NBER WORKING PAPER SERIES

MONEY AND INTEREST IN A CASH-IN-ADVANCE ECONOMY

Robert E. Lucas, Jr.

Nancy L. Stokey

Working Paper No. 1618

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
May 1985

We wish to thank Jean-Michel Grandmont, Rodolfo Manuelli, Edward Prescott and Thomas Sargent for helpful comments. This work was supported by grants from the National Science Foundation, and the Center for Advanced Studies in Managerial Economics and Decision Sciences at Northwestern University. The research reported here is part of the NBER's research program in Economic Fluctuations. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Working Paper #1618
May 1985

Money and Interest in a
Cash-in-Advance Economy

ABSTRACT

In this paper we analyze an aggregative general equilibrium model in which the use of money is motivated by a cash-in-advance constraint, applied to purchases of a subset of consumption goods. The system is subject to both real and monetary shocks, which are economy-wide and observed by all. We develop methods for verifying the existence of, characterizing, and explicitly calculating equilibria. A main result of the analysis is that current money growth affects the current real allocation only insofar as it affects expectations about future money growth, i.e., only through its value as a signal.

Robert E. Lucas, Jr.
Department of Economics
University of Chicago
1126 East 59th Street
Chicago, IL 60637

Nancy L. Stokey
Department of Managerial Economics
J.L. Kellogg Graduate School of Management
Northwestern University
Evanston, IL 60201

1. Introduction

In this paper we analyze an aggregative general equilibrium model in which the use of money is motivated by a Clower [1967]-type cash-in-advance constraint, applied to purchases of a subset of consumption goods. This system is subject to both real and monetary shocks, which are economy-wide and observed by all. The model is designed to study how the behavior of equilibrium quantities and prices (including interest rates) depends upon the stochastic processes generating the rate of growth of the money supply and the level of real output.

One way to think of the paper is as a contribution to the theory of interest. The model captures the real and nominal determinants of interest rates in a way that reproduces the familiar Fisherian formulas in deterministic contexts, and also shows how these formulas need to be modified in a wide class of stochastic environments. This motivation is shared with Lucas [1982] and Svensson [1983], but in those two papers the equilibrium resource allocations were determined entirely by the exogenously given goods endowments, so that the analysis involved determining the behavior of prices *given* quantities. In this paper, as in Lucas and Stokey [1983] and Lucas [1984], agents have possibilities for substituting against money that were not present in Lucas [1982] or Svensson [1983]. Therefore, money shocks induce real distortions, so that equilibrium quantities and prices must be determined simultaneously. A main objective of this paper is to deal with the technical problems raised by this simultaneity.

Another closely related paper is Townsend [1984]. In the model there, as in the one here, agents hold cash to carry out transactions, and monetary policy affects the real allocation. The models differ in that Townsend's allows capital accumulation, while ours provides richer possibilities for the timing of information arrival and trade. In addition, the techniques used differ considerably: the recursive methods used here require somewhat stronger assumptions on preferences, but allow a sharper characterization of the equilibrium and are very amenable to numerical simulations.

To keep the study of positive questions simple, we will be abstracting from most of the issues that make the normative study of monetary policy difficult (and interesting). In particular, business cycles originating in monetary disturbances will not be studied, and fiscal authorities will be assumed to have access to lump-sum, non-distorting taxes. The only distortion present in the system will be the "inflation tax" so that optimal monetary policies will, in all cases, be those that set this "tax" equal to zero in all circumstances. Characterizing these will be a relatively simple by-product of the analysis, the main focus of which will be on determining the allocative consequences of arbitrary policies.

In section 2, the model is set out and the problem of solving for the equilibrium is reduced to the study of a functional equation in a variable that may be thought of as the value of cash balances. Two existence theorems for solutions to this equation are offered in section 3. One is based on the Arzela-Ascoli lemma and the Schauder fixed point theorem. The other, using stronger conditions on preferences, is based on the contraction mapping theorem, and also establishes the uniqueness of equilibria. Section 4 develops some properties of equilibria. In section 5, a number of examples are studied under more specific assumptions about shocks and preferences. Concluding remarks are given in section 6.

2. The Model

The model[1] is formulated in discrete time with an infinite horizon. Each period is in turn divided into two subperiods, which will correspond to the structure of trading. Shocks to the system in any period are denoted by $(s_1,s_2) = s \in S = S_1 \times S_2 \subseteq \mathbb{R}^n$, where s_1 and s_2 are shocks that occur in the first and second subperiods respectively. The shocks form a first-order Markov process with a stationary transition function $\pi(s,A)$. Specifically, let $\mathbf{S_1}$ and $\mathbf{S_2}$ denote the families of Borel sets of S_1 and S_2, respectively, let s and s' denote shocks in successive periods, and let

$$\pi_1(s,A_1) = \Pr\{s_1' \in A_1 | s\}, \qquad s \in S,\ A_1 \in \mathbf{S}_1,$$

and

$$\pi_2(s,s_1',A_2) = \Pr\{s_2' \in A_2 | s,s_1'\}, \qquad s \in S,\ s_1' \in S_1,\ A_2 \in \mathbf{S_2}.$$

Then

$$\pi(s,\ A_1 \times A_2) = \Pr\{s' \in A_1 \times A_2 | s\}$$

$$= \int_{A_1} \pi_1(s,ds_1')\pi_2(s,s_1',A_2), \quad s \in S,\ A_1 \in \mathbf{S_1},\ A_2 \in \mathbf{S_2},$$

defines the transition function for the process.

Within each period two rounds of trading occur. In the first subperiod agents trade securities, and in the second they trade goods. In the securities market agents make portfolio decisions, including a decision about the size of their cash balances, and in the goods market they make consumption decisions. There are two consumption goods available each period: "cash goods," which are subject to a Clower (cash-in-advance) constraint, and "credit goods," which are not. Thus, an agent's consumption decision in the goods market is constrained by the fact that his purchases of cash goods must be financed out of currency acquired during securities trading earlier in that period. It is not possible to acquire additional currency once goods trading has begun, or to use currency acquired from the contemporaneous sale of endowment goods. For sellers, cash goods sales result in currency receipts that simply accumulate during the period and are carried as overnight balances, while credit goods sales result in invoices. Both overnight balances and invoices become cash available for spending during securities trading on the following day.

There is a single, infinitely-lived "representative consumer." His consumption of cash and credit goods are c_{1t} and c_{2t} respectively, and his preferences are

$$E\left\{\sum_{t=0}^{\infty} \beta^t U(c_t)\right\},$$

where $0 < \beta < 1$, $c_t = (c_{1t}, c_{2t})$, and the expectation is over realizations of the shocks. We assume that U is continuously differentiable, strictly increasing, and strictly concave. Other restrictions will be added in the next section, when existence and uniqueness of an equilibrium are discussed.

Goods are not storable, and the technology each period is simply

$$c_1 + c_2 \leq y,$$

where y(s), the endowment, is a function of the current shock. Because receipts from the sale of either good in any period are carried over to the securities market in the following period, it is clear that in each period cash and credit goods will sell at the same nominal price.

The only activity of the government in this economy is to supply money, and the money growth factor in any period t is a fixed function $g(s_1)$ of the shock s_1 in the securities market in that period. This convention fixes the timing of monetary injections and withdrawals, which always occur at the time of securities trading, and are accomplished via lump-sum transfers and taxes.[2] Therefore, in terms of the previous period's money supply, the transfer received (tax paid if negative) in the securities market in any period is $g(s_1) - 1$.

Note that since s is a vector of arbitrary (but finite) length, the specification of the endowment process and monetary "policy" is extremely flexible. In particular, s_1 and/or s_2 may include lagged values of the endowment and the rate of money growth, signals about future values of these variables, and pure "noise" components that serve as randomizing devices.

We will motivate a definition of a stationary equilibrium, in which prices and quantities are fixed functions of the state of the system. To do so, we begin with the decision problem facing an agent engaged in securities trading. Suppose that his assets, after the current tax or transfer, are $\underline{a}$ relative to the economy-wide average, which we have normalized to unity. His information about current and future states consists of last period's state, $\bar{s}$, and his knowledge about the current state, s_1. His immediate problem is to divide his assets between cash balances $m \geq 0$ and purchases (sales) at the price $q(\bar{s},s_1)$ of claims to dollars one period hence. His budget constraint for this portfolio problem is then

$$(2.1) \qquad m + q(\bar{s},s_1)\, b - a = 0.$$

After securities trading is concluded, the agent holds the portfolio (m,b). Before trading in goods, he learns s_2, so that $s = (s_1,s_2)$ is sufficient for forecasting future states and last period's state $\bar{s}$ becomes redundant information. At this point, he purchases goods $c = (c_1,c_2)$ at a price $p(s)$ (expressed as a ratio to the current period's money supply) subject to the cash constraint

$$p(s)c_1(s) - m \leq 0, \qquad \text{all } s. \tag{2.2}$$

These purchases together with the sale of his endowment $y(s)$, also determine his asset position, $z(s)$, before the tax or transfer in next period's securities market, so that his budget constraint in the goods market is:

$$z(s) - m - b - p(s)[y(s) - c_1(s) - c_2(s)] = 0, \quad \text{all } s. \tag{2.3}$$

Let $F(a,\bar{s},s_1)$ be the value of the maximized objective function for a consumer beginning securities trading with assets $\underline{a}$ when the economy is in state $(\bar{s},s_1)$. Then F must be the value function for the two-stage maximum problem

$$\begin{aligned} F(a,\bar{s},s_1) \equiv \max_{m,b} \int_{S_2} \Big[& \max_{c(s),z(s)} \Big\{ U(c(s)) \\ & + \beta \int_{S_1} F([z(s) + g(s_1') - 1]/g(s_1'),s,s_1')\, \pi_1(s,ds_1')\Big\}\Big] \pi_2(\bar{s},s_1,ds_2), \end{aligned} \tag{2.4}$$

where for each s, the choice $(c(s),z(s))$ must satisfy (2.2) and (2.3) given goods prices $p(s)$, and the choice (m,b) must satisfy (2.1), given the bond price $q(\bar{s},s_1)$.

A stationary equilibrium for this system consists of bond prices $q(\bar{s},s_1)$, bond holdings $b(\bar{s},s_1)$, cash balances $m(\bar{s},s_1)$, goods prices $p(s)$, and consumption allocations $c(s) = (c_1(s),c_2(s))$, defined for all $\bar{s} \in S$, $s \in S$, and satisfying the following conditions:

(2.5a) $c_1(s) + c_2(s) = y(s), \quad \text{all } s;$

(2.5b) $m(\bar{s},s_1) = 1, \quad \text{all } (\bar{s},s_1),$

(2.5c) $b(\bar{s},s_1) = 0, \quad \text{all } (\bar{s},s_1),$

and

(2.5d) for $a = 1$, and for each $\bar{s} \in S$ and $s_1 \in S_1$, $(m(\bar{s},s_1),\ b(\bar{s},s_1),\ c(s))$ maximizes (2.4) subject to (2.1)-(2.3), given $q(\bar{s},s_1)$ and $p(s)$.

These conditions are standard: (2.5d) requires that m, b, and c be the demands of a "representative consumer," (that is, one with relative assets equal to unity) at the equilibrium prices, and conditions (2.5a-c) require that with these demands, the goods, money and bond markets clear.

The first-order conditions for the two maximum problems in (2.4) (with the market-clearing condition (2.5b) imposed) are:

(2.6) $U_1(c(s)) - p(s)[v(s) + w(s)] = 0, \qquad \text{all } s;$

(2.7) $U_2(c(s)) - p(s)v(s) = 0, \qquad \text{all } s;$

(2.8) $p(s)c_1(s) - 1 \leq 0$, with equality if $w(s) > 0$, all s;

(2.9) $\beta\int_{S_1} F_a\left(\frac{[z(s) + g(s_1') - 1]}{g(s_1')}, s, s_1'\right)\frac{1}{g(s_1')}\,\pi_1(s,ds_1') - v(s) = 0, \qquad \text{all } s;$

(2.10) $-\lambda + \int_{S_2} [v(s) + w(s)]\pi_2(\bar{s},s_1,ds_2) = 0, \qquad \text{all } (\bar{s},s_1);$

$$(2.11) \qquad -\lambda q(\bar{s},s_1) + \int_{S_2} v(s)\pi_2(\bar{s},s_1,ds_2) = 0, \qquad \text{all } (\bar{s},s_1);$$

where λ, $w(s)$ and $v(s)$ are the multipliers associated with (2.1), (2.2), and (2.3), respectively. In addition, the envelope condition for (2.4) is

$$(2.12) \qquad F_a(a,\bar{s},s_1) = \lambda.$$

As a first step in solving for an equilibrium, use (2.12) to eliminate λ from (2.10)

$$(2.13) \qquad F_a(a,\bar{s},s_1) = \int_{S_2} [v(s) + w(s)]\pi_2(\bar{s},s_1,ds_2).$$

Then, substitute from (2.5a)-(2.5c) into (2.3) to find that $z(s) = 1$, all s, and recall that $a = 1$ as well. Therefore, substituting from (2.13) into (2.9) we find that

$$(2.14) \qquad v(s) = \beta\int_S \frac{[v(s') + w(s')]}{g(s_1')} \pi(s,ds').$$

Equation (2.14), together with conditions (2.6)-(2.8) and (2.5a) form a system of five equations in the five unknown functions $v(s)$, $w(s)$, $p(s)$, $c_1(s)$ and $c_2(s)$. (Equation (2.11) then determines $q(\bar{s},s_1)$ in terms of these other variables.) In the next section, we turn our attention to the existence and uniqueness of functions satisfying this system.

3. Existence of Equilibrium

The economy described in the preceding section is specified by the current period utility function $U: \mathbb{R}_+^2 \to \mathbb{R}$, the discount factor $\beta \in (0,1)$, the state space S, the transition function π: $S \times S \to [0,1]$, and the functions g: $S_1 \to \mathbb{R}_+$ and y: $S \to \mathbb{R}_+$ governing money growth and endowments. We will study the existence and nature of equilibria under the following restrictions.

Assumption I: S is compact.

Assumption II: Both g and y are continuous in s and bounded away from zero.

Note that under Assumptions I and II, $g(s_1)$ and $y(s)$ take values in closed intervals $[\underline{g},\bar{g}]$ and $[\underline{y},\bar{y}]$, with $\underline{g} > 0$ and $\underline{y} > 0$.

Assumption III: For any $\varepsilon > 0$ there exists some $\delta(\varepsilon) > 0$ such that

$$\|s - s'\| < \delta(\varepsilon) \Rightarrow \int_S |\Delta(s,s',ds'')| < \varepsilon,$$

where Δ: $S \times S \times S \to [-1,1]$ is defined by

$$\Delta(s,s',A) = \pi(s,A) - \pi(s',A).$$

Assumption III implies that for any continuous function $f:S \to \mathbb{R}$, $\int_S f(s')\pi(s,ds')$ is a continuous function of s.

Assumption IV: For each $s \in S$, $0 < \beta\int_{S_1} \frac{1}{g(s_1')} \pi_1(s,ds_1') \leq 1$.

Assumption V: U is continuously differentiable, strictly increasing and strictly concave, and for all $y \in [\underline{y},\bar{y}]$

$$\lim_{c\to 0} cU_2(c,y - c) = 0,$$

$$\lim_{c\to y} cU_2(c,y - c) = \infty,$$

$$\max_{\substack{c \in [0,y] \\ y \in [\underline{y},\bar{y}]}} cU_1(c,y-c) \equiv A < \infty,$$

and

$$\lim_{c \to 0} U_1(c,y-c)/U_2(c,y-c) > 1.$$

Assumption VI: For all $y \in [\underline{y},\bar{y}]$, $cU_2(c,y-c)$ is strictly increasing in c.

Our strategy for proving existence of an equilibrium is first to use (2.5a), (2.6)-(2.8) to eliminate w(s') from (2.15) as described in Lemmas 1 and 2. Then (2.15) becomes a functional equation in the single function v(s), the properties of which we will develop in Theorems 1-5.

For fixed $v > 0$ and $y \in [\underline{y},\bar{y}]$, equations (2.5a), (2.6), (2.7), and (2.8) are simply four equations in c_1, c_2, w, and p: the values of the equilibrium functions (c(s), w(s), p(s)) when v = v(s) and y = y(s). Use (2.7) to eliminate p and (2.5a) to eliminate c_2, so that for each s, (y,v,w,c_1) must satisfy

$$(3.1) \qquad \frac{U_1(c_1, y-c_1)}{U_2(c_1, y-c_1)} = \frac{v+w}{v},$$

$$(3.2) \qquad c_1 U_2(c_1, y-c_1) \leq v, \text{ with equality if } w > 0.$$

Therefore, an equilibrium is characterized by functions v(s), w(s), and $c_1(s)$ satisfying (2.14), (3.1) and (3.2). In Lemmas 1 and 2, this system is further simplified.

Define the function $\hat{c}: \mathbb{R}_+ \times [\underline{y},\bar{y}] \to [0,\bar{y}]$ by

$$(3.3) \qquad \hat{c}(v,y)U_2(\hat{c}(v,y), y - \hat{c}(v,y)) = v.$$

Under Assumptions V and VI, $\hat{c}$ is well defined (see Figure 1). It is continuous and strictly increasing in v and y.

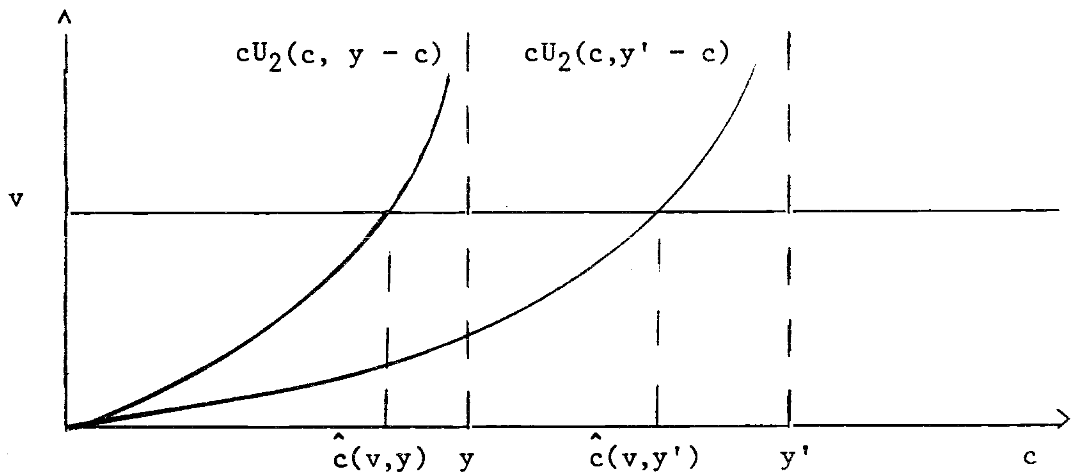

Figure 1

Next, define $c^*: [\underline{y},\bar{y}] \to [0,\bar{y}]$ by

$$(3.4) \qquad U_1(c^*(y),\, y - c^*(y))/U_2(c^*(y),\, y - c^*(y)) = 1.$$

Under Assumption V, c^* is well defined, continuous, and strictly increasing. We are now ready to prove

<u>Lemma 1:</u> Under Assumptions V and VI, for any $v > 0$ and $y \in [\underline{y},\bar{y}]$, there is a unique pair (w,c_1) satisfying (3.1) and (3.2). This solution is:

$$(3.5) \qquad \text{if} \quad U_1(\hat{c}(v,y),\, y - \hat{c}(v,y))/U_2(\hat{c}(v,y),\, y - \hat{c}(v,y)) > 1,$$

$$(3.6) \qquad \text{then} \qquad c_1 = \hat{c}(v,y),$$

(3.7) and $w = c_1 U_1(c_1,\ y - c_1) - v,$

(3.8) and if $U_1(\hat{c}(v,y),\ y - \hat{c}(v,y))/U_2(\hat{c}(v,y),\ y - \hat{c}(v,y)) \leq 1,$

(3.9) then $c_1 = c^*,$

(3.10) and $w = 0.$

Proof: First let (v,y) be given and suppose that $\hat{c}(v,y)$ satisfies (3.5). From (3.6) and (3.3), it follows that (3.2) holds with equality, and then (3.7) implies (3.1). Next, suppose that $\hat{c}(v,y)$ satisfies (3.8). From (3.9), (3.10) and (3.4), it follows that (3.1) holds. Moreover, from (3.8) it follows that

$$U_1(\hat{c},\ y - \hat{c})/U_2(\hat{c},\ y - \hat{c}) \leq 1 = U_1(c^*,\ y - c^*)/U_2(c^*,\ y - c^*).$$

Hence, by the concavity of U it follows that $\hat{c} \geq c^*$ so that by Assumption VI,

$$\hat{c}U_2(\hat{c},\ y - \hat{c}) \geq c^*U_2(c^*,\ y - c^*).$$

Therefore, using (3.3),

$$v = \hat{c}U_2(\hat{c},\ y - \hat{c})$$

$$\geq c^*U_2(c^*,\ y - c^*)$$

$$= c^* U_1(c^*, y - c^*),$$

so that (3.2) holds.

Next let (v,y) be given and suppose that (c_1,w) satisfies (3.1) and (3.2). If $w > 0$, then (3.2) and (3.3) imply that (3.6) holds. Then (3.7) follows from (3.1). But (3.6) and $w > 0$ are consistent with (3.1) only if (3.5) holds. If $w = 0$, then (3.1) and (3.4) imply that (3.9) holds. Then (3.2) implies:

$$c^* U_2(c^*, y - c^*) \leq v = \hat{c} U_2(\hat{c}, y - \hat{c}).$$

Using Assumption VI and the concavity of U, and reversing the argument above, this implies that (3.8) holds. $\square$

Next define $h\colon \mathbb{R}_+ \times [\underline{y},\bar{y}] \to \mathbb{R}_+$ by

$$h(v,y) = \hat{c}(v,y) U_1(\hat{c}(v,y),\ y - \hat{c}(v,y)). \tag{3.11}$$

Lemma 2: Under Assumptions V and VI, $v(s)$, $w(s)$ and $c_1(s)$ satisfy (2.14), (3.1) and (3.2) if and only if $v(s)$ satisfies

$$v(s) = \beta \int_S \max[v(s'),\ h(v(s'), y(s'))] \frac{1}{g(s_1')} \pi(s, ds'), \tag{3.12}$$

and for each s, $w(s)$ and $c_1(s)$ are given by (3.5)-(3.10).

Proof: From Lemma 1 it follows that given $v(s)$, $w(s)$ and $c_1(s)$ satisfy (3.1) and (3.2) if and only if they satisfy (3.5)-(3.10). Choose any s. If $(v(s), y(s))$ satisfies (3.5), then

$$v(s) < v(s) + w(s)$$
$$= c_1(s)U_1(c_1(s),\ y(s) - c_1(s))$$
$$= h(v(s),y(s)),$$

while if $(v(s),y(s))$ satisfies (3.8), then

$$v(s) + w(s) = v(s)$$
$$= \hat{c}(s)U_2(\hat{c}(s),\ y(s) - \hat{c}(s))$$
$$\geq h(v(s),y(s)).$$

Hence

$$v(s) + w(s) = \max[v(s),\ h(v(s),y(s))], \qquad \text{all } s \in S,$$

so that (2.14) can be rewritten as (3.12). $\square$

We are now ready to prove

Theorem 1: Under Assumptions I-VI, there exists a bounded, continuous function v satisfying (3.12), where h is as defined in (3.11). Moreover, v is nonnegative and $\|v\| \leq A$, where A is defined in Assumption V.

Proof:[3] Let $\mathcal{F}$ be the space of bounded, continuous functions $f\colon S \to \mathbb{R}_+$, with the norm $\|f\| = \sup_{s \in S} |f(s)|$. Let $D \subset \mathcal{F}$ be the subset of functions f that are nonnegative and have $\|f\| \leq A$. Define the operator T on D by

$$(Tf)(s) = \beta \int_S \max[f(s'),h(f(s'),y(s'))]\ \frac{1}{g(s_1')}\ \pi(s,ds').$$

Under Assumptions I–II, y is bounded and continuous, and by hypothesis so is f. Under Assumptions V and VI, h is well-defined, is continuous as a function of s', and is bounded below by zero and above by A. Under Assumptions I–II, $1/g(s_1')$ is also bounded and continuous. Hence, for any $f \in D$

$$\beta \max[f(s'), h(f(s'), y(s')]/g(s_1')$$

is a bounded, continuous function of s'. Clearly then, Tf is bounded. Specifically, it is nonnegative, and under Assumption IV it is bounded above by A. Also, from Assumption III it follows that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any $f \in D$,

$$|s - s'| < \delta(\varepsilon) \Rightarrow$$

$$|Tf(s) - Tf(s')|$$

$$= \beta \left| \int_S \max[f(s''), h(f(s''), y(s''))] \frac{1}{g(s_1'')} \Delta(s, s', ds'') \right|$$

$$\leq A \frac{\beta}{g} \int_S |\Delta(s, s', ds'')|$$

$$\leq \varepsilon.$$

Therefore, Tf is continuous in s, so that $T: D \to D$.

Moreover, the last argument also establishes that the family D is equicontinuous; and clearly it is also bounded. Then by the Arzela-Ascoli theorem, D is relatively compact, and consequently every subset of D is relatively compact.

Summing up, D is a nonempty, closed, bounded, convex subset of the Banach space $\mathcal{F}$, and $T: D \to \mathcal{F}$ maps D into itself. Moreover, T is continuous and maps every subset of D into a relatively compact set. Hence, T is a compact operator and, by the Schauder theorem, has a fixed point v in D. Clearly, v satisfies (3.12). □

Theorem 1 does not rule out the possibility of a trivial solution $v(s) \equiv 0$ to (3.12), nor does it insure that any nontrivial solutions exist. A zero solution, which is consistent with Assumptions I-VI, has an economic interpretation as a "barter" equilibrium. It occurs if

$$\lim_{c \to 0} cU_1(c, y - c) = 0,$$

in which case $\hat{c}(0,y) = 0$, all y, and hence $h(0,y) \equiv 0$.[4] The next result is a sufficient condition to rule out trivial solutions.

<u>Theorem 2:</u> Let Assumptions I-VI hold, and assume in addition that for all $y \in [\underline{y},\bar{y}]$

$$\lim_{c \to 0} cU_1(c, y - c) > 0. \tag{3.13}$$

Then, $v \equiv 0$ is not a solution to (3.12).

<u>Proof:</u> Under (3.13), h(v,y) is bounded away from zero, so that if $v \equiv 0$, $Tv > 0$. □

Theorem 1 guarantees the existence of a solution to (3.12), but says nothing about the number of solutions and/or how to compute them. These questions can be answered, at least in part, by exploiting the fact that under additional hypotheses the operator T defined in the proof of Theorem 1 is monotone. In particular, T is monotone if h(v,y) is increasing in v. To insure this, we add

<u>Assumption VII:</u> For each $y \in [\underline{y},\bar{y}]$, $cU_1(c,y - c)$ is nondecreasing in c.

Since under Assumptions V and VI, $\hat{c}(v,y)$ is increasing in v, the addition of Assumption VII implies that $h(v,y)$ is nondecreasing in v.

<u>Theorem 3:</u> Let Assumptions I-VII hold and define the sequences $\{\underline{v}_n\}$ and $\{\bar{v}_n\}$ in D by

$$\underline{v}_0(s) \equiv 0 \quad \text{and} \quad \underline{v}_{n+1} = T\underline{v}_n, \quad n = 0,1,2,\ldots$$

$$\bar{v}_0(s) \equiv A \quad \text{and} \quad \bar{v}_{n+1} = T\bar{v}_n, \quad n = 0,1,2,\ldots$$

Then $\{\underline{v}_n\}$ and $\{\bar{v}_n\}$ converge pointwise to solutions to (3.12) in D, $\underline{v}$ and $\bar{v}$ say, and for any solution v to (3.12),

$$\underline{v} \leq v \leq \bar{v}.$$

<u>Proof:</u> Under Assumptions V-VII, the function h is nondecreasing in v, so that the operator T is monotone: $u,v \in D$ and $u \geq v$ imply $Tu \geq Tv$. Moreover, for all $s \in S$

$$\underline{v}_1 = T\underline{v}_0 \geq 0 \equiv \underline{v}_0$$

and

$$\bar{v}_1 = T\bar{v}_0 \leq A \equiv \bar{v}_0.$$

Hence, by induction, $\bar{v}_{n+1} \geq \bar{v}_n$ and $\underline{v}_{n+1} \leq \underline{v}_n$, all n, and since both sequences take values in [0,A], both converge. As shown in the proof of Theorem 1, both $\{\underline{v}_n\}$ and $\{\bar{v}_n\}$ are equicontinuous families, so that the limit functions $\underline{v}$ and

$\bar{v}$ are both in D.

Finally, if v is any fixed point of T it must satisfy

$$\underline{v}_0 = 0 \leqslant v \leqslant A = \bar{v}_0.$$

Then the monotonicity of T implies

$$\underline{v}_1 = T\underline{v}_0 \leqslant Tv = v \leqslant T\bar{v}_0 = \bar{v}_1,$$

and hence, by induction,

$$\underline{v} = \lim_{n\to\infty} \underline{v}_n \leqslant v \leqslant \lim_{n\to\infty} \bar{v}_n = \bar{v}. \qquad \square$$

Theorem 3 is useful computationally because it provides a way of constructing two solutions, $\underline{v}$ and $\bar{v}$, of (3.12) and, if $\underline{v}$ and $\bar{v}$ should coincide, of verifying that their common value is the only solution.

Our next theorem uses Assumptions I-VII plus one additional restriction on preferences, to establish a sufficient condition for (3.12) to have a nontrivial solution.

<u>Theorem 4.</u> Let Assumptions I-VII hold, and suppose that

$$(3.14) \qquad \lim_{c\to 0} U_2(c,\underline{y} - c)/U_1(c,\underline{y} - c) \;<\; \beta\Gamma \;\equiv\; \min_{s} \beta \int_{S_1} \frac{1}{g(s_1')} \pi(s,ds_1').$$

Then (3.12) has a solution with $v(s) > 0$, all $s \in S$.

<u>Proof:</u> From (3.14) and Assumption V it follows that there exists $\underline{c}$ satisfying

$$U_2(\underline{c},\underline{y} - \underline{c})/U_1(\underline{c},\underline{y} - \underline{c}) = \beta\Gamma.$$

Define $v^* \equiv \underline{c}U_2(\underline{c},\underline{y} - \underline{c})$, so that $\hat{c}(v^*,\underline{y}) \equiv \underline{c}$. Then since $h(v^*,y) \leq A$ and $\beta\Gamma < 1$,

$$\begin{aligned} v^* &\equiv \beta\Gamma\underline{c}U_1(\underline{c},\underline{y} - \underline{c}) \\ &= \beta\Gamma h(v^*,\underline{y}) \\ &\leq A. \end{aligned}$$

We show that the function $\bar{v} = \lim_{n\to\infty} \bar{v}_n$ defined in Theorem 3 is bounded below by v^*. For each n, let

$$a_n = \min_{s\in S} \bar{v}_n(s).$$

Since h is increasing in v and y, it follows that for all n,s,

$$\begin{aligned} \bar{v}_{n+1}(s) &= \beta\int_S \frac{\max[\bar{v}_n(s'),h(\bar{v}_n(s'),y(s'))]}{g(s_1')} \pi(s,ds') \\ &\geq \beta\int_S \frac{h(a_n,\underline{y})}{g(s_1')} \pi(s,ds') \\ &\geq \beta\Gamma h(a_n,\underline{y}). \end{aligned}$$

Since $a_0 \equiv A \geq v^*$, it follows by induction that

$$a_{n+1} \geq \beta\Gamma h(a_n,\underline{y}) \geq \beta\Gamma h(v^*,\underline{y}) = v^*, \qquad \text{all } n,$$

and hence $\bar{v}(s) \geq v^*$, all s. []

Theorems 2 and 4 still allow the coexistence of both zero and strictly positive solutions, as the following example shows. Let

$$U(c_1, c_2) = c_1^{1/2} + c_2^{1/2}.$$

Then

$$\lim_{c \to 0} cU_1(c, y - c) = \lim_{c \to 0} \frac{1}{2} c^{1/2} = 0,$$

so that $v(s) \equiv 0$ is a solution. But

$$\lim_{c \to 0} \frac{U_2(c, y - c)}{U_1(c, y - c)} = \lim_{c \to 0} \left(\frac{c}{y - c}\right)^{1/2} = 0,$$

so that (3.14) holds for any $\beta\Gamma$ and a positive solution also exists.

Our final result gives sufficient conditions for the operator T defined in the proof of Theorem 1 to be a contraction. This will insure the uniqueness of the solution to (3.12). It requires strengthening Assumption IV to

Assumption IV': For each $s \in S$,

$$0 < \beta \int_{S_1} \frac{1}{g(s_1')} \pi(s, ds_1') < 1.$$

It also requires adding an assumption on preferences that guarantees that the slope of $h(v,y)$ in the v direction is less than unity, i.e., that $h(v,y) - v$

is nonincreasing in v. Using the definitions of $\hat{c}$ and h in (3.3) and (3.11), we find that a sufficient condition is

Assumption VIII: For each $y \in [\underline{y}, \bar{y}]$,

$$(3.15) \qquad c[U_1(c, y - c) - U_2(c, y - c)]$$

is a nonincreasing function of c.

Note that (3.15), evaluated at $\hat{c}(v,y)$ is just $h(v,y) - v$. Since under Assumptions V-VI, $\hat{c}(v,y)$ is increasing in v, the addition of Assumption VIII insures that $h(v,y) - v$ is nonincreasing in v.

Theorem 5: Let Assumptions I-III, IV' and V-VIII hold. Then (3.12) has a unique solution $v \in D$ and for all $v_0 \in D$, $\lim_{n\to\infty} \|T^n v_0 - v\| = 0$.

Proof. We will show that under these additional hypotheses, the operator T defined in the proof of Theorem 1 satisfies Blackwell's [1965] Theorem 5, sufficient conditions for a contraction. As observed in the proof of Theorem 3, under Assumptions V-VII, h is nondecreasing in v, so that T is monotone. We need only to verify that for some $\delta \in [0,1)$, $T(v + k) \leq Tv + \delta k$, for any $v \in \mathcal{F}$ and constant $k > 0$. Under Assumption VIII, $h(v,y) - v$ is nonincreasing in v: for any $v \in \mathcal{F}$ and $k > 0$,

$$h(v + k, y) - (v + k) \leq h(v,y) - v$$

or

$$h(v + k) \leq h(v,y) + k.$$

Then

$$T(v + k) = \beta \int_S \max[v(s') + k,\ h(v(s') + k,\ y(s'))] \frac{1}{g(s_1')} \pi(s,ds')$$

$$\leqslant \beta \int_S \max\ [v(s') + k,\ h(v(s'),y(s')) + k] \frac{1}{g(s_1')} \pi(s,ds')$$

$$= Tv + \beta k \int_S \frac{1}{g(s_1')} \pi(s,ds').$$

Now if Assumption IV' holds, then it follows from Assumptions I and III that

$$\beta \int_S \frac{1}{g(s')} \pi(s,ds') \leqslant \delta, \qquad \text{for all } s \in S,$$

for some $\delta < 1$, so that T is a contraction with modulus δ. The conclusion then follows from the contraction mapping theorem. []

Theorems 1 - 5 apply to the case in which the state space S consists of a finite number of points and the transition function is described by a Markov matrix

$$\Pi = [\pi_{ij}] \quad \text{where } \pi_{ij} = \Pr\{s' = s_j | s = s_i\}.$$

In this case (3.12) defines an operator T taking the set D = $\{v \in \mathbb{R}^n \mid 0 \leqslant v_i \leqslant A,\ i = 1,..,n\}$ into itself. Since D is compact and convex, Theorem 1 would in this case be an application of Brouwer's Theorem.[5]

This completes our analysis of equation (3.12). Given a solution v(s) to (3.12), Lemma 2 guarantees that there is exactly one corresponding solution w(s) and $c(s) = (c_1(s),\ y(s) - c_1(s))$ to (3.1) and (3.2). The corresponding (normalized) equilibrium price level is given by (2.7). The price $q(\bar{s},s_1)$ of a one-period nominal bond is given by (2.11), (2.12) and (2.13). The nominal interest rate $r(\bar{s},s_1)$, defined by $q = (1 + r)^{-1}$, is then given in terms of v by

$$(3.16) \qquad 1 + r(\bar{s},s_1) = \frac{\int_{S_2} \max[v(s),\ h[v(s),y(s)]]\ \pi_2(\bar{s},s_1,ds_2)}{\int_{S_2} v(s)\ \pi_2(\bar{s},s_1,ds_2)}.$$

4. Properties of Equilibria

From (2.6) and Lemma 2, we see that

$$\max[v(s),\ h(v(s),\ y(s))]$$

is the marginal utility of a unit of cash, available at the time of goods trading, when the state is s, and from (2.7) we see that v(s) is the marginal utility of a unit of wealth (not in the form of cash), at the same time. Then (3.12), written as

$$(4.1) \qquad v(s) = \beta E\{\max[v(s'),\ h(v(s'),\ y(s'))]\ \frac{1}{g(s_1')}\ |s\},$$

equates the marginal utility of wealth at the time of goods trading in one period, with the (discounted) expected marginal utility of cash at the time of goods trading in the subsequent period. This reflects the fact that the consumer can, say, cut his consumption of credit goods slightly, and have the proceeds available in the form of cash to purchase cash goods in the next period's goods market.

A similar tradeoff is reflected in (3.16) which determines the nominal interest rate. Using (4.1), we can write (3.16) as

$$(4.2) \qquad 1 + r(\bar{s}, s_1) = \frac{E\{\max[v(s),\ h(v(s),\ y(s))]\ |\bar{s},\ s_1\}}{E\{v(s)|\bar{s},\ s_1\}}$$

$$(4.2') \qquad = \frac{E\{\max[v(s),\ h(v(s),\ y(s))]|\bar{s},\ s_1\}}{\beta E\{\max[v(s'),\ h(v(s'),\ y(s'))]\ \frac{1}{g(s_1')}\ |\bar{s},\ s_1\}}.$$

Thus, the nominal interest rate in state $(\bar{s}_2, s_1)$ is the ratio of the expected marginal utility of cash during goods trading later in the period, to the (discounted) expected marginal utility of cash during goods trading in the subsequent period.

Using (4.1) and (4.2'), we can study the effect of the timing of information. In particular, we will show that the equilibrium real allocation does not depend on the accuracy of advance information about real income available at the time of securities trading, while the variability of the nominal interest rate does.

To see this, consider any family of economies all with the following characteristics. The securities market shock s_1 has two components, $s_1 = (s_{11}, s_{12})$. Money growth depends only on s_{11}, and s_{12} is a signal about s_2. Real income, $y(s_2)$, is assumed to depend only on s_2. The spaces S_{11} and S_2 and the functions $g(s_{11})$ and $y(s_2)$ are identical for all members of the family, but S_{12}, the signal space, varies. The transition functions also vary, but all must have the following two features.

First, for each economy, there exists a transition function $\hat{\pi}$: $S_{11} \times S_2 \times S \to [0,1]$ such that

$$(4.3) \qquad \pi(s_{11},s_{12},s_2,A) = \hat{\pi}(s_{11},s_2,A), \quad \text{all } s_{11},s_{12},s_2,A.$$

This condition says that s_{12} is a signal only about s_2, i.e., does not provide any information about later events. Second, we require that there exist a transition function π^*: $S_{11} \times S_2 \times S_{11} \times S_2 \to [0,1]$, such that for all economies in the family,

$$\hat{\pi}(s_{11}, s_2, A_{11} \times S_{12} \times A_2) = \pi^*(s_{11}, s_2, A_{11} \times A_2), \quad \text{all } s_{11}, s_2, A_{11}, A_2. \tag{4.4}$$

This condition says that the underlying joint distribution π^* of (s_{11}, s_2) is the same for all economies in the family.

Consider any such family of economies, and choose one member. Using (4.3), we find that the solution(s) $v(s)$ to (3.12) for that economy satisfy

$$v(s) = \beta \int_S \max[v(s'),\ h(v(s'),\ y(s_2'))] \frac{1}{g(s_1')} \hat{\pi}(s_{11}, s_2, ds'). \tag{4.5}$$

Since s_{12} soes not appear on the right side of (4.5), $v(s)$ does not depend on s_{12}. Therefore we can integrate out s_{12}', and use (4.4) to find that

$$v(s) = \beta \int_{S_{11}} \int_{S_2} \max[v(s'),\ h(v(s'),\ y(s_2'))] \frac{1}{g(s_1')} \pi^*(s_{11}, s_2, ds_{11}', ds_2')$$

$$= \beta E\left\{\max[v(s'),\ h(v(s'),\ y(s_2'))] \frac{1}{g(s_1')} \,\middle|\, s_{11},\ s_2\right\}. \tag{4.6}$$

Since v does not depend on s_{12}', the right side of (4.6) is identical for all members of the family, and the set of solutions $v(s)$ will be too. Then, since $y(s_2)$ is common to all, the equilibrium allocations will also be identical.

The behavior of interest rates will differ, however, depending on the informativeness of the signal s_{12}. Using (4.2), we see that

$$(4.7) \qquad 1 + r(\bar{s},s_{11},s_{12}) = \frac{E\{\max[v(s),h(v(s),y(s_2))] \,|\, \bar{s},s_{11},s_{12}\}}{\beta E\{\max[v(s'),\ h(v(s'),\ y(s_2'))] \frac{1}{g(s_1')} \,|\, \bar{s},\ s_{11},s_{12}\}}.$$

Thus, the variability of interest rates depends on the informativeness of the signal s_{12}, even though the equilibrium allocation does not. This point is further illustrated in Example 3 in the next section.

Another observation about the equilibrium real allocation follows directly from (3.12): current money growth affects the current allocation <u>only</u> insofar as it affects expectations about future money growth, i.e, <u>only</u> through its value as a signal. In particular, if s and s' are two states for which

$$(4.8) \qquad \pi(s,A) = \pi(s',A),\ \text{all } A \in \mathbf{S},$$

then states s and s' have the same informational content. In this case, it follows directly from (3.12) that v(s), the marginal utility of income, is the same in s and s'. Therefore, using (2.7) we find that

$$\frac{U_2(c(s))}{p(s)} = v(s) = v(s') = \frac{U_2(c(s'))}{p(s')}\ .$$

This is the sense in which only the informational content of money growth matters. In particular, if (4.8) holds and in addition the endowments are the same in the two states, y(s) = y(s'), then it follows from Lemma 1 that the allocations are the same, c(s) = c(s'). This is true even if the associated money growth rates differ, $g(s_1) \neq g(s_1')$. Conversely, if the current endowments and money growth rates are the same in the two states, but their information contents differ--(4.8) fails--then in general $v(s) \neq v(s')$, and the allocations will differ. In short, with income constant, two states yield the same allocation if (4.8) holds and in general yield different allocations if it fails. The current rate of money growth plays no direct role in determining the current allocation--only expectations about money growth (and income) matter.

The equilibria we have described are related to the traditional theory of money demand. The connections are easiest to see in the case when all information is available at the time of securities trading (i.e., when given s_1, the conditional distribution of s_2 is degenerate), and the cash-in-advance constraint is always binding. When these conditions hold, $c_1(s)$ is equal to equilibrium real balances, and the solution for nominal interest rates is, from (3.16) and Lemmas 1 and 2,

$$1 + r(s) = \frac{h(v(s),y(s))}{v(s)}$$

$$= \frac{U_1(c_1(s),\ y(s) - c_1(s))}{U_2(c_1(s),\ y(s) - c_1(s))} .$$

This relationship between three variables, c_1, y and r, can be solved to give real balances as a function of income and the rate of interest, and since the form of this function depends only on preferences, it does not do too much violence to ordinary usage to call it a demand function.

When the timing of information becomes more complex, this connection becomes looser. During securities trading, the nominal interest rate is set and agents are committed to holdings of <u>nominal</u> balances. Later, during goods trading, income is realized and a nominal price level is established. Equation (3.16) still holds, but it is no longer accurate to think of agents as demanding <u>real</u> balances at either trading stage.

5. Examples

The theory developed in sections 2 and 3 admits a wide variety of possibilities for the equilibrium behavior of interest rates and real balances, depending on what is assumed about the behavior of real and monetary shocks.[6] This section illustrates some of these possibilities with specific examples.

The first example simply shows that results familiar from deterministic monetary theory carry over to the present model.

Example 1. Let $S = \{s\} = \{(s_1,s_2)\}$, $\pi(s,s) = 1$, $g(s_1) \equiv g$ and $y(s) \equiv y$.

In this case, a solution $v(s)$ to (3.12) is a constant v satisfying

$$v = \frac{\beta}{g} \max[v,\ h(v,y)].$$

If $\beta/g < 1$, then a solution is any v satisfying:

$$v = \frac{\beta}{g} h(v,y).$$

If $\beta/g = 1$, then a solution is any v satisfying $v \geq h(v,y)$. From Theorem 1 it follows that in either case there is at least one solution $v \geq 0$, from Theorem 2 that if (3.13) holds $v = 0$ is not a solution, and from Theorem 4 that if (3.14) holds there is a positive solution. Provided $v > 0$, it follows from (3.16) that

$$1 + r = \frac{h(v,y)}{v} = \frac{g}{\beta} \equiv (1 + \rho)(1 + \delta) = 1 + \rho + \delta + \rho\delta,$$

so that the nominal interest rate r is approximately the sum of the rate of money growth, $\rho \equiv g - 1$, and the subjective rate of time preference, $\delta \equiv \beta^{-1} - 1$.

If the income and monetary shocks are independent of each other, and each is independently and identically distributed over time, this example is only slightly changed.

Example 2: Let $y(s) \equiv y(s_2)$ and let

$$\pi_1(s_1,s_2,A_1) = \pi_1(A_1), \quad \text{all } s_1,\ s_2,\ A_1,$$

and

$$\pi_2(s_2,s_1,A_2) = \pi_2(A_2), \quad \text{all } s_2,\ s_1,\ A_2.$$

It follows immediately that $\pi(s_1,s_2,A) = \pi(A)$, all s_1, s_2, A. Hence the right side of (3.12) is independent of s, so that v(s) is also. Thus, a solution is any constant v satisfying

$$v = \beta E\{1/g(s_1)\}\ E\{\max[v,\ h(v,y(s_2))]\}. \tag{5.1}$$

If $v > 0$, it follows from (4.2) that the associated interest rate is also constant:

$$1 + r = [\beta E\{1/g(s_1')\}]^{-1}.$$

Examples 1 and 2 show that if real income is i.i.d., then any monetary policy g(s) consisting of i.i.d. money growth rates yields the same interest rate behavior and resource allocation as does the deterministic policy $\bar{g}$ with $1/\bar{g} = E\{1/g(s_1)\}$. Thus, they illustrate the point, made more generally in section 4, that only expectations about future money growth have allocative consequences.

The next example illustrates the importance of the timing of information about interest rates, discussed at a more general level in section 4. The securities market shock s_1 will have two components, $s_1 = (s_{11}, s_{12})$, where money growth depends only on s_{11}, and s_{12} is a signal about s_2. In addition, the monetary and real shocks are as in Example 2--independent of each other and i.i.d. over time, with income depending on s_2 only. Thus, the <u>only</u> information about the value of real income available at the time of securities trading is the signal s_{12}.

<u>Example 3</u>: Let

$$s_1 = (s_{11}, s_{12}) \in S_{11} \times S_{12} = S_1,$$

$$\pi_1(s_1, s_2, A_{11} \times A_{12}) = \pi_{11}(A_{11})\pi_{12}(A_{12}),$$

$$\pi_2(\bar{s}_2, s_1, A_2) = \pi_2(s_{12}, A_2),$$

so that

$$\pi(s_1, s_2, A_{11} \times A_{12} \times A_2) = \pi_{11}(A_{11}) \int_{A_{12}} \pi_{12}(ds'_{12})\pi_2(s'_{12}, A_2).$$

Then as in Example 2, the right side of (3.12) is independent of s, so that v(s) is too. Thus, a solution is any constant v satisfying (5.1). If $v > 0$, interest rates are given by (4.2). They <u>do</u>, in general, depend on the current securities market shock. Specifically,

$$(5.2) \qquad 1 + r(s_{12}) = \frac{1}{\beta E\{1/g(s'_1)\}} \frac{E\{\max[v, h(v, y(s_2))] \mid s_{12}\}}{E\{\max[v, h(v, y(s'_2))]\}},$$

so that the interest rate depends on s_{12}, the information about real income

$y(s_2)$ in the goods market later in the period. The nominal interest rate is higher when the conditional expected marginal utility of cash balances, given s_{12}, is above the average, and conversely.

The effect of advance information about income is illustrated by considering a family of economies, constructed as in section 4. That is, all have the same state spaces S_{11} and S_2, and the same monetary and real income shocks $g(s_{11})$ and $y(s_2)$. The signal spaces S_{12} may differ, but the distributions π_1 and π_2 satisfy (4.3) and (4.4). In addition, the distributions satisfy the conditions of Example 3. Hence, (4.4) requires that π_{11} and

$$\int_{S_{12}} \pi_{12}(ds_{12})\pi_2(s_{12}, A_2)$$

be the same for all members of this family.

Clearly the solution v given by (5.1), and hence the allocations, are identical in all these economies. Specifically, the consumption allocation in each period depends only on the level of real income, $y(s_2)$ in that period, and is identical, state by state, for all the economies. However, the behavior of interest rates may be quite different, as can be seen from (5.2).

If the signal s_{12} contains no information about s_2, then the situation is as in Example 2: the interest rate is constant. At the other extreme, if s_{12} is a perfect predictor of s_2, then interest rates fluctuate in a way that reflects the marginal utility of a dollar (in cash) in the subsequent goods market. If s_{12} is an imperfect signal about s_2, then interest rates will fluctuate, but in a less extreme fashion. Note, too, that the average interest rate, $E\{1 + r(s_{12})\}$, is the same for all economies in a family.

A device we have found useful in generating additional examples that illuminate the connections between shocks and resource allocations is to place assumptions directly on equilibrium allocations and/or prices and then to "work backwards" to characterize the monetary policies (if any) that <u>implement</u> this allocation. This approach uses (3.12) in an inverse way, and makes no use of Theorems 1-5. The remaining examples in this section give some idea of the possibilities.

<u>Example 4</u>: Interest-stabilizing policies.

Assume that all information becomes available at the time of securities trading, so that given s_1, the conditional distribution of s_2 is degenerate. Let preferences assume the homethetic, constant relative risk-aversion form

$$U(c_1,c_2) = \frac{1}{\sigma}([u(c_1,c_2]^{\sigma} - 1),$$

where u is homogeneous of degree one and $\sigma \leqslant 1$. We wish to characterize the allocations and policies consistent with the maintenance of a constant value $r > 0$, for the interest rate. Since with $r > 0$ the cash-in-advance constraint is binding in all states, we have

$$v(s) = c_1(s)U_2(c(s)),$$

and

$$\max\ [v(s),h(v(s),y(s))] = h(v(s),y(s)) = c_1(s)U_1(c(s)).$$

Hence, using (4.2) and the assumption that s_1 provides perfect information about s_2, we find that the allocation c(s) must satisfy

$$1 + r = \frac{U_1(c(s))}{U_2(c(s))} .$$

Since U is homothetic, this in turn implies that the associated allocation takes the form $(c_1(s),c_2(s)) = (ay(s), (1 - a)y(s))$, where a is the unique solution to

$$(5.3) \qquad 1 + r = \frac{u_1(a,1 - a)}{u_2(a,1 - a)} .$$

Then, using the fact that u is homogeneous of degree one, the functions v(s) and h(v(s),y(s)) are simply

$$v(s) = c_1(s)U_2(c(s))$$

$$= ay^{\sigma}(s)\ [u(a,1 - a)]^{\sigma-1}u_2(a,1 - a),$$

and

$$h(v(s),y(s)) = c_1(s)U_1(c(s))$$

$$= ay^{\sigma}(s)\ [u(a,1 - a)]^{\sigma-1}u_1(a,1 - a).$$

Then (4.1) simplifies to

$$y^{\sigma}(s)u_2(a,1 - a) = \beta E\{\ y^{\sigma}(s')\ u_1(a,1 - a)\frac{1}{g(s')}\ |\ s\}.$$

Hence, using (5.3), we find that

$$(5.4) \qquad \frac{1}{1 + r} = \frac{u_2(a,1 - a)}{u_1(a,1 - a)} = \beta E\{ (\frac{y(s')}{y(s)})^{\sigma} \frac{1}{g(s')} \mid s \}.$$

Monetary policies consistent with a constant positive interest rate must, then, maintain the constancy of the expression on the right of (5.4). One policy that will do this is

$$g(s') = \beta(1 + r)(\frac{y(s')}{y(s)})^{\sigma},$$

with probability 1. Evidently there are others that will achieve the same end, since (5.4) is an expected value restriction.

If (5.4) is to hold, monetary policy must to react to contemporaneous output movements, except in the borderline case of $\sigma = 0$ (logarithmic utility). If $0 < \sigma < 1$, money growth must be positively correlated with output; if $\sigma < 0$, negatively.

We will continue in a more general way to calculate policies that implement particular equilibria by restricting discussion to finite state spaces. Number the states $1,2,\ldots,n$, and let $y = (y_1,\ldots,y_n)$, $g = (g_1,\ldots,g_n)$ and $v(s) = (v_1, \ldots, v_n)$ be the corresponding values for $y(s)$, $g(s)$, and $v(s)$. Let $\Pi = [\pi_{ij}]$ be the transition matrix. Then (3.12) becomes

$$v_i = \beta \sum_{j=1}^{n} \frac{\max[v_j,h(v_j,y_j)]}{g_j} \pi_{ij}, \quad i = 1,\ldots,n.$$

We consider the implementation of given vectors $v \in \mathbb{R}^n_+$. Let (v,Π) be given, and consider solving

$$(5.5) \qquad v = \Pi x$$

for the unknown vector x. A monetary policy g implements a given (v,Π) if and only if

$$(5.6) \qquad g_i = \frac{\beta}{x_i} \max\,[v_i,\ h(v_i,y_i)], \quad \text{all } i$$

for some x satisfying (5.5).

It is clear that given the transition probabilities Π, not all vectors $v > 0$ can be implemented. First, (5.5) has a solution if and only if $w\Pi = 0$ implies $w \cdot v = 0$, for any $w \in \mathbb{R}^n$. In other words, to be implementable by some g, v must lie in the subspace of $\mathbb{R}^n$ spanned by the columns of Π. For example, if the shocks are independently and identically distributed, then Π has rank one, and v must be a constant vector. (This conclusion was reached by a different route in Example 2.) In general, if Π has rank $m \leq n$, implementable v's must lie in an m-dimensional subspace, and for given v, the solutions x lie in an (n-m)-dimensional subspace. This conclusion can be interpreted as follows.

If θ is a probability vector interpreted as a distribution of s_t, then $\theta\Pi$ is the implied distribution of s_{t+1}. Then full rank for Π means that two different s_t distributions, θ and $\tilde{\theta}$ say, with $\theta - \tilde{\theta} \neq 0$, always imply different s_{t+1} distributions: $(\theta - \tilde{\theta})\Pi \neq 0$, so that different states always have different "information effects." In this model, it is different information effects that induce differences in resource allocations, so that the rank of Π is critical in determining the variety of v-values that can be implemented.

A second restriction comes from the fact that the vector g, and thus the vector x, must be positive. This also restricts the set of implementable v's.

Example 5: Two states, symmetric transitions.

Let n = 2 and let the transition matrix be given by

$$\Pi = \begin{vmatrix} \frac{1+\alpha}{2} & \frac{1-\alpha}{2} \\ \\ \frac{1-\alpha}{2} & \frac{1+\alpha}{2} \end{vmatrix},$$

where $-1 < \alpha < 1$. The symmetry of Π means that in a stationary distribution the system spends half the time in state 1 and half in state 2.

If $\alpha = 0$, Π is singular and (5.5) has a solution x if and only if $v_1 = v_2$ (as in Example 2, above). Given their common value v, the corresponding solutions x and g must satisfy

$$v = (\tfrac{1}{2} \ \ \tfrac{1}{2})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\beta}{2g_1}\max[v,h(v,y_1)] + \frac{\beta}{2g_2}\max[v,h(v,y_2)].$$

Hence there is a one-dimensional manifold of solutions.

If $\alpha \neq 0$, Π can be inverted and the unique solution x to (5.5) is

$$x_1 = \frac{1+\alpha}{2\alpha} v_1 - \frac{1-\alpha}{2\alpha} v_2,$$

$$x_2 = -\frac{1-\alpha}{2\alpha} v_1 + \frac{1+\alpha}{2\alpha} v_2.$$

However, x_1 and x_2 must be positive, so that if $\alpha > 0$, we must have

$$\text{(5.7)} \qquad \frac{1-\alpha}{1+\alpha} < \frac{v_1}{v_2} < \frac{1+\alpha}{1-\alpha},$$

while if $\alpha < 0$,

$$(5.8) \qquad \frac{1+\alpha}{1-\alpha} < \frac{v_1}{v_2} < \frac{1-\alpha}{1+\alpha} .$$

For any pair (v_1, v_2) satisfying these bounds, there will be a unique (g_1, g_2) that implements it.

To illustrate in more detail, let preferences be

$$U(c_1, c_2) = -\frac{1}{c_1} + \ln(c_2),$$

and let $y_1 = y_2 = 2$, so that states 1 and 2 differ only due to different money growth rates. Then $h(v,y) = y^{-1}(1 + \frac{1}{v}) = \frac{1}{2}(1 + \frac{1}{v})$. We seek to construct a policy (g_1, g_2) to implement the equilibrium $v_1 = 2$ and $v_2 = 2/3$. For these v and y values, $h_1 = 3/4$ and $h_2 = 5/4$. In state 1, the cash constraint is slack $(h_1 < v_1)$, so the associated allocation $(c_{11}, 2 - c_{11})$ is given by

$$1 = \frac{U_1(c_{11}, 2 - c_{11})}{U_2(c_{11}, 2 - c_{11})} = \frac{2 - c_{11}}{c_{11}^2},$$

or $c_{11} = 1$ and $y - c_{11} = 1$. In state 2, the cash constraint is binding and c_{21} is given by

$$\frac{2 - c_{21}}{c_{21}^2} = \frac{h_2}{v_2} = \frac{15}{8},$$

or $c_{21} = 4/5$ and $y - c_{21} = 6/5$.

If $\alpha > 0$, this allocation can be implemented only if $v_1/v_2 = 3$ satisfies the bounds in (5.7), i.e. only if $\alpha > 1/2$. For example, if $\alpha = 3/4$, then

$$x_1 = \frac{7}{6} v_1 - \frac{1}{6} v_2 = \frac{20}{9}$$

$$x_2 = -\frac{1}{6} v_1 + \frac{7}{6} v_2 = \frac{4}{9}$$

From (5.5), then,

$$\frac{\beta}{g_1} = \frac{1}{2} x_1 = \frac{10}{9}$$

and

$$\frac{\beta}{g_2} = \frac{4}{5} x_2 = \frac{16}{45}$$

describe the required money growth rates, relative to the discount factor β.

To achieve the high consumption of cash goods and the slack Clower constraint in state 1, the monetary authority must signal a relatively low rate of monetary growth between this period and the next. With positive serial correlation between states ($\alpha > 0$), this is done by having a relatively low current money growth rate. With negative serial correlation ($\alpha < 0$), the signal would work the opposite way, as the reader is invited to convince himself by repeating these calculations for a negative α satisfying (5.8).

6. Conclusion

In this paper we have developed methods for verifying the existence of, characterizing, and explicitly calculating equilibria for a simple monetary economy, and we have illustrated these methods on a variety of examples. The model analyzed allows situations in which different monetary policies induce different real resource allocations: this is its main novelty over earlier models using similar methods. On the other hand, the methods used rely heavily on the state variables of the system being exogenous: it is not yet known whether models incorporating capital accumulation, like the one in Townsend [1984], can be studied with recursive methods.

As a theory of nominal interest rates, the model captures two forces long believed to be critical. First, shocks to real endowments, by altering marginal rates of substitution, affect real interest rates. Second, current monetary shocks, or more precisely, any variables conveying useful information about future monetary shocks, affect the "inflation premiums" on interest rates. We have shown how the effects of these two very different forces can be analyzed, even when they are permitted to interact in complicated ways (as we think they do in reality).

The theory's implications do not take the form of predicting definite signs for the correlations between movements in money and other variables, either contemporaneous or lagged. There is no doubt that a bumper crop of apples depresses the price of apples, and that a useful model of the apple market should reproduce this feature. But money is not at all like apples. In a model in which information is common, a monetary change is irrelevant history as soon as it has occurred, and it affects real resource allocations only insofar as it conveys information about the future. In such a model, one can obtain tight predictions about the entire joint distributions of money, interest rates, and other variables, current and lagged, given the joint distribution of all exogenous variables, including those that are purely informational. However, no general predictions about individual moments involving endogenous variables are possible. This reflects the fact that the information content of the current value of any variable can be understood only if the entire stochastic environment is specified.

Notes

[1]This model is a special case of the one discussed in Lucas [1984] and is very closely related to Lucas and Stokey [1983] and Townsend [1984]; the reader is referred there for further discussion.

[2]With infinitely-lived agents and recourse to lump-sum taxes, the timing of taxes and subsidies is immaterial, and there is no distinction between an injection of money through a fiscal transfer payment and an injection through an "open-market" purchase of government bonds. Hence, this convention will not affect the results. See Lucas and Stokey [1983] for a parallel discussion in which taxes are assumed to distort and this distinction is central.

[3]See Hutson and Pym [1980], chapter 8, for the terminology used and results cited in this proof.

[4]Since $U_1(0,y)/U_2(0,y) > 1$, the cash-in-advance constraint is binding in this solution, so $p(s)c_1(s) = 1$ and the price level, $p(s)$, is "infinite." A condition like (3.13), below, is used in Brock and Scheinkman [1980] and Scheinkman [1980] to rule out non-stationary equilibria that converge to "barter," as well as stationary barter equilibria in overlapping generations models.

[5]This is the route taken by Labadie [1984], Theorem 1, in a problem that is technically very similar to ours.

[6]Equilibrium also depends, of course, on the nature of preferences. Preferences affect the basic functional equation (3.13) through the function h, which is in turn determined entirely by the utility function U. If U takes the form $U(c_1,c_2) = \ln(c_1) + g(c_2)$ then $h(v,y) \equiv 1$ and (3.12) may be solved immediately for a constant function $v(s)$. It would be useful to exhibit this case as a "borderline" within some family of perference functions, but we have not found an interesting parametric family for which h can be calculated using "pencil and paper" methods. It seems clear that a wide variety of functions h are consistent with preferences satisfying Assumptions V and VI.

References

Blackwell, David, "Discounted Dynamic Programming," Annals of Mathematical Statistics, 36 (1965), 226-35.

Brock, William A. and Jose A. Scheinkman, "Some Remarks on Monetary Policy in an Overlapping Generations Model," in John H. Karaken and Neil Wallace (eds.), Models of Monetary Economies (Minneapolis: Federal Reserve Bank of Minneapolis) 1980, pp. 211-232.

Clower, Robert W., "A Reconsideration of the Microfoundations of Monetary Theory," Western Economic Journal, 6 (1967), 1-9.

Hutson, V. and J. S. Pym, Applications of Functional Analysis and Operator Theory (London: Academic Press) 1980.

Labadie, Pamela, "A Test of Risk Premia Behavior in an Overlapping Generations Model," Columbia University working paper, 1984.

Lucas, Robert E., Jr., "Interest Rates and Currency Prices in a Two-Country World," Journal of Monetary Econmics, 10 (1982), 335-360.

__________, "Money in a Theory of Finance," Journal of Monetary Economics, 1984, forthcoming.

Lucas, Robert E., Jr. and Nancy L. Stokey, "Optimal Fiscal and Monetary Policy in an Economy Without Capital," Journal of Monetary Economics, 12 (1983), 55-93.

Scheinkman, Jose A., "Discussion," in John H. Karaken and Neil Wallace (eds.), Models of Monetary Economies (Minneapolis: Federal Reserve Bank of Minneapolis) 1980, pp. 91-96.

Svensson, Lars E. O., "Money and Asset Prices in a Cash-in-Advance Economy," Seminar Paper No. 267, Institute for International Economic Studies, Stockholm, December, 1983.

Townsend, Robert M., "Asset-Return Anomalies in a Monetary Economy," Carnegie-Mellon University working paper, 1984.