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ABSTRACT

Linear RE models typically possess a multiplicity of solutions. Consider, however, the requirement that the solution coefficients must not be infinitely discontinuous in the model's structural parameters. In particular, we require that the solutions should be continuous in the limit as those parameters, which express quantitatively the extent to which expectations affect endogenous variables, go to zero. The paper shows that under this condition there is, for a very broad class of linear RE models, only a single solution.

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1. Introduction

It is very widely recognized that linear rational expectations (RE) models typically feature a multiplicity of solutions, i.e., processes for endogenous variables that satisfy all the model's equations and the orthogonality conditions for RE. Various "selection criteria" or "solution refinements" have been proposed over the years, by writers including Taylor (1977), Blanchard and Kahn (1980), Whiteman (1983), McCallum (1983), Evans (1986), Evans and Honkapohja (2001), Driskill (2006), Ellison and Pearlman (2011), and Cho and Moreno (2011). None of these proposals, however, has been generally accepted by researchers. The most prominent approach—that of Blanchard-Kahn and Whiteman—is to assume that if there is only a single solution that is dynamically stable then it will prevail; otherwise each stable solution represents a possible outcome. But it too has a number of critics, with various objections being voiced by the other writers listed above plus Bullard (2006), Bullard and Mitra (2002), and Cochrane (2007).[1]

In monetary economics the Blanchard-Kahn-Whiteman "determinacy" approach is by far the most popular, partly due to the enormous influence of Woodford (2003, pp. 77-85, 90-96, 252-261). That the issues generated by solution multiplicities are central to the logical foundations of today's mainstream New-Keynesian approach to monetary policy analysis, and that they remain unsettled, is clearly evidenced by the recent exchange between Cochrane (2009) and McCallum (2009b,c).[2] Obviously, this situation is highly unsatisfactory. The purpose of the present paper, consequently, is to propose a

[1] It should be noted that the current revised version of Cochrane's NBER Working Paper 13409, a version of which has been published as Cochrane (2011), has eliminated most of the discussion that is mentioned critically in McCallum (2009b).

[2] For a brief account, see Appendix D.

criterion or refinement, one that is based on continuity of solution coefficients with respect to structural parameters. The spirit of the undertaking is that the basic objective in economic modelling is not primarily to conform to some particular definition of equilibrium, but to develop models that are plausible, in terms of their predictions about the consequences of alternative economic arrangements and policies.[3] Throughout the present discussion, the analysis will be limited to linear models.

2. Basic Univariate Case

Consider the following univariate model, assumed to be structural:

(1) $$y_t = aE_t y_{t+1} + cy_{t-1} \qquad ac < 0.25.$$

Here we have for simplicity omitted the constant term and exogenous shocks, which are inessential to the argument.[4] The "fundamental" solutions are of the form

(2) $$y_t = \phi y_{t-1}$$

so $E_t y_{t+1} = \phi^2 y_{t-1}$. Then substitution of the latter and (2) into (1) followed by undetermined-coefficient (UC) reasoning indicates that ϕ must satisfy

(3) $$a\phi^2 - \phi + c = 0.$$

Thus the fundamental solutions are given by (2) with the following two values for ϕ:

(4a) $$\phi^{(-)} = \frac{1-\sqrt{1-4ac}}{2a}$$

(4b) $$\phi^{(+)} = \frac{1+\sqrt{1-4ac}}{2a}.$$

The proposed refinement is that ϕ must be continuous in the parameters a and c.

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[3] The latter is indeed a motivation for requiring agent maximization in the usual definition of equilibrium.
[4] With respect to exogenous variables, see Appendix A. Note that the condition ac<0.25 is imposed so as to require real-valued solutions, which we now also assume for the multivariate cases below.

In particular, ϕ must be continuous in 'a' over intervals of values that include a = 0. The rationale is that in this extreme case expectational variables are entirely absent from the system so the solution is unambiguously $y_t = cy_{t-1}$. In addition, small values of 'a' reflect cases in which expectational effects are small, so they should imply solutions with ϕ close to c. Furthermore, continuity of solution parameters is necessary for impulse-response functions to be well behaved when exogenous variables are included in the model.[5]

Clearly, this requirement implies that the solution for model (1) is given by (2) with the limiting value, as $a \to 0$, of $\phi^{(-)} = c$, as in (4a), and not by $\phi^{(+)} = \pm\infty$, i.e., an infinite discontinuity.[6] It is useful to note that (with $a \neq 0$) as the parameter $c \to 0$ we have $\phi^{(-)} \to 0$, whereas $\phi^{(+)} \to 1/a$. Thus the refinement leads to the same solution as the minimum state variable (MSV) solution suggested by McCallum (1983).[7]

From the foregoing we see that the proposed refinement leads to a single solution when we are limiting consideration to fundamental solutions. But suppose we admit solutions of the general "sunspot" form

(5) $$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 \xi_t ,$$

where ξ_t is a stationary stochastic process with the property $E_t \xi_{t+1} = \rho \xi_t$.[8] Then we have

(6) $$E_t y_{t+1} = \phi_1(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 \xi_t) + \phi_2 y_{t-1} + \phi_3 \rho \xi_t$$

and substitution into (1) leads to the following UC conditions:

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[5] Again see Appendix A.
[6] The first of these two limits is obtained by means of l'Hôpital's rule; see Appendix A.
[7] It should perhaps be mentioned that I am using the MSV terminology in the manner of McCallum (1983), not in that of Evans and Honkapohja (2001, p. 194), which does not imply that the MSV solution is unique.
[8] It is the case that almost any RE solution to model (1) can be expressed in this form. See appendix B below.

(7a) $\phi_1 = a\phi_1^2 + a\phi_2 + c$

(7b) $\phi_2 = a\phi_1\phi_2$

(7c) $\phi_3 = a\phi_1\phi_3 + a\rho\phi_3$.

Now the third of these requires that either $\phi_3 = 0$ or $1 = a\phi_1 + a\rho$. If we take the latter as relevant we have the condition that $\phi_1 = (1 - a\rho)/a$, which clearly has an infinite discontinuity at a = 0. Thus we reject that possibility and require $\phi_3 = 0$. Then (7a) and (7b) apply and the latter of these implies that either $\phi_2 = 0$ or $\phi_1 = 1/a$. Since the latter has an infinite discontinuity at a = 0, we conclude that $\phi_2 = 0$ is relevant.[9] But then (7a) becomes the same condition as (3) and we are back in the case, already discussed, with only fundamental solutions and with those corresponding to the expressions (4a) and (4b). Then, as before, only the former gives a solution, which is (4a), that satisfies our continuity requirement; i.e., the proposed refinement rules out all solutions except (2) with ϕ given by $\phi^{(-)}$. Obviously, to be useful this result must extend to richer models. Even in the present context, however, it is interesting that there is only a single solution that satisfies the continuity principle. Also significantly, perhaps, it is the same solution as the one that utilizes a "direction of causality" criterion as developed in McCallum (2009a), as well as the "minimum state variable" solution from McCallum (1983).[10]

One reader has asked why an analyst should be concerned with properties of $\phi^{(-)}$ and $\phi^{(+)}$ prevailing in the vicinity of a = 0 in cases in which it is likely that 'a' is not

[9] The coefficient on y_{t-1} in (5) implies that a tiny change in the expectational parameter a—say, from 0.01 to –0.01—could have an implausibly large effect on the dynamic behavior of y_t.
[10] The latter has been recognized as a notable concept by numerous writers, including Evans (1986), Evans and Honkapohja (2001), Driskill (2006), Ellison and Pearlman (2011), and Cho and Moreno (2011).

close to 0 (e.g., the model has a = 0.95). My response is that what we are concerned with, at this methodological step, is whether (for example) the solution with $\phi^{(-)}$ or the one with $\phi^{(+)}$ is appropriate for models of the specification at hand, which is presumed by the modeler to be relevant for a wide range of parameter values. Thus we ask whether the functions $\phi^{(-)}$ and $\phi^{(+)}$ imply plausible solutions for all values of 'a' and c that are permitted by the model (here, all values such that $ac < 0.25$). For example, how would the two solutions (functions of 'a' and c) behave in the case that a = 0.01? And the case that a = –0.01? Clearly the solution $\phi^{(+)}$ says that ϕ would be huge and positive in one case but huge and negative in the other. It is highly implausible, I suggest, that this small change in model calibration could have such a major effect on the implied economic behavior of variable y_t, especially since the small absolute value of 'a' in both cases indicates that the influence of expectations about the next-period value should be very small—as is, in fact, implied by the solution $\phi^{(-)}$. Methodologically, that is, it is desirable to decide which solution *function* is appropriate for the general model before turning to the specific parameter values relevant to the economy at hand—in this case 'a' and c—for the applied step of forecasting or policy analysis.

3. Multivariate Extension

We now consider richer models of the form

(10) $y_t = A\, E_t y_{t+1} + C\, y_{t-1} + D\, u_t,$

where y_t is a m×1 vector of endogenous variables, A and C are m×m matrices of real numbers, D is m×n, and u_t is a n×1 vector of exogenous variables generated by a dynamically stable process

(11) $u_t = R u_{t-1} + \varepsilon_t,$

with ε_t a white noise vector and R a matrix with all eigenvalues less than 1.0 in modulus. It will not be assumed that A is invertible. In this formulation the endogenous variables in y_t are jump variables whereas their lagged values in y_{t-1} are predetermined, that is, dependent only on lagged values of exogenous or endogenous variables. This specification is useful for various reasons, the main one with respect to the issue at hand being that it is very broad and inclusive. In particular, any model satisfying the formulations of King and Watson (1998) or Klein (2000), can (with the use of auxiliary variables) be written in this form—and the form will accommodate any finite number of lags, expectational leads, and lags of expectational leads.[11] In that context, we consider fundamental solutions to the model (10)-(11), which are of the form

(12) $\quad y_t = \Omega\, y_{t-1} + \Gamma\, u_t.$

in which Ω is required to be real.[12] Then we have that $E_t y_{t+1} = \Omega(\Omega y_{t-1} + \Gamma u_t) + \Gamma R u_t$ and straightforward undetermined-coefficient reasoning shows that Ω and Γ must satisfy

(13) $\quad A\Omega^2 - \Omega + C = O$

(14) $\quad \Gamma = A\Omega\Gamma + A\Gamma R + D.$

For any given Ω, (14) yields a unique Γ generically,[13] but there may be many $m\times m$ real matrices that solve (13) for Ω. Accordingly, the following analysis centers around (13), setting D = O for notational simplicity. For reference below, note that, from (13), each solution will satisfy

(15) $\quad \Omega = (I - A\Omega)^{-1}C\,,$

[11] See McCallum (2007, p. 1379).
[12] A constant term can be defined by the coefficient on an exogenous variable that is a driftless random walk with innovation variance of zero.
[13] Generically, $I - R'\otimes[(I - A\Omega)^{-1}A]$ will be invertible, permitting solution for vec(Γ) using the identity vec(ABC) = $[C'\otimes A]$vec(B) that holds for any conformable A, B, and C.

provided that the inverse exists (a regularity condition that is generally assumed).

In order to accommodate singular A matrices, we write

(16) $$\begin{bmatrix} A & O \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} I & -C \\ I & O \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix},$$

in which the first row reproduces the matrix quadratic (13). Let the 2m×2m matrices on the left and right sides of (16) be denoted $\bar{A}$ and $\bar{C}$, respectively. Then we solve for the (generalized) eigenvalues of the matrix pencil $[\bar{C} - \lambda\bar{A}]$, alternatively termed the (generalized) eigenvalues of $\bar{C}$ with respect to $\bar{A}$ (e.g., Uhlig (1999)). Specifically, the Schur generalized decomposition theorem establishes that there exist unitary matrices Q and Z of order 2m×2m such that $Q\bar{C}Z = T$ and $Q\bar{A}Z = S$ with T and S triangular.[14]

Then (generalized) eigenvalues of the matrix pencil $[\bar{C} - \lambda\bar{A}]$ are defined as t_{ii}/s_{ii}. Some of these eigenvalues may be "infinite," in the sense that some s_{ii} may equal zero. This will be the case, indeed, whenever A and therefore $\bar{A}$ are of less than full rank since then S is also singular. All of the foregoing is true for any ordering of the eigenvalues and associated columns of Z (and rows of Q). For the moment, let us *temporarily* focus on the arrangement that places the t_{ii}/s_{ii} in order of decreasing modulus, which will be referred to as the MOD ordering.[15]

To begin the analysis, premultiply (16) by Q. Since $Q\bar{A} = SH$ and $Q\bar{C} = TH$, where $H \equiv Z^{-1}$, the resulting equation can be written as

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[14] Provided only that there exists some λ for which $\det[\bar{C} - \lambda\bar{A}] \neq 0$. See Klein (2000) or Golub and Van Loan (1996). In what follows, the term eigenvalues will be used to refer also to generalized eigenvalues.

[15] The discussion proceeds as if none of the t_{ii}/s_{ii} equals 1.0 exactly. If one does, the model can be adjusted, by multiplying some relevant coefficient by (e.g.) 0.9999 or by eliminating the variable in favor of its first difference.

(17) $$\begin{bmatrix} S_{11} & O \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} T_{11} & O \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}.$$

The first row of (17) reduces to

(18) $S_{11}(H_{11}\Omega + H_{12})\Omega = T_{11}(H_{11}\Omega + H_{12})$.

Then if H_{11} is invertible the latter can be used to solve for Ω, which is $m \times m$, as

(19) $\Omega = -H_{11}^{-1} H_{12} = -H_{11}^{-1}(-H_{11}Z_{12}Z_{22}^{-1}) = Z_{12}Z_{22}^{-1}$,

where the second equality uses the upper right-hand submatrix of the identity HZ = I, provided that H_{11} is invertible, which we assume without significant loss of generality.[16]

As mentioned above, there are many fundamental solutions Ω to (13). These correspond to the $(2m)!/(m!)^2$ different combinations of the 2m eigenvalues taken m at a time, which result in different groupings of the columns of Z and therefore different compositions of the submatrices Z_{12} and Z_{22}. Here, with the eigenvalues t_{ii}/s_{ii} temporarily arranged in order of decreasing modulus, the diagonal elements of S_{22} will all be non-zero provided that S has at least m non-zero eigenvalues, which we assume to be the case.[17] For any solution under consideration to be dynamically stable, all the eigenvalues of Ω must of course be smaller than 1.0 in modulus. To evaluate them in terms of the ratios t_{ii}/s_{ii}, note that with Ω given by (19), the second row of (17) becomes

(20) $S_{22}(H_{21}\Omega + H_{22})\Omega = T_{22}(H_{21}\Omega + H_{22})$,

or

(21) $S_{22}(H_{22} - H_{21}H_{11}^{-1}H_{12})\Omega = T_{22}(H_{22} - H_{21}H_{11}^{-1}H_{12})$.

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[16] This invertibility condition, also required by King and Watson (1998) and Klein (2000), obtains except for degenerate special cases of (1) that can be solved by simpler methods than considered here. Note that the invertibility of H_{11} implies the invertibility of Z_{22}, given that H and Z are unitary.

[17] It is obvious that $\bar{A}$ has at least m nonzero eigenvalues so, with Q and Z unitary, S must have rank of at least m. This is not sufficient for S to have at least m nonzero eigenvalues, however; hence the assumption.

The latter, by virtue of the lower right-hand submatrix of HZ = I, is equivalent to

(22) $S_{22}Z_{22}^{-1}\Omega = T_{22}Z_{22}^{-1}$.

Therefore we have the result

(23) $\Omega = Z_{22}S_{22}^{-1}T_{22}Z_{22}^{-1}$,

so Ω has the same eigenvalues as $S_{22}^{-1}T_{22}$. The latter is triangular, moreover, so the relevant eigenvalues are (given the MOD ordering) the m smallest of the 2m ratios t_{ii}/s_{ii}. For dynamic stability, the modulus of each of these ratios must then be less than 1. (In many cases, some of the m smallest moduli will equal zero.)

Now we consider alternative fundamental solutions, which will involve departures from the MOD ordering. There are $(2m)!/(m!)^2$ different groupings of system eigenvalues (and associated eigenvectors) that include two groups of m each.[18] It is well known (and is shown in (23)) that m of the system eigenvalues will be the eigenvalues of Ω. The other half of the system eigenvalues are the inverses of the eigenvalues of the matrix F, where $F = (I - A\Omega)^{-1}A$.[19] According to our refinement, we now select the solution for which $\Omega \to C$ as $A \to O$. By continuity of eigenvalues with respect to structural parameters (Horn and Johnson, 1985, pp. 539-540), this is the same solution as the MSV solution for which $\Omega \to O$ as $C \to O$.[20] It can be identified operationally by replacing C by κC in all equations and then letting the scalar κ decrease continuously from 1 to 0. Examination of a plot or table of the eigenvalues for various κ values will indicate which solution—i.e., which Ω—has this property. [This procedure is mentioned

[18] Some of these may yield complex values for solution coefficients; these are of course eliminated.
[19] This result, which is somewhat tedious, is developed in McCallum (2007, pp. 1382-3).
[20] This generalization of the univariate case regarding (4a) and (4b) is implied by the analysis on p. 165 of McCallum (1983), with an extension to singular A matrices provided by noting that they imply eigenvalues of F equal to zero. For details, see Appendix C.

in McCallum (2004) and is illustrated in McCallum (2009a).] Let us use Ω_0 to denote the Ω matrix for this particular solution, with the related F matrix being $F_0 = (I - A\Omega_0)^{-1}A$. Furthermore, we observe crucially that any other fundamental solution will have one or more of the eigenvalues of Ω approaching $\pm\infty$ as $A \rightarrow O$. See Appendix C. Thus we have developed the multivariate extension of the argument for fundamental solutions.

It is important to note that the Ω_0, F_0 solution does not necessarily coincide with the MOD solution. In most cases these two solutions will coincide, but in a few they will differ. This fact, which is mentioned by Uhlig (1999, p. 46), is illustrated in McCallum (2004) and (2009a).[21]

Continuing the analysis, sunspot solutions can be considered for model (10)(11) by looking for solutions of the form

(24) $$y_t = \Omega y_{t-1} + \Phi_1 y_{t-2} + \Phi_2 \xi_t,$$

where I have used D = O to avoid clutter but have added y_{t-2} and a m×1 sunspot vector ξ_t that has the property $E_t\xi_{t+1} = G\xi_t$ for all t. Then in this case we have

(25) $$E_t y_{t+1} = \Omega(\Omega y_{t-1} + \Phi_1 y_{t-2} + \Phi_2 \xi_t) + \Phi_1 y_{t-1} + \Phi_2 G\xi_t$$

and substitution into (10) gives

(26) $$\Omega y_{t-1} + \Phi_1 y_{t-2} + \Phi_2 \xi_t = A[\,\Omega(\Omega y_{t-1} + \Phi_1 y_{t-2} + \Phi_2 \xi_t) + \Phi_1 y_{t-1} + \Phi_2 G\xi_t\,] + Cy_{t-1}.$$

Consequently, the implied undetermined coefficient equations are

(27a) $\Omega = A\Omega^2 + A\Phi_1 + C$

(27b) $(I - A\Omega)\Phi_1 = O$

(27c) $(I - A\Omega)\Phi_2 = A\Phi_2 G$.

[21] Uhlig's objection that "…uniqueness is lost once two or more such paths cross each other …" is handled by presuming that the slopes of the paths are continuous.

Now the argument is an extension of that for the univariate case of Section 3. Equations (27b) and (27c) imply that if $(I - A\Omega)$ is nonsingular, then $\Phi_1 = O$, $\Phi_2 = O$, and we again have the solution with $\Omega = \Omega_0$ with no sunspot terms. If on the other hand $(I - A\Omega)$ is singular, Φ_2 is not determined and (27) admits an infinity of sunspot solutions. But in this case (27b) implies $\det(\Phi_1) = \det(A)\det(\Omega)\det(\Phi_1)$ or $\det(\Omega) = 1/\det(A)$, so as $A \to O$ we have $\det(\Omega) \to \infty$ which implies a discontinuity in the solution for y_t. Therefore, the sunspot solutions do not satisfy our refinement criterion.

4. Conclusion

We conclude with a very brief description of the paper's argument. Linear RE models typically have more than one solution and fairly often possess more than one dynamically stable solution. Consider, however, the requirement that the solution coefficients should be continuous in the model's structural parameters. In particular, we require that the solution coefficients should be continuous in the limit as certain parameters, which express the extent to which expectations affect endogenous variables, go to zero. (If expectations enter the structural equations very weakly, they should not have much effect on the solution expressions.) The paper shows that, for a very broad class of linear RE models,[22] this requirement is satisfied by only a single solution.

[22] The class is one that permits any finite (i) number of endogenous variables, (ii) lag length, (iii) expectational lead length, and (iv) lag length for expectational leads.

Appendix A

The purpose here is to illustrate that it is sufficient for this paper's argument to examine cases in which exogenous variables are not explicitly included. For the simplest case, suppose that we extend equation (1) to include an exogenous shock z_t that is generated by a stable first-order autoregressive process with coefficient ρ. Then we have

(A-1) $$y_t = aE_t y_{t+1} + cy_{t-1} + dz_t$$

(A-2) $$z_t = \rho z_{t-1} + \varepsilon_t$$

where ε_t is white noise. Then fundamental solutions are of the form

(A-3) $$y_t = \phi_1 y_{t-1} + \phi_2 z_t.$$

Consequently, $E_t y_{t+1} = \phi_1^2 y_{t-1} + (\phi_1 + \rho)\phi_2 z_t$. Using the latter and (A-3) in (A-1) we have

(A-4) $$\phi_1 y_{t-1} + \phi_2 z_t = a[\phi_1^2 y_{t-1} + \phi_2(\phi_1 + \rho) z_t] + cy_{t-1} + dz_t$$

so equating coefficients yields

(A-5) $$\phi_1 = a\phi_1^2 + c$$

(A-6) $$\phi_2 = a\phi_2(\phi_1 + \rho) + d.$$

As in Section 2, therefore, from (A-5) we have

(A-7a) $$\phi_1^{(-)} = \frac{1 - \sqrt{1 - 4ac}}{2a}$$

(A-7b) $$\phi_1^{(+)} = \frac{1 + \sqrt{1 - 4ac}}{2a}.$$

Thus as $a \to 0$ we have $\phi_1^{(-)} \to c$ and $\phi_1^{(+)} \to \pm\infty$ as with equations (4a) and (4b).[23]

Continuing, for $\phi_2^{(-)}$ we have $\phi_2^{(-)}[1 - a\phi_1^{(-)} - a\rho] = d$ or $\phi_2^{(-)} = d / [1 - a\phi_1^{(-)} - a\rho]$

[23] In the case of (A-7a), both numerator and denominator approach zero as a approaches zero, but their derivatives approach 2c and 2 respectively, so L'Hospital's rule gives a limit of c.

so that $\phi_2^{(-)} = \dfrac{d}{1-\dfrac{1-\sqrt{1-4ac}}{2}-a\rho}$ and

(A-8) $$\lim_{a\to 0}\phi_2^{(-)} = \frac{d}{1-\frac{1-1}{2}-0} = d\,.$$

By contrast, $$\lim_{a\to 0}\phi_2^{(+)} = \frac{d}{1-\frac{1+1}{2}-0} = \pm\infty\,.$$

Next, the non-fundamental solutions—permitting sunspots—are of form

(A-9) $y_t = \phi_1 y_{t-1} + \phi_2 z_t + \phi_3 z_{t-1} + \phi_4 \xi_t + \phi_5 y_{t-2}$

for any ξ_t with $E_{t-1}\xi_t = g\xi_{t-1}$. Then $E_t y_{t+1} = \phi_1(\phi_1 y_{t-1} + \phi_2 z_t + \phi_3 z_{t-1} + \phi_4 \xi_t + \phi_5 y_{t-2})$ + $\phi_2\rho z_t$ + $\phi_3 z_t$ +$\phi_4\xi_t$ whereby substitution into (A-1) plus equating of coefficients yields

(A-10) $\phi_1 = a\phi_1^2$

(A-11) $\phi_2 = d + a\phi_1\phi_2 + a\rho\phi_2 + a\phi_1\phi_3$

(A-12) $\phi_3 = a\phi_1\phi_3$

(A-13) $\phi_4 = a\phi_1\phi_4$

(A-14) $\phi_5 = a\phi_1\phi_5$

Thus we have either $\phi_1 = \phi_3 = \phi_4 = \phi_5 = 0$ in which case $\phi_2 = d/(1-a\rho)$ and yields the well-behaved MSV solution (2) with (4a); or else we have $\phi_1 = 1/a$ in which case $\phi_2 = -(d+\phi_3)/a\rho$ and thus for any arbitrary ϕ_3 (except the zero-measure value of $-d$) there is an infinite discontinuity in the limit as $a \to 0$. Accordingly, permitting sunspot solutions does not alter the conclusion that only the MSV solution avoids the implication of an infinite discontinuity in the implied impulse response function.

Appendix B

The object here is to demonstrate that the form of sunspot solution considered in equation (5) is highly general for the model (1), which we repeat as

(B-1) $y_t = aE_t y_{t+1} + cy_{t-1}$.

We now consider solutions of the form

(B-2) $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \psi_t$

where ψ_t is any remaining term representing a covariance-stationary stochastic process. Then $E_t y_{t+1} = \phi_1(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \psi_t) + \phi_2 y_{t-1} + E_t \psi_{t+1}$ and the undetermined-coefficient requirement is that the following must hold for all realizations:

(B-3) $\phi_1 y_{t-1} + \phi_2 y_{t-2} + \psi_t = a[\phi_1(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \psi_t) + \phi_2 y_{t-1} + E_t \psi_{t+1}] + cy_{t-1}$.

Thus we have UC conditions (7a), (7b), and

(B-4) $\psi_t = a\phi_1 \psi_t + aE_t \psi_{t+1}$.

The latter requires, however, that $E_t \psi_{t+1} = \frac{1 - a\phi_1}{a} \psi_t$, which establishes the claim in Section 2 of the text.

In the more general case of model (10) (11), a similar argument leads to the following counterpart of (B-4):

(B-5) $(I - A\Omega)\Psi_t = AE_t \Psi_{t+1}$.

Therefore, provided that $(I - A\Omega)^{-1}$ exists—a regularity condition that is presumed in virtually all multivariate analyses—we have the condition $\Psi_t = GE_t \Psi_{t+1}$ as assumed in equations (25)-(27).

Appendix C

The object here is to show that the solution (12) to model (10)(11) for which $\Omega \to C$ as $A \to O$ is the same solution as the one for which $\Omega \to O$ as $C \to O$. Let us begin with the case in which A is nonsingular. Then we can express the crucial matrix quadratic (13) as

$$\text{(B-1)} \qquad \begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}C \\ I & O \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}.$$

Let M denote the square matrix of order 2m × 2m. Clearly its eigenvalues are the numbers denoted λ that satisfy

$$\text{(B-2)} \qquad \det(M - \lambda I) = \det \begin{bmatrix} A^{-1} - \lambda I & -A^{-1}C \\ I & O - \lambda I \end{bmatrix} = 0.$$

An identity for partitioned matrices reported by Johnston (1972, eqn. 4-37, p. 95) is as follows. If the matrix $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ with B_{11} nonsingular, then

$$\text{(B-3)} \qquad \det(B) = \det(B_{11}) \ \det[B_{22} - B_{21}B_{11}^{-1}B_{12}].$$

The latter then implies that

$$\text{(B-4)} \qquad \det(M - \lambda I) = \det(A^{-1} - \lambda I)\ \det[-\lambda I + (A^{-1} - \lambda I)^{-1} A^{-1} C] = 0.$$

Thus we see, from the latter, that half of the eigenvalues of M are the eigenvalues of A^{-1}, while the other half, the eigenvalues of Ω, depend upon both A and C (see McCallum 2007, pp. 1382-3). Then by further inspection of (B-4) we see that when C = O, the second half of the λs are all equal to 0. Thus the single solution given by the particular arrangement, for which all eigenvalues of Ω approach zeros as $C \to O$, simultaneously has the other half of the eigenvalues of M approaching the eigenvalues of A^{-1}.

. Now consider the same arrangement but with C held fixed and consider the implication of $A \rightarrow O$. Then the eigenvalues that approached zeros before now approach the eigenvalues of C while the eigenvalues that approached those of A^{-1} before now approach $\pm\infty$. This establishes the result at issue for the case in which A is nonsingular. For any other solution—i.e., any other arrangement—as $A \rightarrow O$ we would have a different Ω and one or more of its eigenvalues would approach $\pm\infty$.

When instead A is singular, similar results obtain but with the matrix A being replaced in the argument by the matrix $F = (I - A\Omega)^{-1}A$.[24] The system eigenvalues then include those of Ω and the inverses of the eigenvalues of F, instead of those of Ω and A^{-1}. As $C \rightarrow O$, we have the m eigenvalues of Ω approaching zeros and the other m eigenvalues approaching the inverses of the eigenvalues of F. Then with the same arrangement, i.e., the solution such that the eigenvalues of Ω approach zeros as $C \rightarrow O$, we find that as $A \rightarrow O$ (with C fixed) also $F \rightarrow O$ and the inverses of the eigenvalues of F each approach $\pm\infty$.

[24] Again, see McCallum (2007, pp. 1381-1383) for this result.

Appendix D

In recent monetary policy analysis, it has been common practice to view models as possessing *determinacy* if they feature a single RE solution that is dynamically stable. Cochrane (2007) has usefully emphasized that this single-stable-solution (SSS) condition is not sufficient as a criterion of determinacy, however, because in typical New Keynesian models, if the Taylor Principle is satisfied, there exists a dynamically explosive solution for the inflation rate that is not ruled out by any transversality condition and accordingly can be eliminated only by an arbitrary dictum. McCallum (2009b) agrees with this specific proposition, but shows that in these models it is typically the case that the explosive solutions in question are not least-squares learnable in the sense of Evans and Honkapohja (2001). Further, he argues that such learnability should be considered a necessary condition for a solution to be regarded as a model's prediction of the depicted economy's behavior since it amounts to a *feasibility* condition that pertains to quantitative information available to individual agents.[25] Consequently, he argues that, despite Cochrane's important point, the solution typically utilized in recent policy analysis is in many (perhaps not all) cases the appropriate one. Cochrane's (2009) response contends that there are three weaknesses in McCallum's argument. McCallum's (2009c) rejoinder claims that in all three cases Cochrane's argument is analytically incorrect or inapplicable, as follows. First, the presence of unobserved exogenous shocks does not, in contrast to Cochrane's presumption, overturn learnability conclusions.[26] Second, Cochrane's argument about "hyperinflationary threats" is not consistent with the analytical setting in which the argument is normally conducted,

[25] See the discussion in McCallum (2012).
[26] This is established in McCallum (2009b) by drawing on results of Evans and Honkapohja (1998).

namely, one in which the central bank is following a specified policy rule for an interest-rate instrument. Third, the point—that a particular structural parameter, concerning the central bank's policy behavior, is not identifiable by an econometrician studying the economy-plus-policy process—is not relevant to the learning process for the private-sector agents in the model. Their learning concerns forecasting of inflation and output in the model economy from a reduced form perspective; the identification of a structural parameter by these agents is not necessary for this step.

In his revised WP13409 and in (2011), Cochrane emphasizes a distinct argument to the effect that the reasoning utilized by Clarida, Gali, and Gertler (2000), among others, who contend that empirical estimates show that the Taylor Principle was not satisfied in the United States during the "Great Inflation" period of the 1970s, is invalid because the crucial policy parameter is not identified. I agree with this significant point as applied to the particular studies discussed by Cochrane, but withhold judgment on the universality of a general conclusion regarding non-identification.

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