This solution implies that when there is a rise in aggregate demand shock of size 1, both output and inflation increase, which is the case that would be expected from economic theory.

**Case 2**. When $\phi_\pi = 0.95$, the FCC holds and $r(\Omega^*) = 0.60$ and $r(F^*) = 1.13$. Hence, the model is indeterminate. The forward solution exists and it is qualitatively similar to the one under determinacy. There exists, however, another stable fundamental solution:

$$x_t = \begin{bmatrix} 0 & 2.10 \\ 0 & 0.88 \end{bmatrix} x_{t-1} + \begin{bmatrix} -29.53 \\ -2.59 \end{bmatrix} z_t .$$

In contrast to the forward solution, both coefficients in $\Gamma$ are negative. This is clearly counterintuitive because it implies that an exogenous increase in aggregate demand decreases both inflation and the output gap. We rule out this solution on the ground of the NBC. Cho and Moreno (2011) show that other solution selection criteria pick up the forward solution as well. However, it is not the case in the following model, which fails to satisfy the FCC.

**Case 3**. When $\phi_\pi = 0.9$, the FCC does not hold: whereas $\Omega^*$ (and $F^*$) still exists, $\Gamma_k$ explodes as $k \to \infty$. From equation (11b), one can see that the matrix governing convergence of $\Gamma_k$ is $\rho F^*$. In this example, $r(F^*) = 1.33$ and $r(\rho F^*) = 1.20 > 1$, implying that $\Gamma_k$ grows without bound. Nevertheless, there are two stable fundamental solutions and hence technically, the model is indeterminate. Unlike Case 2, however, none of the solutions is economically sensible. The MOD solution is given by:

$$x_t = \begin{bmatrix} 0 & 0.59 \\ 0 & 0.67 \end{bmatrix} x_{t-1} + \begin{bmatrix} -39.08 \\ -9.15 \end{bmatrix} z_t .$$

The other fundamental solution is also qualitatively similar to the MOD solution. Just like the fundamental solution in Case 2 other than the forward solution, a rise in aggregate demand decreases both inflation and the output gap. Surprisingly, E-stability, one of the most popular solution selection criteria, does not distinguish Case 2 and Case 3, and fails to refine away the MOD solution in the latter case as it turns out to be E-stable.[9] This example illustrates the importance of the forward convergence requirement.

## 7. Conclusion

The forward convergence and no-bubble conditions generalize and modify the process consistency criterion of Flood and Garber (1980) and the restrictions on the expectations of the variables in the future in Blanchard and Kahn (1980), respectively, both of which have often been assumed to hold but not examined for general RE models in the literature.

We demonstrate the importance of forward convergence as a powerful model refinement scheme for linear RE models. Within the class of forward convergent models, we completely characterize the set of economically sensible equilibria to a given model based on the solution refinement scheme. Our analysis indicates that the FCC holds for almost all determinate economic models and, consequently, provides some economic justification for emphasis on a determinate equilibrium. In the case of indeterminacy, moreover, the forward convergence and no-bubble condition detect whether a given model is by itself economically reasonable, and if so, the forward method provides the set of relevant equilibria. We show, through a standard New-Keynesian example, that some indeterminate models may not be forward-convergent and, hence, they have no economically sensible solutions. Therefore, the FCC must be verified, not assumed, and should be both useful and important in practice as a model refinement in such cases.

[9] To compare the E-stability criterion with the FCC (and the NBC as well) on the same basis, we assume that a vector of constants is not included in the perceived law of motion (PLM).

## Appendix

**Proof of Proposition 1** Consider an arbitrary stochastic process $w_t$ and let $V$ be a $n \times k$ matrix with $0 \le k \le n$ where its columns are orthonormal, spanning the support of $w_t$ for all $t$. This implies that $E_t w_{t+1} \in Col(V)$ almost surely as well. Since $w_t$ is a solution to $w_t = FE_t w_{t+1}$, $w_t$ must be in the column space of $F$. Hence, the columns of $V$ can be interpreted as the members of an orthonormal basis for $Col(F)$ without loss of generality. Note that the number of columns of $V$ cannot be greater than that of $F$, i.e., $1 \le k \le m \le n$. Next, we apply the real Schur decomposition theorem to the matrix $F$ to find an $n \times m$ orthonormal matrix $U$ and an $m \times m$ block diagonal matrix $\Psi$ of full rank such that $F = U\Psi U'$. This can be written as

(A1) $\quad FU\Psi^{-1}U' = UU'$.

Note that since $\Psi$ is block-diagonal, so is $\Psi^{-1}$. Since the columns of $V$ are some or all of the columns of $U$, we can partition $U$ into two parts such that $U = [V\ V_0]$ where $V_0$ has the remaining columns of $U$. We also partition $\Psi^{-1}$ into two parts such that $\Psi^{-1} = \begin{bmatrix} \Phi & O \\ O & \Phi_0 \end{bmatrix}$ where $\Phi$ is the $k \times k$ block-diagonal matrix associated with $V$ and $\Phi_0$ is the $(m-k)\times(m-k)$ block-diagonal matrix associated with $V_0$. Since we consider a real-valued stochastic process $w_t$ only, if $\Phi$ contains complex-valued eigenvalues, their conjugate members must also be the eigenvalues of $\Phi$.

Then, for any $V$, $U\Psi^{-1}U'V = [V\ V_0]\Psi^{-1}\begin{bmatrix} I_k \\ 0_{m-k} \end{bmatrix} = [V\ V_0]\begin{bmatrix} \Phi \\ 0_{m-k} \end{bmatrix} = V\Phi$. Now post-multiply $V$ to both sides of equation (A1) to yield $FU\Psi^{-1}U'V = UU'V = V$. Note that

$U\Psi^{-1}U'V = [V\ V_0]\Psi^{-1}\begin{bmatrix} I_k \\ 0_{m-k} \end{bmatrix} = [V\ V_0]\begin{bmatrix} \Phi \\ 0_{m-k} \end{bmatrix} = V\Phi$. Therefore, we have:

(A2) $FV\Phi V' = F\Lambda = VV'$,

where $\Lambda = V\Phi V'$. Post-multiplying $V\Phi^{-1}V'$ to both sides of (A2) yields $FVV' = V\Phi^{-1}V'$.[10] Note that $E_t w_{t+1} = VV'E_t w_{t+1}$ almost surely. Therefore, the model (7) is almost surely identical to $w_t = FVV'E_t w_{t+1} = V\Phi^{-1}V'E_t w_{t+1}$. Pre-multiplying $\Lambda$ to both sides of (7), we have

(A3) $\Lambda w_t = VV'E_t w_{t+1}$.[11]

Finally, we define the vector of rational expectations errors $\eta_{t+1} \equiv w_{t+1} - E_t w_{t+1}$. Since $w_{t+1}$ and $E_t w_{t+1}$ are in the column space of $V$, $\eta_{t+1}$ must also be in $Col(V)$, implying that $\eta_{t+1}$ can be written as $\eta_t = V\xi_{t+1}$. Therefore, from (A3), $VV'E_t w_{t+1} = w_{t+1} - V\xi_{t+1} = \Lambda w_t$, which is equation (8). *Q.E.D.*

**Proof of Proposition 3**

We basically repeat the formal proof given in Cho and Moreno (2011, p. 266) as it is a crucial step for establishing our main result. Consider the model (1) and suppose that the FCC holds. Then the forward solution exists and is given by equation (13) $x_t = \Omega^* x_{t-1} + \Gamma^* z_t$. Since the pair ($\Omega_k, \Gamma_k$) in equation (11) is unique and real-valued, so are the limiting values ($\Omega^*, \Gamma^*$). Since the forward solution is a fundamental solution, it must solve the forward representation of the model (9) as $k$ goes to infinity:

[10] This operation does not lose any information as we can recover (A2) by post-multiplying $V\Phi V'$ to $FVV' = V\Phi^{-1}V'$.

[11] This operation does not lose any information as we can recover (7) by multiplying $F$ to (A3).

(A4) $x_t = \lim_{k\to\infty} M_k E_t x_{t+k} + \Omega^* x_{t-1} + \Gamma^* z_t$.

Therefore, it must be true that $\lim_{k\to\infty} M_k E_t x_{t+k} = 0_{n\times 1}$ when expectations are formed with the forward solution, implying that the forward solution satisfies the NBC. Suppose that the NBC holds for another (fundamental or non-fundamental) solution. Since the solution must solve (A4), (A4) becomes the forward solution, which contradicts the supposition that this solution differs from the forward solution. When the FCC does not hold, $(\Omega_k, \Gamma_k)$ does not converge or is not well-defined if the regularity condition is violated. Consequently, for any other solution, $\lim_{k\to\infty} M_k E_t x_{t+k}$ is not well-defined, implying the violation of the NBC. *Q.E.D.*

**Proof of Proposition 4**

**Assertion 1** Note that the eigenvalues of $\Omega^*$ and the inverses of the eigenvalues of $\mathrm{F}^*$ constitute the generalized eigenvalues of the model. Therefore, if $r(\Omega^*) < 1$ and $r(F^*) \le 1$, then there are exactly $n$ generalized eigenvalues inside the unit circle. Therefore, $\Omega^*$ is $\Omega^{MOD}$ and the model is determinate from Proposition 2.

**Assertion 2** Suppose that $r(\Omega^*) < 1$ and $r(F^*) > 1$. From Proposition 3, the forward solution component of (15) satisfies the NBC and all other fundamental solutions violate the NBC from Proposition 2. Note that $F^*$ must contain at least one root outside the unit circle. Let $m$ be the number of some or all of such unstable roots $(1 \le m \le n)$. Following Proposition 1, one can construct a stationary process $w^*_t = \Lambda E_t w^*_{t+1} + V\xi_t$ where the non-zero eigenvalues of $\Lambda$ are the inverses of the chosen unstable roots of $F^*$. *Q.E.D.*

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