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ABSTRACT

The long-run risks (LRR) asset pricing model emphasizes the role of low-frequency movements in expected growth and economic uncertainty, along with investor preferences for early resolution of uncertainty, as an important economic-channel that determines asset prices. In this paper, we estimate the LRR model. To accomplish this we develop a method that allows us to estimate models with recursive preferences, latent state variables, and time-aggregated data. Time-aggregation makes the decision interval of the agent an important parameter to estimate. We find that time-aggregation can significantly affect parameter estimates and statistical inference. Imposing the pricing restrictions and explicitly accounting for time-aggregation, we show that the estimated LRR model can account for the joint dynamics of aggregate consumption, asset cash flows and prices, including the equity premia, risk-free rate and volatility puzzles.

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1 Introduction

The long-run risks (LRR) model developed by Bansal and Yaron (2004) has received significant attention in the macroeconomics and finance literatures. The model captures the intuition that risks embodied in low-frequency movements in the expected growth and conditional volatility of consumption are important for understanding asset prices. In this paper, we use macro and financial data to estimate and empirically evaluate the LRR model. Estimation of the model has to confront three challenges. First, the return on the aggregate consumption asset, a key input of the LRR model based on Epstein and Zin (1989) preferences, is not observable. A second challenge is in extracting low-frequency expected growth and volatility movements in the observed consumption data. A third challenge is dealing with time-aggregation that emanates from a potential mismatch between the decision interval of the agent and the sampling frequency of the data, which could distort inference and parameter estimates.

We develop an estimation method that addresses the aforementioned challenges. To make estimation of the model feasible, we exploit consumption dynamics and the model pricing restrictions to derive the unobservable return on the consumption claim in terms of the state variables and underlying parameters. We show how the decision interval of the agent, a parameter we estimate (modeled in number of days), influences the dynamics of time-aggregated data and the model's implications for asset prices. We extract the latent state variables – the long-run growth component and conditional volatility of consumption growth – from the observed price-dividend ratio and the risk-free rate by imposing the model-implied cross-equation restrictions. Incorporating all these pieces together, we estimate the model using a GMM framework along the lines of Hansen (1982) by exploiting a set of moment restrictions of the joint dynamics of time-aggregated consumption, dividends and asset prices.

Overall, our empirical findings provide considerable support for the LRR model. Our estimation results, which utilize the longest available sample of annual data from 1930 till 2009, suggest that: (i) investors have a preference for early resolution of uncertainty, (ii) shocks to the expected growth component of consumption have a long-run effect that persists beyond typical business cycle frequencies, (iii) although variation in consumption volatility is relatively small, the effect of volatility shocks is long-lasting, (iv) agents' decision interval corresponds to a period of about 33 days, and accounting for temporal aggregation has important quantitative effects on estimation (v) the model is not rejected by the overidentifying restrictions and can account for the observed risk premia and volatility of equity returns, the risk-free rate dynamics and other stylized features of macro and asset market data.

The estimated values of risk aversion and intertemporal elasticity of substitution (IES) in the benchmark LRR model are 7.4 and 2.1 respectively. Both estimates have relatively tight standard errors – 1.6 for risk aversion and 0.8 for the IES. We find that the long-run growth and volatility components of consumption are highly persistent, with implied annual autocorrelations of 0.80 and 0.98, respectively.[1] These estimates underscore the long-run nature of expected growth and volatility shocks that manifest into high equity premia and high volatility of asset prices. The estimated model implies a market risk premium of 6.4% and a 19.6% volatility of stock market returns, and generates a low risk-free rate of about 1.1%. The estimated number of decision periods within a year is 11 which translates to a decision interval of approximately 33 days. Importantly, the benchmark LRR specification is not rejected by the overidentifying moments of the joint dynamics of consumption, market dividends and returns – the p-value associated with the J-test statistic is 13%.

We also examine a specification of the LRR model in which the stochastic volatility component is shut down. Many of the structural parameters are similar to those estimated under the benchmark LRR model although with larger standard standard errors. Moreover, the model is formally rejected by the overidentifying restrictions. The larger standard errors and model rejection are largely driven by predictability moments such as the correlations between future returns (consumption growth) and current price-dividend ratio. Such moments naturally require time variation in fundamentals which in the model emanate from stochastic volatility. In all our analysis shows that stochastic volatility in consumption and dividend growth is important for confronting the model with asset pricing data.

A prominent feature of our analysis is time-aggregation. To analyze the effect of time-aggregation, we also estimate an annual version of the LRR model, that assumes that the decision interval of the agent and the data sampling frequency are both annual and, therefore, ignores restrictions of temporal aggregation. We find this model specification to be strongly rejected in the data. Similar to the benchmark LRR case, the estimates of the the annual model imply high persistence in the expected consumption growth. However, the contribution of long-run risks to the volatility of consumption growth in the two specifications is very different – it is much higher in the benchmark model than in the annual specification.

[1] Annual persistence is computed by raising the estimate of monthly autocorrelation to the 12-th power.

These differences are driven by time-aggregation effects. In the annual specification, the entire shock to annual consumption growth is identified as a short-run risk, while under the null of the benchmark model, a portion of this shock comes from long-run risk fluctuations. Thus, the annual model is misspecified, which leads to distortions in parameter estimates and, in particular, a much larger estimate of risk aversion of about 19. Using simulations, we further document that when time-aggregation of monthly dynamics is ignored, the model is overly rejected, the risk aversion estimate rises, and the contribution of long-run growth risks diminishes, all of which is consistent with our empirical findings. Our evidence suggests that when the restrictions of time-aggregation are not imposed in the estimation, a sizable portion of the low-frequency growth shock tends to be attributed to the short-run shock, which lowers the role of long-run risks and makes it hard for the model to match the volatility of asset returns and prices. Overall, our evidence suggests that accounting for temporal aggregation in estimating the model and measuring the contribution of different risk sources is extremely important, particularly in the presence of low-frequency fluctuations in consumption.

In addition to time-series dynamics, we evaluate the cross-sectional implications of our benchmark LRR model for size and book-to-market sorted portfolios. We show that assets with large mean returns, such as value and small market capitalization, are more sensitive to long-run. Similar to the implications for the market portfolio, we find that low-frequency growth risks are the key source of risk premia in the cross section. Importantly, we show that the LRR model is also able to replicate the failure of the CAPM — our benchmark LRR specification generates low market betas and high CAPM alphas of the value-minus-growth and small-minus-large strategies, of the same magnitudes as in the data.

Earlier work by Epstein and Zin (1989) relies on the GMM of Hansen and Singleton (1982) to estimate a recursive preference based model. However, they replace the return on the consumption asset with the value-weighted market return. Our approach, as discussed above, allows us to infer the dynamics of the wealth return from the observed data and obviates the need to substitute it with the stock market return. More recently, a series of papers explore the ability of long-run growth risks to account for asset market data. Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2008) show that long-run risks in cash flows are important in understanding cross-sectional variation in risk premia. Bekaert, Engstrom, and Xing (2005), Bansal, Gallant, and Tauchen (2007), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2009), Lettau and Ludvigson (2005), Parker and Julliard (2005), Jagannathan and Wang (2010), and Constantinides and Gosh (2008) exploit features of the recursive preferences and/or of long-run risks to account for various features of asset returns. Distinct from these papers, we estimate and evaluate the LRR model in the GMM framework while imposing the model restrictions on the joint dynamics of consumption, dividends, and prices that appropriately account for temporal aggregation.

The paper continues as follows. Section 2 presents the model and its testable restrictions. Section 3 provides details of our estimation methodology. Section 4 describes the data. We report and discuss results of our empirical analysis in Section 5. Section 6 provides concluding remarks.

2 Model

In this section we specify the long-run risks model based on Bansal and Yaron (2004). The underlying environment is one with complete markets and a representative agent that has Epstein and Zin (1989) type preferences, which allow for a separation of risk aversion and the elasticity of intertemporal substitution. Specifically, the agent maximizes her life-time utility, which is defined recursively as,

$$V_t = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\Big(E_t\big[V_{t+1}^{1-\gamma}\big]\Big)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \tag{1}$$

where C_t is consumption at time t, $0 < \delta < 1$ reflects the agent's time preferences, γ is the coefficient of risk aversion, $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, and ψ is the elasticity of intertemporal substitution (IES). Utility maximization is subject to the budget constraint,

$$W_{t+1} = (W_t - C_t)R_{c,t+1}\,, \tag{2}$$

where W_t is the wealth of the agent, and $R_{c,t}$ is the return on all invested wealth.

Consumption growth has the following dynamics:

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\ x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\ \sigma_{t+1}^2 &= \sigma_0^2 + \nu(\sigma_t^2 - \sigma_0^2) + \sigma_w w_{t+1}\,, \end{aligned} \tag{3}$$

where Δc_{t+1} is the growth rate of log consumption, and the three shocks, η, e, and w are assumed to be $i.i.d$ Normal and uncorrelated. The conditional expectation of consumption growth is given by $\mu_c + x_t$, where x_t is a small but persistent component that captures long-run risks in consumption growth. The parameter ρ determines the persistence in the conditional mean of consumption growth. For parsimony, as in Bansal and Yaron (2004), we have a common time-varying volatility in consumption, which, as shown in their paper, leads to time-varying risk premia. The unconditional variance of consumption is σ_0^2 and ν governs the persistence of the volatility process.

2.1 The Long-Run Risks Model's IMRS

For these preferences, the log of the IMRS, $m_{t+1} = \log(M_{t+1})$, is

$$m_{t+1} \;=\; \theta \log \delta \;-\; \frac{\theta}{\psi}\Delta c_{t+1} \;+\; (\theta - 1) r_{c,t+1}\,, \tag{4}$$

where $r_{c,t+1}$ is the continuous return on the consumption asset, which is endogenous to the model. Thus, in order to characterize the intertemporal marginal rate of substitution, one needs to solve for the unobservable return on the consumption claim. To solve for $r_{c,t+1}$, we use the dynamics of the consumption growth and the log-linear approximation of the continuous return, namely,

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} + \Delta c_{t+1} - z_t\,, \tag{5}$$

where $z_t = \log(P_t/C_t)$ is the log price-consumption ratio (i.e., the valuation ratio corresponding to a claim that pays aggregate consumption), and κ's are constants of log-linearization,

$$\kappa_1 \;=\; \frac{\exp(\bar{z})}{1 + \exp(\bar{z})} \tag{6}$$

$$\kappa_0 \;=\; \log(1 + \exp(\bar{z})) - \kappa_1 \bar{z}\,, \tag{7}$$

where $\bar{z}$ denotes the mean of the log price-consumption ratio.

To derive the time series for $r_{c,t+1}$, we require a solution for log price-consumption ratio, which we conjecture follows,

$$z_t = A_0 + A_1 x_t + A_2 \sigma_t^2\,. \tag{8}$$

The solution coefficients A's depend on all the preference parameters and the parameters that govern the dynamics of consumption growth. For notational ease, let $z_t = \mathbf{A}'Y_t$, where $Y_t' = [1 \quad x_t \quad \sigma_t^2]$ is the vector of state variables, and $\mathbf{A}' = [A_0 \quad A_1 \quad A_2]$, which is given by

$$\mathbf{A}' = \left[A_0 \qquad \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho} \qquad -\frac{(\gamma-1)(1-\frac{1}{\psi})}{2\ (1-\kappa_1\nu)} \left[1 + \left(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho}\right)^2 \right] \right] \tag{9}$$

be the corresponding vector of price-consumption elasticities.[2] As discussed in Bansal and Yaron (2004), the elasticities of the price-consumption ratio with respect to the expected growth component, x_t, and volatility, σ_t, depend on the preference configuration. In particular, for the elasticity A_1 to be positive, the IES parameter has to be greater than one. Moreover, for the price-consumption ratio to exhibit a negative response to an increase in economic uncertainty, the IES again has to be larger than one, given that risk aversion is greater than one.

Note that the derived solutions depend on the approximating constants, κ_0 and κ_1, which, in turn, depend on the endogenous mean of the price-consumption ratio, $\bar{z}$. In order to solve for $\bar{z}$, we first substitute expressions for κ's (equations (6) and (7)) into the expressions for A's and solve for the mean of the price-consumption ratio. Specifically, $\bar{z}$ can be found numerically by solving a fixed-point problem,

$$\bar{z} = \mathbf{A}(\bar{z})'\bar{Y}\,, \tag{10}$$

where the dependence of the A's on $\bar{z}$ is given above, and $\bar{Y}$ is the mean of the state vector Y. This is quite easy to implement in practice.

Given the solution for z_t, the IMRS can be stated in terms of the state variables and innovations,

$$m_{t+1} = \mathbf{\Gamma}'Y_t - \mathbf{\Lambda}'\zeta_{t+1}\,, \tag{11}$$

where the three sources of risks are

$$\zeta_{t+1}' = \left[\sigma_t\eta_{t+1} \quad \sigma_t e_{t+1} \quad \sigma_w w_{t+1} \right]\,, \tag{12}$$

and the three dimensional vectors $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ are given by,

$$\mathbf{\Gamma}' = \left[\Gamma_0 \qquad -\frac{1}{\psi} \qquad -(\gamma-1)(\gamma-\tfrac{1}{\psi})\tfrac{1}{2}\left[1+\left(\tfrac{\kappa_1\varphi_e}{1-\kappa_1\rho}\right)^2\right] \right]\,, \tag{13}$$

[2]The expressions for A_0 and Γ_0 in equation (13) below, as well as the derivations of all other expressions, are given in Appendix A.1.

$$\mathbf{\Lambda}' = \begin{bmatrix} \gamma & (\gamma - \frac{1}{\psi})\frac{\kappa_1 \varphi_e}{1-\kappa_1 \rho} & -(\gamma-1)(\gamma - \frac{1}{\psi})\frac{\kappa_1}{2\,(1-\kappa_1 \nu)}\left[1 + (\frac{\kappa_1 \varphi_e}{1-\kappa_1 \rho})^2\right] \end{bmatrix}. \tag{14}$$

Note that the stochastic discount factor in equation (11) is exact up to an approximation error emanating from the linearization around the theoretical value of average price-consumption ratio. We find that this approximation error is quite small and does not materially affect our empirical results that follow. Appendix A.5 provides a detailed discussion of the magnitude of the approximation error and a comparison of the above log-linear solution with a solution based on numerical methods.

Other assets can easily be priced using the IMRS given in equation (11). We assume that the dividend dynamics for any other asset j follow

$$\Delta d_{j,t+1} = \mu_j + \phi_j x_t + \varphi_j \sigma_t u_{j,t+1} \tag{15}$$

where ϕ_j and φ_j determine asset j's exposure to the long-run and volatility risks, respectively. Exposure to short-run consumption risks is determined by the correlation between dividend and consumption innovations, $u_{j,t+1}$ and η_{t+1}, which we denote by ϱ_j. We let Δd_{t+1} denote the dividend growth rate of the aggregate market portfolio, and reserve d-subscript for various quantities of the stock market index. Specifically, we use μ_d, ϕ_d, and ϱ_d for the aggregate market dividend and let $z_{d,t}$ and $r_{d,t+1}$ denote the price-dividend ratio and the return on the aggregate market portfolio.

The first-order condition yields the following asset pricing Euler condition,

$$E_t \left[\exp\left(m_{t+1} + r_{j,t+1}\right)\right] = 1 \ , \tag{16}$$

where $r_{j,t+1}$ is the log of the gross return on asset j. Similar to the claim to consumption, the price-dividend ratio for any asset j, $z_{j,t} = \mathbf{A_j'} Y_t$, with the solutions given in Appendix A.2. Furthermore, given the expression for the IMRS, it follows that the risk premium on asset j is,

$$E_t[r_{j,t+1} - r_{f,t} + 0.5\sigma^2_{t,r_j}] = \beta_{\eta,j}\lambda_\eta \sigma_t^2 + \beta_{e,j}\lambda_e \sigma_t^2 + \beta_{w,j}\lambda_w \sigma_w^2, \tag{17}$$

where $\beta_{i,j}$ is the return beta for asset j with respect to the i^{th} risk source where $i = \{\eta, e, w\}$, and λ_i is the corresponding entry of the vector of market prices of risks, $\mathbf{\Lambda}$. Under the structure of the model, the return β's and market price of risks λ's will be functions of the preference parameters and the underlying parameters of consumption and dividend dynamics, details of which are given in Appendix A.2. Finally, it is easy to verify that the risk free rate can be represented as,

$$r_{f,t} = F_0 + F_1 x_t + F_2 \sigma_t^2 = \mathbf{F}' Y_t \tag{18}$$

where the loading coefficients are given in Appendix A.3.

Intertemporal elasticity of substitution is a critical parameter in the LRR model. Work by Giovannini and Weil (1989), Tallarini (2000), Hansen, Heaton, and Li (2008), and Hansen and Sargent (2006) considers the special case where the IES parameter is one. Our estimation methodology nests this special case in a continuous fashion (details are given in Appendix A.4). Namely, the IMRS components as given in equation (11) adjust in a continuous way as one takes the limit of the IES parameter at one.[3] That is,

$$\lim_{\psi \to 1} \kappa_1 = \delta \quad \lim_{\psi \to 1} \Gamma' = \Gamma'(\psi = 1, \kappa_1 = \delta) \quad \lim_{\psi \to 1} \Lambda' = \Lambda'(\psi = 1, \kappa_1 = \delta)\,. \tag{19}$$

The discussion above highlights the fact that the generalized pricing kernel (11) does not confine an econometrician to a prespecified value of the IES. That is, in estimation the IES is a free parameter.

3 Estimation Method

The key point of this paper is to evaluate and test the LRR model's ability to jointly match consumption, dividend, and asset price dynamics. Specifically, we are interested in the estimates of the model's parameters, $\Theta = \{\gamma, \psi, \delta, \mu_c, \rho, \varphi_e, \sigma_0^2, \nu, \sigma_w, \mu_d, \phi_d, \varphi_d, \varrho_d\}$, as well as the *estimated* decision interval of the agent – an issue we discuss further below. Economically, we are interested in knowing whether at these estimates the model can account for aggregate macro and asset price data. This would provide direct evidence about the importance and magnitude of long-run growth and volatility risks in consumption and dividends, and the magnitude of preference parameters (IES and risk aversion). Our empirical results, as discussed below, shed light on the broader questions regarding the role of long-run versus shorter horizon risks (e.g., business cycle risks), the duration of long-run shocks, and the interaction of these dynamics with preferences in understanding risks that drive financial markets.

[3] Evaluating the pricing kernel (11) under the above restrictions gives exactly the same solution as in Giovannini and Weil (1989), Tallarini (2000), and Hansen, Heaton, and Li (2008).

Our estimation strategy is a moment-based, GMM approach. We provide analytical expressions for moments of consumption and dividend dynamics, asset prices, and joint dynamics between consumption and asset prices. More specifically, we recover the state-variables x_t and σ_t^2 from the observed data, then provide analytical expressions for the conditional moments in terms of the state-variables, and then derive unconditional moments purely in terms of the parameters of interest, Θ.

Since time-aggregation is of considerable importance in the LRR model, the aforementioned estimation procedure also includes estimation of the decision interval. Our estimation therefore searches *jointly* for the best parameter Θ and decision frequency that fits the data. The longest and best quality (in the sense of measurement error) observed data is annual. However, it is natural to assume the agent's decision interval is shorter. We account for this potential time-aggregation and are still able to provide analytical moments in terms of the parameters of interest, Θ. Earlier papers that account for time-aggregation in estimation in an asset-pricing context include Hansen and Sargent (1983) and Heaton (1995). We show that time-aggregation has important effects on model estimation and inference. In the presence of time-aggregation, the shocks in IMRS (equation(11)) cannot be recovered and hence the standard Euler Equation-based estimation approach, as in Hansen and Singleton (1982), cannot be used.[4] Our moments-based approach allows for estimation even when the shocks and the IMRS are not available to the econometrician.

3.1 Recovering the state variables

To recover the state variables x_t and σ_t^2, we use the fact that the price dividend ratio and the risk free rate are affine in these state variables. That is, equations (8) and (18) constitute, for each date t, the following system

$$\mathcal{S}_t(\Theta) = \begin{bmatrix} z_{d,t} \\ r_{f,t} \end{bmatrix} - \begin{bmatrix} \mathbf{A_d}(\boldsymbol{\Theta})' \\ \mathbf{F}(\boldsymbol{\Theta})' \end{bmatrix} \begin{bmatrix} Y_t \end{bmatrix}. \tag{20}$$

[4]Using equation (11), the log of the h-period time-aggregated IMRS follows:

$$m_{t+h,h} \equiv \sum_{j=1}^{h} m_{t+j} = \tilde{\mathbf{\Gamma}}' Y_t - \sum_{j=1}^{h} \left[\lambda_\eta \sigma_{t+j-1} \eta_{t+j} + \lambda_e \sigma_{t+j-1} e_{t+j} + \lambda_w \sigma_w w_{t+j} \right]$$

It is easily shown that long-run and volatility shocks, e_t and w_t, can be extracted from the available high-frequency financial data. However, short-run consumption innovations, η_t, cannot be recovered unless consumption data are observed at a fine (monthly) frequency.

Given the observed financial variables, the price-dividend ratio and the risk-free rate, it is possible to recover x_t and σ_t. In practice we solve, for each date t and given Θ, the pair of state variables that minimizes the system given in (20) above, while ensuring positivity of the variance. This has the added virtue of minimizing any measurement errors in the observed price-dividend ratio and the ex-ante real risk-free rate. It is worth noting that we utilize here the monthly price-dividend ratio and risk free rate on an annual frequency corresponding to the appropriate beginning of year information. Moreover, it is important to recognize that this extraction step of state variables is ultimately done simultaneously with the overall GMM estimation of the moment conditions characterizing asset prices—see further discussion on the estimation below.

3.2 Time Aggregation

Given the state variables, we focus on moments that capture several key features of the consumption, dividend, and asset data. First, we focus on the consumption and dividend growth transition moments which ensures that the consumption and dividend dynamics are consistent with the data. The second set of moments focuses on the level of returns, and the third set of moments focuses on predictability of asset returns and consumption. More specifically, the list of moments we use for consumption and dividend growth are their respective volatility, autocorrelation, and joint correlations. In terms of return-based moments, we utilize the level of equity and risk-free rates, and the price-dividend ratio. In addition, we use the volatility of the market return, which exposes our estimation to both the asset return level puzzles as well as the volatility puzzles. To account for predictability, we use as moments the correlations of the price-dividend ratio with future consumption growth and the market return respectively. We also use the contemporaneous correlation between the price-dividend ratio and consumption volatility as a moment capturing their negative relationship. Finally, we impose the model implications for the orthogonality between annual consumption growth innovations and x – the expected consumption growth state, as well as the orthogonality between the innovation to the squared consumption growth innovation and the volatility state σ^2 as two additional moments. A detailed description of all the moment conditions is given in Table III, under the consumption & dividend moments, asset return moments, and predictability moments respectively. Our estimation approach, which uses first and second moments, allows us to impose model-restrictions and goes beyond the mean return restrictions that follow from using only $E(M_{t+1}R_{t+1}) = 1$. This is important as the second moment restrictions (such as the volatility puzzles) contain considerable information about model parameters of interest.[5]

To derive the unconditional moments, we first provide an analytical expression for the conditional moments based on the state-variables. We can express these moments for general frequencies of time aggregation. We let τ denote the time index for the econometrician's sampling frequency – in our baseline configuration the data is observed annually and therefore τ accumulates in annual increments. We let h denote the agent's (integer) number of decision periods within a sampling period. For example, if the econometrician uses annual data and the decision interval is monthly (weekly), τ increments annually while $h = 12(52)$. Similarly, if τ increments at quarterly frequency, and $h = 3$ the available data is assumed to be quarterly and the decision interval is monthly. We let calender time, t, accumulate at the same frequency as the agent's decision interval; it follows that at a point in time, calender time, which moves at the decision frequency is related to the sampling frequency by $t = \tau \cdot h$. The available sample for *observable* data is given by $\tau = 1, 2, ...\mathcal{T}$. We assume consumption, C_t, is unobserved and the only observable consumption is total consumption over the year, namely, $\sum_{i=1}^{h} C_{t-h+i}$. Based on the notation above, we denote Δc_τ^a, as *observed* log annual consumption growth, which is shown (see Appendix B) to be well approximated as

$$\Delta c_\tau^a \equiv \log \frac{\sum_{i=1}^{h} C_{t-h+i}}{\sum_{i=1}^{h} C_{t-2h+i}} \approx \sum_{j=2}^{h} \frac{j-1}{h} \Delta c_{t-2h+j} + \sum_{j=1}^{h} \frac{h-j+1}{h} \Delta c_{t-h+j} \quad \forall t = \tau \cdot h \tag{21}$$

where the superscript on the left-hand side indicates observed aggregated data which for example if available annual as in our case would imply τ increments at yearly frequency.

To better understand the way the estimation method accounts for time-aggregation it is instructive to present several moment conditions explicitly. To do so for the consumption growth moments, it is first useful to write the annual consumption growth rate in terms of the state variables available at the beginning month of the base year as well as a sequence of innovations which are mean zero conditional on that information set,

$$\begin{aligned} \Delta c_\tau^a &= h\mu_c + \frac{\rho(1-\rho^h)^2}{h(1-\rho)^2} x_{t-2h} + \sum_{j=1}^{h-1} a_j \varphi_e \sigma_{t-2h-1+j} e_{t-2h+j} + \sum_{j=1}^{h} b_j \varphi_e \sigma_{t-1-j} e_{t-j} \\ &\quad + \sum_{j=0}^{h-1} \frac{j+1}{h} \sigma_{t-1-j} \eta_{t-j} + \sum_{j=0}^{h-2} \frac{h-j-1}{h} \sigma_{t-h-1-j} \eta_{t-h-j} \quad \forall t = \tau \cdot h \end{aligned} \tag{22}$$

[5] E.g., with an *i.i.d* growth model, the price-dividend ratio is constant and has zero variance, which is clearly at odds with the data.

where $a_j = \frac{1}{h\rho^{j-1}}[(\frac{1-\rho^h}{1-\rho}) - \frac{1}{1-\rho}(\frac{1-\rho^{j-1}}{1-\rho} - (j-1)\rho^{j-1})]$ and $b_j = \frac{1}{h\rho^{j-1}}[j - \rho\frac{1-\rho^j}{1-\rho}]$.

3.3 Moments

The relevant state variables for computing any conditional moment of consumption growth (and other variables) are x_{t-2h} and σ^2_{t-2h}. Using equation (22)) it follows the conditional mean of consumption growth is $h\mu_c + \frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}x_{t-2h}$, and consequently the unconditional mean and variance of *observed* annual consumption growth follows:

$$\begin{aligned} E[\Delta c^a_\tau] &= h\mu_c \quad \forall t = \tau \cdot h \qquad (23) \\ Var[\Delta c^a_\tau] &= \left[\frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}\right]^2 var(x_{t-2h}) \\ &+ \sum_{j=1}^{h-1}\left[(a_j\varphi_e)^2 + (\frac{h-j}{h})^2\right]\sigma_0^2 + \sum_{j=1}^{h}\left[(b_j\varphi_e)^2 + (\frac{j}{h})^2\right]\sigma_0^2. \quad \forall t = \tau \cdot h \end{aligned}$$

In the case of annual data and a monthly decision interval, $h\mu_c$ is equal to $12\mu_c$ and μ_c is interpreted as a monthly quantity. In the case of an annual sampling frequency and an annual decision interval this simply reduces to μ_c, since $h = 1$ and τ moves at annual increments.

The k-th autocovariance of annual consumption growth can be readily computed in analogous fashion. In the estimation, we utilize the first two autocorrelations of consumption growth as moments. Appendix B provides more details for derivations of the moments used in our estimation. The dynamics for annual dividend growth can be written in a similar fashion as those given in equation (22) for consumption growth (that is dividend growth can be represented in terms of x_{t-2h} and σ^2_{t-2h} and a sequence of shocks orthogonal to time $t-2h$ information). Hence, the variance, the first autocorrelation of dividend growth, as well as the covariation between annual dividend and consumption growth also serve as moments in our estimation.

Turning to moments that include asset prices, it is useful to consider first the annual price-dividend ratio $z^a_{d,\tau} \equiv \log\frac{P_t}{\sum_{j=0}^{h-1}D_{t-j}}$ defined as the log of the end of year price over the twelve-month trailing sum of dividends. Recall that the solution to the monthly price-dividend ratio takes the form, $z_{d,t} = A_{0,d} + A_{1,d}x_t + A_{2,d}\sigma^2_t$ with the solutions for A_ds given in Appendix A.2. Using the definition of the annual price dividend ratio, it can be shown that its dynamics follow,

$$\begin{aligned} z^a_{d,\tau} &= A_{0,d} + A_{2,d}\sigma_0^2(1-\nu^h) - \log(h) + 0.5\mu_d(h-1) \\ & [\pi + A_{1,d}\rho]x_{t-h} + A_{2,d}\nu^h\sigma^2_{t-h} + \sum_{j=1}^{h}[q_j + A_{1,d}\rho^j]\varphi_e\sigma_{t-h-1+j}e_{t-h+j} \\ & \sum_{j=1}^{h-1}\frac{h-j}{h}\varphi_d\sigma_{t-j}u_{t-j+1} + \sum_{j=1}^{h}A_{2,d}\sigma_w\nu^{h-j}w_{t-h+j} \end{aligned} \tag{24}$$

where π and q_j are given in Appendix B. It follows that the mean and variance of the annual price-dividend ratio are,

$$\begin{aligned} E[z^a_{d,\tau}] &= A_{0,d} + A_{2,d}\sigma_0^2 - \log(h) - 0.5\mu_d(h-1) \quad \forall t = \tau\cdot h \\ Var[z^a_{d,\tau}] &= [\pi + A_{1,d}\rho^h]^2 var(x_t) + [A_{2,d}\nu^h]^2 var(\sigma_t^2) + \sum_{j=1}^{h}[q_j + A_{1,d}\rho^{h-j}]^2(\varphi_e\sigma_0)^2 \\ & \quad + \sum_{j=1}^{h-1}[\frac{h-j}{h}\varphi_d]^2\sigma_0^2 + \sum_{j=1}^{h}[A_{2,d}\sigma_w\nu^{h-j}]^2. \quad \forall t = \tau\cdot h \end{aligned} \tag{25}$$

The k-th order autocovariance of the price-dividend growth can be computed similarly (see details in Appendix B), and in the estimation we make use of the first and second autocorrelation of the price-dividend ratio.

The remaining moments involve the risk free rate and the market return. Given that both have a known representation in terms of Θ and the state variables (e.g., see equation (18) for the risk free rate), we utilize the same methodology to compute their respective unconditional means as well as the volatility of the market return. Finally, these representation allow us also to compute the covariations between the price-dividend ratio and the market return, consumption growth, and consumption volatility respectively, which serve as our last set of moments.

3.4 Estimation

Let $\mathcal{M}(\Theta; h, \{Data\})$ denote the difference between the model based moment conditions (evaluated at Θ) and their data counterpart when there are h decision periods within a sampling interval. That is, an element in this vector is one of the moments described above minus the same moment based purely on the data. We choose the parameter vector Θ

by evaluating the annual based moment conditions $\mathcal{M}(\Theta; h, \{Data\})$ while *simultaneously* choosing h, and the state variables x_t and σ_t^2 as described above. The parameter vector Θ_T is estimated by minimizing the standard GMM criteria,

$$\{\Theta; h\} \quad = \quad argmin_{\Theta;h}\mathcal{M}(\Theta; h, \{Data_\tau\})'W(\Theta; h)\mathcal{M}(\Theta; h, \{Data_\tau\}) \qquad (26)$$

where $Data_\tau$ pertains to observed annual data used in evaluating the moment conditions.[6] The weighting matrix $W(\Theta; h)$ used in estimation is the diagonal inverse of the variance-covariance matrix of the moment conditions and is updated continuously, motivated by Hansen, Heaton, and Yaron (1996). To construct the chi-squared test for over-identifying restrictions, we compute J-statistic using Lemma 4.2 in Hansen (1982) which holds for a general weighting matrix. The variance-covariance matrix is computed using the Newey and West (1987) estimator.

4 Data

We use data on consumption and asset prices for the time period from 1930 till 2009. We take the view that this sample better represents the overall variation in asset and macro economic data. Importantly, the long span of the data helps in achieving more reliable statistical inference. We work with the data sampled on an annual frequency as they are less prone to errors that arise from seasonalities and other measurement problems highlighted in Wilcox (1992).

To estimate the model, we exploit the dynamics of the observed aggregate consumption, the stock market portfolio, and the risk-free rate. Consumption data represent per-capita series of real consumption expenditure on non-durables and services from the NIPA tables

[6] In the estimation of (26), and for each candidate decision interval h, the extraction of state variables x_t and σ_t^2 follows by choosing a pair of state variables for each date t by minimizing a weighted sum of squared errors, of

$$\{x_t, \sigma_t^2\} = \text{argmin } \mathcal{S}_t(\Theta)W_T\mathcal{S}_t(\Theta), \text{ for } \forall\ t,$$

where W_T is a diagonal matrix of second moments of the observed price-dividend ratio and the risk-free rate. To guarantee positivity of the variance component, we solve the minimization problem by searching over a two-dimensional grid in the $\{x_t, \sigma_t^2\}$-space. We allow for a grid of 4,000 possible pairs, and make sure that the permissible space is wide enough to ensure that solutions lie in the interior region. We find that refining the grid further or expanding the boundary of the state space does not affect the implied state dynamics and our GMM estimates. Note that, alternatively, one could extract the states by using a constrained least-squares solver that has the advantage of allowing for a continuum of values for x and σ^2. Given that our grid is quite fine, our estimation yields almost identical results as the one based on the standard constrained OLS.

available from the Bureau of Economic Analysis. Aggregate stock market data consist of annual observations of returns, dividends, and prices of the CRSP value-weighted portfolio of all stocks traded on the NYSE, AMEX, and NASDAQ. Price and dividend series are constructed on the per-share basis as in Campbell and Shiller (1988), Bansal, Dittmar, and Lundblad (2005), and Hansen, Heaton, and Li (2008). Market data are converted to real using the consumer price index (CPI) from the Bureau of Labor Statistics. Growth rates of consumption and dividends are constructed by taking the first difference of the corresponding log series. Finally, the ex-ante real risk-free rate is constructed as a fitted value from a projection of the ex-post real rate on the current nominal yield and inflation over the previous year. To run the predictive regression, we use monthly observations on the three-month nominal yield from the CRSP Fama Risk Free Rate tapes and CPI series. The annual real risk-free rate is defined as the annualized predicted value as of the beginning of year. Table I provides key sample statistics for aggregate consumption growth, the stock market index, and the risk-free rate. As well known, the data feature a sizable equity premium of about 7%, high volatility of equity returns and low and relatively stable interest rates.

To explore the cross-sectional implications of the model, we employ portfolios with opposite size and book-to-market characteristics that are known to have provided investors with quite different premia over the years. The construction of portfolios is standard (see Fama and French (1993)). In particular, for the size sort, we allocate individual firms into 5 portfolios according to their market capitalization at the end of June of each year. Book-to-market quintiles are likewise re-sorted at the end of June by ranking all the firms based on their book-to-market ratios, defined as book equity at the last fiscal year end of the prior calendar year divided by market equity at the end of December of the previous year. NYSE breakpoints are used in both sorts. For each portfolio, we construct value-weighted monthly returns, as well as per-share price and dividend series. Monthly data are then time-aggregated to an annual frequency and converted to real using the consumer price index. Over the sample period, small stocks have outperformed large firms by about 7% and the spread in returns on value and growth firms has averaged almost 6% (see Table VII). Heterogeneity in risk premia across size and book-to-market portfolios is known to present a challenge for the standard CAPM since the market betas show almost no cross-sectional variation. Below, we will evaluate the ability of the LRR model to account for the observed size and value premia as well as the failure of the CAPM betas.

5 Empirical Findings

In this section, we present the estimates of the LRR model and discuss its implications for the joint dynamics of aggregate consumption, dividends and prices of the market portfolio. We also highlight the effect of time-aggregation on parameter estimates and inference, and evaluate the cross-sectional predictions of the model.

5.1 Estimation Evidence

Using the methodology outlined above, we estimate two model specifications: the LRR model and a nested specification, labeled "No-Vol", that restricts the conditional volatility process to be constant. The "No-Vol" specification is estimated by exploiting the same set of moments as in the LRR case, except for the conditional moment of the volatility dynamics, which is not defined when time-variation in the conditional second moment is ruled out. We use data sampled on the annual frequency and take into account potential time-aggregation effects by estimating the decision frequency (h) along with the structural parameters for preferences and the dynamics of consumption and dividends. Table II presents GMM estimates of the two models, their standard errors and the χ^2-test of overidentifying restrictions. The sample moments, their model-implied counterparts and the t-statistics for the difference between the two are provided in Table III.[7]

The parameter estimates of the LRR model, reported in the right panel of Table II, provide strong evidence of (i) time-aggregation, and (ii) a persistent predictable component in growth rates and persistent time-varying uncertainty. The estimate of the decision frequency ($\hat{h}$) is 11 which corresponds to a decision interval of approximately 33 days (365/11). Note that the estimate of h is significantly greater that one, i.e., the decision interval is much shorter than a year. We show below that time-aggregation consequent to a discrepancy between the (true) decision frequency and the sampling frequency of the data, and if ignored, may lead to significant biases in the estimates of the key model parameters and distorted inference.

The estimate of ρ, which governs the autocorrelation of the conditional mean of consumption growth, is 0.98 and is significantly different from zero. The magnitude of long-

[7]Small differences in the sample statistics reported in Tables I and III are due to the loss of few initial observations in constructing GMM moment conditions.

run risks is quite small, $\hat{\varphi}_e = 0.031$ (SE = 0.016), suggesting that the predictable growth component contributes relatively little to the overall variation of consumption growth. These parameter values illustrate the difficulty in detecting long-run risks solely from the available consumption data. The estimated long-run risk component, along with realized values of annual consumption growth, are illustrated in Figure 1.[8] The conditional volatility process is also highly persistent, $\hat{\nu} = 0.998$ (SE = 0.002), but is driven by quantitatively small shocks. The extracted volatility component exhibits a pronounced variation across time and a considerable decline in the 90's. Consistent with the model, the volatility component is a strong predictor of future returns. The regression of 5-year ahead excess returns of the market portfolio on the extracted variance yields an R^2 of 12%. As further shown in Table II, dividends of the market portfolio are significantly exposed to long-run consumption risks with $\hat{\phi}_d = 4.45$ (SE = 1.6). The short-run correlation between consumption and dividend growth rates, on the other hand, is not estimated precisely, with the point estimate and standard error of 0.49 and 0.33, respectively.

The estimate of risk aversion in the LRR specification is about 7.4, which is relatively low from the perspective of the asset pricing literature. The IES estimate is above one ($\hat{\psi} = 2.05,\ \mathrm{SE} = 0.84$), which is essential for a negative price of volatility risks and a positive relationship between the price-consumption ratio and expected growth.[9] Our estimate of the IES is driven by the negative correlation between the price-dividend ratio and consumption volatility, as well as by low level of the real risk-free rate, an observation also underscored in Hansen, Heaton, and Li (2008). Hall (1988) and Beeler and Campbell (2012) report a smaller magnitude of the IES, but they do not impose a broad set of equilibrium asset pricing restrictions in their estimation of the IES parameter.[10]

The estimates of the parameters governing the dynamics of expected consumption growth and volatility capture the economic mechanism highlighted in the LRR literature, first discussed in Bansal and Yaron (2004). Because the extracted expected growth and volatility shocks are long-lasting, they have an economically significant impact on growth expectations and future uncertainty, and therefore on assets' valuations. As Table III shows, the LRR model is indeed able to account for the level and volatility of the market returns, while

[8]For expositional purposes, the two series are plotted on different scales.

[9]It is worth noting that the parameter estimates presented in Table II, including the estimate of the decision frequency, are largely within one standard error of the values commonly used in calibrations of the LRR model and, in particular, are close to those in Bansal, Kiku, and Yaron (2012).

[10]Moreover, Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012) show that the approach of estimating the IES solely based on the risk free rate (e.g., Hall (1988)) can yield sizably downward biased estimates when the underlying shocks exhibit stochastic volatility as is the case in the LRR framework.

matching the low mean of the observed risk-free rate. Consistent with the data, the model generates a persistent and volatile price-dividend ratio with a first-order autocorrelation of 0.92 and a standard deviation of 0.45. As argued above, the magnitude of the long-run consumption risks is relatively small, which allows the model to match the time-series dynamics of consumption growth. The standard deviation and the first-order autocorrelation of annual consumption growth at the point estimates are 2.3% and 0.45, respectively. As further shown in the table, the model is successful in explaining other key moments of the joint distribution of consumption, dividends and asset returns. This success manifests itself formally through the χ^2-test of overidentifying restrictions that indicates that the LRR model is not rejected at the conventional 5% significance level.

In contrast to the LRR model, the restricted "No-Vol" specification is strongly rejected. The estimation results for the constant-volatility specification are presented in the left panel of Table II. Overall, the estimates of the (unrestricted) parameters of the "No-Vol" specification are similar in magnitude to the corresponding estimates of the fully specified LRR model. The estimate of the decision frequency is 10 (SE = 1.97), which is still statistically different from one; the estimates of risk aversion and IES are 7.1 and 2.3, respectively. Notice, however, that the preference parameters in this case are estimated less precisely, especially the time-discount factor and the IES. The large standard errors point out difficulties in separately identifying the rate of time preferences and IES when time-variation in risk premia is ruled out. The "No-Vol" specification confirms the presence of a small persistent component in consumption growth. Moreover, since volatility risks are now shut off, the contribution of long-run (and short-run) consumption risks under the "No-Vol" specification is amplified relative to the estimated dynamics of the LRR model. This allows the constant-volatility set-up to generate a sizable equity premium and high variation in asset returns. However, as Table III shows, the "No-Vol" specification fails to match several empirical moments and, in particular, evidence on predictability of consumption growth rates and returns. With no variation in uncertainty, the price-dividend ratio is driven solely by the long-run risk component generating excessive predictability of future growth rates. This specification is also unable to account for predictability of asset returns as it implies a constant risk premium. The rejection of the constant-volatility specification reveals the importance of time-variation in consumption volatility and the ensuing variation in risk premia.

One essential feature of the LRR framework is predictability of consumption growth rates. Our benchmark estimation incorporated the correlations between the price-dividend ratio and consumption growth and market return at a one year horizon. Beeler and Campbell (2012) argue that the LRR model may imply high (low) long-horizon predictability of consumption growth (return) relative to the data. Bansal, Kiku, and Yaron (2012) document that these data-features are very imprecisely estimated, and therefore not likely to receive significant weight in estimation. Nevertheless to account for the predictability of consumption and return carefully, we also consider an augmented set of moment conditions which includes predictability moments at the three-year horizon for both consumption growth rates and market returns. The parameter estimates based on the extended set of moments and the model fit are presented in Table VIII. As the table shows, the estimates and the model implications are robust to the inclusion of the additional moment restrictions, indicating that indeed the longer horizon predictability moments do not alter the parameter estimates. Consistent with our benchmark estimation, we find strong evidence of persistent variations in the conditional mean and volatility of consumption and dividend growth rates.

It is worth noting that a broader perspective on consumption growth predictability provides strong evidence for consumption predictability. The Beeler and Campbell (2012) claim for consumption predictability is based on using only the price-dividend ratio as a predictor of future consumption growth. Bansal, Kiku, and Yaron (2012) show that in the data the R^2's for consumption growth implied by a first-order VAR that includes consumption growth, the price-dividend ratio and the risk-free rate are 23%, 15% and 13% at horizons of one, five and ten years, respectively. These results are statistically significant both at short and long horizons. That is multi-variable based prediction of consumption does imply significant long-horizon predictability. Our estimated model is also consistent with this multi-variable based long horizon predictability.

To evaluate the role of recursive preferences, we also estimate a specification based on the same dynamics as in the LRR model but with time-separable (CRRA) preferences. That is, we impose the restriction that risk aversion and the IES parameter are reciprocals of each other, $\gamma = 1/\psi$. To save space, we discuss this estimation evidence without showing detailed output (which is available upon request). The estimates of risk aversion and IES in the power-utility specification are 3.1 and 0.32, respectively. Consequently, the CRRA specification implies a high level of the risk-free rate with a mean of 3%, an essentially zero risk premia, and low volatilities of the market return and price-dividend ratio of only 8.9% and 0.07, respectively. As all of these implications are sharply inconsistent with the data, the power-utility specification is overwhelmingly rejected with a p-value of the χ^2-test statistic of virtually zero. This evidence highlights the importance of recursive preferences in transmitting persistent risks in cash flows into asset prices.

5.2 The Effect of Time-Aggregation

Our GMM estimates of the LRR model presented above suggest that the frequency of the model dynamics (roughly monthly) is significantly shorter than the sampling frequency of the data (annual). It has been recognized in the time-series literature that temporal aggregation may cause a substantial loss of information about the underlying dynamics and, if not appropriately taken into account, may systematically bias inference (e.g., Working (1960), Hansen and Sargent (1983), Christiano, Eichenbaum, and Marshall (1991), Marcet (1991), Drost and Nijman (1993), Heaton (1995)). This issue may be particularly relevant in the context of the LRR model, in which the conditional distribution of consumption and cash-flow growth rates is time-varying and is driven by small persistent risks.

To assess biases resulting from a misspecification of the model's frequency, we re-estimate the model imposing the restriction that the decision interval is annual. In our notation, we set $h = 1$, and run estimation using the same set of annual data, now with no need to account for temporal aggregation. In other words, we consider an econometrician who is entirely ignorant of the issue of time-aggregation and assumes that the frequency of the model dynamics coincides with the frequency of the data. The left panel of Table IV provides the parameter estimates of this "Annual" specification along with their standard errors and the χ^2-test statistic. The most apparent difference between the estimates of the "Annual" specification and those of the LRR model, presented in Table II, is in the estimate of risk aversion. Ignoring time-aggregation due to the misspecification of the decision frequency results in a substantially higher estimate of risk aversion of about 14, which is almost twice that of the time-aggregated LRR model. Importantly, Table IV shows that the "Annual" specification is strongly rejected by the data.

To understand the rejection of this specification, in Table V we report population moments of the joint distribution of consumption, dividends and asset prices implied by the "Annual" estimates and the corresponding statistics in the data. As the table shows, the rejection of the "Annual" specification comes primarily from its failure to account for the dynamics of equity prices and returns. It significantly underestimates variation in the price-dividend ratio, generates only 11% volatility in equity returns, and despite the large estimate of risk aversion, is unable to explain the high level of risk premia.

The failure of the "Annual" specification is driven by its inability to identify the magnitude and the contribution of persistent, long-run and volatility, risks. Conceptually, if the true decision interval of the agent is shorter than annual, the "Annual"-based moment restrictions that ignore time-aggregation are severely misspecified. This misspecification shifts the emphasis of the model from long-run risks to short-run innovations in consumption. To highlight the intuition, we refer to equation (22) that describes the dynamics of the time-aggregated consumption growth. For concreteness, assume that the true model is monthly while the data are sampled annually. Note that the innovation in annual consumption growth is a mixture of the underlying long- and short-run monthly shocks. While the LRR model appropriately accounts for this composite innovation structure by allowing for temporal aggregation in estimation, the restricted "Annual" specification has no means to separate out the two shocks and tends to attribute the whole innovation to short-run fluctuations in consumption growth. In other words, the "Annual" specification amplifies the contribution of short-run risks at the expense of low-frequency movements in consumption. Consequently, the misspecified "Annual" set-up appear similar to a specification based on the i.i.d. dynamics. Quantitatively, the variance decomposition of annual consumption growth reveals that the contribution of long-run risks implied by the estimates of the LRR model is around 27%, whereas under the "Annual" specification, long-run risks account for only 14% of the overall variation of consumption growth. Further, consistent with Drost and Nijman (1993), we find that it is much harder to detect time-variation in the conditional volatility of growth rates using low-frequency data and disregarding restrictions of temporal aggregation.

To corroborate the above argument, we compare impulse responses of consumption growth and its conditional variance implied by the LRR model and the "Annual" specification. The impulse response functions are constructed by fitting an ARMA model to the data simulated at the point estimates of the two specifications.[11] To make the model comparison meaningful, we use annual consumption growth rates, and variance observations sampled at the end of each simulation year. Note that for the LRR model, we simulate consumption at a frequency of 33 days (i.e., 11 decision periods per year), then aggregate it to the annual frequency by summing-up consumption levels within a year, and compute annual growth rates. The cumulative response of annual consumption growth for each specification is presented in Figure 2. While the contemporaneous responses in the two cases are the same, the subsequent response of consumption growth in the "Annual" specification

[11] To highlight differences in population, we simulate a long sample of data for each specification. We use eight autoregressive and eight moving-average terms for both consumption growth and its variance. The impulse responses are robust to changes in the ARMA specification as long as there are enough terms to account for predictable variations.

is much lower than that in the LRR model. In the limit, a one-percent ARMA-shock raises the level of annual consumption by 2.5% in the LRR model and by only 1.9% in the "Annual" specification. The implications of the two specifications for the variance dynamics are significantly different as well. Figure 3 reveals that variance risks in the "Annual" specification, although persistent, taper off much more rapidly than those in the LRR model. These differences allow us to better understand the limitations of the "Annual" specification and its ultimate rejection in the data. To be able to fit the dynamics of observed consumption and dividends when temporal aggregation is ruled out, this specification has to suppress the contribution of low-frequency and volatility risks and, instead, magnify short-run fluctuations in cash-flow growth rates. However, short-run risk provide little help in explaining the dynamics of asset prices.

To further highlight the consequences of ignoring time-aggregation effects, we simulate the LRR model using the parameter estimates reported in Table II. We aggregate the simulated data to construct annual consumption, dividends and prices and use those to estimate the "Annual" specification. This experiment is designed to illustrate what happens if the true model frequency is shorter than annual but an econometrician, equipped with annual data, assumes a yearly decision interval and, therefore, ignores restrictions of temporal aggregation. The output of this simulation exercise is reported in the right panel of Table IV. We present finite-sample distributions of the parameter estimates as well as population values computed using a long sample of simulated data. Overall, we find that the estimated parameters in these simulations are quite close to the "Annual" estimates based on the observed data. In particular, the misspecification of the model frequency results in a considerably weaker contribution of long-run and volatility risks, and consequently, in a significantly biased estimate of risk aversion. Note that the estimate of risk aversion under the "Annual" specification almost doubles (from its true value of 7.4 to about 14.1, on average) and this bias, although reduced, does not vanish asymptotically. In simulations, as in the data, the "Annual" specification fails to account for high volatility of prices and returns. This evidence confirms difficulties in detecting long-run and volatility risks if one neglects temporal aggregation and ignores how the underlying model dynamics are integrated to lower-frequency data employed in estimation.

In Table VI, we vary the time-aggregation parameter h to show how different assumptions about the length of the decision interval affect the model's estimates and inference. In addition to the already discussed "Annual" set-up, we consider three specifications: "Bi-weekly" (with $h = 26$), "Monthly" ($h = 12$), and "Quarterly" ($h = 4$). Note that the farther the decision frequency deviates from its optimal value of 11, the more the model fit deteriorates. At the monthly frequency, both the estimates and the pricing implications are quite similar to those of the unrestricted LRR model, and the p-value of the overidentifying restrictions test is borderline significant. The "Bi-weekly" specification features a further decline in the p-value, and is rejected largely by the risk-free rate and return predictability moments. As the frequency declines to quarterly, the model loses its ability to account for the level and variation in asset returns and prices, and is strongly rejected. These implications, including the increase in the estimate of risk aversion, are similar to the predictions of the "Annual" specification and follow from the failure to properly identify long-run and volatility risks. These results underscore the importance of incorporating the restrictions of temporal aggregation for drawing inferences about the underlying risk dynamics and investors' preferences.

5.3 Cross-Sectional Implications

One of the important dimensions of financial data is the cross-sectional heterogeneity in mean returns, in particular along size and book-to-market dimensions. Table VII shows that the average return of the high book-to-market portfolio is higher than that of the low book-to-market portfolio by 5.6% per annum. This is the well-known value premium. Similarly, the portfolio of small market-capitalization firms outperforms the large-firm portfolio by about 7%, on average. The observed dispersion in mean returns on size and book-to-market sorted portfolios is known to present a challenge for the standard CAPM. In the data, the market betas for the value-minus-growth and small-minus-large portfolios are quite small, while the market-adjusted returns (i.e., CAPM α's) are large and significant.

In this section, we evaluate the implications of the LRR model for the cross-section of size and book-to-market portfolios. We estimate the dynamics of dividend growth rates of the four portfolios: small, large, value and growth, and assess the ability of the LRR model to simultaneously account for the value and size premia, and the magnitudes on the CAPM betas and alphas. To keep the estimation problem manageable, the cross-sectional parameters are estimated using the state variables extracted in our benchmark estimation of the model, and holding preferences, consumption, and market-dividend parameters fixed at the point estimates reported in Table II. For each portfolio, we use the following set of moment conditions to estimate its dynamics: the mean and volatility of dividend growth rates, their correlation with consumption growth, the risk premium, the volatility of returns, the mean and volatility of the price-dividend ratio, and the market beta. The cross-sectional moments are evaluated using annual data and incorporating restrictions of temporal aggregation.

Panel A of Table VII presents the cross-sectional estimates and, in particular, dividends' exposure to consumption risks for the four portfolios. Consistent with empirical evidence in Bansal, Dittmar, and Lundblad (2005), and Hansen, Heaton, and Li (2008), we find that the value portfolio feature much higher exposure to low-frequency risks in consumption relative to the growth portfolio (8.2 versus 4.3). Similarly, long-run risk exposure of the small-size portfolio exceeds that of the large portfolio (7.9 versus 4.5). Small and value portfolios are also characterized by a higher short-run correlation of their dividends with consumption relative to portfolios with opposite size and book-to-market characteristics, although short-run risk dynamics are estimated with large standard errors. The estimate of φ_j, which governs dividend exposure to volatility risks, is higher for the small portfolio compared to the large portfolio, and is almost uniform across book-to-market sorted portfolios. The bottom line of Panel A presents the model-implied risk premium for each of the four portfolios. Comparable to the premia observed in the data, the model predicts a sizable value premium of about 5% and a large size premium of 6.5%. The risk-premium decomposition reveals that, across portfolios, about 60-65% of the premium comes as a compensation for long-run risks, about 25-30% is accounted for by volatility risks, and the remaining fraction is due to asset exposure to short-run consumption risks.

Panel B of Table VII shows the CAPM implications of the long-run risk model. It presents the implied market betas and alphas for the small-minus-large and value-minus-growth portfolios. As in the data, the model-implied CAPM betas of the spread portfolios are quite low: 0.7 and 0.6 for the small–large and value–growth strategies, respectively. Consequently, the LRR model is able to replicate the failure of the CAPM by generating quantitatively sizable alphas of the arbitrage portfolios. The model-implied alphas of the small–large and value–growth portfolios are about 2% and 1.4%, respectively. The ability of the LRR model to account for a significant portion of the value- and size-premium puzzles comes from the fact that in the model, the market beta is not a sufficient risk statistics (i.e., the market return is not perfectly correlated with the SDF). In particular, the market exposure to long-run risks is significantly lower than that of the underlying pricing kernel. Therefore, the model features high market-adjusted alphas of the small–large and value–growth portfolios as those are highly exposed to long-run risks in consumption.

6 Conclusions

This paper develops a method for estimating asset pricing models with recursive preferences and generalized consumption and cash-flow dynamics while accounting for time-aggregation in the observed data. Specifically, we estimate the long-run risks model as well as the decision interval of the agent which accounts for the observed time-aggregated data. The paper shows how to estimate the short-run, long-run and volatility risk components in aggregate consumption and utilize these to construct the unobservable return on aggregate wealth - a key input in estimating models with Kreps and Porteus (1978), Epstein and Zin (1989)-Weil (1989) preferences.

Empirically we find that the long-run risks model is able to successfully capture the time-series and cross-sectional variation in returns. The estimation identifies a persistent long run risk growth and volatility component in the consumption and dividend data. The model is not rejected by the over-identifying restrictions test. We provide evidence that time-aggregation can result in substantially biased estimates for risk aversion and that ignoring time-aggregation leads to false rejections of the model and to large estimates of risk aversion. Remarkably, we find that the best estimate for the investor's decision interval is nearly a month (33 days).

At the estimated values for the preference parameters, and the decision interval, the long run growth risks and volatility contribute about the same amount to the risk premia. Overall, the model accounts for the low risk free rate, and the level of the market, value, and size premia, as well as the volatility of the market return, the risk free rate, and the price-dividend ratio. In all, this evidence provides empirical support for the economic risk channels highlighted by the LRR model.

Appendix

A.1 Consumption Claim

To derive asset prices we use the IMRS together with consumption and dividend dynamics given in (3) and (15). The Euler condition in equation (16) implies that any asset j in this economy should satisfy the following pricing restriction,

$$E_t\Big[\exp\big(\theta \ln \delta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r_{c,t+1} + r_{j,t+1}\big)\Big] = 1 \ , \tag{A-1}$$

where $r_{j,t+1} \equiv \log(R_{j,t+1})$ and $r_{c,t+1}$ is the log return on wealth. Notice that the solution to (A-1) depends on time-series properties of the unobservable return r_c. Therefore, we first substitute $r_{j,t+1} = r_{c,t+1}$ and solve for the return on the aggregate consumption claim; after that, we present the solution for the return on a dividend-paying asset.

We start by conjecturing that the logarithm of the price to consumption ratio follows, $z_t = A_0 + A_1 x_t + A_2 \sigma_t^2$. Armed with the endogenous variable z_t, we plug the approximation $r_{c,t+1} = \kappa_0 + \Delta c_{t+1} + \kappa_1 z_{t+1} - z_t$ into the Euler equation above. The solution coefficients, A's, can now be easily derived by collecting the terms on the corresponding state variables. In particular,

$$\begin{aligned}
A_0 &= \frac{1}{1-\kappa_1}\left[\log\delta + \kappa_0 + \Big(1 - \frac{1}{\psi}\Big)\mu_c + \kappa_1 A_2 (1-\nu)\sigma_0^2 + \frac{\theta}{2}\Big(\kappa_1 A_2 \sigma_w\Big)^2\right] \\
A_1 &= \frac{1-\frac{1}{\psi}}{1-\kappa_1\rho} \\
A_2 &= -\frac{(\gamma-1)(1-\frac{1}{\psi})}{2\,(1-\kappa_1\nu)}\left[1 + \Big(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho}\Big)^2\right]
\end{aligned} \tag{A-2}$$

For more details, see the the appendix in Bansal and Yaron (2004).

Notice that the derived solutions depend on the approximating constants, κ_0 and κ_1, which, in their turn, depend on the unknown mean of the price to consumption ratio, $\bar{z}$. In order to solve for the price of the consumption asset, we first substitute expressions for κ's (equations (6) and (7)) into the expressions for A's and solve for the mean of the price to consumption ratio. Specifically, $\bar{z}$ can be found by numerically solving a fixed-point problem:

$$\bar{z} = A_0(\bar{z}) + A_2(\bar{z})\sigma_0^2 \ ,$$

where the dependence of A's on $\bar{z}$ is given above.

The solution for the price-consumption ratio, z_t, allows us to write the pricing kernel as a function of the state variables and the model parameters,

$$m_{t+1} = \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \sigma_t^2 - \lambda_\eta \sigma_t \eta_{t+1} - \lambda_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1} \ , \tag{A-3}$$

where

$$\begin{aligned} \Gamma_0 &= \log\delta - \frac{1}{\psi}\mu_c - 0.5\,\theta(\theta-1)\big(\kappa_1 A_2 \sigma_w\big)^2 \\ \Gamma_1 &= -\frac{1}{\psi} \\ \Gamma_2 &= (\theta-1)(\kappa_1\nu - 1)A_2 \end{aligned} \tag{A-4}$$

and

$$\begin{aligned} \lambda_\eta &= \gamma \\ \lambda_e &= (1-\theta)\kappa_1 A_1 \varphi_e \ = \ \big(\gamma - \frac{1}{\psi}\big)\frac{\kappa_1\varphi_e}{1-\kappa_1\rho} \\ \lambda_w &= (1-\theta)\kappa_1 A_2 \ = \ -(\gamma-1)\big(\gamma - \frac{1}{\psi}\big)\frac{0.5\,\kappa_1}{1-\kappa_1\nu}\Big[1 + \Big(\frac{\kappa_1\varphi_e}{1-\kappa_1\rho}\Big)^2\Big] \end{aligned} \tag{A-5}$$

Note that λ's represent market prices of transient (η_{t+1}), long-run (e_{t+1}) and volatility (w_{t+1}) risks respectively. For more detailed discussion see Bansal and Yaron (2004).

A.2 Dividend Paying Assets

The solution coefficients for the valuation ratio of a dividend-paying asset j can be derived in a similar fashion as for the consumption asset. In particular, the price-dividend ratio for a claim to dividends dynamics, given in (15), $z_{j,t} = A_{0,j} + A_{1,j}x_t + A_{2,j}\sigma_t^2$, where

$$\begin{aligned} A_{0,j} &= \frac{1}{1-\kappa_{1,j}}\Big[\Gamma_0 + \kappa_{0,j} + \mu_j + \kappa_{1,j}A_{2,j}(1-\nu)\sigma_0^2 + \frac{1}{2}\Big(\kappa_{1,j}A_{2,j} - \lambda_w\Big)^2\sigma_w^2\Big] \\ A_{1,j} &= \frac{\phi_j - \frac{1}{\psi}}{1-\kappa_{1,j}\rho} \\ A_{2,j} &= \frac{1}{1-\kappa_{1,j}\nu}\Big[\Gamma_2 + \frac{1}{2}\Big(\varphi_j^2 + \lambda_\eta^2 - 2\varrho_j\varphi_j\lambda_\eta + (\kappa_{1,j}A_{1,j}\varphi_e - \lambda_e)^2\Big)\Big] \end{aligned} \tag{A-6}$$

It follows then that the innovation into the asset return is given by,

$$r_{j,t+1} - E_t[r_{j,t+1}] = \beta_{u,j}\sigma_t u_{j,t+1} + \beta_{e,j}\sigma_t e_{t+1} + \beta_{w,j}\sigma_w w_{t+1}\ , \tag{A-7}$$

where the asset's betas are defined as,

$$\beta_{u,j} = \varphi_j, \quad \beta_{e,j} = \kappa_{1,j}A_{1,j}\varphi_e, \quad \beta_{w,j} = \kappa_{1,j}A_{2,j}$$

The risk premium for any asset is determined by the covariation of the return innovation with the innovation into the pricing kernel. Thus, the risk premium for $r_{j,t+1}$ is equal to the product of the asset's exposures to systematic risks and the corresponding risk prices,

$$\begin{aligned} E_t[r_{j,t+1} - r_{f,t}] + 0.5\sigma_{t,r_j}^2 &= -Cov_t\Big(m_{t+1} - E_t(m_{t+1}), r_{j,t+1} - E_t(r_{j,t+1})\Big) \\ &= \lambda_\eta\sigma_t^2\beta_{\eta,j} + \lambda_e\sigma_t^2\beta_{e,j} + \lambda_w\sigma_w^2\beta_{w,j}\ , \end{aligned}$$

where the exposure of asset j return to short-run consumption innovation is $\beta_{\eta,j} = \varphi_j\varrho_j$.

A.3 Risk Free Rate

The solution coefficients for the risk-free dynamics follow directly from the state-representation of the SDF. In particular,

$$\begin{aligned} F_0 &= -\Gamma_0 - 0.5\left[\lambda_w \sigma_w)\right]^2 \\ F_1 &= -\Gamma_1 \\ F_2 &= -\Gamma_2 - 0.5\left[\lambda_\eta^2 + \lambda_e^2\right] \end{aligned} \tag{A-8}$$

A.4 IES=1

When $\psi = 1$, the log of the IMRS is given in terms of the value function normalized by consumption, $vc_t = \log(V_t/C_t)$,

$$m_{t+1} = \log\delta - \gamma\Delta c_{t+1} + (1-\gamma)vc_{t+1} - \frac{1-\gamma}{\delta}vc_t$$

Conjecturing that $vc_t = \tilde{A}_0 + \tilde{A}_1 x_t + \tilde{A}_2\sigma_t^2$ and using the evolution of vc_t:

$$vc_t = \frac{\delta}{1-\gamma}\log E_t\Big[\exp\{(1-\gamma)(vc_{t+1} + \Delta c_{t+1}\}\Big],$$

the solution coefficients are given by,

$$\begin{aligned} \tilde{A}_0 &= \frac{\delta}{1-\delta}\left[\mu_c + \tilde{A}_2(1-\nu)\sigma_0^2 + \frac{1}{2}(1-\gamma)(\tilde{A}_2\sigma_w)^2\right] \\ \tilde{A}_1 &= \frac{\delta}{1-\delta\rho} \\ \tilde{A}_2 &= -(\gamma-1)\frac{0.5\,\delta}{1-\delta\nu}\left[1 + \left(\frac{\delta\varphi_e}{1-\delta\rho}\right)^2\right] \end{aligned} \tag{A-9}$$

As above, the pricing kernel can be expressed in terms of underlying preference parameters, state variables and systematic shocks,

$$m_{t+1} = \Gamma_0 + \Gamma_1 x_t + \Gamma_2\sigma_t^2 - \lambda_\eta\sigma_t\eta_{t+1} - \lambda_e\sigma_t e_{t+1} - \lambda_w\sigma_w w_{t+1} \tag{A-10}$$

where:

$$\begin{aligned}
\Gamma_0 &= \log\delta \;-\; \mu_c \;-\; 0.5\,(1-\gamma)^2(\tilde{A}_2\sigma_w)^2 \\
\Gamma_1 &= -1 \qquad\qquad (\text{A-11}) \\
\Gamma_2 &= -\frac{(\gamma-1)^2}{2}\left[1+\left(\frac{\delta\varphi_e}{1-\delta\rho}\right)^2\right]
\end{aligned}$$

and

$$\begin{aligned}
\lambda_\eta &= \gamma \qquad\qquad (\text{A-12}) \\
\lambda_e &= (\gamma-1)\frac{\delta\varphi_e}{1-\delta\rho} \\
\lambda_w &= -(\gamma-1)^2\frac{0.5\,\delta}{1-\delta\nu}\left[1+\left(\frac{\delta\varphi_e}{1-\delta\rho}\right)^2\right] \qquad\qquad (\text{A-13})
\end{aligned}$$

Finally, note that in the IES=1 case, the wealth to consumption ratio is constant, namely, $\frac{W_t}{C_t} = \frac{1}{1-\delta}$. The price to consumption ratio, therefore, is equal $\frac{P_t}{C_t} = \exp(\bar{z}) = \frac{\delta}{1-\delta}$. Consequently, the parameter of the log-approximation of the log-wealth return,

$$\kappa_1 \;=\; \frac{\exp(\bar{z})}{1+\exp(\bar{z})} = \frac{\frac{\delta}{1-\delta}}{1+\frac{\delta}{1-\delta}} = \delta.$$

Plugging $\kappa_1 = \delta$ and $\psi = 1$ into equations (A-3), (A-4) and (A-5), yields exactly equation (A-10), (A-11) and (A-12). It then follows that

$$\lim_{\psi\to 1}\kappa_1 = \delta \quad \lim_{\psi\to 1}\Gamma' = \Gamma'(\psi = 1, \kappa_1 = \delta) \quad \lim_{\psi\to 1}\Lambda' = \Lambda'(\psi = 1, \kappa_1 = \delta) \qquad (\text{A-14})$$

A.5 Pricing Kernel Approximation Error

In our empirical work, we rely on the approximate analytical solutions of the model presented above. In this section, we evaluate the accuracy of the log-linear approximation by comparing the approximate analytical solution for the price to consumption ratio to its numerical counterpart. The magnitude of the approximation error allows us to assess the reliability of the log-linear solution for the stochastic discount factor, and consequently, model implications based on the log-linear approximation.

Notice that the value function in the Epstein-Zin preferences is given by,

$$V_t = (1-\delta)^{\frac{\psi}{\psi-1}} W_t (W_t/C_t)^{\frac{1}{\psi-1}}, \tag{A-15}$$

i.e., the life-time utility of the agent, normalized by the level of either consumption or wealth, is proportional to the wealth to consumption ratio. Hence, the solution to the wealth-consumption ratio (or, alternatively, price to consumption) based on the log-linearization of the wealth return in equation (5) determines the dynamics of the value function. Recall also that the evolution of the IMRS (see equation (4)), through the return on wealth, depends on the valuation of the consumption claim. Thus, the log-linear solution for the IMRS, as well, hinges on the accuracy of the log-linear approximation of the price-consumption ratio.

Our numerical solutions are based on the approach proposed by Tauchen and Hussey (1991). This method relies on a discrete representation of the conditional density of the state variables, x and σ^2, which allows us to solve the pricing equation by approximating the integral in (16) with a finite sum using the Gauss-Hermite quadrature. Note that the resulting solutions are subject to a discretization error. In order to minimize the error and ensure the high quality of the benchmark numerical solutions, we use a sufficiently large number of grid points in the quadrature rule.[12] In addition, in this exercise we shut-off the channel of time-varying consumption volatility. Aside from this restriction, we evaluate and compare numerical and log-linear analytical solutions using the parametrization of consumption growth dynamics given in caption of Table A.1. The table presents the mean level of the price-consumption ratio and its volatility for various combinations of risk aversion and IES; the time-discount preference parameter δ is set at 0.9989.

Overall, we find approximate analytical and numerical solutions to be remarkably close to each other. In particular, for risk aversion of 10 and IES of 2, the mean and the volatility of the log price to consumption ratio implied by the log-linear approximation are 4.716 and 0.0321. Numerical solutions yield 4.724 and 0.0318, respectively.[13] The approximation error, expressed as a percentage of the corresponding numerical value, is about 0.17% for the mean and 0.86% for the standard deviation of the log price-consumption ratio. As the elasticity of

[12]Specifically, we discretize the dynamics of the expected growth component, x_t, using a 100-point rule. We find that increasing the number of grid points leads to virtually identical numerical solutions.

[13]All the numbers reported in this section are in monthly terms.

intertemporal substitution decreases to 0.5, the percentage error falls to about 0.02% for $\bar{z}$ and 0.42% for σ_z. Although the accuracy of the log-linearization slightly deteriorates as the magnitude of risk aversion increases, deviations between analytical and numerical solutions remain relatively small. For example, holding IES at 2 and varying risk aversion between 5 and 15 results in 0.03%–0.51% error band for the mean and 0.17%–2.17% for the standard deviation of the log price to consumption ratio.

As discussed above, the dynamics of the price to consumption ratio has a direct bearing on the time-series properties of the IMRS. The fairly small approximation error in the price-consumption ratio that we document guarantees the accuracy of the pricing implications based on the log-linear solutions. Indeed, we find that approximate analytical and numerical solutions deliver very similar quantitative implications along all dimensions of the model, including levels and variances of the risk-free rate, price-dividend ratios, returns on consumption and dividend claims, and the pricing kernel.[14] This evidence confirms that empirical findings presented in the paper are robust to the log-linearization of the model.

B Time Aggregated Moments

The mean and variance for annual consumption growth are already given in the text. Based again on equation (22), the first and second autocovariances of annual consumption growth can be written as,

$$\begin{aligned}
AC1(\Delta c_{t+h,h}) &= \rho^h \left[\frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}\right]^2 var(x_{t-2h}) \\
&\quad + \sum_{j=1}^{h-1} a_j b_j \varphi_e^2 \sigma_0^2 + \sum_{j=0}^{h-2} [(\frac{h-j}{h})(\frac{j}{h})]\sigma_0^2 + \sum_{j=0}^{h} (\rho^j \varphi_e)^2 \sigma_0^2 \\
AC2(\Delta c_{t+h,h}) &= \rho^{2h} \left[\frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}\right]^2 var(x_{t-2h}) \\
&\quad + \sum_{j=1}^{h-1} a_j b_j \varphi_e^2 \sigma_0^2 + \sum_{j=0}^{h-2} [(\frac{h-j}{h})(\frac{j}{h})]\sigma_0^2 + \sum_{j=0}^{h} (\rho^j \varphi_e)^2 \sigma_0^2
\end{aligned}$$

Given the monthly dynamics for dividends, equation (15), the dynamics for annual

[14] Available upon request, the detailed evidence is not reported here for brevity.

dividend growth can be written in a similar fashion as those for annual consumption growth,

$$\begin{aligned}\Delta d_{t+h,h} &= h\mu_d + \phi_d \frac{\rho(1-\rho^h)^2}{h(1-\rho)^2} x_{t-h} + \phi_d \sum_{j=1}^{h-1} a_j \varphi_e \sigma_{t-h-1+j} e_{t-h+j} + \phi_d \sum_{j=1}^{h} b_j \varphi_e \sigma_{t+h-1-j} e_{t+h-j} \\ &\quad + \varphi_d \sum_{j=0}^{h-1} \frac{j+1}{h} \sigma_{t+h-1-j} u_{t+h-j} + \varphi_d \sum_{j=0}^{h-2} \frac{h-j-1}{h} \sigma_{t-1-j} u_{t-j}. \qquad \text{(A-16)}\end{aligned}$$

Hence, the mean, variance, and first autocovariance of annual dividend growth are easily computed as for consumption growth. Finally, the unconditional covariation between dividend and consumption growth is given by

$$\begin{aligned}cov(\Delta c_{t+h,h}, \Delta d_{t+h,h}) &= \left[\frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}\right]^2 \phi_d var(x_{t-h}) \qquad \text{(A-17)} \\ &\quad + \sum_{j=1}^{h-1} \left[\phi_d (a_j \varphi_e)^2 + \varphi_d \varrho_d (\frac{h-j}{h})^2\right] \sigma_0^2 + \sum_{j=1}^{h} \left[\phi_d (b_j \varphi_e)^2 + \varphi_d \varrho_d (\frac{j}{h})^2\right] \sigma_0^2\end{aligned}$$

The volatility dynamics moments are based on equation (**??**).

The text discusses the first and second moment of the log annualized price-dividend ratio $z_{d,t,h}$. Based on equation (25), the k-th autocovariance of $z_{d,t,h}$ is,

$$\begin{aligned}cov(z_{d,t+kh,h}, z_{d,t,h}) &= [(A_{1,d}^2 + \pi^2)\rho^{kh} + A_{1,d}\pi\rho^{k-1}(1+\rho^{2h})]var(x_t) + A_{2,d}^2 \nu^{kh} var(\sigma_t^2) \\ &\quad + \sum_{j=1}^{h-1} q_j (\varphi_e \sigma_0)^2 [\pi \rho^{kh-j} + A_{1,d} \rho^{(k+1)h-j}] \qquad \text{(A-18)}\end{aligned}$$

where $\pi = \frac{\phi_d}{h(1-\rho)}[\rho\frac{1-\rho^{h-1}}{1-\rho} - (h-1)\rho^h]$ and $q_j = \frac{\phi_d}{h\rho^{j-1}(1-\rho)}[\frac{1-\rho^{h-1}}{1-\rho} - (h-1)\rho^{h-1} - \frac{1-\rho^{j-1}}{1-\rho} + (j-1)\rho^{j-1}]$.

To solve for the annualized return on the dividend paying asset and risk free rate, start with the monthly dynamics for these assets,

$$\begin{aligned}r_{f,t} &= F_0 + F_1 x_t + F_2 \sigma_t^2 \\ r_{d,t+1} &= B_{0,d} + B_{1,d} x_t + B_{2,d} \sigma_t^2 + \beta_{e,d} \sigma_t e_{t+1} + \beta_{u,d} \sigma_t u_{t+1} + \beta_{w,d} \sigma_w w_{t+1}\end{aligned}$$

where F's are given in equations (A-9) and B's follow directly from the solution for the price-dividend ratio. The moments for the annual risk free rate, $r_{f,t,h} \equiv \sum_{j=0}^{h-1} r_{f,t+j}$, can now be easily derived,

$$\begin{aligned}
E[r_{f,t,h}] &= h[F_0 + F_2\sigma_0] \\
var[r_{f,t,h}] &= [F_1\frac{1-\rho^h}{1-\rho}]^2 var(x_t) + [F_2\frac{1-\nu^h}{1-\nu}]^2 var(\sigma_t^2) \\
&+ \sum_{j=1}^{h-1}[F_1\varphi_e\frac{1-\rho^{h-j}}{1-\rho}]^2\sigma_0^2 + \sum_{j=1}^{h-1}[F_2\frac{1-\nu^{h-j}}{1-\nu}]^2\sigma_w^2
\end{aligned}$$

Similarly the market return, $r_{d,t+h,h} \equiv \sum_{j=0}^{h-1} r_{d,t+1+j}$, can be written as,

$$\begin{aligned}
E[r_{d,t+h,h}] &= h[B_{0,d} + B_{2,d}\sigma_0^2] \\
var[r_{d,t+h,h}] &= [B_{1,d}\frac{1-\rho^h}{1-\rho}]^2 var(x_t) + B_{2,d}[\frac{1-\nu^h}{1-\nu}]^2 var(\sigma_t^2) \\
&+ \sum_{j=1}^{h}[B_{1,d}\varphi_e\frac{1-\rho^{h-j}}{1-\rho} + \beta_{e,d}]^2\sigma_0^2 + \sum_{j=1}^{h}[B_{2,d}\frac{1-\nu^{h-j}}{1-\nu} + \beta_{w,d}]^2\sigma_w^2 + h\beta_{u,d}^2\sigma_0^2
\end{aligned}$$

Finally, using the formulae for the annual price-dividend ratio, consumption growth, and the market return, the moments characterizing their covariation are,

$$\begin{aligned}
cov(\Delta c_{t+h,h}, z_{d,t+h,h}) &= \frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}\left(\pi + A_{1,d}\rho^h\right) var(x_t) + \sum_{j=1}^{h-1} a_j(\varphi_e\sigma_0)^2[q_j + A_{1,d}\rho^{h-j}] \\
&\quad \frac{A_{1,d}}{h(1-\rho)}[h - \rho\frac{1-\rho^h}{1-\rho}](\varphi_e\sigma_0)^2 + \sum_{j=1}^{h-1}[\frac{h-j}{h}]^2\varphi_d\varrho_d\sigma_0^2 \\
cov(r_{d,t+h,h}, z_{d,t+h,h}) &= B_{1,d}\frac{1-\rho^h}{1-\rho}\rho^h(\pi + A_{1,d}\rho^h)var(x_t) + B_{2,d}A_{2,d}\frac{1-\nu^h}{1-\nu}\nu^{2h}var(\sigma_t^2) \\
&+ \sum_{j=1}^{h}\rho^{h-j}B_{1,d}\frac{1-\rho^h}{1-\rho}(\varphi_e\sigma_0)^2[q_j + A_{1,d}\rho^{h-j}] + A_{2,d}B_{2,d}\frac{1-\nu^h}{1-\nu}\sigma_w^2\sum_{j=1}^{h}\nu^{2(h-j)}
\end{aligned}$$

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Table I

Summary Statistics

	Mean	StdDev
Consumption Growth	0.019	0.022
Dividend Growth	0.009	0.113
Market Return	0.079	0.020
Log(P/D)	3.369	0.453
Risk-Free Rate	0.005	0.029

Table I presents descriptive statistics for aggregate consumption growth, returns, dividend growth and the logarithm of the price-dividend ratio of the stock market portfolio, and the risk-free rate. Returns are value-weighted, dividends and price-dividend ratios are constructed on the per-share basis, growth rates are measured by taking the first difference of the logarithm of the corresponding series. All data are real, sampled on an annual frequency and cover the period from 1930 to 2009.

Table II

Model Estimates: Parameters

Parameters	**No-Vol Model**		**LRR Model**	
	Estimate	SE	Estimate	SE
Preferences				
γ	7.05	1.85	7.42	1.55
ψ	2.33	3.11	2.05	0.84
δ	0.9986	0.0022	0.9989	0.0010
Cash Flows				
μ_c	0.0016	0.0008	0.0012	0.0007
ρ	0.9813	0.0103	0.9812	0.0086
φ_e	0.0392	0.0229	0.0306	0.0160
σ_0	0.0081	0.0010	0.0073	0.0015
ν			0.9983	0.0021
σ_w			2.62e-6	3.10e-6
μ_d	0.0024	0.0023	0.0020	0.0017
ϕ_d	4.34	0.73	4.45	1.63
φ_d	4.96	1.31	5.00	1.39
ϱ_d	0.42	0.42	0.49	0.33
Aggregation				
h	10	1.97	11	2.17
χ^2-test	59.60		9.98	
p-value	0.00		0.13	

Table II presents parameter estimates and χ^2-test of overidentifying restrictions for two models. The "No-Vol Model" allows for time-variation in conditional means but rules out variation in conditional second moments of consumption and dividend growth rates. The "LRR Model" is the long-run risk model that incorporates both persistent expected growth and time-varying volatility in cash flows. The models are estimated via GMM using annual data from 1930 till 2009 and taking into account the effect of time-aggregation. The set of moment conditions used in estimation is given in Table III.

Table III
Model Estimates: Moments

Moments	**No-Vol Model**			**LRR Model**		
	Sample	Model	t(diff)	Sample	Model	t(diff)
Consumption & Dividends						
vol(Δc^a_τ)	0.019	0.026	-1.07	0.019	0.023	-0.64
AC1(Δc^a_τ)	0.386	0.484	-0.45	0.386	0.424	-0.17
AC2(Δc^a_τ)	0.160	0.272	-0.55	0.160	0.199	-0.20
vol(Δd^a_τ)	0.111	0.125	-0.34	0.111	0.112	-0.03
AC1(Δd^a_τ)	0.205	0.445	-1.52	0.205	0.396	-1.20
corr($\Delta c^a_\tau, \Delta d^a_\tau$)	0.531	0.614	-0.27	0.531	0.620	-0.29
E(η^a_τ)	0.003	0.000	0.66	0.001	0.000	0.09
E(u^a_τ)	-0.018	0.000	-0.94	-0.038	0.000	-0.95
E($\eta^a_\tau x_{\tau-2}$)	-5.2e-5	0.000	-3.54	-2.8e-4	0.000	-2.00
E($[\eta^{a^2}_\tau - E_{\tau-2}\eta^{a^2}_\tau]\sigma^2_{\tau-2}$)				1.0e-6	0.000	1.18
vol($\eta^{a^2}_\tau$)	0.001	0.000	2.45	0.010	0.001	1.08
AC1($\eta^{a^2}_\tau$)	0.240	0.082	1.57	0.593	0.173	0.81
Asset Prices						
E($z^a_{d,\tau}$)	3.390	3.367	0.29	3.390	3.369	0.26
vol($z^a_{d,\tau}$)	0.419	0.330	0.85	0.419	0.455	-0.34
AC1($z^a_{d,\tau}$)	0.832	0.803	0.12	0.832	0.920	-0.38
E($R^a_{d,\tau} - R^a_{f,\tau}$)	0.086	0.069	0.88	0.086	0.064	1.18
vol($r^a_{d,\tau}$)	0.178	0.220	-1.28	0.178	0.196	-0.54
E($r^a_{f,\tau}$)	0.003	0.012	-2.12	0.003	0.011	-1.78
Predictability						
corr($r^a_{d,\tau}, z^a_{d,\tau-1}$)	-0.230	0.029	-2.41	-0.230	-0.071	-1.48
corr($\Delta c^a_\tau, z^a_{d,\tau-1}$)	0.184	0.626	-2.53	0.184	0.303	-0.69

Table III presents sample- and model-based moments, computed at the parameter estimates reported in Table II, and t-statistics for the difference between sample and model-implied moments. The "No-Vol Model" allows for time-variation in conditional means but rules out variation in conditional second moments of consumption and dividend growth rates. The "LRR Model" is the long-run risk model that incorporates both persistent expected growth and time-varying volatility in cash flows. E($\cdot$), vol($\cdot$), AC1($\cdot$), AC2($\cdot$), corr($\cdot,\cdot$) denote the mean, standard deviation, first- and second-order autocorrelations, and correlation respectively. Δc^a_τ and Δd^a_τ denote time-aggregated annual consumption and dividend growth rates. η^a_τ and u^a_τ correspond to innovations into annual consumption and dividend growth, respectively. The annual price-dividend ratio, $z^a_{d,\tau}$, is defined as the log of the end of year price over the twelve-month trailing sum of dividends. $r^a_{d,\tau} \equiv \log(R^a_{d,\tau})$ is the continuously compounded annual return of the aggregate market, and $r^a_{f,\tau} \equiv \log(R^a_{f,\tau})$ is the logarithm of the annual risk-free rate. x_τ and σ_τ are end of year-τ expected growth and conditional volatility of consumption growth, respectively.

Table IV

No Time-Aggregation: Parameter Estimates of the Annual Specification

Parameters	**Empirical Data**		**Simulated Data**			
	Estimate	SE	Population	5%	50%	95%
Preferences						
γ	14.00	7.32	12.76	6.30	14.10	18.46
ψ	1.10	1.48	1.15	0.80	1.18	1.71
δ	0.9922	0.0320	0.9967	0.9872	0.9939	0.9967
Cash Flows						
μ_c	0.0155	0.0080	0.0189	0.0141	0.0179	0.0244
ρ	0.9097	0.1025	0.8917	0.6617	0.8927	0.9524
φ_e	0.1677	0.1355	0.2012	0.0566	0.1702	0.2415
σ_0	0.0242	0.0028	0.0197	0.0036	0.0200	0.0255
ν	0.9195	0.1637	0.7657	0.6729	0.8215	0.8904
σ_w	8.47e-6	1.32e-5	1.65e-5	8.92e-6	1.55e-5	2.06e-5
μ_d	0.0144	0.0165	0.0026	0.0000	0.0077	0.0171
ϕ_d	2.60	2.00	2.14	1.87	2.83	4.04
φ_d	3.98	0.86	4.25	3.73	4.59	6.54
ϱ_d	0.36	0.30	0.30	0.19	0.31	0.48
χ^2-test	100.66			18.94	105.04	676.98
p-value	0.000			0.000	0.000	0.008

Table IV presents the estimated parameters and the χ^2-test of overidentifying restrictions of the annual specification of the long-run risks model. The estimation does *not* account for time-aggregation of the data, and both the sampling frequency and the decision interval are assumed to be annual. The first set of columns, under the heading "Empirical Data", provides the 1930-2009 sample estimates and their standard errors. The second set of columns reports the corresponding estimates in time-aggregated simulated data that have been generated at the parameter estimates of the LRR model in Table II. Population estimates are based on a long sample of simulated data, the 5, 50 and 95 percentiles characterize the finite-sample distribution of annual estimates.

Table V

No Time-Aggregation: Moments of the Annual Specification

Moments	**Annual Specification**		
	Sample	Model	t(diff)
<u>Consumption & Dividends</u>			
$\mathrm{vol}(\Delta c^a_\tau)$	0.019	0.026	-1.05
$\mathrm{AC1}(\Delta c^a_\tau)$	0.386	0.128	1.19
$\mathrm{AC2}(\Delta c^a_\tau)$	0.160	0.116	0.22
$\mathrm{vol}(\Delta d^a_\tau)$	0.111	0.100	0.29
$\mathrm{AC1}(\Delta d^a_\tau)$	0.205	0.059	0.92
$\mathrm{corr}(\Delta c^a_\tau, \Delta d^a_\tau)$	0.531	0.422	0.35
$\mathrm{E}(\eta^a_\tau)$	-0.001	0.000	-0.08
$\mathrm{E}(u^a_\tau)$	-0.004	0.000	-0.74
$\mathrm{E}(\eta^a_\tau x_{\tau-2})$	-0.002	0.000	-2.22
$\mathrm{E}([\eta^{a^2}_\tau - E_{\tau-2}\eta^{a^2}_\tau]\sigma^2_{\tau-2})$	1.0e-5	0.000	1.11
$\mathrm{vol}(\eta^{a^2}_\tau)$	0.009	0.024	-1.95
$\mathrm{AC1}(\eta^{a^2}_\tau)$	0.433	7.3e-7	1.09
<u>Asset Prices</u>			
$\mathrm{E}(z^a_{d,\tau})$	3.390	3.351	0.49
$\mathrm{vol}(z^a_{d,\tau})$	0.419	0.137	2.69
$\mathrm{AC1}(z^a_{d,\tau})$	0.832	0.910	-0.34
$\mathrm{E}(R^a_{d,\tau} - R^a_{f,\tau})$	0.086	0.042	2.35
$\mathrm{vol}(r^a_{d,\tau})$	0.178	0.111	2.09
$\mathrm{E}(r^a_{f,\tau})$	0.003	0.013	-2.37
<u>Predictability</u>			
$\mathrm{corr}(r^a_{d,\tau}, z^a_{d,\tau-1})$	-0.230	0.079	-2.88
$\mathrm{corr}(\Delta c^a_\tau, z^a_{d,\tau-1})$	0.184	0.374	-1.09

Table V presents sample and population moments of the annual specification, and t-statistics for their differences. Population moments are computed at the parameter estimates reported in Table IV. The estimation does *not* account for time-aggregation, and both the sampling frequency and the decision interval are assumed to be annual. $\mathrm{E}(\cdot)$, $\mathrm{vol}(\cdot)$, $\mathrm{AC1}(\cdot)$, $\mathrm{AC2}(\cdot)$, $\mathrm{corr}(\cdot,\cdot)$ denote the mean, standard deviation, first- and second-order autocorrelations, and correlation respectively. Δc^a_τ and Δd^a_τ denote annual consumption and dividend growth rates. η^a_τ and u^a_τ correspond to innovations into annual consumption and dividend growth, respectively. $z^a_{d,\tau}$ is the log of the annual price-dividend ratio, $r^a_{d,\tau} \equiv \log(R^a_{d,\tau})$ is the continuously compounded annual return of the aggregate market, and $r^a_{f,\tau} \equiv \log(R^a_{f,\tau})$ is the logarithm of the annual risk-free rate. x_τ and σ_τ are end of year-τ expected growth and conditional volatility of consumption growth, respectively.

Table VI
Model Estimates with Fixed Decision Frequency

Parameters	Bi-weekly	Monthly	Quarterly	Annual
Preferences				
γ	6.45	7.13	9.20	14.00
ψ	1.79	2.08	1.42	1.10
δ	0.9995	0.9990	0.9967	0.9922
Cash Flows				
μ_c	0.0008	0.0016	0.0044	0.0155
ρ	0.9932	0.9822	0.9720	0.9097
φ_e	0.0124	0.0293	0.0460	0.1677
σ_0	0.0050	0.0073	0.0144	0.0242
ν	0.9994	0.9987	0.9870	0.9195
σ_w	6.15e-7	2.05e-6	3.64e-6	8.47e-6
μ_d	0.0008	0.0016	0.0040	0.0144
ϕ_d	3.97	3.83	3.10	2.60
φ_d	4.63	4.49	4.98	3.98
ϱ_d	0.48	0.43	0.43	0.36
χ^2-test	16.61	13.97	47.22	100.66
p-value	0.02	0.05	0.00	0.00

Table VI presents parameter estimates and χ^2-test of overidentifying restrictions for alternative frequency specifications of the long-run risks model. In "Bi-weekly", "Monthly", "Quarterly" and "Annual" specifications the time-aggregation parameter, h, is set at 26, 12, 4 and 1 respectively. The model parameters are estimated via GMM using annual data from 1930 till 2009 and, save for the annual specification, taking into account the effect of time-aggregation.

Table VII

Model Implications for the Cross-Section of Returns

Panel A: Cross-Sectional Estimates and Risk Premia

	Small		**Large**		**Growth**		**Value**	
Parameters								
$\mu_j(\%)$	0.47	(0.06)	0.12	(0.04)	0.18	(0.05)	0.33	(0.05)
ϕ_j	7.92	(1.87)	4.45	(0.80)	4.25	(0.73)	8.23	(2.40)
φ_j	7.30	(3.76)	3.88	(1.34)	5.58	(1.41)	5.46	(5.89)
ϱ_j	0.48	(1.02)	0.22	(0.24)	0.21	(0.24)	0.47	(1.34)
Risk Premia(%)								
Data	13.83		6.92		6.74		12.34	
Model	11.77		5.29		5.41		10.53	

Panel B: CAPM Implications

	Small–Large		**Value–Growth**	
	Data	Model	Data	Model
β^{CAPM}	0.57	0.71	0.31	0.59
$\alpha^{CAPM}(\%)$	2.67	1.96	3.28	1.35

Panel A of Table VII presents estimated parameters of dividend dynamics for the top and bottom quintile portfolios sorted by size (large and small) and book-to-market characteristic (value and growth), and their risk premia in the data and implied by the model. Panel B provides the CAPM betas and alphas for the small-minus-large and value-minus-growth strategies.

Table VIII

LRR Model Estimates based on Extended Set of Moments

Panel A: Parameter Estimates

Parameters	Estimate	SE
Preferences		
γ	8.19	3.58
ψ	1.96	0.77
δ	0.9991	0.0011
Cash Flows		
μ_c	0.0010	0.0010
ρ	0.9809	0.0154
φ_e	0.0283	0.0100
σ_0	0.0066	0.0022
ν	0.9986	0.0020
σ_w	2.49e-06	2.60e-06
μ_d	0.0014	0.0016
ϕ_d	4.25	2.08
φ_d	4.76	1.75
ϱ_d	0.47	0.38
Aggregation		
h	11	3.72
χ^2-test	14.06	
p-value	0.08	

Panel B: Moments

Moments	Sample	Model	t(diff)
Consumption & Dividends			
vol(Δc^a_τ)	0.019	0.020	-0.27
AC1(Δc^a_τ)	0.386	0.403	-0.08
AC2(Δc^a_τ)	0.160	0.175	-0.08
vol(Δd^a_τ)	0.111	0.095	0.40
AC1(Δd^a_τ)	0.205	0.378	-1.09
corr($\Delta c^a_\tau, \Delta d^a_\tau$)	0.531	0.590	-0.19
E(η^a_τ)	0.000	0.000	-0.03
E(u^a_τ)	-0.047	0.000	-1.25
E($\eta^a_\tau x_{\tau-2}$)	-2.6e-04	0.000	-2.11
E($[\eta^{a^2}_\tau - E_{\tau-2}\eta^{a^2}_\tau]\sigma^2_{\tau-2}$)	8.3e-07	0.000	1.19
vol($\eta^{a^2}_\tau$)	0.009	0.001	1.09
AC1($\eta^{a^2}_\tau$)	0.610	0.213	0.75
Asset Prices			
E($z^a_{d,\tau}$)	3.390	3.367	0.29
vol($z^a_{d,\tau}$)	0.419	0.446	-0.26
AC1($z^a_{d,\tau}$)	0.832	0.943	-0.48
E($R^a_{d,\tau} - R^a_{f,\tau}$)	0.086	0.055	1.67
vol($r^a_{d,\tau}$)	0.178	0.165	0.43
E($r^a_{f,\tau}$)	0.003	0.008	-1.14
Predictability			
corr($r^a_{d,\tau}, z^a_{d,\tau-1}$)	-0.230	-0.096	-1.24
corr($\Delta c^a_\tau, z^a_{d,\tau-1}$)	0.184	0.230	-0.27
corr($r^a_{d,\tau}, z^a_{d,\tau-3}$)	-0.379	-0.166	-1.30
corr($\Delta c^a_\tau, z^a_{d,\tau-3}$)	0.067	0.242	-0.95

Table VIII presents model estimates based on the moment restrictions augmented by long-run predictability moments. The model is estimated via GMM using annual data from 1930 till 2009 and taking into account the effect of time-aggregation.

Table A.1

Approximation Error

Panel A: Approximate Analytical Solutions

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	3.592	4.754	5.058	0.059	0.021	0.032
RA	**10**	3.789	4.572	4.716	0.060	0.021	0.032
	15	4.055	4.421	4.470	0.062	0.021	0.032

Panel B: Numerical Solutions

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	3.594	4.755	5.060	0.059	0.021	0.032
RA	**10**	3.788	4.576	4.724	0.060	0.021	0.032
	15	4.033	4.436	4.493	0.061	0.021	0.031

Panel C: Approximation Error (as a % of numerical values)

		Mean log(P/C)			Vol log(P/C)		
		IES			IES		
		0.5	**1.5**	**2**	**0.5**	**1.5**	**2**
	5	0.05	0.01	0.03	0.04	-0.16	-0.17
RA	**10**	-0.02	0.10	0.17	-0.42	-0.83	-0.86
	15	-0.54	0.32	0.51	-1.84	-2.16	-2.17

Table A.1 illustrates differences between analytical and numerical solutions for the log of the price to consumption ratio for various configurations of risk aversion (RA) and intertemporal elasticity of substitution (IES).

Table A.2

LRR Model Estimates using Quarterly Data: Parameters

Parameters	Estimate	SE
Preferences		
γ	7.02	6.59
ψ	1.62	0.66
δ	0.9989	0.0006
Cash Flows		
μ_c	0.0016	0.0016
ρ	0.9833	0.0139
φ_e	0.0470	0.0295
σ_0	0.0035	0.0008
ν	0.9963	0.0022
σ_w	3.38e-6	6.45e-6
μ_d	0.0018	0.0023
ϕ_d	3.95	2.14
φ_d	5.43	1.20
ϱ_d	0.08	0.10
Aggregation		
h	2	1.05
χ^2-test	13.94	
p-value	0.03	

Table A.2 presents parameter estimates and the χ^2-test of overidentifying restrictions of the long-run risk model based on the post-war quarterly data. The model is estimated via GMM using quarterly data from 1948 till 2009 and taking into account the effect of time-aggregation.

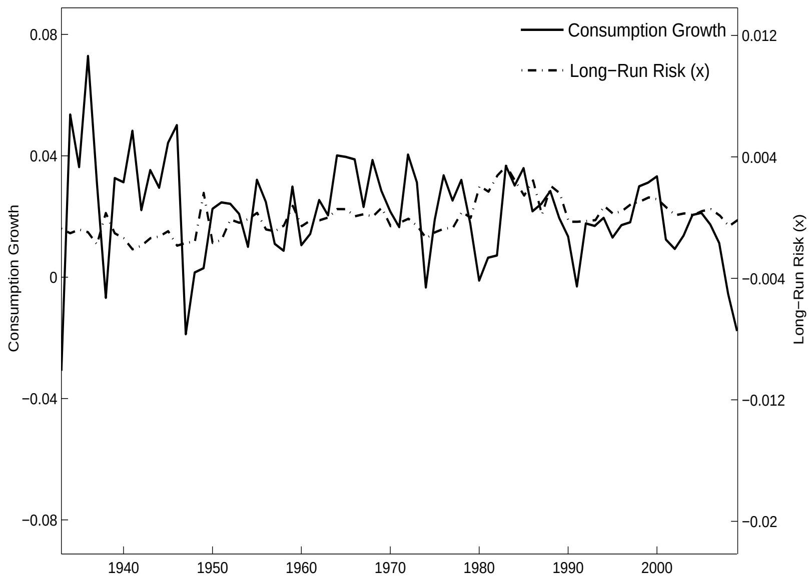

Figure 1. Realized and Expected Growth of Consumption

Figure 1 plots time series of realized (solid line) and expected (dash line) growth in consumption. Consumption is defined as the per-capita expenditure on non-durables and services. The data are real, sampled on an annual frequency and cover the period from 1930 to 2009.

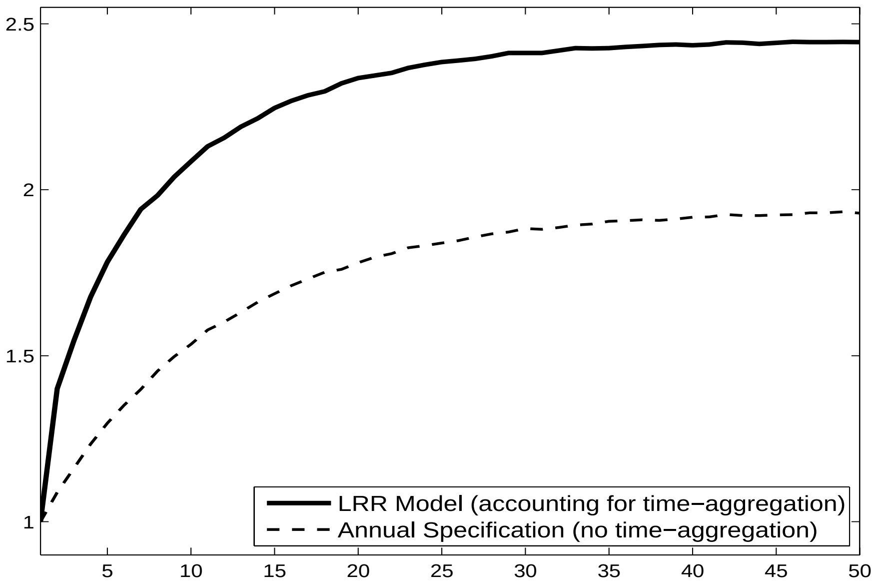

Figure 2. Cumulative Impulse Response of Consumption Growth

Figure 2 plots the cumulative impulse response of annual consumption growth to a unit shock constructed by fitting an ARMA model to a long sample of simulated annual data. The data are simulated at the point estimates of the LRR model (solid line) and those of the annual specification (dashed line).

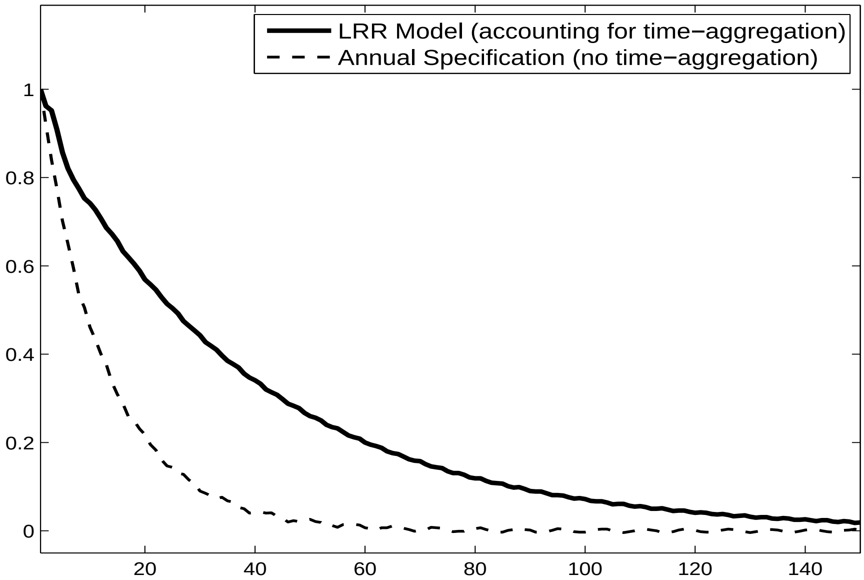

Figure 3. Impulse Response Function of the Variance of Consumption Growth

Figure 3 plots the impulse response of the conditional variance of consumption growth to a unit shock constructed by fitting an ARMA model to a long sample of simulated data. The variance of consumption growth is simulated at the point estimates of the LRR model (solid line) and those of the annual specification (dashed line).