THE OPTIMAL INFLATION RATE IN NEW KEYNESIAN MODELS

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ABSTRACT

We study the effects of positive steady-state inflation in New Keynesian models subject to the zero bound on interest rates. We derive the utility-based welfare loss function taking into account the effects of positive steady-state inflation and show that steady-state inflation affects welfare through three distinct channels: steady-state effects, the magnitude of the coefficients in the utility-function approximation, and the dynamics of the model. We solve for the optimal level of inflation in the model and find that, for plausible calibrations, the optimal inflation rate is low, less than two percent, even after considering a variety of extensions, including price indexation, endogenous price stickiness, capital formation, model-uncertainty, and downward nominal wage rigidities. On the normative side, price level targeting delivers large welfare gains and a very low optimal inflation rate consistent with price stability.

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"The crisis has shown that interest rates can actually hit the zero level, and when this happens it is a severe constraint on monetary policy that ties your hands during times of trouble. As a matter of logic, higher average inflation and thus higher average nominal interest rates before the crisis would have given more room for monetary policy to be eased during the crisis and would have resulted in less deterioration of fiscal positions. What we need to think about now is whether this could justify setting a higher inflation target in the future."

Olivier Blanchard, February 12th, 2010

I   Introduction

One of the defining features of the current economic crisis has been the zero bound on nominal interest rates. With standard monetary policy running out of ammunition in the midst of one of the sharpest downturns in post-World War II economic history, some have suggested that central banks should consider allowing for higher target inflation rates than would have been considered reasonable just a few years ago. We contribute to this question by explicitly incorporating positive steady-state (or "trend") inflation in a New Keynesian model as well as the zero lower bound (ZLB) on nominal interest rates. We derive the effects of non-zero steady-state inflation on the loss function, thereby laying the groundwork for welfare analysis. In particular, we show that steady-state inflation affects welfare through three distinct channels: steady-state effects, the magnitude of the coefficients in the utility-function approximation, and the dynamics of the model. Our baseline finding is that the optimal rate of inflation is low, less than two percent a year, even when we allow for a variety of features that would tend to lower the costs or to raise the benefits of positive steady-state inflation.

Despite the importance of quantifying the optimal inflation rate for policy-makers, modern monetary models of the business cycle, namely the New Keynesian framework, have been strikingly ill-suited to address this question because of their near exclusive reliance on the assumption of zero steady-state inflation, particularly in welfare analysis.[1] Our first contribution is to address the implications of positive steady-state inflation for welfare analysis by solving for the micro-founded loss function in an otherwise standard New Keynesian model with labor as the only factor of production.   We identify two costs of positive trend inflation due to price dispersion, one well-known and the other less so, if at all. The first is the steady-state effect: with staggered price setting, higher inflation leads to greater price dispersion which causes an inefficient allocation of resources among firms, thereby lowering aggregate

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[1] Most papers incorporating positive steady-state inflation into the New Keynesian framework have focused on the implications for dynamics and determinacy issues. For example, Cogley and Sbordone (2008) show that accounting for positive steady-state inflation significantly improves the fit of the New Keynesian Phillips Curve. Kiley (2007) and Ascari and Ropele (2009) show that the Taylor principle is not sufficient to guarantee a unique rational expectations equilibrium in New Keynesian models for even moderate levels of inflation. Coibion and Gorodnichenko (2009) show that once this feature of New Keynesian models is incorporated into historical monetary policy analysis, the pre-Volcker monetary policy rule ensured the presence of self-fulfilling expectational fluctuations despite likely satisfying the Taylor principle, a reflection of the high target rate of inflation over this time period.

welfare. The second is that positive steady-state inflation raises the welfare cost of a given amount of inflation volatility. This reflects the fact that inflation variations create distortions in relative prices given staggered price setting. Since positive trend inflation already generates some inefficient price dispersion, the additional distortion in relative prices from an inflation shock becomes more costly as firms have to compensate workers for the increasingly high marginal disutility of sector-specific labor. Thus, the increased distortion in relative prices due to an inflation shock becomes costlier as we increase the initial price dispersion which makes the variance of inflation costlier for welfare as the steady-state level of inflation rises. In addition to the two costs from relative price dispersion, a third cost of inflation in our model comes from the dynamic effect of positive inflation on pricing decisions. Greater steady-state inflation induces more forward-looking behavior when sticky-price firms are able to reset their prices because the gradual depreciation of the relative reset price can lead to larger losses than under zero inflation. As a result, inflation is more volatile which lowers aggregate welfare. This cost of inflation due to the positive relationship between the level and volatility of inflation has been well-documented empirically but is commonly ignored in quantitative analyses because of questions as to the source of the relationship.[2] As with the first two costs of inflation, this cost arises endogenously in the New Keynesian model when one incorporates positive steady-state inflation.

The key benefit of positive inflation in our model is a reduced frequency of hitting the zero bound on interest rates. As emphasized in Christiano et al (2009), hitting the zero bound induces a deflationary mechanism which leads to increased volatility and hence large welfare costs. A higher steady-state level of inflation implies a higher level of nominal interest rates and therefore, as suggested by Blanchard, a reduced incidence of zero-bound episodes. Our approach for modeling the zero bound follows Bodenstein et al (2009) by solving for the duration of the zero bound endogenously, unlike in Christiano et al (2009) or Eggertsson and Woodford (2004). This is important because the welfare costs of inflation are a function of the variance of inflation and output, which themselves depend on the frequency at which the zero bound is reached as well as the duration of zero bound episodes.

After calibrating the model to broadly match the historical incidence of hitting the zero lower bound in the U.S., we then solve for the rate of inflation that maximizes welfare. We show numerically that the welfare loss function is generally concave with respect to steady-state inflation, such that the optimal rate of inflation is positive as a result of the zero bound. However, for plausible calibrations of the structural parameters of the model and the properties of the shocks driving the economy, the optimal inflation rate is quite low: less than two percent per year. This result is remarkably robust to changes in parameter values, as long as these do not dramatically increase the implied frequency of being at the zero

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[2] For example, Mankiw's undergraduate Macroeconomics textbook notes that "in thinking about the costs of inflation, it is important to note a widely documented but little understood fact: high inflation is variable inflation." Similar statements can be found in other prominent texts.

lower bound. Furthermore, we show that all three costs of inflation – the steady state effect, the increasing cost of inflation volatility, and the positive link between the level and volatility of inflation – are quantitatively important: each is sufficient to bring the optimal inflation rate down to 2.5% or lower when the ZLB is present.

This low optimal rate of inflation is unlikely to be driven by a low cost of business cycles: our model implies that eliminating business cycle fluctuations at the optimal inflation rate would improve welfare by the equivalent of permanently raising consumption by 1.5%, in line with much of the literature on business cycle costs (Barlevy (2004)). In addition, hitting the ZLB is quite costly in the model: our calibration implies that at an annual inflation rate of 3%, a period of 8 quarters spent at the zero bound reduces welfare by the equivalent of a permanent decrease in consumption of more than 4%, above and beyond the welfare effects of the shocks which pushed the economy into the zero bound. Furthermore, while the model implies that the optimal weight on the variance of the output gap in the welfare loss function is small, we show that to push the optimal inflation rate above the typical inflation targets of central banks would require this coefficient to be more than ten times larger than the weight on the annualized inflation variance. Such weights would imply a welfare cost of being at the ZLB for 8 quarters roughly equal to a permanent decline in consumption of 40% ($\approx$$4 trillion per year), a magnitude which strikes us as too large to be plausible.

To further investigate the robustness of this result, we extend our baseline model to consider several mechanisms which might raise the optimal rate of inflation. For example, in the presence of uncertainty about the true parameter values, policy-makers might want to choose a higher inflation rate as a buffer against the possibility that the true parameters imply more frequent and costly incidence of the zero bound. We address this possibility in two ways. First, we calculate the optimal inflation rate taking into account the uncertainty about parameter values and find that this raises the optimal inflation rate only modestly, from 1.1% to 1.4% per year. Second, we repeatedly draw from the distribution of parameters and calculate the optimal inflation rate for each draw. We find that the 90% confidence interval of optimal inflation rates ranges from 0.1% to 2.2% a year, which closely mirrors the target range for inflation of most modern central banks.

Similarly, one might be concerned that our findings hinge on modeling price stickiness as in Calvo (1983). First, because this approach implies that some firms do not change prices for extended periods of time, it could overstate the cost of price dispersion and therefore understate the optimal inflation rate. To address this possibility, we reproduce our analysis using Taylor (1977) staggered price setting of fixed durations. The latter implies significantly less price dispersion than the Calvo assumption, yet using this alternative pricing structure has no significant impact on the optimal inflation rate. Second, the degree of price rigidity in both Calvo and Taylor pricing is commonly treated as a

structural parameter, yet it is unlikely that the frequency of price setting is completely independent of the inflation rate, even for low inflation rates like those experienced in the U.S. As a result, we consider a specification of the model in which the degree of price rigidity varies systematically with the trend level of inflation but find that this modification also does not qualitatively change the optimal inflation rate.

Tobin (1972) suggests downward nominal wage rigidity as an additional factor which might push the optimal inflation rate higher. By facilitating the downward adjustment of real wages in the presence of downward nominal wage rigidity, positive inflation can be beneficial. We incorporate this "greasing the wheels" effect by constraining changes in the aggregate nominal wage index to be non-negative. Strikingly, this addition significantly *lowers* the optimal inflation rate. The intuition for this somewhat surprising finding is that downward wage rigidity lowers the volatility of marginal costs and hence of inflation. In addition, in the face of a negative demand shock, marginal costs decline by less in the presence of downward-wage rigidity, leading to a smaller decline in inflation and thus a smaller change of interest rates. Hence, the ZLB binds less frequently which further reduces the benefits of positive inflation.

Our analysis abstracts from several other factors which might affect the optimal inflation rate. For example, Friedman (1969) argued that the optimal rate of inflation must be negative to equalize the marginal cost and benefit of holding money. Because our model is that of a cashless economy, this cost of inflation is absent, but would tend to lower the optimal rate of inflation even further, as emphasized by Khan et al (2003), Schmitt-Grohe and Uribe (2007, 2010) and Aruoba and Schorfheide (2009). Similarly, a long literature has studied the costs and benefits of the seigniorage revenue to policymakers associated with positive inflation, a feature which we also abstract from since seigniorage revenues for countries like the U.S. are quite small, as are the deadweight losses associated with it.[3] Feldstein (1997) emphasizes an additional cost of inflation arising from fixed nominal tax brackets, which would again lower the optimal inflation rate. Furthermore, while our model includes an inflation cost arising from the positive link between the level and the volatility of inflation, it is likely that we still understate this cost of inflation because we abstract from the possibility that higher inflation volatility will raise risk premiums due to the increased risk of redistribution among borrowers and lenders. In addition, the relationship between the level and the volatility of inflation could be even stronger than in our model because higher steady-state inflation moves the economy closer to the indeterminacy region where sunspot shocks would further raise inflation volatility. Finally, because we do not model the possibility of endogenous countercyclical fiscal policy nor do we incorporate the possibility of nonstandard monetary policy actions during ZLB episodes, it is likely that we overstate the costs of hitting the ZLB and therefore again overstate the optimal rate of inflation.

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[3] See for example Cooley and Hansen (1991) and Summers (1991).

This paper is closely related to recent work that has also emphasized the effects of the zero bound on interest rates for the optimal inflation rate, such as Walsh (2009), Billi (2009), and Williams (2009). A key difference between the approach taken in this paper and such previous work is that we explicitly model the effects of positive trend inflation on the steady-state, dynamics, and loss function of the model. Billi (2009) and Walsh (2009), for example, use a New Keynesian model log-linearized around zero steady-state inflation and therefore do not explicitly incorporate the positive relationship between the level and volatility of inflation, while Williams (2009) relies on a non-microfounded model. In addition, these papers do not take into account the effects of positive steady-state inflation on the approximation to the utility function and thus do not fully incorporate the costs of inflation arising from price dispersion.[4]

An advantage of working with a micro-founded model and its implied welfare function is the ability to engage in normative analysis. In our baseline model, the endogenous response of monetary policy-makers to macroeconomic conditions is captured by a Taylor rule. Thus, we are also able to study the welfare effects of altering the systematic response of policy-makers to endogenous fluctuations (i.e. the coefficients of the Taylor rule) and determine the new optimal steady-state rate of inflation.[5] The most striking finding from this analysis is that even modest price-level targeting would raise welfare by non-trivial amounts for any steady-state inflation rate. In addition, by reducing the volatility of inflation and output, and therefore the frequency of being at the zero lower bound on interest rates, price-level targeting leads to much lower optimal rates of inflation than in our baseline findings. In a companion paper (Coibion et al, 2010), we show that similar results obtain under the Ramsey-optimal policy, consistent with the finding of Eggertsson and Woodford (2003) and Wolman (2005) that PLT closely approximates the Ramsey-optimal policy. In short, the optimal policy rule for the model can be characterized by the name of "price stability" as typically stated in the legal mandates of most central banks.

Section 2 presents the baseline New Keynesian model and derivations when allowing for positive steady-state inflation, including the associated loss function. Section 3 includes our calibration of the model as well as the results for the optimal rate of inflation while section 4 investigates the robustness of our results to parameter values. Section 5 then considers extensions of the baseline model which could

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[4] Fuchi et al (2008) study the optimal inflation rate for Japan allowing for the zero-lower bound on interest rates, price stickiness, nominal wage rigidity and the opportunity cost of holding money and find a range between 0.5% and 2%. However, they also do not explicitly take into account the effects of positive steady-state inflation on the dynamics of the model nor on the utility function approximation.

[5] Billi (2009) solves for the optimal policy, including the steady-state inflation rate, under commitment, discretion, and Taylor rules while we focus only on the latter case in this paper and consider the fully optimal policy under commitment in a companion paper. While his results under commitment and Taylor rules point to an optimal inflation rate of under 1%, he argues that under discretion the optimal inflation rate can be as high as 17%.

potentially lead to higher estimates of steady-state inflation. Section 6 considers additional normative implications of the model, including price level targeting, while section 7 concludes.

II   A New Keynesian Model with Positive Steady-State Inflation

We consider a standard New Keynesian model with a representative consumer, a continuum of monopolistic producers of intermediate goods, a fiscal authority and a central bank.

2.1   Model

The representative consumer aims to maximize the present discounted value of the utility stream from consumption and leisure

$$\max E_t \sum_{j=0}^{\infty} \beta^j \left\{ \log C_{t+j} - \frac{\eta}{\eta+1} \int_0^1 N_{t+j}(i)^{1+1/\eta} di \right\} \tag{1}$$

where C is consumption of the final good, $N_t(i)$ is labor supplied to individual industry i, η is the Frisch labor supply elasticity and β is the discount factor. The budget constraint each period t is given by

$$\xi_t : C_t + S_t/P_t \leq \int_0^1 (N_t(i)W_t(i)/P_t)di + S_{t-1}q_{t-1}R_{t-1}/P_t + T_t \tag{2}$$

where S is the stock of one-period bonds held by the consumer, R is the gross nominal interest rate, P is the price of the final good, $W(i)$ is the nominal wage earned from labor in industry i, T is transfers and profits from ownership of firms, q is a risk premium shock, and ξ is the shadow value of wealth.[6] The first order conditions from this utility-maximization problem are then:

$$C_t^{-1} = \xi_t \tag{3}$$
$$N_t(i)^{1/\eta} = \xi_t W_t(i)/P_t \tag{4}$$
$$\xi_t/P_t = \beta E_t[\xi_{t+1}q_t R_t/P_{t+1}] \tag{5}$$

Production of the final good is done by a perfectly competitive sector which combines a continuum of intermediate goods into a final good per the following aggregator

$$Y_t = \left[\int_0^1 Y_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)} \tag{6}$$

where Y_t is the final good and $Y(i)$ is intermediate good i, while θ denotes the elasticity of substitution across intermediate goods, yielding the following demand curve for goods of intermediate sector i

$$Y_t(i) = Y_t(P_t(i)/P_t)^{-\theta} \tag{7}$$

and the following expression for the aggregate price level

$$P_t = \left[\int_0^1 P_t(i)^{(1-\theta)} di \right]^{1/(1-\theta)}. \tag{8}$$

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[6] As discussed in Smets and Wouters (2007), a positive shock to q, which is the wedge between the interest rate controlled by the central bank and the return on assets held by the households, increases the required return on assets and reduces current consumption. The shock q has similar effects as net-worth shocks in models with financial accelerators (see Bernanke et al (1999) for a survey).

The production of each intermediate good is done by a monopolist facing a production function linear in labor

$$Y_t(i) = A_t N_t(i) \tag{9}$$

where A denotes the level of technology, common across firms. Each intermediate good producer has sticky prices, modeled as in Calvo (1983) where $1 - \lambda$ is the probability that each firm will be able to reoptimize its price each period. We allow for indexation of prices to steady-state inflation by firms who do not reoptimize their prices each period, with ω representing the degree of indexation (0 for no indexation to 1 for full indexation). Denoting the optimal reset price of firm i by $B(i)$, re-optimizing firms solve the following profit-maximization problem

$$\max E_t \sum_{j=0}^{\infty} \lambda^j Q_{t,t+j} \left[Y_{t+j}(i) B_t(i) \bar{\Pi}^{j\omega} - W_{t+j}(i) N_{t+j}(i) \right] \tag{10}$$

where Q is the stochastic discount factor and $\bar{\Pi}$ is the gross steady-state level of inflation. The optimal relative reset price is then given by

$$\frac{B_t(i)}{P_t} = \frac{\theta}{\theta - 1} \frac{E_t \sum_{j=0}^{\infty} \lambda^j Q_{t,t+j} Y_{t+j} (P_{t+j}/P_t)^{\theta+1} \bar{\Pi}^{-j\omega\theta} (MC(i)_{t+j}/P_{t+j})}{E_t \sum_{j=0}^{\infty} \lambda^j Q_{t,t+j} Y_{t+j} (P_{t+j}/P_t)^{\theta} \bar{\Pi}^{-j\omega\theta}} \tag{11}$$

where firm-specific marginal costs can be related to aggregate variables using

$$\frac{MC_{t+j}(i)}{P_{t+j}} = \left(\frac{C_{t+j}}{A_{t+j}} \right) \left(\frac{Y_{t+j}}{A_{t+j}} \right)^{1/\eta} \left(\frac{B_t(i)}{P_t} \right)^{-\theta/\eta} \left(\frac{P_{t+j}}{\bar{\Pi}^{j\omega} P_t} \right)^{\theta/\eta}. \tag{12}$$

Given these price-setting assumptions, the dynamics of the price level are governed by

$$P_t^{1-\theta} = (1 - \lambda) B_t^{1-\theta} + \lambda P_{t-1}^{1-\theta} \bar{\Pi}^{\omega(1-\theta)} \tag{13}$$

We allow for government consumption of final goods (G_t), so the goods market clearing condition for the economy is

$$Y_t = C_t + G_t. \tag{14}$$

We define the aggregate labor input as

$$N_t = \left[\int_0^1 N_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)}. \tag{15}$$

2.2 Steady-state and log-linearization

Following Coibion and Gorodnichenko (2009), we log-linearize the model around the steady-state in which inflation need not be zero. Since positive trend inflation may imply that the steady state and the flexible price level of output differ, we adopt the following notational convention. Variables with a bar denote steady state values, e.g. $\bar{Y}_t$ is the steady state level of output. We assume that technology is a random walk and hence we normalize all non-stationary real variables by the level of technology. Lower-case letters denote the log of a variable, e.g. $y_t = \log Y_t$ is the log of current output. We let hats on lower case letters denote deviations from steady state, e.g. $\hat{y}_t = y_t - \bar{y}_t$ is the approximate percentage deviation of output from steady state. Since we define the steady state as embodying the current level of

technology, deviations from the steady state are stationary. Finally, we denote deviations from the flexible price level steady state with a tilde, e.g. $\tilde{y}_t = y_t - \bar{y}_t^F$ is the approximate percentage deviation of output from its flexible price steady state, where the superscript F denotes a flexible price level quantity. Define the net steady-state level of inflation as $\bar{\pi} = \log(\bar{\Pi})$. The log-linearized consumption Euler equation is

$$-\hat{c}_t = E_t[-\hat{c}_{t+1} + \hat{r}_t - \hat{\pi}_{t+1} + \hat{q}_t] \tag{16}$$

and the goods market clearing condition becomes

$$\hat{y}_t = \bar{c}_y \hat{c}_t + \bar{g}_y \hat{g}_t \tag{17}$$

where $\bar{c}_y$ and $\bar{g}_y$ are the steady-state ratios of consumption and government to output respectively. Also, integrating over firm-specific production functions and log-linearizing yields

$$\hat{y}_t = \hat{n}_t \tag{18}$$

Allowing for positive steady-state inflation (i.e., $\bar{\pi} > 0$) primarily affects the steady-state and price-setting components of the model. For example, the steady-state level of the output gap (which is defined as the deviation of steady state output from its flexible price level counterpart $\bar{X}_t = \bar{Y}_t / \bar{Y}_t^F$) is given by

$$\bar{X}^{(\eta+1)/\eta} = \frac{1 - \left[1 - \lambda\beta^{-1}\bar{\Pi}^{(1-\omega)\theta}\right]\left(\frac{1 - \lambda\bar{\Pi}^{(1-\omega)(\theta-1)}}{1-\lambda}\right)^{(\eta+\theta)/(\eta(\theta-1))}}{\lambda\beta^{-1}\bar{\Pi}^{(1-\omega)\theta(\eta+1)/\eta}}. \tag{19}$$

Note that the steady-state level of the gap is equal to one when steady-state inflation is zero (i.e., $\bar{\Pi} = 1$) or when the degree of price indexation is exactly equal to one. As emphasized by Ascari and Ropele (2007), there is a non-linear relationship between the steady-state levels of inflation and output. For very low but positive trend inflation, $\bar{X}$ is increasing in trend inflation but the sign is quickly reversed so that $\bar{X}$ is falling with trend inflation for most positive levels of trend inflation.

Secondly, positive steady-state inflation affects the relationship between aggregate inflation and the re-optimizing price. Specifically, the relationship between the two in the steady state is now given by

$$\overline{(B/P)} = \left(\frac{1-\lambda}{1-\lambda\bar{\Pi}^{(1-\omega)(\theta-1)}}\right)^{1/(\theta-1)} \tag{20}$$

and the log-linearized equation is described by

$$\hat{\pi}_t = \left(\frac{1 - \lambda\bar{\Pi}^{(1-\omega)(\theta-1)}}{\lambda\bar{\Pi}^{(1-\omega)(\theta-1)}}\right)\hat{b}_t \quad \Rightarrow \quad \hat{b}_t = M\hat{\pi}_t \tag{21}$$

which implies that inflation is less sensitive to changes in the re-optimizing price as steady-state inflation rises. This effect reflects the fact that, with positive steady-state inflation, firms which reset prices have higher prices than others and receive a smaller share of expenditures, thereby reducing the sensitivity of inflation to these price changes. Indexation of prices works to offset this effect however, with full-indexation completely restoring the usual relationship between reset prices and inflation.

Similarly, positive steady-state inflation has important effects on the log-linearized optimal reset price equation, which is given by

$$\left(1 + \tfrac{\theta}{\eta}\right)\hat{b}_t = (1 - \gamma_2)\sum_{j=0}^{\infty}\gamma_2^j\left(\tfrac{1}{\eta}E_t\hat{y}_{t+j} + E_t\hat{c}_{t+j}\right) + E_t\sum_{j=1}^{\infty}(\gamma_2^j - \gamma_1^j)(\widehat{gy}_{t+j} + \hat{r}_{t+j-1}) +$$

$$\sum_{j=1}^{\infty}\left[\gamma_2^j\left(1 + \tfrac{\theta(\eta+1)}{\eta}\right) - \gamma_1^j\theta\right]E_t\hat{\pi}_{t+j} + \hat{m}_t \tag{22}$$

where $\hat{m}_t$ is a cost-push shock, $\gamma_1 = \lambda\beta^{-1}\overline{\Pi}^{(1-\omega)(\theta-1)}$ and $\gamma_2 = \gamma_1\overline{\Pi}^{(1-\omega)(1+\theta/\eta)}$ so that without steady-state inflation or full indexation we have $\gamma_1 = \gamma_2$. When $\omega < 1$, a higher $\bar{\pi}$ increases the coefficients on future output and inflation but also leads to the inclusion of a new term composed of future differences between output growth and interest rates. Each of these effects makes price-setting decisions more forward-looking.[7] The increased coefficient on expectations of future inflation, which reflects the expected future depreciation of the reset price and the losses associated with it, plays a particularly important role. In response to an inflationary shock, a firm which can reset its price will expect higher inflation today and in the future as other firms update their prices. Given this expectation, the higher the trend rate of inflation, the greater the optimal reset price must be to offset the expected future depreciation in the reset price. Thus, reset prices become more responsive to shocks with higher $\bar{\pi}$. This effect dominates the reduced sensitivity of inflation to the reset price in equation (21), thereby endogenously generating a positive relationship between the level and the volatility of inflation.

To close the model, we assume that the log deviation of the desired gross interest rate from its steady state value, $\hat{r}_t^*$ follows a Taylor rule

$$\hat{r}_t^* = \rho_1\hat{r}_{t-1}^* + \rho_2\hat{r}_{t-2}^* + (1 - \rho_1 - \rho_2)\left[\phi_\pi(\pi_t - \pi^*) + \phi_y(y_t - y^*) + \phi_{gy}(gy_t - gy^*) + \phi_p(p_t - p_t^*)\right] + \varepsilon_t^r$$

where $\phi_\pi, \phi_y, \phi_{gy}, \phi_p$ capture the strength of the policy response to deviations of inflation, the output gap, the output growth rate and the price level from their respective targets, parameters ρ_1 and ρ_2 reflect interest rate smoothing, while ε_t^r is a policy shock. We set $\pi^* = \bar{\pi}, p_t^* = \pi^*t = \bar{\pi}t, y^* = \bar{y}$ and $gy^* = \overline{gy}$ so that the central bank has no inflationary or output bias. The growth rate of output is related to the output gap by

$$\widehat{gy}_t = \hat{y}_t - \hat{y}_{t-1} + (a_t - a_{t-1} - \mu) \tag{23}$$

where a_t is the log level of technology and μ its trend growth rate. Since the actual level of the net interest rate is bounded by zero, the log deviation of the gross interest rate is bounded by $\hat{r}_t = \log(R_t) - \log(\bar{R}) \geq -\log(\bar{R}) = -\bar{r}$ and the dynamics of the actual interest rate are given by

$$\hat{r}_t = \max\{\hat{r}_t^*, -\bar{r}\}. \tag{24}$$

2.3   Shocks

We assume that technology follows a random walk process with drift:

$$a_t = a_{t-1} + \mu + \varepsilon_t^a \tag{25}$$

Each of the risk premium, government, and Phillips Curve shocks follow AR(1) processes

$$\hat{q}_t = \rho_q\hat{q}_{t-1} + \varepsilon_t^q, \tag{26}$$

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[7] See Coibion and Gorodnichenko (2009) for a discussion of each of these effects.

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \varepsilon_t^g, \tag{27}$$
$$\hat{m}_t = \rho_m \hat{m}_{t-1} + \varepsilon_t^m. \tag{28}$$

We assume that $\varepsilon_t^a, \varepsilon_t^q, \varepsilon_t^g, \varepsilon_t^m, \varepsilon_t^r$ are mutually and serially uncorrelated.

2.4 Welfare function

To quantify welfare for different levels of steady-state inflation, we use a second-order approximation to the household utility function as in Woodford (2003).[8] We show our main results in a series of lemmas culminating in Proposition 1. All proofs are in Appendix A.

First of all, we decompose utility described in equation (1) into utility due to consumption and (dis)utility due to labor supply. Lemmas 1 and 2 provide second order approximations for each component.

Lemma 1. Utility from consumption in equation (1) is given by
$$u(C_t) = \tilde{c}_t + t.i.p. + h.o.t. \tag{29}$$
where $\tilde{c}_t = \log(C_t/C_t^F)$ is the percent deviation of consumption from its flexible-price level *t.i.p.* stands for terms independent from policy, and *h.o.t.* means higher order terms.

Lemma 2. Using production function (9), define $\tilde{v}(Y_t(i)) \equiv v(N_t(i)) \equiv N_t(i)^{1+1/\eta}$. Then
$$\tilde{v}(Y_t(i)) \approx \bar{Y}_t^F \tilde{v}_{\bar{Y}_t^F}' \left\{ \tilde{y}_t(i) + \tfrac{1}{2}(1 + \eta^{-1}) \tilde{y}_t^2(i) \right\} + t.i.p. + h.o.t. \tag{30}$$
where $\tilde{y}_t(i) = \log(Y_t(i)/\bar{Y}_t^F)$ is the deviation of firm *i*'s output from the flexible-price level of output $\bar{Y}_t^F$. Correspondingly, the total disutility from labor supply is
$$\int_0^1 \tilde{v}(Y_t(i)) di = \bar{Y}_t^F \tilde{v}_{\bar{Y}_t^F}' \left\{ E_i \tilde{y}_t(i) + \tfrac{1}{2}(1 + \eta^{-1}) \int_0^1 \tilde{y}_t^2(i) di \right\} + t.i.p. + h.o.t. \tag{31}$$
Proof: See Proposition 6.3 in Woodford (2003).

The key insight from Lemmas 1 and 2 is that welfare is diminished when consumption is low relative to its flexible-price level and when the cross-sectional dispersion of output is large. To understand and assess the implications of cross-sectional output dispersion, we need to examine the cross-sectional dispersion of prices.

Denote the cross-sectional dispersion of prices at time *t* with $\Delta_t = \text{var}_i(\log(P_t(i)))$ and let $\bar{\Delta}$ be the cross-sectional dispersion of prices in the non-stochastic steady state. It is straightforward to show that $\bar{\Delta} = \bar{\pi}^2 \frac{\lambda(1-\omega)^2}{(1-\lambda)^2}$ where $\bar{\Delta}$ is increasing in price stickiness λ and steady-state inflation $\bar{\pi}$ and decreasing in the degree of indexation ω. Define $\bar{P}_t = E_i \log P_t(i)$ as the average (across firms) log price of goods.

Lemma 3. The difference between the log price index P_t and the average log price across firms $\bar{P}_t$ is given by

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[8] In our welfare calculations, we use the second order approximation to the consumer utility function while structural relationships in the economy are approximated to first order. As discussed in Woodford (2010), this approach is valid if distortions to the steady state are small so that the first order terms in the utility approximation are premultiplied by coefficients that can also be treated as first order terms. Since given our parameterization the distortions from imperfect competition and inflation are small (as in Woodford 2003), this condition is satisfied in our analysis.

$$\log P_t - \bar{P}_t = Q_p^0 + \tfrac{1-\theta}{2}Q_p^1(\Delta_t - \bar{\Delta}) + h.o.t. \tag{32}$$
where $Q_p^0 = \tfrac{1-\theta}{2}\bar{\Delta}\big/[1 + \tfrac{1}{2}(1-\theta)^2\bar{\Delta}]^2$ and $Q_p^1 = [1 - \tfrac{1}{2}(1-\theta)^2\bar{\Delta}]\big/[1 + \tfrac{1}{2}(1-\theta)^2\bar{\Delta}]^3$.

Lemma 3 is a manifestation of Jensen's inequality. Note that since $\bar{\Delta}$ is quadratic in $\bar{\pi}$, the dispersion of prices $\bar{\Delta}$ is approximately zero when $\bar{\pi} \approx 0$ and therefore $Q_p^0 \approx 0$, $Q_p^1 \approx 1$ so that $\log P_t - \bar{P}_t \approx \tfrac{1-\theta}{2}\Delta_t$, which is the standard result. Again, since $\bar{\Delta}$ is quadratic in $\bar{\pi}$, one can show that $\partial Q_p^0/\partial\bar{\pi} \approx 0$, $\partial Q_p^1/\partial\bar{\pi} \approx 0$ when $\bar{\pi} \approx 0$.

Using Lemma 3, we describe the dynamic properties of the price dispersion in Lemma 4.

Lemma 4. Let $\Xi_t = \Delta_t - \bar{\Delta}$ be the deviation of cross-section price dispersion from its non-stochastic steady state level $\bar{\Delta}$. Then
$$\Xi_t = \Gamma_2(\pi_t - \bar{\pi}) + \lambda\Gamma_1\Xi_{t-1} + (\pi_t - \bar{\pi})^2\Gamma_0\Big\{(1-\lambda)M^2 + \lambda + \tfrac{(1-\theta)^2}{4}[Q_p^1]^2\,\Gamma_2^2 + (1-\lambda)(1-$$
$$\theta)MQ_p^1\,\Gamma_2 - \lambda(1-\theta)Q_p^1\,\Gamma_2\Big\} + h.o.t. \tag{33}$$
where
$$\Gamma_0 = \{1 + (\theta - 1)Q_p^1[(1-\lambda)(\bar{b} + Q_p^0) - \lambda(1-\omega)\bar{\pi}]\}^{-1},$$
$$\Gamma_1 = \{1 - (\theta - 1)(1-\omega)\bar{\pi}Q_p^1\}\Gamma_0,$$
$$\Gamma_2 = 2\{(1-\lambda)M(\bar{b} + Q_p^0) + \lambda(1-\omega)\bar{\pi}\}\Gamma_0,$$
$\bar{b}$ is the log of the optimal reset price in the non-stochastic steady state.

This lemma shows that the cross-sectional price dispersion is a function of its past values as well as the deviation of inflation from its steady state level. In the vicinity of $\bar{\pi} = 0$, $\Gamma_0 \approx 1$, $\Gamma_1 \approx 1$, $\Gamma_2 \approx 0$ and thus cross-sectional price dispersion varies very little over time since it is only a function of the variance of inflation. This is the standard result for welfare calculations in a zero steady state inflation environment (see e.g. Proposition 6.3 in Woodford (2003)). However, $\partial\Gamma_0/\partial\bar{\pi} > 0$, $\partial\Gamma_1/\partial\bar{\pi} < 0$, $\partial\Gamma_2/\partial\bar{\pi} > 0$ locally at $\bar{\pi} \approx 0$. Hence, deviations of inflation from its steady state level have an increasingly strong effect on the cross-sectional price dispersion as $\bar{\pi}$ rises and, as a result, the dynamics of price dispersion can become first-order when $\bar{\pi}$ is sufficiently high. However, given our parameter values and for the levels of trend inflation that we consider, Γ_2 remains very small so that price dispersion effectively remains of second order as in Woodford (2003) and thus we can use a linear approximation of the structural relationships in the economy and a second order approximation of consumer utility for welfare calculations.[9] Henceforth, we will treat Γ_2 as approximately zero.

Using the demand condition (7), we can link the cross-sectional dispersion of output to the cross-sectional dispersion of prices:
$$\tilde{y}_t(i) = \log Y_t(i) - \log \bar{Y}_t^F = \log Y_t - \log \bar{Y}_t^F - \theta\{\log P_t(i) - \log P_t\} \tag{34}$$
and hence

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[9] In our baseline calibration the highest value for Γ_2 is 0.044 which is reached at 6% annual inflation.

$$\Upsilon_t \equiv \text{var}_i \hat{y}_t(i) = \text{var}_i \tilde{y}_t(i) = \theta^2 \text{var}_i \big(\log(P_t(i))\big) = \theta^2 \Delta_t. \tag{35}$$

Let $\overline{\Upsilon}$ be the cross-sectional dispersion of output in the non-stochastic steady state. The remaining piece in the second-order approximation of household's utility is $E_i \tilde{y}_t(i)$, which is the average deviation of output from flexible price level at the firm level. Using the insight of Lemma 3, we can relate $E_i \tilde{y}_t(i)$ to the deviation of output from its flexible-price level at the aggregate level.

Lemma 5. If the deviation of output from its flexible-price level at the aggregate level is defined as $\tilde{y}_t = \log(Y_t / \overline{Y}_t^F)$, then

$$E_i \tilde{y}_t(i) = \tilde{y}_t - Q_y^0 - \tfrac{\theta-1}{2\theta} Q_y^1 (\Upsilon_t - \overline{\Upsilon}) + h.o.t. \tag{36}$$

where $Q_y^0 = \tfrac{\theta-1}{2\theta} \overline{\Upsilon} / [1 + \tfrac{1}{2}\big(\tfrac{1-\theta}{\theta}\big)^2 \overline{\Upsilon}]^2$ and $Q_y^1 = [1 - \tfrac{1}{2}\big(\tfrac{1-\theta}{\theta}\big)^2 \overline{\Upsilon}] / [1 + \tfrac{1}{2}\big(\tfrac{1-\theta}{\theta}\big)^2 \overline{\Upsilon}]^3$.

Similar to the cross-sectional price dispersion, one can show that, since $\overline{\Upsilon}$ is quadratic in $\bar{\pi}$, $Q_y^0 \approx 0$, $Q_y^1 \approx 1$ and $\partial Q_y^0 / \partial \bar{\pi} \approx 0$, $\partial Q_y^1 / \partial \bar{\pi} \approx 0$ when $\bar{\pi} \approx 0$.

The central result can be summarized with the following proposition.

Proposition 1. Given Lemmas 1-5, the second order approximation to per period utility in eq. (1) is[10]

$$\Theta_0 + \Theta_1 \text{var}(\hat{y}_t) + \Theta_2 \text{var}(\hat{\pi}_t) \tag{37}$$

where parameters $\Theta_i, i = \{0,1,2\}$ depend on the steady state inflation $\bar{\pi}$ and are given by

$\Theta_0 = \Big[1 - \tfrac{(1-\Phi)}{(1-\bar{g}_y)}\big(1 - (1 + \eta^{-1})Q_y^0\big)\Big] \log \overline{X} - \tfrac{(1-\Phi)}{(1-\bar{g}_y)} \Big\{ (1 + \eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta} Q_y^1 [\tfrac{1}{2} - (1 + $

$\quad \eta^{-1})Q_y^0]\overline{\Upsilon}\Big\} - \tfrac{(1-\Phi)(1+\eta^{-1})}{2(1-\bar{g}_y)} [\log \overline{X}]^2 - \tfrac{\theta^2}{2(1-\bar{g}_y)} [Q_y^1(\theta^{-1} - 1) - Q_y^1 + (1 + \eta^{-1})\big(1 + \tfrac{\theta-1}{\theta} Q_y^0 Q_y^1\big)]\overline{\Delta},$

$\Theta_1 = -\tfrac{1}{2}(1 + \eta^{-1})/(1 - \bar{g}_y),$

$\Theta_2 = -\tfrac{\theta^2}{2(1-\bar{g}_y)} \Gamma_3 \big\{ [Q_y^1(\theta^{-1} - 1) + (1 + \eta^{-1})\big(1 + \tfrac{\theta-1}{\theta} Q_y^0 Q_y^1\big)] - (1 + \eta^{-1})\tfrac{\theta-1}{\theta} Q_y^1 \log \overline{X} \big\},$

$\Gamma_3 = \tfrac{\Gamma_0}{1 - \lambda \Gamma_1} \{(1 - \lambda)M^2 + \lambda\},$

$\Phi = -\log(\tfrac{\theta-1}{\theta}).$

The loss function in Proposition 1 illustrates the three mechanisms via which trend inflation affects welfare: the steady-state effects, the effects on the coefficients of the utility-function approximation, and the dynamics of the economy via the second moments of macroeconomic variables.[11] First, the term Θ_0 captures the steady-state effects from positive trend inflation, which hinge on the increase in the cross-sectional steady-state dispersion in prices (and therefore in inefficient allocations of resources across sectors) associated with positive trend inflation.[12] Note that as $\bar{\pi}$ approaches zero, Θ_0 converges to zero.

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[10] The complete approximation also contains two linear terms, the expected output gap and expected inflation. Since the distortions to the steady state are small for the levels of trend inflation we consider, the coefficients that multiply these terms can be considered as first order so we can evaluate these terms using the first order approximation to the laws of motion as in Woodford (2003). We confirmed in numeric simulations that they can be ignored.

[11] When $\bar{\pi} = 0$, equation (41) reduces to the standard second-order approximation of the utility function as in Proposition 6.4 of Woodford (2003). There is a slight difference between our approximation and the approximation in Woodford (2003) since we focus on the per-period utility while Woodford calculated the present value.

[12] The parameter Φ measures the deviation of the flexible-price level of output from the flexible-price perfect-competition level of output. See Woodford (2003) for derivation.

As shown by Ascari and Ropele (2009), when $\bar{\pi} = 0$, $\partial\Theta_0/\partial\bar{\pi} > 0$, but the sign of the slope quickly reverses at marginally positive inflation rates. In our baseline calibration, Θ_0 is strictly negative and $\partial\Theta_0/\partial\bar{\pi} < 0$ when trend inflation exceeds 0.04% per annum. Thus for quantitatively relevant inflation rates, the welfare loss from steady-state effects is increasing in the steady-state level of inflation. This is intuitive since, except for very small levels of inflation, the steady state level of output declines with higher $\bar{\pi}$ because the steady state cross-sectional price dispersion rises. The steady-state cost of inflation from price dispersion is one of the best-known costs of inflation and arises naturally from the integration of positive trend inflation into the New Keynesian model. Consistent with this effect being driven by the increase in dispersion, one can show that the steady-state effect is eliminated with full indexation of prices and mitigated with partial indexation.

Second, the coefficient on the variance of output around its steady state $\Theta_1 < 0$ does not depend on trend inflation. This term is directly related to the disutility of cross-sectional dispersion of labor supply. A part of this disutility is brought about by the variance of output around its steady state which is not a direct function of trend inflation. However, this does not imply that positive $\bar{\pi}$ does not impose any output cost. Rather, trend inflation reduces the steady state level of output, which is already captured by Θ_0. Once this is taken into account, then log utility implies that a given level of output variance around the (new) steady state is as costly as it was before. Furthermore, the variance of output around its steady state depends on the dynamic properties of the model which are affected by the level of trend inflation.

The coefficient on the variance of inflation $\Theta_2 < 0$ captures the sensitivity of the welfare loss due to the cross-sectional dispersion of prices. One can also show analytically that for $\bar{\pi} \approx 0$, $\partial\Theta_2/\partial\bar{\pi} < 0$ so that the cross-sectional dispersion of prices becomes *ceteris paribus* costlier in terms of welfare. Recall that an inflationary shock creates distortions in relative prices. Given that positive trend inflation already generates some price dispersion and hence an inefficient allocation of resources, firms operating at an inefficient level have to compensate workers for the increasingly high marginal disutility of sector-specific labor. Because of this rising marginal disutility, the increased distortion in relative prices due to an inflation shock becomes costlier as we increase the initial price dispersion which makes the variance of inflation costlier for welfare as the trend level of inflation rises. Thus, this is a second, and to the best of our knowledge previously unidentified, channel through which the price dispersion arising from staggered price setting under positive inflation reduces welfare. Finally, Θ_2 increases in the Frisch labor supply elasticity and decreases in the elasticity of substitution across goods θ and the Calvo parameter λ.

III   Calibration and Optimal Inflation

Having derived the approximation to the utility function, we now turn to solving for the optimal inflation rate. Because utility depends on the volatility of macroeconomic variables, this will be a function of the

structural parameters and shock processes. Therefore, we first discuss our parameter selection and then consider the implications for the optimal inflation rate in the model. We investigate the robustness of our results to parameter values in subsequent sections.

3.1   Parameters

Our baseline parameter values are illustrated in Table 1. For the utility function, we set η, the Frisch labor supply elasticity, equal to one. The steady-state discount factor β is set to 0.998 to match the real rate of 2.3% per year on 6-month commercial paper or assets with similar short-term maturities given that we set the steady-state growth rate of real GDP per capita to be 1.5% per year ($\overline{GY} = 1.015^{0.25}$), as in Coibion and Gorodnichenko (2009). We set the elasticity of substitution across intermediate goods θ to 10, so that steady-state markups are equal to 11%. This size of the markup is consistent with estimates presented in Burnside (1996) and Basu and Fernald (1997). The degree of price stickiness (λ) is set to 0.55, which amounts to firms resetting prices approximately every 7 months on average. This is midway between the micro estimates of Bils and Klenow (2004), who find that firms change prices every 4 to 5 months, and those of Nakamura and Steinsson (2008), who find that firms change prices every 9 to 11 months.

The degree of price indexation is assumed to be zero in the baseline for three reasons. First, the workhorse New Keynesian model is based only on price stickiness, making this the most natural benchmark (Clarida et al (1999), Woodford (2003)). Second, any price indexation implies that firms are constantly changing prices, a feature strongly at odds with the empirical findings of Bils and Klenow (2004) and more recently Nakamura and Steinsson (2008), among many others. Third, while indexation is often included to replicate the apparent role for lagged inflation in empirical estimates of the New Keynesian Phillips Curve (NKPC; see Gali and Gertler 1999), Cogley and Sbordone (2008) show that once one controls for steady-state inflation, estimates of the NKPC reject the presence of indexation in price setting decisions. However, we relax the assumption of no indexation in the robustness checks in section 4.1.

The coefficients for the Taylor rule are taken from Coibion and Gorodnichenko (2009). These estimates point to strong long-run responses by the central bank to inflation and output growth (2.5 and 1.5 respectively) and a moderate response to the output gap (0.43).[13]   The steady-state share of consumption is set to 0.80 so that the share of government spending is twenty percent. The calibration of the shocks is primarily taken from the estimated DSGE model of Smets and Wouters (2007) with the exception of the persistence of the risk premium shocks for which we consider a larger value calibrated at

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[13] Because empirical Taylor rules are estimated using annualized rates while the Taylor rule in the model is expressed at quarterly rates, we rescale the coefficient on the output gap in the model such that $\phi_y = 0.43/4 = 0.11$.

0.94 to match the historical frequency of hitting the zero lower bound and the routinely high persistence of risk premia in the financial series.[14]

In our baseline model, positive trend inflation is costly because it leads to more price dispersion and therefore less efficient allocations, more volatile inflation, and a greater welfare cost for a given amount of inflation volatility. On the other hand, positive trend inflation gives policy-makers more room to avoid the ZLB on interest rates. Therefore, a key determinant of the tradeoff between the two depends on how frequently the ZLB is binding for different levels of trend inflation. To illustrate the implications of our parameter calibration for how often we hit the ZLB, Figure 1 plots the fraction of time spent at the ZLB from simulating our model for different steady-state levels of the inflation rate. In addition, we plot the steady-state level of the nominal interest rate associated with each inflation rate, where the steady-state nominal rate in the model is determined by $\bar{R} = \bar{\Pi} \cdot \overline{GY}/\bar{\beta}$. Our calibration implies that with a steady-state inflation rate of approximately 3.5%, the average rate for the U.S. since the early 1950's, the economy should be at the ZLB approximately 4 percent of the time. With U.S. interest rates at the ZLB since late 2008 and expected to remain so until at least the end of 2010, this yields a historical frequency of being at the ZLB of 3.5 percent (i.e. around 2 years out of 60).

In addition, this calibration agrees with the historical changes in interest rates associated with post-WWII U.S. recessions. For example, starting with the 1958 recession and excluding the current recession, the average decline in the Federal Funds Rate during a recession has been 4.76 percentage points.[15] The model predicts that the average nominal interest rate with 3.5% steady-state inflation is around 6%, so the ZLB would not have been binding during the average recession, consistent with the historical experience. Only the 1981-82 recession led to a decline in nominal interest rates that would have been sufficiently large to reach the ZLB (8.66% drop in interest rates), but did not because nominal interest rates and estimates of trend inflation over this period were much higher than their average values. Thus, with 3-3.5% inflation, our calibration (dotted line in Figure 1) implies that it would take unusually large recessions for the ZLB to become binding. In addition, our calibration indicates that at much lower levels of trend inflation $\bar{\pi}$, the ZLB would be binding much more frequently. For example, at a zero steady-state inflation rate, the ZLB would be binding 25% of the time. Given the historical experience of the U.S., this seems conservative, as it exceeds the historical frequency of recessions. The model predicts a steady-state level of interest rates of less than 2.5% when $\bar{\pi} = 0$, and six out the last eight recessions (again excluding the current episode) were associated with decreases in interest rates that exceeded this

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[14] This is, for example, consistent with the persistence of the spread between Baa and Aaa bonds which we estimate to be 0.945 between 1920:1 and 2009:2 and 0.941 between 1950:1 and 2009:2 at the quarterly frequency.

[15] This magnitude is calculated by taking the average level of the fed funds rate (FFR) over the last 6 months prior to the start of each recession as defined by the NBER and subtracting the minimum level of the FFR reached in the aftermath of that recession.

value (specifically the 1969, 1973, 1980, 1981, 1990 and 2001 recessions). Our calibration is also largely in line with the frequency of the ZLB we would have observed given historical declines in nominal interest rates during recessions and counterfactual levels of trend inflation (broken line in Figure 1). Thus, we interpret our parameterization as providing a reasonable representation of the likelihood of hitting the ZLB for different inflation rates given the historical experience of the U.S.

3.2   Optimal Inflation

Having derived the dynamics of the model, parameterized the shocks, and obtained the second-order approximation to the utility function, we now simulate the model for different levels of trend inflation $\bar{\pi}$ and compute expected utility for each $\bar{\pi}$. The results taking into account the ZLB and in the case when we ignore the ZLB are plotted in Panel A of Figure 2. Note first that, when the ZLB is not taken into account, the optimal rate of inflation is zero because there are only costs to inflation and no benefits.[16] Figure 2 also plots the other extreme when we include the ZLB but do not take into account the effects of positive steady-state inflation on the loss function or the dynamics of the model. In this case, there are no costs to inflation so utility is strictly increasing as steady-state inflation rises and the frequency of the ZLB diminishes. Our key result is the specification which incorporates both the costs and benefits of inflation. As a result of the presence of the ZLB, we find that utility is increasing at very low levels of inflation so that that zero inflation is not optimal when the zero bound is present. Second, *the peak level of utility is reached when the inflation rate is 1.1% at an annualized rate*. This is at the bottom end of the target range of most central banks, which are commonly between 1% and 3%. Thus, our baseline results imply that taking into account the zero bound on interest rates raises the optimal level of inflation, but with no additional benefits to inflation included in the model, the optimal inflation rate is within the standard range of inflation targets. Third, inflation rates above the optimal level monotonically lower utility. Fourth, the costs of even moderate inflation can be nontrivial. For example, a 4% annualized inflation rate would lower utility by nearly 2% relative to the optimal level, which given log utility in consumption is equivalent to a permanent 2% decrease in the level of consumption.

In addition, Panel B of Figure 2 quantifies the importance of each of the three costs of inflation – the steady state effect, the increasing cost of inflation volatility, and the positive link between the level and volatility of inflation – by calculating the optimal inflation rate subject to the zero lower bound when only one of these costs, in turn, is included. The first finding to note is that allowing for any of the three inflation costs is sufficient to bring the optimal inflation rate to 2.5% or below. Thus, all three inflation costs incorporated in the model are individually large enough to prevent the ZLB from pushing the optimal inflation rate above the current target range of most central banks. Second, the steady-state cost

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[16] We determine optimal inflation in a grid-space with a width of 0.04% per year. It is therefore possible, although quantitatively not important, that the optimal inflation rate without the ZLB is positive but less than 0.04% per year.

of price dispersion is the largest cost of inflation out of the three, bringing the optimal inflation rate down to 1.3% by itself.

To get a sense of which factors drive these results, the top row of Figure 3 plots the coefficients of the second-order approximation to the utility function from Proposition 1. In short, the results confirm the analytical derivations in section 2.4. First, rising inflation has important negative steady-state effects on utility, as the increasing price dispersion inefficiently lowers the steady-state level of production and consumption. For example, 6% inflation lowers utility by nearly 4% relative to zero inflation via the steady-state effects. Second, the coefficient on the variance of output around its steady state is independent of $\bar{\pi}$ even though the new steady state level of output is lower. This reflects our assumption of log-utility in consumption. Third, the coefficient on inflation variance is decreasing in $\bar{\pi}$, i.e., holding the inflation variance constant, higher $\bar{\pi}$ raises the utility cost of the variance in inflation. This reflects the fact that when the steady state level of price dispersion is already high then a temporary increase in price dispersion due to higher or lower inflation is even more costly. Moving from zero inflation to six percent inflation raises the coefficient on inflation variance by over 40% in absolute value. Thus, as $\bar{\pi}$ rises, policy-makers should place an increasing weight on the variance of inflation relative to the variance of the output gap.

The middle row of Figure 3 plots the effects of $\bar{\pi}$ on the variance of inflation and the output gap, i.e. the dynamic effects of steady-state inflation and the zero bound on interest rates. In addition, we plot the corresponding moments in the absence of the zero-bound on interest rates to characterize the contribution of the zero-bound on macroeconomic dynamics. A notable feature of the figure is that output volatility rises much more rapidly as $\bar{\pi}$ falls when the ZLB is present. Intuitively, the ZLB is hit more often at a low $\bar{\pi}$. With the nominal rate fixed at zero the central bank cannot stabilize the economy by cutting interest rates further and thus macroeconomic volatility increases. As we increase $\bar{\pi}$, macroeconomic volatility (especially for output) diminishes. This is the benefit of higher $\bar{\pi}$ in the model.

The effect of changes in $\bar{\pi}$, however, is non-linear for the variance of inflation when we take into account the zero-bound on interest rates. At low levels of inflation, increasing $\bar{\pi}$ reduces the volatility of inflation for the same reason as for output: the reduced frequency of hitting the zero bound. On the other hand, higher $\bar{\pi}$ also tends to make price-setting decisions more forward-looking, so that, absent the zero bound, inflation volatility is consistently rising with $\bar{\pi}$, a feature emphasized in Kiley (2007) and consistent with a long literature documenting a positive relationship between the level and variance of inflation (Okun (1971), Taylor (1981) and Kiley (2000)). When $\bar{\pi}$ rises past a specific value, the latter effect starts to dominate and the variance of inflation begins to rise with $\bar{\pi}$. Given our baseline values, this switch occurs at an annualized trend inflation rate of approximately 2.5%. These results show the importance of modeling both the zero-bound and the effects of $\bar{\pi}$ on the dynamics of the model.

The bottom row of Figure 3 then plots the contribution of these different effects on the welfare costs of inflation, i.e. each of the terms in Proposition 1. These include the steady-state effects of $\bar{\pi}$ as well as the interaction of the effects of $\bar{\pi}$ on the coefficients of the utility function approximation and the dynamics of the economy. The most striking result is that the welfare costs and benefits of positive $\bar{\pi}$ are essentially driven by only two components: the steady-state effect and the contribution of inflation variance to utility. In particular, the U-shape pattern of the inflation variance combined with decreasing Θ_2 plays the key role in delivering a positive level of the optimal inflation rate, while the effects of the ZLB on the contribution of the output gap variability are an order of magnitude smaller and therefore play a limited role in determining the optimal inflation rate.

3.3 Are the costs of business cycles and the ZLB too small in the model?

The minor contribution of output gap volatility to the optimal inflation rate might be interpreted as an indication that the model understates the costs of business cycles in general and the ZLB in particular. For the former, the implied welfare costs of business cycles in our model are approximately 1.5% of steady-state consumption at the optimal inflation rate, in line with DeLong and Summers (1988), Barlevy (2004), and much larger than in Lucas (1987). To assess the cost of hitting the ZLB, we compute the average welfare loss net of steady-state effects from simulating the model under different inflation rates both with and without the zero bound. The difference between the two provides a measure of the additional welfare cost of business cycles due to the presence of the ZLB. We can then divide this cost by the average frequency of being at the zero bound from our simulations, for each level of steady-state inflation, to get a per-quarter average welfare loss measure conditional on being at the ZLB which is plotted in Figure 4. With a steady-state inflation rate of zero, the average cost of a quarter spent at the ZLB is equivalent to a permanent 3.5% reduction in consumption. As steady-state inflation rises, this per-period cost declines because the average duration of ZLB episodes gets shorter and the output losses during the ZLB are increasing non-linearly with the duration of the ZLB (see Christiano et al (2009)). At a steady-state inflation rate of 3%, the average per-quarter cost of the ZLB is approximately equal to a permanent 0.5% reduction in consumption. The additional cost conditional on being restrained by the zero lower bound on interest rates for 8 quarters would then exceed a permanent 4% reduction in consumption, or \$400 billion per year based on 2008 consumption data.[17,18] For comparison, Williams

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[17] In this calculation, we treat as negligible the fact that in our model the presence of the ZLB can reduce the average level of output because the product of the average reduction in output, the frequency of hitting the ZLB and the utility weight on the first order term corresponding to the output gap is small. However, conditional on hitting the ZLB, the reduction in output could be substantial. For example, the average per-quarter reduction in output when the ZLB is binding at three percent trend inflation is about one percent. Assuming the binding ZLB lasts for eight quarters, the fall in output is then worth \$1.2 trillion dollars over 2 years, or 33% more than Williams' (2009) estimate.

(2009) uses the Federal Reserve's FRB/US model to estimate that the ZLB between 2009 and 2010 cost $1.8 trillion in lost output over four years, or roughly $300 billion per year in lost consumption over four years if one assumes that the decline in consumption was proportional to the decline in output. Thus, the costs of both business cycles and the ZLB in the model cannot be described as being uncharacteristically small.

3.4   How does optimal inflation depend on the coefficient on the variance of the output gap?

The fact that the large decline in the volatility of the output gap associated with the decreased frequency of hitting the ZLB as steady state inflation rises has so little effect on optimal inflation reflects the small relative weight assigned to output gap fluctuations in the utility function. At $\bar{\pi} = 0$, the coefficient on the output gap variance in the loss function is less than one-hundredth that on the quarterly inflation variance (or one-tenth for the annualized inflation variance), and this difference becomes even more pronounced as $\bar{\pi}$ rises. The low optimal weight on output gap volatility is standard in New Keynesian models (see Woodford (2003)) and could reflect the lack of involuntary unemployment, which causes substantial hardship to a small fraction of the population and whose welfare effects may therefore be poorly approximated by changes in aggregate consumption and employment, or the absence of distributional considerations, since business cycles disproportionately affect low income/wealth individuals who are likely to have higher marginal utilities of consumption than the average consumer.[19]

To assess how sensitive the optimal inflation rate is to the coefficient on the output gap variance, we increase this coefficient by a factor ranging from 1 to 100 and reproduce our results for the optimal inflation rate for each factor (see Figure 4). As expected, raising the coefficient on the variance of the output gap pushes the optimal inflation rate higher. However, the coefficient on the output gap variance needs to be strikingly large to qualitatively affect our findings. For example, much of the traditional literature on optimal monetary policy assumed an equal weight on output and annualized inflation variances in the loss function. With inflation being measured at an annualized rate, this equal weighting obtains at zero steady-state inflation when Θ_1 is multiplied by a factor of approximately 10. Yet this weighting would push the optimal inflation rate up only modestly, to less than 1.4% per year.

Raising the optimal inflation rate substantially above 2.5% would require increasing the coefficient on output gap variance in the loss function by around 100, thereby making the weight placed on output volatility approximately ten times as large as that on annualized inflation volatility, a weighting scheme which seems difficult to justify. Furthermore, such a loss function would imply that the costs of business cycles are equivalent to a permanent drop in consumption of 10% when evaluated at the optimal

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[18] In our baseline calibration, the 90% confidence interval for the duration of ZLB episodes is (2,12) at 3.5% inflation.

[19] Christiano et al (2010) introduce involuntary unemployment into a New Keynesian model and find that this does not significantly raise the cost of business cycles.

inflation rate, which is much larger than commonly found in the literature. Finally, placing such weight on output volatility would also substantially increase the per-quarter cost of having the ZLB bind. Evaluated at the optimal inflation rate, each quarter with a binding ZLB would lower utility by an average amount equivalent to a 5% permanent reduction in consumption, such that an episode of 8 consecutive quarters at the ZLB would deliver welfare losses equivalent to roughly 40% of steady-state consumption, above and beyond the costs of the shock in the absence of the ZLB. Thus, while one can mechanically raise the optimal inflation rate by appealing to larger weights on output fluctuations than implied by the model, weighting schemes which meaningfully raise the optimal inflation rate will point to welfare costs of business cycles, and particularly episodes at the ZLB, that depart from the conventional wisdom.

IV   Robustness of the Optimal Inflation Rate to Alternative Parameter Values

In this section, we investigate the robustness of the optimal inflation rate to our parameterization of the model. We focus particularly on the pricing parameters, the discount factor, and the risk premium shock.[20]

4.1   Pricing Parameters

Figure 5 plots the utility-approximation for different levels of $\bar{\pi}$ for alternative pricing parameters, as well as the optimal inflation rates associated with these parameter values. First, we consider the role of the elasticity of substitution θ. Note that the welfare costs of inflation are larger when θ is high. This result captures the fact that a higher elasticity of substitution generates more real rigidity and therefore more persistence in fluctuations, thereby raising the welfare cost of fluctuations for any $\bar{\pi}$. However, the effects of this parameter on the optimal $\bar{\pi}$ are relatively small: using a value of $\theta = 5$, half of our baseline and an upper bound on how big markups are in the economy, raises the optimal $\bar{\pi}$ to about 1.6% from our baseline of 1.1%. This is well within the range of inflation targets employed by modern central banks.

We also investigate the role of price indexation. In our baseline, we assumed $\omega = 0$, based on the fact that firms do not change prices every period in the data, as documented by Bils and Klenow (2004) and Nakamura and Steinsson (2008), as well as the results of Cogley and Sbordone (2008) who argue that once one controls for time-varying trend inflation, we cannot reject the null that $\omega = 0$ for the US. However, because price-indexation is such a common component of New Keynesian models, we consider the effects of price indexation on our results. Figure 5 indicates that higher levels of indexation lead to higher optimal rates of inflation because indexation reduces the dispersion of prices. Yet with $\omega = 0.5$,

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[20] We also investigated the sensitivity of our results to the Frisch labor supply elasticity and other parameters. These had minor quantitative effects on our results; we omit them from the text in the interest of space. We also added habit formation to the model and found that this had little effect on the optimal inflation rate.

which is most likely an upper bound for an empirically plausible degree of indexation in low-inflation economies like the U.S., the optimal $\bar{\pi}$ remains less than 2%.

Third, we investigate the effects of price stickiness. Our baseline calibration, $\lambda = 0.55$, is midway between the findings of Bils and Klenow (2004) of median price durations of 4-5 months and those of Nakamura and Steinsson (2008) of median price durations of 9-11 months. We now consider values of λ ranging from 0.50 to 0.65. There is little impact on welfare for very low levels of inflation, but as the steady state inflation rate rises past 3%, higher degrees of price stickiness are associated with much larger welfare losses than the baseline.[21] This reflects the fact that with more price stickiness, price dispersion is greater, and this effect is amplified at higher levels of steady-state inflation, thereby generating much larger welfare losses. Nonetheless, this has only minor effects on the optimal inflation rate.

4.2 Discount Factor and Risk Premium Shocks

We also consider the sensitivity of our results to the discount factor and the parameters governing the risk premium shocks. First, we reproduce our baseline welfare figure for different levels of the persistence to risk premium shocks. The results are quite sensitive to this parameter, which reflects the fact that these shocks play a crucial role in hitting the zero lower bound. For example, Figure 6 illustrates that when we raise the persistence of the shock from 0.94 to 0.96, the optimal inflation rate rises from 1.2% to 2.7% because this increase in the persistence of the shock has a large effect on the frequency and duration of being at the ZLB. At 3.5% inflation, this frequency more than doubles relative to our baseline scenario, thereby raising the benefit of higher steady-state inflation. The reverse occurs with lower persistence of risk premium shocks: the frequency of being at the ZLB declines sharply as does the optimal inflation rate. Second, similar results obtain when we vary the volatility of the risk premium shock. When we increase the standard deviation of these shocks to 0.0035 from our baseline of 0.0024, the optimal inflation rate again rises to slightly over two percent. As with the persistence of the shocks, this is driven by a higher frequency of being at the ZLB: at 3.5% inflation, this alternative shock volatility implies the economy would be at the ZLB more than twice as often as under our baseline calibration.

Third, we consider the sensitivity of our results to the steady-state level of the discount factor β. This parameter is also important in determining the frequency at which the economy is at the ZLB since it affects the steady-state level of nominal interest rates. This is particularly important at low levels of $\bar{\pi}$. As with the risk premium shock variables, a higher value of β is associated with a lower steady-state level of nominal interest rates, so that the ZLB will be binding more frequently. For example, with β=0.9999 (which corresponds to a real rate of 1.54% per year), the ZLB is binding approximately 6% of the time

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[21] In fact, if we increase λ above 0.65 the model starts to run into indeterminacy regions which are potentially associated with unbounded volatility.

when steady-state inflation is 3.5%. At the maximum, however, the optimal $\bar{\pi}$ is only 0.5% higher than implied by our baseline results.

These robustness checks clearly illustrate how important the frequency at which the economy hits the ZLB is for our results. Naturally, parameter changes which make the ZLB binding more often raise the optimal rate of inflation because a higher $\bar{\pi}$ lowers the frequency of hitting the ZLB. Thus, the key point is not the specific values chosen for these parameters but rather having a combination of them that closely reproduces the historical frequency of hitting the ZLB for the U.S. In any case, even if we consider parameter values that double the frequency of hitting the ZLB at the historical average rate of inflation for the U.S., the optimal inflation rate rises only to about 2-2.5%, which is still well within the target range of most central banks.

V   What Could Raise the Optimal Inflation Rate?

While our baseline model emphasizes the tradeoff between higher trend inflation to ensure against the zero-bound on nominal interest rates versus the utility costs associated with higher trend inflation, previous research has identified additional factors beyond the lower-bound on nominal interest rates which might lead to higher levels of optimal inflation. In this section, we extend our analysis to assess their quantitative importance. First, we include capital formation in the model. Second, we allow for uncertainty about parameter values on the part of policy-makers. Third, we consider the possibility that the degree of price stickiness varies with $\bar{\pi}$. Fourth, we explore whether our results are sensitive to using Taylor pricing. Fifth, we integrate downward nominal wage rigidity, i.e. "greasing the wheels," into the model.

5.1   Capital

First, we consider how sensitive our results are to the introduction of capital. We present a detailed model in Appendix B and only provide a verbal description in this section. In this model, firms produce output with a Cobb-Douglas technology (capital share $\alpha = 0.33$). All capital goods are homogeneous and can be equally well employed by all firms. Capital is accumulated by the representative consumer subject to a quadratic adjustment cost to capital ($\psi = 3$ as in Woodford (2003)) and rented out in a perfectly competitive rental market. The aggregate capital stock depreciates at rate $\delta = 0.02$ per quarter. We calculate the new steady state level of output relative to the flexible price level output and derive the analogue of Proposition 1 in Appendix B with proofs in Appendix C.

By allowing capital to freely move between firms we reduce the steady state welfare cost from trend inflation. Now firms that have a relatively low price can hire additional capital rather than sector-specific workers to boost their output. Thus the disutility of labor does not increase by as much as it did in the labor-only model and this will be a force to raise the optimal inflation level. However, capital also

reduces the likelihood of hitting the ZLB. Unlike the labor-only model, including capital permits positive savings and so we are less likely to be in a situation where an increase in q_t pushes interest rates to zero. This channel will lower the optimal inflation rate in the capital model relative to the labor-only model. We isolate the first channel by setting $\rho_q = 0.93$ to match the historical frequency at the ZLB. As shown in Figure 7, utility peaks at a trend inflation rate of 1.7% per annum suggesting that capital does not lower the cost of inflation substantially, yielding only a small net increase in the optimal inflation rate.

5.2   Model Uncertainty

An additional feature that could potentially lead to higher rates of optimal inflation is uncertainty about the model on the part of policy-makers. If some plausible parameter values lead to much higher frequencies of hitting the ZLB or raise the output costs of being at the ZLB, then policy-makers might want to insure against these outcomes by allowing for a higher $\bar{\pi}$. To quantify this notion, we consider two exercises. First, we identify $\bar{\pi}$ that maximizes expected utility taking into account parameter uncertainty. Second, we generate a distribution of optimal $\bar{\pi}$ from the distribution of parameters.

Our starting point for both exercises is uncertainty about the parameters of the model. We characterize this uncertainty via the variance-covariance matrix of the estimated parameters from Smets and Wouters (2007). We place an upper bound on parameter values to eliminate draws where the ZLB binds unrealistically often, in excess of 10% at 6% annual trend inflation. To assess the optimal inflation rate given uncertainty about parameter values, we compute the expected utility associated with each level of steady-state inflation by repeatedly drawing from the distribution of parameter values.[22] Figure 8 (Panel A) plots the implied levels of expected utility associated with each steady-state level of inflation. Maximum utility is achieved with an inflation rate of 1.4% per year. As expected, this is higher than our baseline result, which reflects the fact that some parameter draws lead to much larger costs of being at the zero-bound, a feature which also leads to a much more pronounced inverted U-shape of the welfare losses from steady-state inflation. Nonetheless, this optimal rate of inflation remains well within the bounds of current inflation targets of modern central banks.[23]

Secondly, for each draw from the parameter space, we solve for the optimal inflation rate, thereby allowing us to characterize the uncertainty associated with our baseline results. Figure 8 (Panel B) plots the distribution function of these inflation rates. The 90% confidence interval of optimal inflation rates ranges from 0.1% to 2.2% per year, which again is very close to the target range for inflation of most

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[22] Note that some parameter draws yield an indeterminate solution. In this case, we solve for the dynamics using the "continuous" solution as in Lubik and Schorfheide (2003).

[23] Billi (2009) considers the effects of parameter uncertainty on the optimal inflation rate via robust control methods and also finds that parameter uncertainty raises the optimal inflation rate slightly.

central banks. In short, incorporating model uncertainty confirms our baseline finding that, even after taking into account the ZLB, the optimal inflation rate is low and close to current targeted levels.

5.3 Endogenous Price Stickiness

In our baseline analysis, we treat the degree of price rigidity as a structural parameter. However, theory implies that the cost to firms of not changing prices should increase as $\bar{\pi}$ rises (Romer 1990). Higher levels of inflation should therefore be associated with lower levels of price stickiness, which would tend to lower the welfare costs of positive trend inflation. Thus, by ignoring this endogeneity, we might be overstating the costs of positive inflation and thereby underestimating the optimal rate of inflation.

The empirical evidence on the sensitivity of price rigidity to the inflation rate is mixed. Gagnon (2009), for example, finds a statistically significant relationship between the inflation rate and the frequency of price changes for Mexico, but only when the inflation rate exceeds 10% per year. Dhyne et al (2005) find a positive relationship between the two for a cross-section of European countries, but the relationship is not statistically significantly different from zero once they control for a variety of other factors. Nakamura and Steinsson (2008) similarly estimate the relationship between annual inflation and the frequency of price changes using micro-level US data and find only mixed evidence in favor of a statistically significant link between the two.

Despite the absence of a strong empirical link between price stickiness and the steady state level of inflation (at least for plausible U.S. levels of inflation), we consider the sensitivity of our baseline results to a possible systematic link between the two. Specifically, we follow Nakamura and Steinsson's empirical approach and posit a linear relationship between the (monthly) frequency of price changes and the steady state annual rate of inflation, with the coefficient on inflation denoted by β_π. The average estimate of Nakamura and Steinsson across price measures and time periods is approximately $\beta_\pi = 0.5$, and the upper bound of their confidence intervals is approximately $\beta_\pi = 1$. We reproduce our analysis using these values, as well as our baseline assumption of $\beta_\pi = 0$, and plot the results in Figure 9.[24] In each case, we calibrate the degree of price rigidity such that $\lambda=0.55$ (our baseline value) at a steady-state level of annual inflation of 3.5%.

Panel A shows the implied variation in the degree of price stickiness. For the mean estimated degree of endogeneity from Nakamura and Steinsson (2008), the average duration between price changes

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[24] To be clear, we allow the degree of price stickiness to vary with the steady-state rate of inflation, but not with fluctuations in the inflation rate around its steady-state value. An alternative approach would be to model endogenous price setting via "menu" costs. Although we do not consider this approach explicitly here, one would expect two opposing effects on the optimal inflation rate from such an extension. First, paying the fixed costs of changing prices would lower welfare, and this cost would be rising with steady-state inflation as firms would reset their prices more often. On the other hand, because of selection effects, price dispersion should rise less rapidly with inflation than under time-dependent pricing models, as emphasized in Burstein and Hellwig (2008). It is unclear, ex ante, which effect would dominate.

varies from eight months to six months, while the upper bound on endogeneity of price stickiness yields durations ranging from nine months to five and a half months. Panel B shows that the frequency of hitting the ZLB varies little across specifications: for a given inflation rate, a higher frequency of price changes is associated with larger movements in inflation (and therefore hitting the ZLB more frequently) but these changes are less persistent (so the economy exits the ZLB more rapidly), leaving the overall frequency of being at the ZLB largely unchanged across specifications. Panels C-F demonstrate the effects of endogeneity on the optimal inflation rate and the utility associated with different levels of steady-state inflation. First, the optimal inflation rate is remarkably insensitive to endogenous price stickiness, falling only to slightly less than 1.0% at the maximum sensitivity of price stickiness to steady-state inflation. Second, the welfare costs of inflation at the optimal rate are rising with endogenous price setting. This reflects the fact that, given the same low optimal rate of inflation, more endogeneity is associated with higher degrees of price stickiness and therefore a higher cost of inflation. Third, despite the fact that the optimal rate of inflation varies little with endogenous price stickiness, the costs of much higher $\bar{\pi}$ are significantly lower relative to our baseline, because higher inflation leads to more frequent price changes and therefore less price dispersion in the steady state.

5.4   Taylor pricing

An alternative approach to Calvo pricing is the staggered contracts approach of Taylor (1977) in which firms set prices for a pre-determined duration of time. This pricing assumption can loosely be thought of as a lower bound on forward-looking behavior (conditional on price durations) since it imposes zero weight on expected profits beyond those of the contract length in the firm's reset price optimization. The derivation of the utility approximation as well as the structural log-linearized equations of the model is similar to Calvo pricing (Appendix D contains details for the utility approximation when the duration of price spells is equal to three quarters).

Figure 10 compares the results under Taylor pricing (duration of price contracts equal to 3 and 4 quarters) with the results from Calvo pricing. The optimal inflation rates for the Taylor model are 1.4 and 1.1 percent per year for price durations of 3 and 4 quarters respectively, which is close to the 1.1 percent per year found for the baseline Calvo model. Note also that the volatility of inflation and the output gap (Panels C and D) as well as the frequency of hitting ZLB (Panel B) are approximately the same in all models. However, the Taylor model has smaller welfare losses as $\bar{\pi}$ increases above 2 percent per year. For example, the welfare loss at the steady state inflation of 6 percent per year relative to the minimum loss is about 4 percent in the Calvo model but only about 1-2 percent in the Taylor model (Panel A). The key source of this difference is that the Taylor model assigns smaller steady state effects (Panel E) and a

lower weight on inflation variability (Panel F) than the Calvo model does.[25] Intuitively, since firms under Calvo pricing may be stuck with a suboptimal price for a long time, the cost of positive steady state inflation is larger than in the Taylor model where firms are guaranteed to change prices in a fixed number of periods. In summary, although both the Taylor and Calvo models yield similar optimal inflation rates, these models provide different estimates of welfare gains from low steady state inflation.

5.5   Downward Wage Rigidity

A common motivation for positive trend inflation, beside the zero-lower bound and seigniorage, is the "greasing the wheels" effect raised by Tobin (1972). If wages are downwardly rigid, as usually found in the data (e.g., Dickens et al 2007), then positive trend inflation will facilitate the downward-adjustment of real wages required to adjust to negative shocks. To quantify the effects of downward nominal wage rigidity in our model, we integrate it in a manner analogous to the zero-bound on interest rates by imposing that changes in the aggregate nominal wage index be above a minimum bound

$$\Delta \widehat{w}_t = \max \{\Delta \widehat{w}_t^m, \Delta \widehat{w}_t^*\}$$

where Δw_t^* is the change in wages that would occur in the absence of the zero-bound on nominal wages and $\Delta \widehat{w}_t^m$ is the lower-bound on nominal wage changes. Note that even with zero steady-state inflation, steady-state nominal wages grow at the rate of technological progress. Thus, we set $\Delta \widehat{w}_t^m$ to be equal to minus the sum of the growth rate of technology and the steady state rate of inflation.

Figure 11 presents the utility associated with different steady-state inflation rates under both the zero-bound on interest rates and downward-wage rigidity. The result is striking: the optimal inflation rate falls to zero with downward wage rigidity. The reason for this counterintuitive finding is illustrated in Panel B of Figure 11. With downward wage rigidity, marginal costs are much less volatile, so the variance of inflation is substantially reduced relative to the case with flexible wages. In addition, the fact that marginal costs are downwardly rigid means that, in the face of a negative demand shock, inflation will decline by less and therefore interest rates will fall less, reducing the frequency of the ZLB. With $\bar{\pi} = 0$, the ZLB binds approximately 10 percent of the time with downward wage rigidity but over 25 percent of the time with flexible wages. This decrease in the frequency of binding ZLB at low $\bar{\pi}$ reduces the benefit of higher trend inflation and leads to lower estimates of the optimal inflation rate.[26]

VI   Normative Implications

We have so far been treating the question of the optimal inflation rate independently of the systematic response of the central bank to macroeconomic fluctuations. In this section, we consider the sensitivity of

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[25] The coefficient on the output variability is the same in the Taylor and Calvo models.
[26] Kim and Ruge-Murcia (2009) similarly find that downward wage rigidity, by itself, has little positive effect on the optimal inflation rate in an estimated DSGE model.

our baseline results to alternative coefficients for the central bank's response function as well as the normative question of how the central bank can optimize over both optimal inflation and its systematic response to the economy.

6.1 Taylor Rule Parameters

Our baseline Taylor rule parameters are taken from Coibion and Gorodnichenko (2009) based on the post-1982 era. However, as has been emphasized in the empirical literature on central banks' reaction functions, there is robust evidence of time-variation in the Federal Reserve's systematic response to economic fluctuations.[27] In addition, like the steady-state inflation rate, the reaction function is under the control of policymakers so we are interested in studying the interaction of these policy variables on welfare. Thus, we consider the implications of alternative parameter values in the Taylor rule, illustrated in Figure 12. First, we consider long-run responses to inflation by the central bank ranging from 2 to 5. The effects on welfare are unambiguous: stronger long-run responses to inflation raise welfare in the model for all inflation rates. Intuitively, this stronger systematic response reduces inflation and output volatility, thereby leading to a lower frequency of being at the ZLB and higher utility. However, this has little effect on the optimal inflation rate, which ranges from 1.2% when $\phi_\pi = 2$ to 0.9% when $\phi_\pi = 5$.

We also investigate the sensitivity to the central bank's response to the real side of the economy via output growth or output gap. We find that stronger responses to output growth generally lower welfare while higher responses to output gap are welfare-improving. This finding is interesting for two reasons. First, Orphanides (2003) and Coibion and Gorodnichenko (2009) emphasize that one of the primary changes in U.S. monetary policy around the time of the Volcker disinflation was the switch from responding aggressively to the gap toward responding more aggressively to the growth rate of output. One advantage of the latter is that output growth is readily observable whereas the output gap is likely to be subject to much more real-time measurement error, as documented in Orphanides and van Nordern (2002). However, our results indicate that responding strongly to output growth actually reduces welfare in a New Keynesian model. Second, Coibion and Gorodnichenko (2009) show that responding to the output gap can be destabilizing in New Keynesian models under positive steady-state inflation because it can lead to indeterminacy. Responding to output growth, on the other hand, helps achieve determinacy for smaller responses to inflation when steady-state inflation is positive. Figure 12 indicates that conditional on staying in the determinacy region the welfare results go in the other direction. Hence, there is a tradeoff between the two measures in terms of stabilization: responding to the (properly measured) gap is welfare improving as long as the economy remains in the determinacy region, but increases the likelihood of switching to an indeterminate equilibrium with the possibility of sunspot

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[27] See Clarida et al (2000), Orphanides (2003), Boivin (2006), and Coibion and Gorodnichenko (2009) for examples.

fluctuations. Responding to the growth rate of output moves the economy away from the indeterminacy region but leads to lower welfare within the indeterminacy region. However, in terms of the optimal inflation rate, the distinction is minor and neither measure has much quantitative importance in determining the optimal inflation rate within the determinacy region of the parameter space.

6.2   Price-Level Targeting, the Zero Bound, and the Optimal Inflation Rate

While our baseline specification of the Taylor rule restricts the endogenous response of the central bank to inflation and the real side of the economy, an additional factor sometimes considered is price-level targeting (PLT). While the evidence for central banks actually following PLT remains scarce, PLT has nonetheless received substantial attention in the literature for several reasons. First, as emphasized in Woodford (2003), PLT guarantees determinacy under zero trend inflation for any positive response to the price level gap. Second, Coibion and Gorodnichenko (2009) show that PLT ensures determinacy for positive steady-state inflation rates as well, and is not subject to the deterioration of the Taylor principle as a result of positive trend inflation which occurs when the central bank responds only to inflation. Third, Gorodnichenko and Shapiro (2007) show that PLT robustly helps stabilize inflation expectations, thereby yielding smaller inflation and output volatility than would occur in inflation-targeting regimes.

We extend our baseline model to include PLT in the central bank's reaction function ($\phi_p > 0$) in Figure 13 which shows the effects of PLT on welfare for different $\bar{\pi}$ as well as its implications for the optimal rate of inflation. First, PLT strictly increases welfare for any $\bar{\pi}$. Second, PLT leads to much lower levels of optimal inflation than inflation-targeting regimes. Even for moderate responses to the price level gap, the optimal level of inflation is less than 0.1 percent per year, which is close to the zero optimal inflation rate in the Ramsey problem under commitment where we do not impose constraints on policy reaction function.[28] This magnitude practically means *price level stability* (rather than inflation stability) which is, in fact, the mandated objective for most central banks.

The intuition for why PLT delivers such a small optimal inflation rate is straightforward. First, as observed in Gorodnichenko and Shapiro (2007), PLT stabilizes expectations and has a profound effect on output and inflation volatility. In our simulations, the reduction in inflation and output volatility is so substantial that the welfare costs of inflation are almost exclusively driven by the steady-state effects. As a result of reduced volatility, the ZLB binds less frequently. For example, with $\phi_p = 0.3$, the ZLB binds less than two percent of the time at a steady-state level of inflation of 3.5%. Second, even if the nominal rate hits zero, the policy rule remains a potent factor in stimulating the economy despite the ZLB because

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[28] The Ramsey problem under commitment seeks the welfare maximizing policy subject to commitment constraints and structural relationships in the economy. In our analysis, the objective function for the Ramsey problem is the sum of the approximation given in Proposition 1 and a penalty for interest rate variability calibrated at $0.077 * \Theta_2$ as in Woodford (2003). See Coibion et al (2010) for more details.

agents know that the deflationary pressures during the ZLB will have to be offset by above-average inflation in the future. This limits the downward movement in inflationary expectations and therefore the associated increase in real interest rates. In short, PLT limits the extent of deflationary spirals so that the exit from a ZLB episode occurs more rapidly and the welfare costs of the ZLB are substantially reduced. To give a sense of the magnitude of the associated welfare change, we note that by increasing ϕ_p from zero to 0.25 (combined with the appropriate change in the optimal rate of inflation), a policymaker could raise welfare by the equivalent to a permanent increase in consumption of 1.5 percent and approach the welfare gains for the optimal policy in the Ramsey problem under commitment (see Figure 13). Thus, these results provide a new justification for the consideration of PLT by monetary policymakers.

In addition, we consider a closely related issue: the degree of interest smoothing. Coibion and Gorodnichenko (2009) observe that PLT is very similar to a policy rule with high inertia, i.e., $\rho_1 + \rho_2$ close to one.[29] For simplicity, we abstract from the AR(2) interest-smoothing commonly found in empirical studies and instead restrict our attention to AR(1) interest smoothing: i.e. we set the AR(1) coefficient in the interest rate smoothing polynomial equal to $\rho_1 + \rho_2$ and the AR(2) coefficient to zero. Figure 13 confirms that, like PLT, higher interest rate smoothing leads to lower levels of optimal steady-state inflation because it reduces the incidence and damage of hitting the ZLB and therefore lowers the optimal inflation rate.[30] However, unlike PLT, this decrease in the optimal inflation rate is associated with lower welfare after $\rho_1 + \rho_2$ passes 0.89. The inverted U-shape of the welfare results from the tradeoff between strong short-run and strong long-run responses to endogenous variables implied by our Taylor rule specification (26) since we hold the long-run responses to inflation, output gap and output growth rate constant in this experiment but, by raising $\rho_1 + \rho_2$, decrease the short run response.

VII   Concluding remarks

If nothing else, the Great Recession has taught monetary economists one lesson: the zero bound is not a theoretical curiosity of interest only to historians of the Great Depression or as a precautionary tale against overly cautious policy-makers such as the Japanese monetary and fiscal authorities in the early 1990s. Instead, the pervasiveness of the zero bound constraint among major industrial countries has demonstrated the necessity of incorporating this issue into modern macroeconomic models. Indeed, the recent interest in raising the inflation targets of central banks has resurrected a basic question for macroeconomists: what is the optimal inflation rate? Strikingly, New Keynesian models, with their

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[29] Specifically, a Taylor rule with only PLT is observationally equivalent to a super-inertial Taylor rule that responds to contemporaneous inflation. More generally, a Taylor rule with "partial" PLT in which the central bank pushes prices only partly back to the initial target path is observationally equivalent to a Taylor rule with interest smoothing and a contemporaneous response to inflation. See Coibion and Gorodnichenko (2009) for more details.

[30] Billi (2009) similarly finds that higher interest smoothing lowers the optimal rate of inflation.

pervasive reliance on the assumption of zero steady-state inflation, have been ill-equipped to answer this key question for central bankers.

We provide an integrated treatment of the effects of non-zero steady-state inflation in New Keynesian models. Most importantly, we derive an approximation to the utility function of the representative agent which incorporates the various dimensions along which steady-state inflation matters: the steady-state, the dynamics of the model, and the coefficients of the utility-function approximation. This allows us to study the optimal rate of inflation using a welfare criterion derived explicitly from the microfoundations of the model. Combining this with the zero-bound on nominal interest rates, we are then able to study the costs and benefits of steady-state inflation and quantify the optimal rate of inflation in the model. Our baseline result is that this optimal rate of inflation is fairly low: less than two percent a year. We show that this result is robust to a variety of parameter specifications and modifications of the model.

Given that most central banks are targeting inflation rates between 1% and 3% a year, our results can be interpreted as supporting the current regimes and providing little evidence in favor of raising these targets to provide additional insurance against the zero-bound constraint on interest rates. However, from a normative point of view, we also show that welfare could be substantially improved by introducing price-level targeting. The latter helps stabilize economic fluctuations and reduces the probability of hitting the zero-lower bound. As a result, the optimal inflation rate under a price level targeting regime would be close to zero. In other words, optimal monetary policy, characterized as a combination of a low inflation target and a systematic response to deviations of the price-level from its target, can be interpreted as being very close to the "price stability" enshrined in the legal mandates of most central banks.

References

Aruoba, S. Boragan, and Frank Schorfheide, 2009. "Sticky Prices versus Monetary Frictions: An Estimation of Policy Trade-offs," forthcoming in *AEJ Macroeconomics*.

Ascari, Guido, and Tiziano Ropele, 2007. "Trend Inflation, Taylor Principle, and Indeterminacy," *Journal of Money, Credit and Banking* 48(1), 1557-1584.

Basu, Susanto, and John G. Fernald, 1997. "Returns to Scale in U.S. Production: Estimates and Implications," *Journal of Political Economy* 105, 249-283.

Billi, Roberto M., 2009, "Optimal Inflation for the US Economy?" Revised version of Research Working Paper 07-03, Federal Reserve Bank of Kansas City.

Bils, Mark, and Peter J. Klenow, 2004. "Some Evidence on the Importance of Sticky Prices," *Journal of Political Economy* 112(5), 947-985.

Barlevy, Gadi, 2004. "The Costs of Business Cycles and the Benefits of Stabilization: A Survey" NBER Working Paper 10926.

Bernanke, Ben, Mark Gertler, and Simon Gilchrist, 1999. "The financial accelerator in a quantitative business cycle framework," *Handbook of Macroeconomics*, vol 1, Elsevier.

Bodenstein, Martin, Christopher J. Erceg, and Luca Guerrieri, 2009. "The effects of foreign shocks when interest rates are at zero," International Finance Discussion Paper 983, Board of Governors of the Federal Reserve System.

Boivin, Jean, 2006. "Has U.S. Monetary Policy Changed? Evidence from Drifting Coefficients and Real-Time Data," *Journal of Money, Credit and Banking* 38(5), 1149-1173.

Burnside, Craig, 1996. "Production Function Regressions, Returns to Scale, and Externalities," *Journal of Monetary Economics* 37, 177-201.

Burstein, Ariel and Christian Hellwig, 2008. "Welfare Costs of Inflation in a Menu Cost Model," American Economic Review 98(2), 438-443.

Christiano, Lawrence J., Martin Eichenbaum, and Sergio Rebelo, 2009. "When is the government spending multiplier large?" NBER Working Paper 15394.

Christiano, Lawrence J., Mathias Trabandt, and Karl Walentin, 2010. "Involuntary Unemployment and the Business Cycle," NBER Working Paper 15801.

Clarida, Richard, Jordi Galí, and Mark Gertler, 1999. "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Literature* 37(4), 1661-1707.

Clarida, Richard, Jordi Galí, and Mark Gertler, 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics* 115(1), 147-180.

Cogley, Timothy, and Argia Sbordone, 2008. "Trend Inflation, Indexation and Inflation Persistence in the New Keynesian Phillips Curve," *American Economic Review* 98(5), 2101–2026.

Coibion, Olivier and Yuriy Gorodnichenko, 2009, "Monetary Policy, Trend Inflation, and the Great Moderation: An Alternative Interpretation" forthcoming in *American Economic Review.*

Coibion, Olivier, Yuriy Gorodnichenko, and Johannes Wieland, 2010, "Optimal Monetary Policy when Trend Inflation is Positive," manuscript.

Cooley, Thomas F. and Gary D. Hansen, 1991. "The Welfare Costs of Moderate Inflations," *Journal of Money, Credit and Banking* 23(3) 483-503.

DeLong, Bradford and Larry Summers, 1988. "How Does Macroeconomic Policy Affect Output?" *Brookings Papers on Economic Activity*, 2, 433-80.

Dickens, William T., Lorenz Goette, Erica L. Groshen, Steinar Holden, Julian Messina, Mark E. Schweitzer, Jarkko Turunen, Melanie E. Ward, 2007. "How wages change: Micro evidence from the International Wage Flexibility Project," *Journal of Economic Perspectives* 21(2), 195-214.

Dhyne, E., Alvarez, L.J., Le-Bihan, H., Veronese, G., Dias, D., Hoffmann, J., Jonker, N., Lunnemann, P., Rumler, F., Vilmunen, J.. 2005. "Price setting in the euro area: some stylized facts from individual consumer price data," European Central Bank, Working Paper 524.

Eggertsson, Gauti and Michael Woodford, 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity*, 1, 139-211.

Eggertsson, Gauti and Michael Woodford, 2004. "Policy Options in a Liquidity Trap," *American Economic Review* 94(2), 76-79.

Feldstein, Martin S., 1997. "The Costs and Benefits of Going from Low Inflation to Price Stability," in *Reducing Inflation: Motivation and Strategy*, 123-166. NBER, Inc.

Friedman, Milton, 1969. *The Optimum Quantity of Money,* Macmillan.

Gagnon, Etienne, 2009, "Price Setting during Low and High Inflation: Evidence from Mexico," *Quarterly Journal of Economics* 124(3), 1221-1263.

Gali, Jordi and Mark Gertler, 1999. "Inflation Dynamics: A Structural Econometric Analysis," *Journal of Monetary Economics* 44(2), 195-222.

Gorodnichenko, Yuriy, and Matthew D. Shapiro, 2007. "Monetary Policy When Potential Output Is Uncertain: Understanding the Growth Gamble of the 1990s," *Journal of Monetary Economics* 54(4), 1132-1162.

Fuchi, Hitoshi, Nobuyuki Oda, and Hiroshi Ugai, 2008. "Optimal Inflation for Japan's Economy," *Journal of the Japanese and International Economies* 22(4), 439-475.

Kiley, Michael E., 2000. "Price Stickiness and Business Cycle Persistence," *Journal of Money, Credit and Banking* 32(1), 28-53.

Kiley, Michael E., 2007. "Is Moderate-to-High Inflation Inherently Unstable?" *International Journal of Central Banking* 3(2), 173-201.

Khan, Aubhik, Robert G. King and Alexander L. Wolman, 2003. "Optimal Monetary Policy," *Review of Economic Studies* 70(4), 825-860.

Kim, Jinill and Francisco J. Ruge-Murcia, 2009. "How much inflation is necessary to grease the wheels?" *Journal of Monetary Economics* 56(3), 365-377.

Lubik, Thomas A. and Frank Schorfheide, 2003. "Computing Sunspot Equilibria in Linear Rational Expectations Models," *Journal of Economic Dynamics and Control* 28(2), 273-285.

Lucas, Robert, 1987. *Models of Business Cycles*. Oxford: Basil Blackwell.

Mankiw, G. N., 2007. *Macroeconomics*, Sixth Edition. New York: Worth Publishers.

Nakamura, Emi, and Jón Steinsson, 2008. "Five Facts About Prices: A Reevaluation of Menu Cost Models," *Quarterly Journal of Economics* 123(4), 1415-1464.

Okun, Arthur, 1971. "The Mirage of Steady Inflation," *Brookings Papers on Economic Activity* 2, 485-498.

Orphanides, Athanasios, and Simon van Norden, 2002. "The Unreliability of Output Gap Estimates in Real Time," *Review of Economics and Statistics* 84(4), 569-583.

Orphanides, Athanasios, 2003. "Historical Monetary Policy Analysis and the Taylor Rule," *Journal of Monetary Economics* 50(5), 983-1022.

Romer, David, 1990, "Staggered Price Setting with Endogenous Frequency of Adjustment," *Economic Letters* 32(3), 205-210.

Smets, Frank and Rafael Wouters, 2007. "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review* 97(3), 586-606.

Schmitt-Grohe, Stephanie and Martin Uribe, 2007, "Optimal Inflation Stabilization in a Medium-Scale Macroeconomic Model" in *Monetary Policy Under Inflation Targeting*, edited by Klaus Schmidt-Hebbel and Rick Mishkin, Central Bank of Chile, Santiago, Chile, 2007, p. 125-186.

Schmitt-Grohe, Stephanie and Martin Uribe, 2010. "The Optimal Rate of Inflation" in *Handbook of Monetary Economics*, vol. 3B (forthcoming, Elsevier)..

Summers, Lawrence, 1991. "Price Stability: How Should Long-Term Monetary Policy Be Determined?" *Journal of Money, Credit and Banking* 23(3), 625-631.

Taylor, John B., 1977. "Staggered Wage Setting in a Macro Model." *American Economic Review* 69(2), 108-113.

Taylor, John B., 1981. "On the Relationship Between the Variability of Inflation and the Average Inflation Rate," *Carnegie-Rochester Conference Series on Public Policy* 15, 57-85.

Tobin, James, 1972. "Inflation and Unemployment," *American Economic Review* 62(1), 1-18.

Walsh, Carl E., 2009. "Using Monetary Policy to Stabilize Economic Activity," in Federal Reserve Bank of Kansas City *Financial Stability and Macroeconomic Policy* Jackson Hole Symposium, 2010: 245-296

Williams, John C., 2009. "Heeding Deadalus: Optimal Inflation and the Zero Bound" *Brookings Papers on Economic Activity* 2009(2), 1-37.

Wolman, Alexander L., 2005. "Real Implications of the Zero Bound on Interest Rates," *Journal of Money, Credit and Banking* 37(2) 273-296.

Woodford, Michael, 2003. *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton: Princeton University Press.

Woodford, Michael, 2010. "Optimal Monetary Stabilization Policy," *Handbook of Monetary Economics*, vol. 3B (forthcoming, Elsevier).

**Table 1: Baseline Parameter Values**

Parameters of Utility Function		*Steady-State Values*	
η: Frisch Labor Elasticity	1.00	$\overline{gy}$: Growth Rate of RGDP/cap	1.5% p.a.
β: Discount factor	0.998	$\overline{c_y}$: Consumption Share of GDP	0.80
		$\overline{g_y}$: Government Share of GDP	0.20
Pricing Parameters		*Shock Persistence*	
θ: Elasticity of substitution	10	ρ_g: Government Spending Shocks	0.97
λ: Degree of Price Stickiness	0.55	ρ_m: Cost-Push Shocks	0.90
ω: Price indexation	0.00	ρ_q: Risk Premium Shocks	0.94
Taylor Rule Parameters		*Shock Volatility*	
ϕ_π: LR response to inflation	2.50	σ_g: Government Spending Shocks	0.0052
ϕ_{gy}: LR response to output growth	1.50	σ_m: Cost-Push Shocks	0.0024
ϕ_x: LR response to output gap	0.11	σ_q: Risk Premium Shocks	0.0024
ρ_1: Interest smoothing	1.05	σ_a: Technology Shocks	0.0090
ρ_2: Interest smoothing	-0.13	σ_r: Monetary Policy Shocks	0.0043

Note: The table presents the baseline parameter values assigned to the model in section 3.1 and used for solving for the optimal inflation rate in section 3.2. "p.a." means per annum.

**Figure 1. Frequency of being in the Zero Lower Bound and Steady-State Nominal Interest Rate**
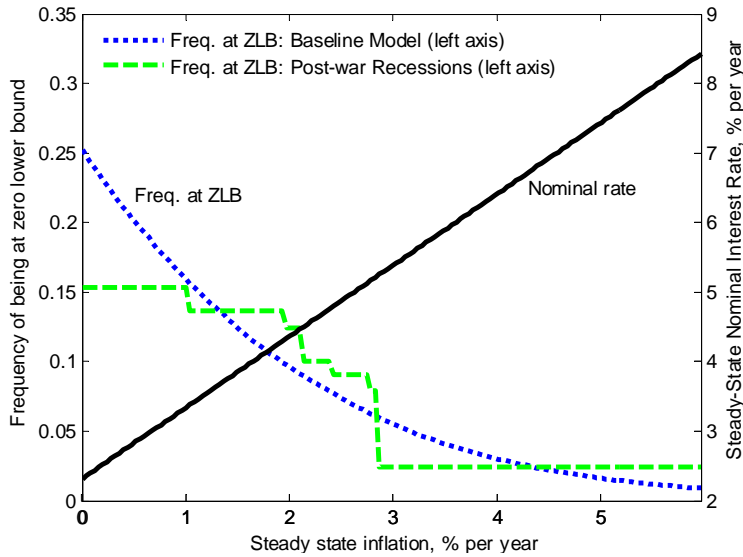
Note: The figure plots the steady-state level of the annualized nominal interest rate (right axis) implied by the baseline model of section 3 for different steady-state inflation rates. In addition, the figure plots the frequency of hitting the zero bound on nominal interest rates (left axis) from simulating the baseline model at different steady-state inflation rates as well as historical frequencies of ZLB for the U.S. for counterfactual inflation rates. See section 3.1 for details.

Figure 2. Utility at Different Levels of Steady-State Inflation
Panel A: Effect of ZLB   Panel B: Effects of positive trend inflation
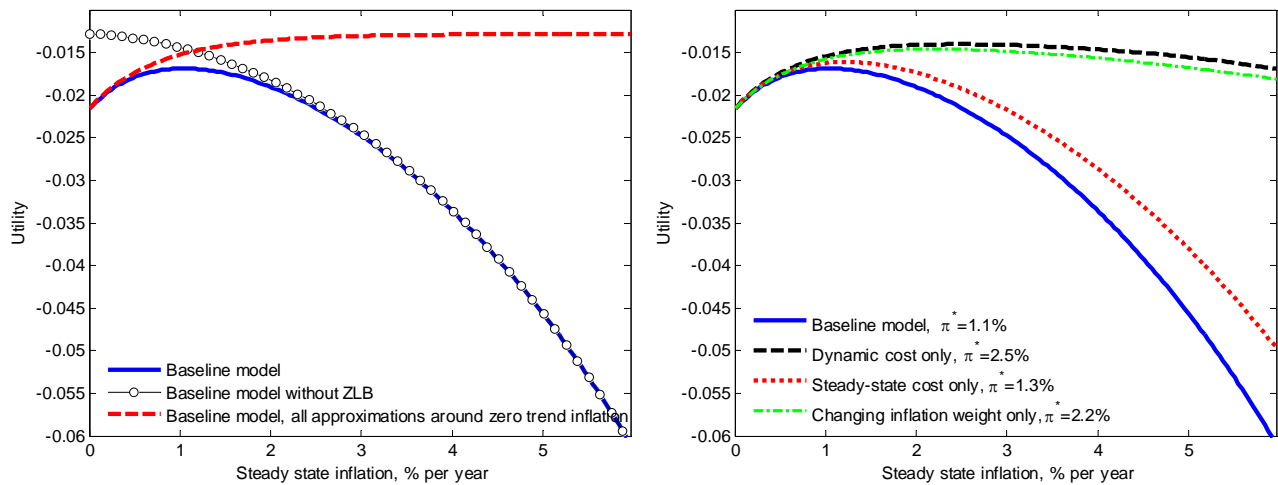
Note: The figures plot the approximation to the utility function in Proposition 1 from simulating the model for different levels of steady-state inflation. Panel A includes results for the baseline model, the baseline model without the ZLB, as well as the model with the ZLB but omitting the three cost channels of inflation: steady-state effects, the changing coefficient on inflation variance in utility and the dynamic effects. Panel B reproduces our baseline with ZLB, then presents results when we restrict the model to include only one cost of inflation and the ZLB. "Dynamic cost only" includes only the dynamic effects of positive inflation and keeps the rest of the model being approximated around zero trend inflation, "Steady-state cost only" includes only the steady-state cost of inflation and keeps the rest of the model being approximated around zero trend inflation, while "Changing inflation weight only" includes only the changing coefficients on inflation variance in the loss function and keeps the rest of the model being approximated around zero trend inflation. See section 3.2 for details.

**Figure 3. The Sources of Utility Costs of Inflation**
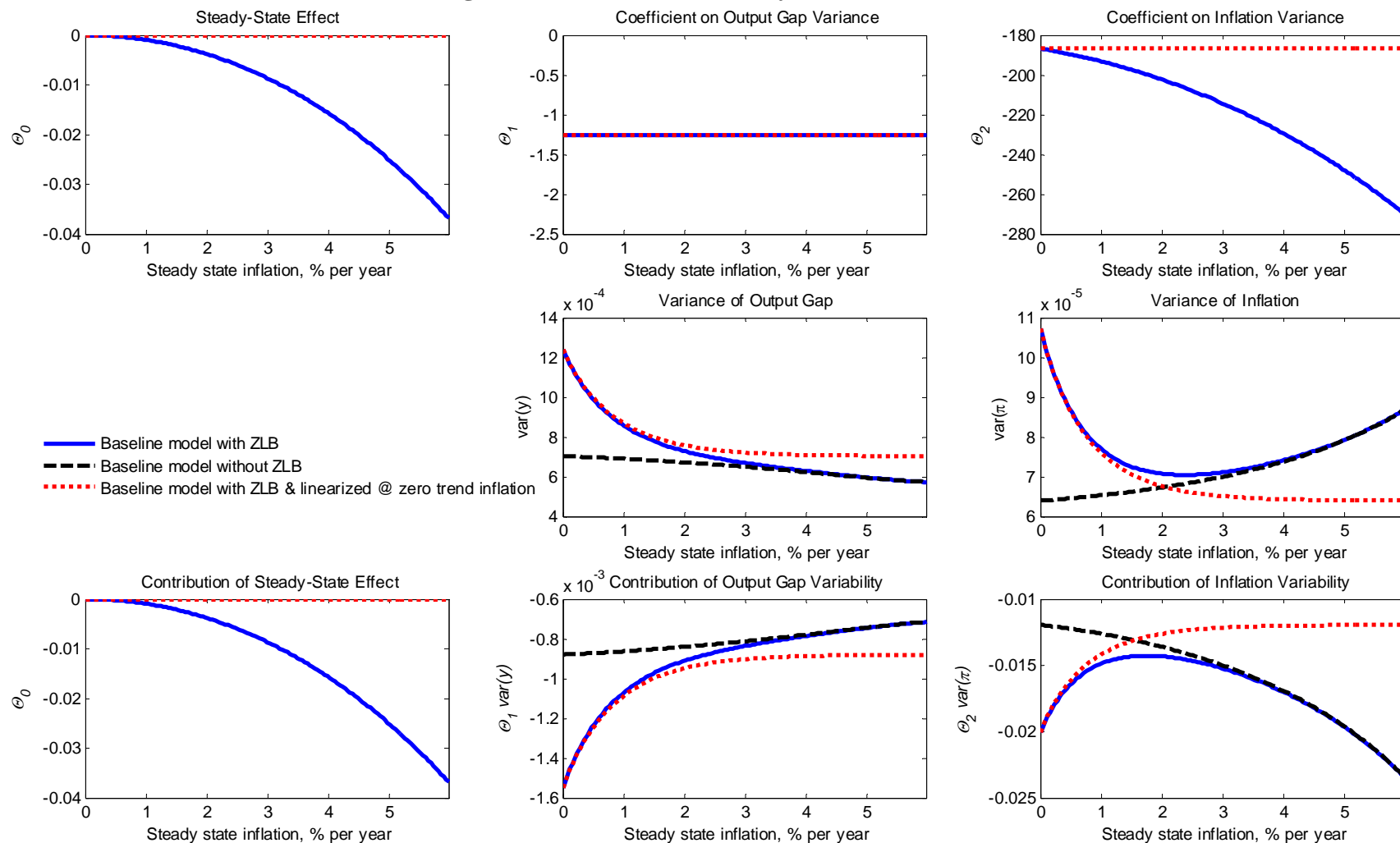
Note: The first row of the figure plots the coefficients of the approximation to the utility function from Proposition 1 for different levels of trend inflation. The second row plots the variance of macroeconomic variables that enter the approximation to the utility function in Proposition 1 from simulating the model subject to the zero bound on nominal interest rates for different levels of steady-state inflation using the baseline parameter values of the model. The dashed black lines are the corresponding moments without the zero-bound on nominal interest rates, while the dotted lines correspond to the moments of the model when the dynamics are approximated at zero trend inflation. The third row plots the contribution of the different components of the approximation to the utility function in Proposition 1. See section 3.2 for details.

Figure 4. The Costs of Business Cycles

Panel A: The costs of the ZLB
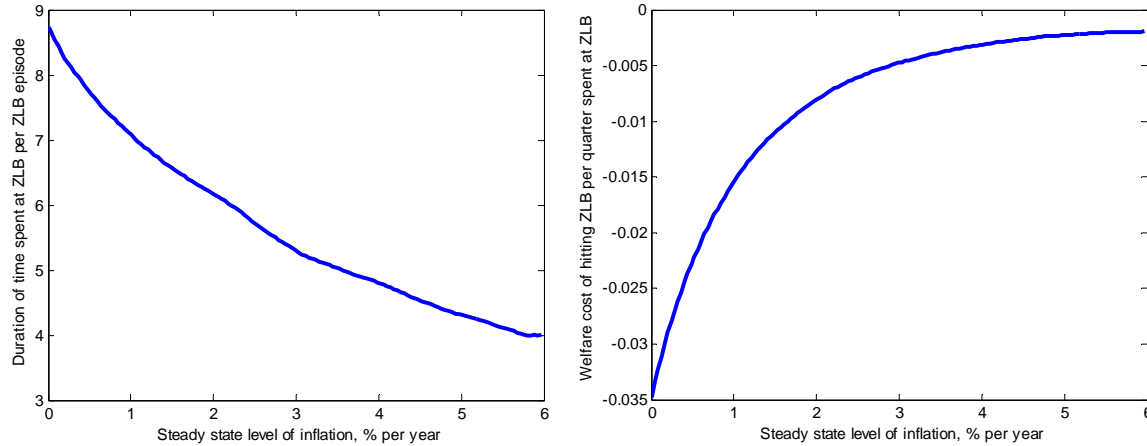
Panel B: Changing the weight on output gap volatility in the loss function
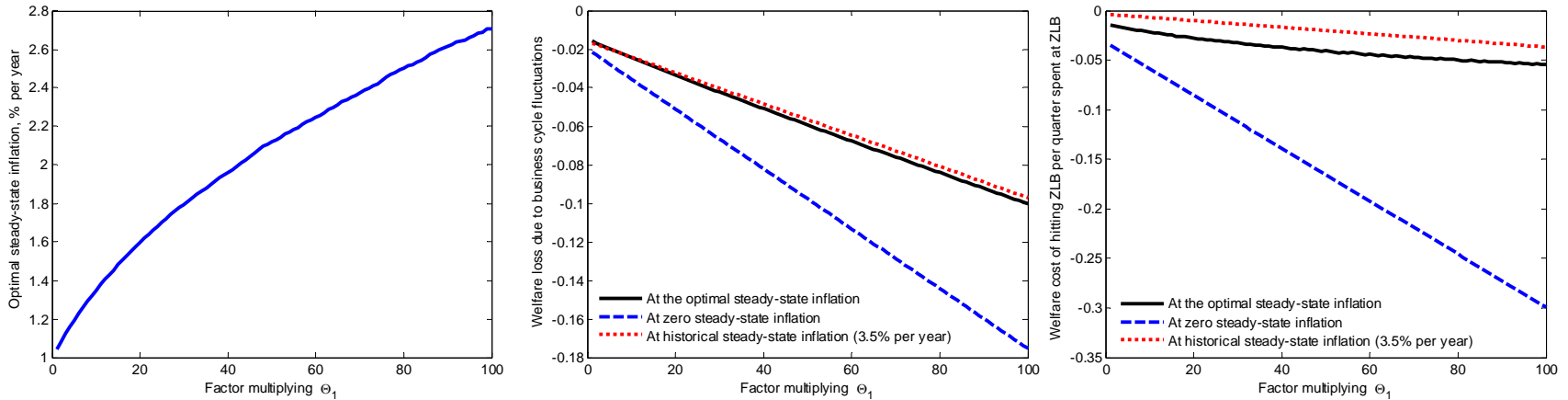

Note: The top panel plots the average duration of ZLB episodes in the baseline calibration and the implied average welfare cost per quarter of being at the ZLB for different levels of trend inflation. The bottom panel plots the effects of changing the coefficient on the variance of the output gap in the utility function approximation of Proposition 1 on the optimal inflation rate (left graph), the welfare costs of business cycle fluctuations (middle graph), and the average welfare costs of hitting the ZLB (right graph). The latter two are measured using Proposition 1 net of the steady-state effects of trend inflation. See sections 3.3 and 3.4 for details.

Figure 5. Robustness: Price Setting Parameters.
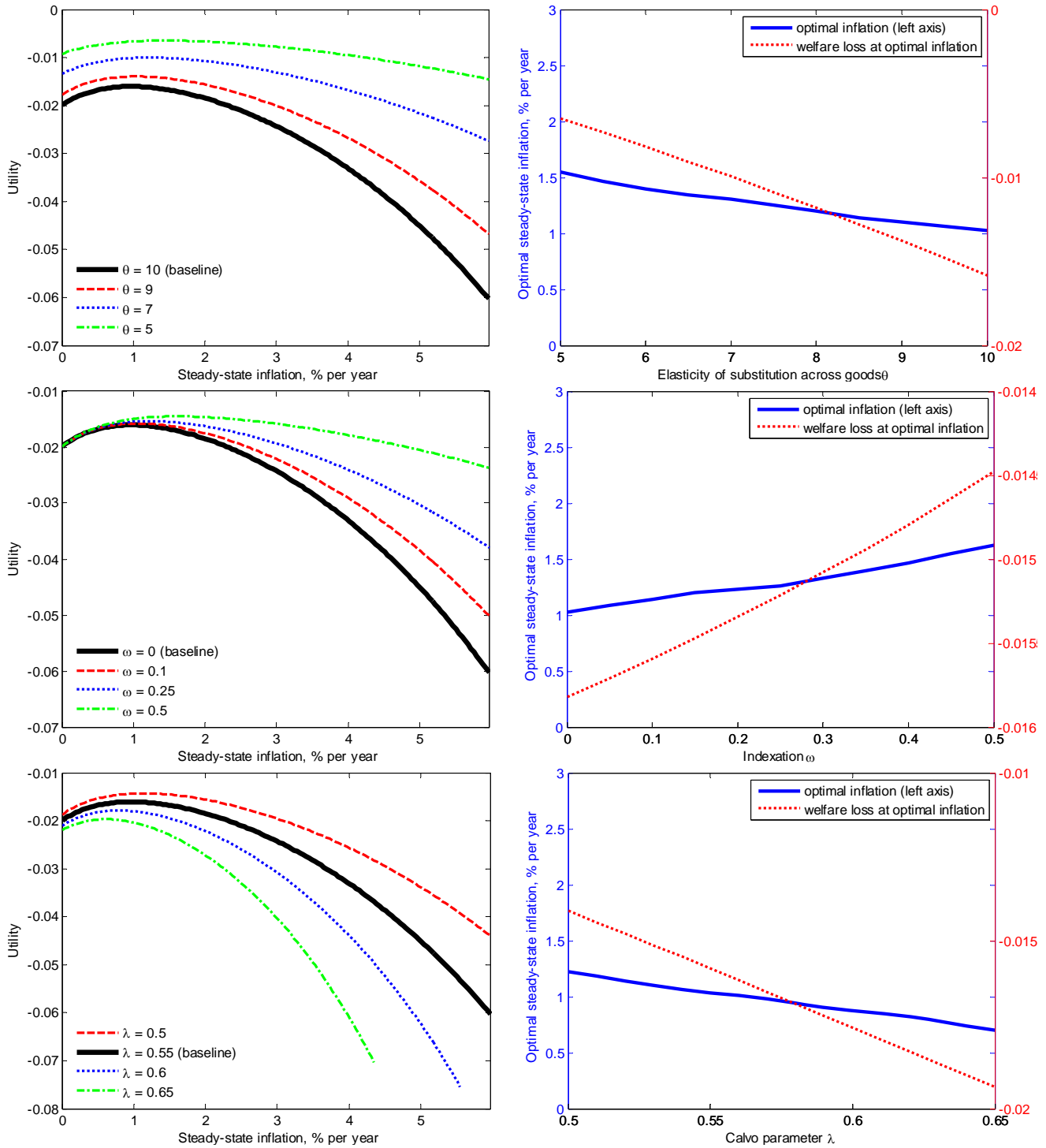
Note: Figures in the left column plot the welfare loss as a function of steady state inflation for alternative values of structural parameters. The solid thick back line corresponds to the baseline parameterization. Figures in the right column plot the optimal level of steady-state inflation and the welfare loss at the optimal steady state level of inflation as a function of a structural parameter.

Figure 6. Robustness: Risk Premium Shocks and the Discount Factor.
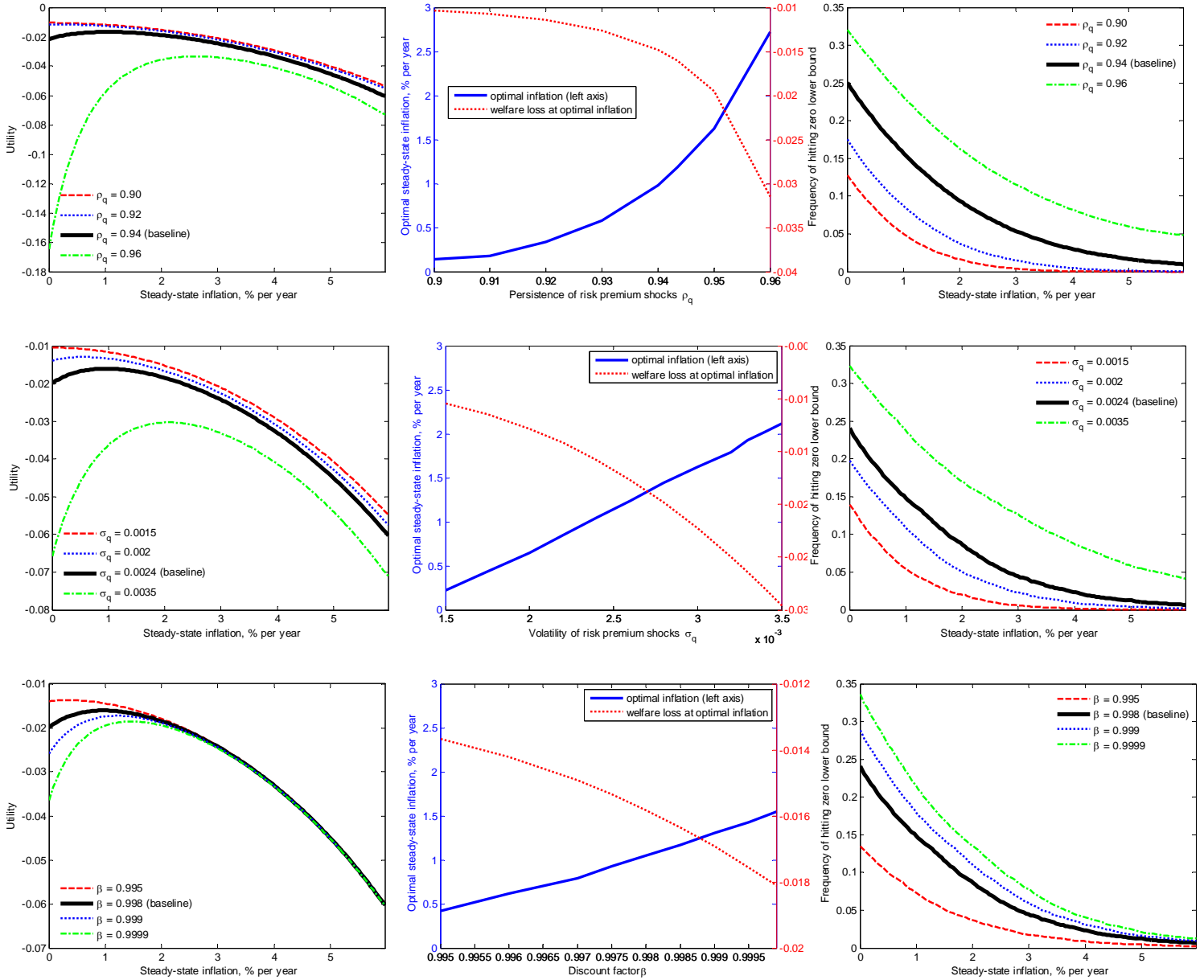
Note: Figures in the left column plot the welfare loss as a function of steady state inflation for alternative values of structural parameters. The solid thick back line corresponds to the baseline parameterization. Figures in the middle column plot the optimal level of steady-state inflation and the welfare loss at the optimal steady state level of inflation as a function of a structural parameter. Figures in the right column plot the frequency of hitting the ZLB for different parameter values.

**Figure 7. Sensitivity Analysis: Capital**
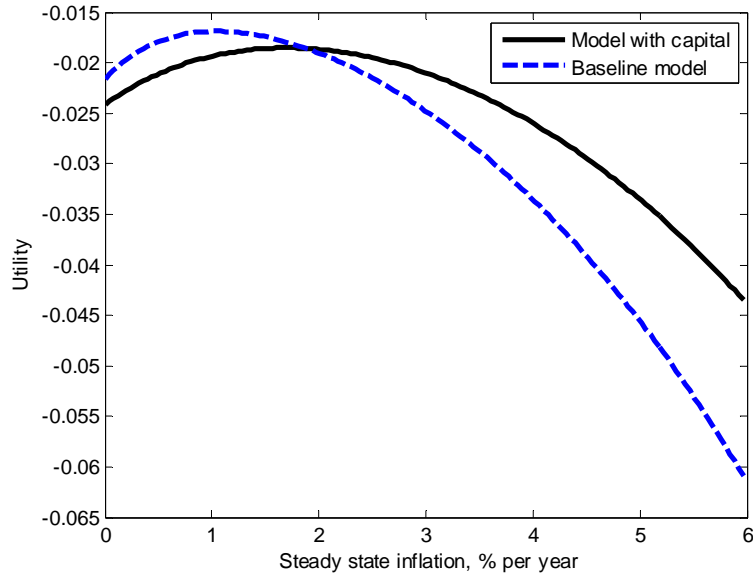
Note: the figure plots the approximation to the utility function in Proposition 2 from simulating the model subject to the zero bound on nominal interest rates for different levels of steady-state inflation using the baseline parameter values of the model with capital. See section 5.1 for details.

**Figure 8. Model Uncertainty**

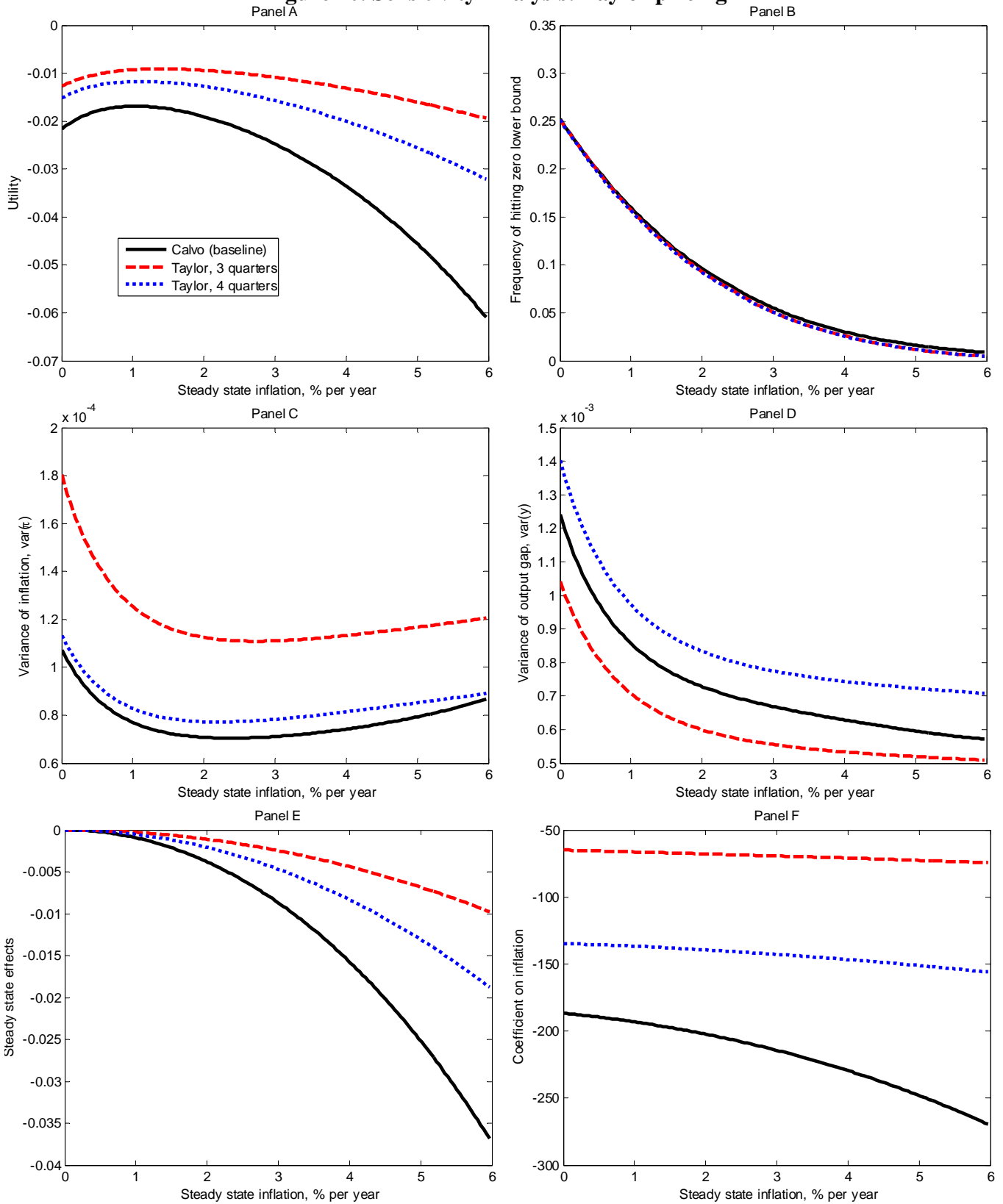Panel A: Optimal Inflation Rate Panel   B: Distribution of Optimal Inflation Rates
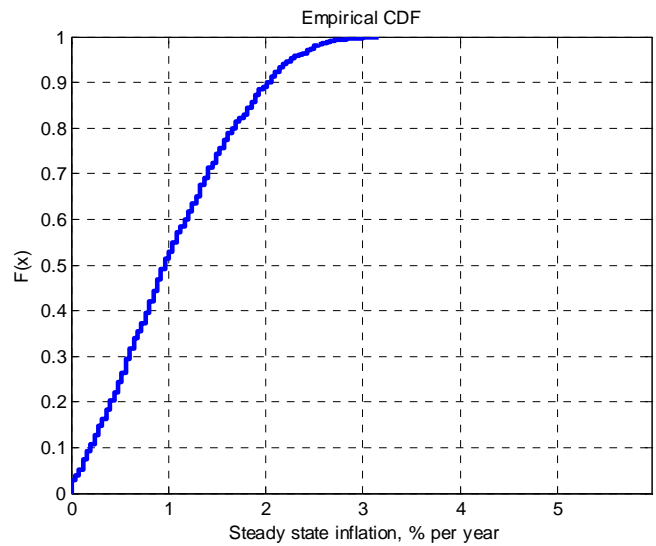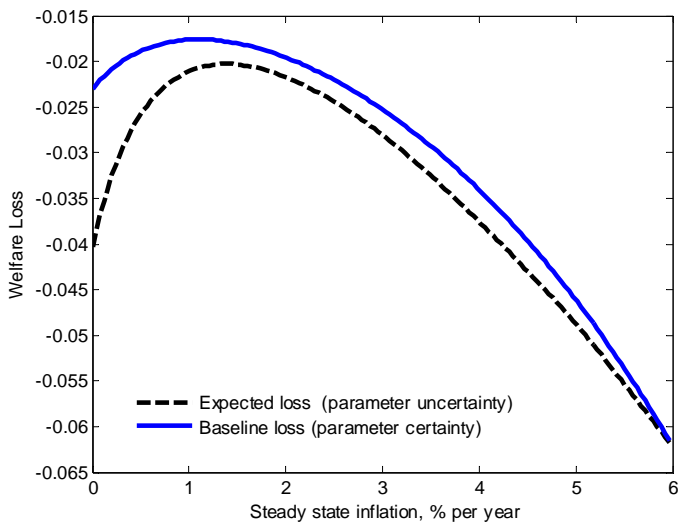
Note: Panel A plots the expected utility for different steady-state inflation rates under baseline parameter values as well as under model-uncertainty. Panel B plots the distribution of optimal inflation rates associated with different draws from the distribution of parameter values. See section 5.2 for details.

Figure 9. Sensitivity Analysis: Endogenous Price Stickiness
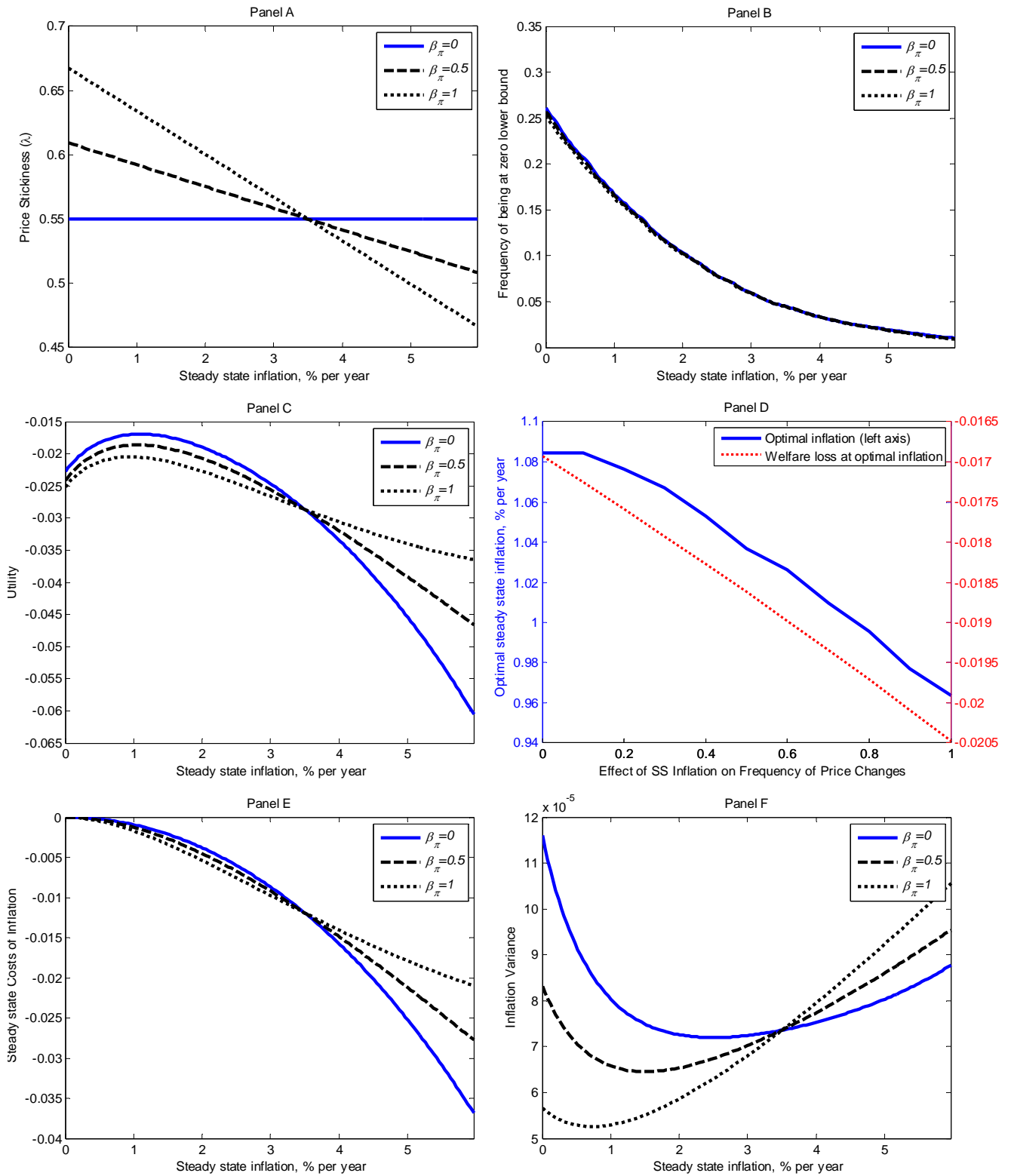
Note: The figures plot the implications of endogenous price stickiness on the model. β_π is the effect of steady state inflation on the frequency of price changes, with $\beta_\pi = 0$ being our baseline case of exogenous price stickiness. See section 5.3 for details.
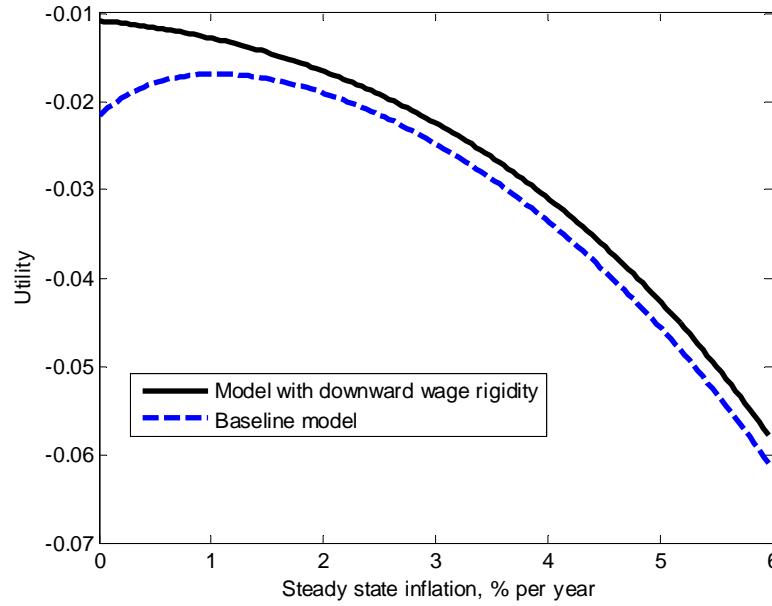
**Figure 10. Sensitivity Analysis: Taylor pricing**

Note: The figures plot the implications of Calvo vs. Taylor price setting for welfare and optimal inflation. *Taylor, X quarters* corresponds to the duration of price contracts equal to *X* quarters. See section 5.4 for details.

Figure 11. Downward Wage Rigidity

Panel A: Optimal Inflation Rate

Panel B: Effects of Downward-Wage Rigidity
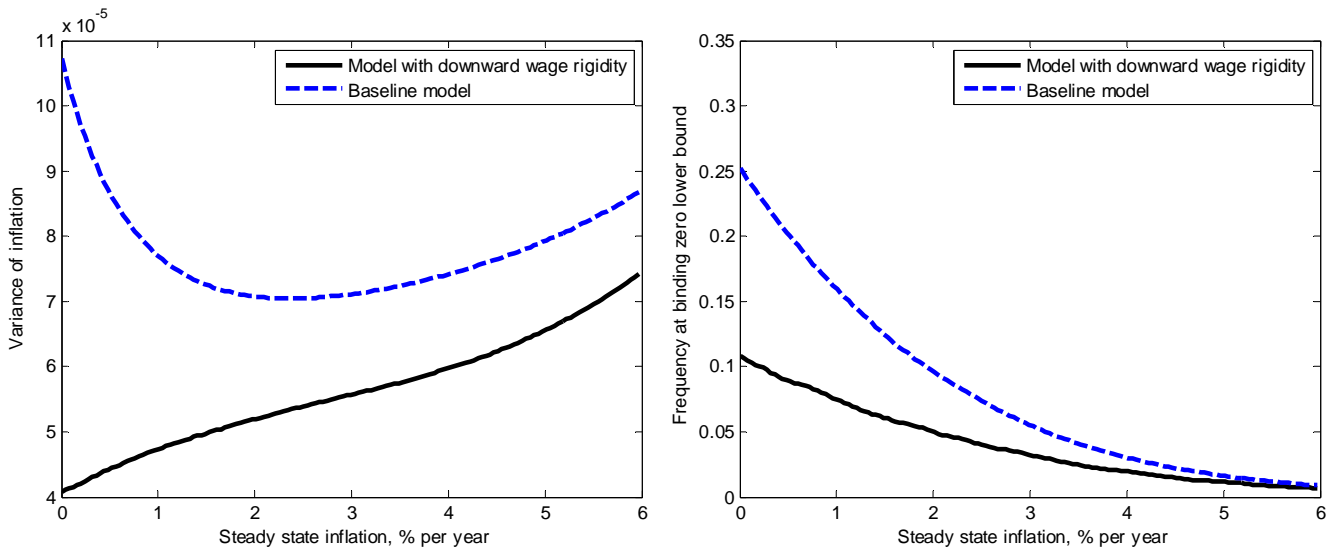
Note: Panel A plots the utility associated with different steady-state inflation rates under the baseline model as well as the model with downward nominal wage rigidity. Panel B figures plot the variance of inflation and the frequency of hitting the zero-bound on interest rates for different steady-state inflation rates using our baseline model and the model with downward nominal wage rigidity. See section 5.5 for details.

Figure 12. Positive Implication: Parameters in the Taylor Rule.
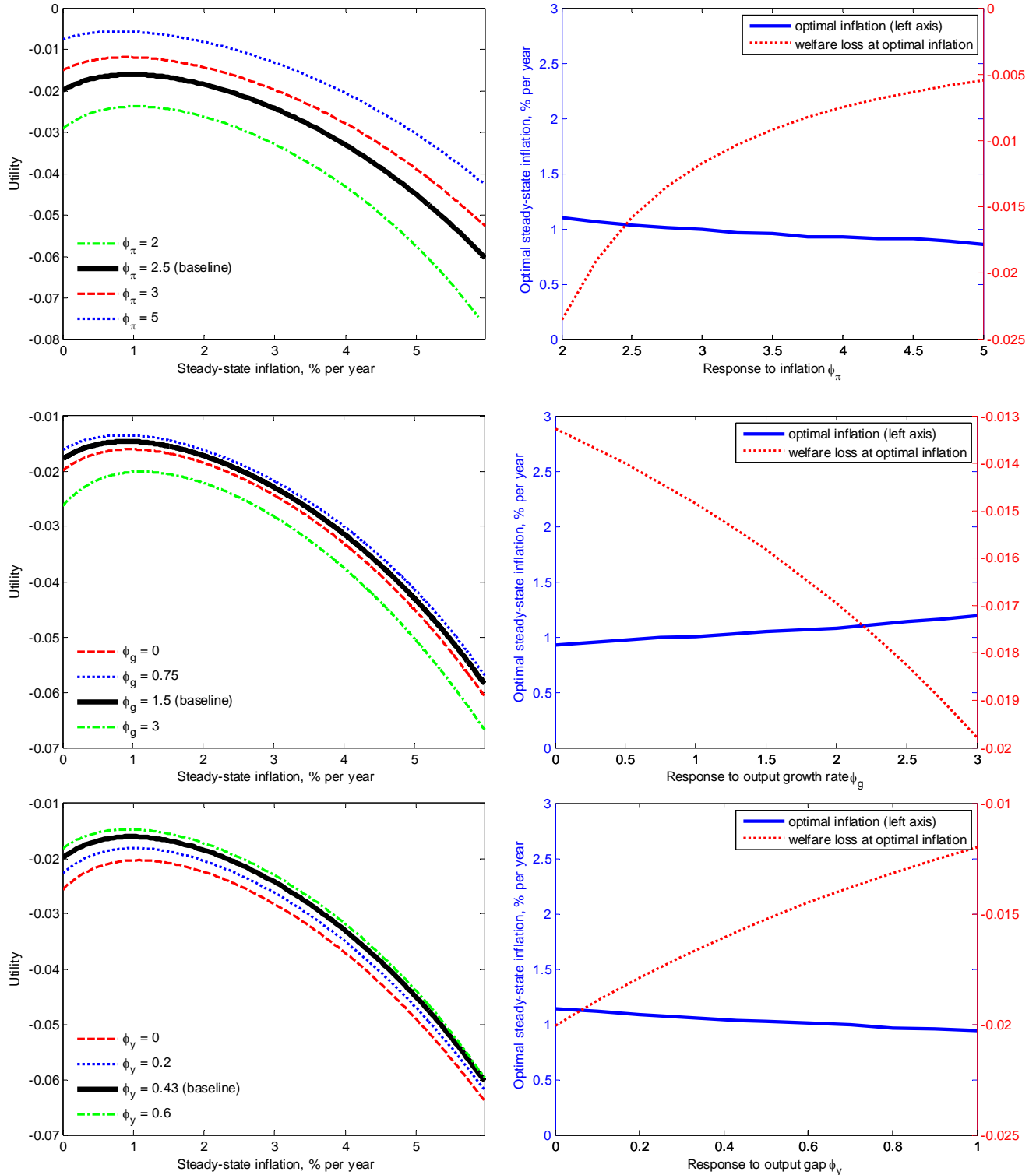
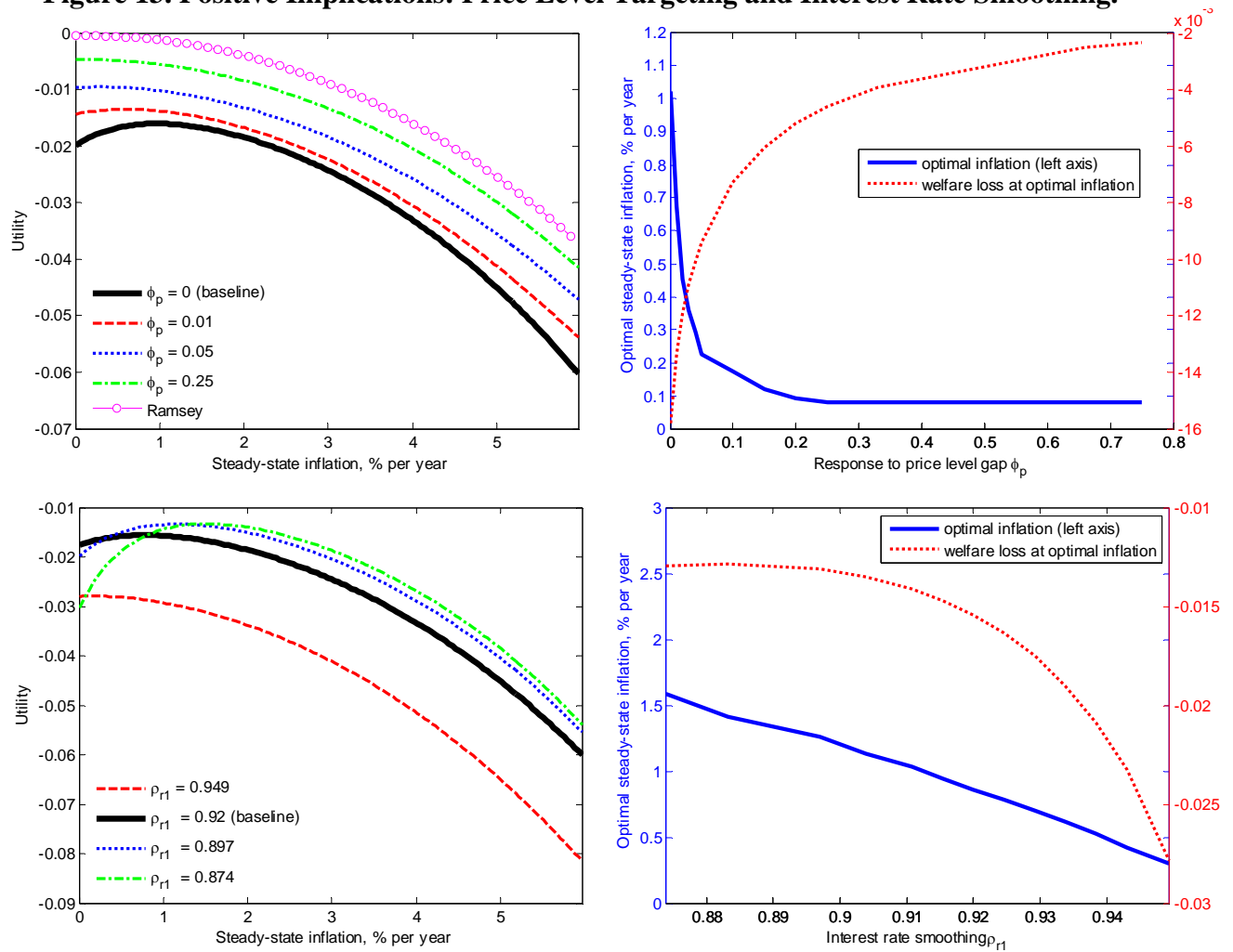Note: Figures in the left column plot the welfare loss as a function of steady state inflation for alternative values of the monetary policy rule parameters. The solid thick back line corresponds to the baseline parameterization. Figures in the right column plot the optimal level of steady-state inflation and the welfare loss at the optimal steady state level of inflation as a function of the monetary policy rule parameters. ϕ_π is the long-run response of interest rates to inflation, ϕ_g is the response to output growth, and ϕ_y is the response to the output gap. See section 6.1 for details.

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Figure 13. Positive Implications: Price Level Targeting and Interest Rate Smoothing.

Note: Figures in the left column plot the welfare loss as a function of steady state inflation for alternative values of the monetary rule parameters. The solid thick back line corresponds to the baseline parameterization. Figures in the right column plot the optimal level of steady-state inflation and the welfare loss at the optimal steady state level of inflation as a function of the monetary policy rule parameters. ϕ_p is the response to the price-level gap while ρ_{r1} is the degree of interest smoothing. See section 6.2 for details.

Appendix A. Proofs for section 2.4.

Lemma 1.
The expansion of the utility derived from consumption around flexible price steady state is

$$u(C_t) \quad = \underbrace{u(\overline{C}_t^F)}_{tip} + u'_{\overline{C}_t^F}(C_t - \overline{C}_t^F) + \tfrac{1}{2}u''_{\overline{C}_t^F}(C_t - \overline{C}_t^F)^2 + hot = \overline{C}_t^F u'_{\overline{C}_t^F}(\tilde{c}_t + \tfrac{1}{2}\tilde{c}_t^2) + \tfrac{1}{2}(\overline{C}_t^F)^2 u''_{\overline{C}_t^F}\tilde{c}_t^2 + hot + tip$$

$$= \overline{C}_t^F u'_{\overline{C}_t^F}\tilde{c}_t + \tfrac{1}{2}(\overline{C}_t^F u'_{\overline{C}_t^F} + u''_{\overline{C}_t^F}(\overline{C}_t^F)^2)\tilde{c}_t^2 + hot + tip = \overline{C}_t^F u'_{\overline{C}_t^F}\tilde{c}_t + \tfrac{1}{2}\overline{C}_t^F u'_{\overline{C}_t^F}(1 + \tfrac{u''_{\overline{C}_t^F}\overline{C}_t^F}{u'_{\overline{C}_t^F}})\tilde{c}_t^2 + hot + tip$$

$$= \overline{C}_t^F u'_{\overline{C}_t^F}\tilde{c}_t + \tfrac{1}{2}\overline{C}_t^F u'_{\overline{C}_t^F}(1 - \sigma^{-1})\tilde{c}_t^2 + hot + tip$$

where σ is the intertemporal elasticity of substitution for consumption, and $\overline{C}_t^F$ is the flexible price level of consumption in the steady state. Tildes denote percent deviations from the flexible price level, *tip* stands for terms independent from policy, *hot* means higher order terms. We assume that utility is logarithmic, $u(C_t) = \log C_t$, $\overline{C}_t^F u'_{\overline{C}_t^F} = 1$ and $\sigma = 1$ so that the expression becomes $u(C_t) = \tilde{c}_t + hot + tip$. ∎

Lemma 3.
Note that

$$\log P_t = \frac{1}{1-\theta}\log(\int_0^1 P_t^{1-\theta}(i)di) = \underbrace{\frac{1}{1-\theta}\int_0^1 \log(P_t^{1-\theta}(i))di}_{\overline{P}_t} + \frac{1}{2}\frac{1}{1-\theta}\frac{1}{\left[EP_t^{1-\theta}(i)\right]^2}\mathrm{var}_i(P_t^{1-\theta}(i)) + hot \qquad (A.1)$$

By the delta method, we have

$$\mathrm{var}_i(P_t^{1-\theta}(i)) \quad = \mathrm{var}_i(\exp\{(1-\theta)\log P_t(i)\})$$

$$\approx (1-\theta)^2[\exp\{(1-\theta)E\log P_t(i)\}]^2\,\mathrm{var}_i(\log P_t(i)) \qquad (A.2)$$

$$= (1-\theta)^2[\exp\{(1-\theta)\overline{P}_t\}]^2\Delta_t$$

Now observe that

$$EP_t^{1-\theta}(i) = E[\exp\{(1-\theta)\log P_t(i)\}] \approx \exp\{(1-\theta)E\log P_t(i)\}$$

$$+ \tfrac{1}{2}(1-\theta)^2\exp\{(1-\theta)E\log P_t(i)\}\underbrace{\mathrm{var}_i(\log P_t(i))}_{\Delta_t} \qquad (A.3)$$

$$= \exp\{(1-\theta)\overline{P}_t\} + \tfrac{1}{2}(1-\theta)^2\exp\{(1-\theta)\overline{P}_t\}\Delta_t = \exp\{(1-\theta)\overline{P}_t\}[1 + \tfrac{1}{2}(1-\theta)^2\Delta_t]$$

Hence it follows that

$$\log P_t \quad = \overline{P}_t + \frac{1}{2}\frac{1}{1-\theta}\frac{(1-\theta)^2[\exp\{(1-\theta)\overline{P}_t\}]^2\Delta_t}{[\exp\{(1-\theta)\overline{P}_t\}]^2\left[1 + \tfrac{1}{2}(1-\theta)^2\Delta_t\right]^2} + hot$$

$$= \overline{P}_t + \frac{1}{2}\frac{(1-\theta)\Delta_t}{\left[1 + \tfrac{1}{2}(1-\theta)^2\Delta_t\right]^2} + hot \qquad (A.4)$$

$$= \overline{P}_t + \frac{1-\theta}{2}\frac{\overline{\Delta}}{\left[1 + \tfrac{1}{2}(1-\theta)^2\overline{\Delta}\right]^2} + \frac{1-\theta}{2}\frac{1 - \tfrac{1}{2}(1-\theta)^2\overline{\Delta}}{\left[1 + \tfrac{1}{2}(1-\theta)^2\overline{\Delta}\right]^3}(\Delta_t - \overline{\Delta}) + hot$$

$$= \overline{P}_t + Q_p^0 + \frac{1-\theta}{2}Q_p^1(\Delta_t - \overline{\Delta}) + hot$$

where $Q_p^0 = \dfrac{\frac{1-\theta}{2}\overline{\Delta}}{\left[1 + \tfrac{1}{2}(1-\theta)^2\overline{\Delta}\right]^2}$ and $Q_p^1 = \dfrac{1 - \tfrac{1}{2}(1-\theta)^2\overline{\Delta}}{\left[1 + \tfrac{1}{2}(1-\theta)^2\overline{\Delta}\right]^3}$. ∎

Lemma 5.

Consider the term $E\hat{y}_t(i)$. From the Dixit-Stiglitz aggregator we have

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \Rightarrow Y_t/\overline{Y}_t^F = \left[\int_0^1 \left(Y_t(i)/\overline{Y}_t^F\right)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \tag{A.5}$$

Note that

$$\log Y_t/\overline{Y}_t^F = \frac{\theta}{\theta-1}\log(\int_0^1 \left(Y_t(i)/\overline{Y}_t^F\right)^{\frac{\theta-1}{\theta}}di) =$$

$$= \frac{\theta}{\theta-1}\int_0^1 \frac{\theta-1}{\theta}\log(Y_t(i)/\overline{Y}_t^F)di + \frac{1}{2}\frac{\theta}{\theta-1}\frac{1}{\left[E(Y_t(i)/\overline{Y}_t^F)^{\frac{\theta-1}{\theta}}\right]^2}\, \mathrm{var}_i((Y_t(i)/\overline{Y}_t^F)^{\frac{\theta-1}{\theta}}) + hot$$

$$= E_i\hat{y}_t(i) + \frac{1}{2}\frac{\theta}{\theta-1}\frac{1}{\left[E(Y_t(i)/\overline{Y}_t^F)^{\frac{\theta-1}{\theta}}\right]^2}\, \mathrm{var}_i((Y_t(i)/\overline{Y}_t^F)^{\frac{\theta-1}{\theta}}) + hot \tag{A.6}$$

Using the delta method, we have

$$\mathrm{var}_i((Y_t(i)/\overline{Y}_t^F)^{\frac{\theta-1}{\theta}}) = \mathrm{var}_i(\exp\{\tfrac{\theta-1}{\theta}\log(Y_t(i)/\overline{Y}_t^F)\})$$

$$\approx (\tfrac{\theta-1}{\theta})^2[\exp\{\tfrac{\theta-1}{\theta}E\log(Y_t(i)/\overline{Y}_t^F)\}]^2\, \mathrm{var}_i(\log(Y_t(i)/\overline{Y}_t^F)) \tag{A.7}$$

Also observe that

$$E(Y_t(i)/\overline{Y}_t^F)^{\frac{\theta-1}{\theta}} = E[\exp\{\tfrac{\theta-1}{\theta}\log(Y_t(i)/\overline{Y}_t^F)\}]$$

$$\approx \exp\{\tfrac{\theta-1}{\theta}E\log(Y_t(i)/\overline{Y}_t^F)\}$$

$$+ \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\exp\{\tfrac{\theta-1}{\theta}E\log(Y_t(i)/\overline{Y}_t^F)\}\, \mathrm{var}_i(\log(Y_t(i)/\overline{Y}_t^F)) \tag{A.8}$$

$$= \exp\{\tfrac{\theta-1}{\theta}E\log(Y_t(i)/\overline{Y}_t^F)\}\left[1 + \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\, \mathrm{var}_i(\log(Y_t(i)/\overline{Y}_t^F))\right]$$

Combine (A.7) and (A.8) to re-write (A.6) as follows

$$\tilde{y}_t = \ln Y_t/\overline{Y}_t^F = E_i\tilde{y}_t(i) + \tfrac{1}{2}\tfrac{\theta}{\theta-1}\frac{(\tfrac{\theta-1}{\theta})^2\, \mathrm{var}_i(\log(Y_t(i)/\overline{Y}_t^F))}{\left[1 + \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\, \mathrm{var}_i(\log(Y_t(i)/\overline{Y}_t^F))\right]^2} + hot$$

$$= E_i\tilde{y}_t(i) + \tfrac{1}{2}\tfrac{\theta-1}{\theta}\, \mathrm{var}_i(\log(y_t(i)/\overline{Y}_t^F)) + hot$$

$$= E_i\tilde{y}_t(i) + \tfrac{1}{2}\tfrac{\theta-1}{\theta}\frac{\overline{\Upsilon}}{\left[1 + \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\overline{\Upsilon}\right]^2} + \tfrac{1}{2}\tfrac{\theta-1}{\theta}\frac{1 - \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\overline{\Upsilon}}{\left[1 + \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\overline{\Upsilon}\right]^3}\left[\Upsilon_t - \overline{\Upsilon}\right] + hot \tag{A.9}$$

$$= E_i\tilde{y}_t(i) + Q_y^0 + \tfrac{1}{2}\tfrac{\theta-1}{\theta}Q_y^1\left(\Upsilon_t - \overline{\Upsilon}\right) + hot$$

where $Q_y^1 = \dfrac{1 - \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\overline{\Upsilon}}{\left[1 + \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\overline{\Upsilon}\right]^3}$ and $Q_y^0 = \dfrac{\tfrac{1}{2}\tfrac{\theta-1}{\theta}\overline{\Upsilon}}{\left[1 + \tfrac{1}{2}(\tfrac{\theta-1}{\theta})^2\overline{\Upsilon}\right]^2}$. ∎

Lemma 4.

Recall that $b_t = \log(B_t/P_t)$ (note that in our notation b_t is the price relative to *current* price level) and $\overline{P}_t = E_i\log P_t(i)$, and $P_t = (\int_0^1 P_t^{1-\theta}(i)di)^{1/(1-\theta)}$. Also keep in mind that the fraction of firms that do not change their price index it to a fraction $0 \leq \omega \leq 1$ of steady-state inflation, $P_t(i) = P_{t-1}(i)\overline{\Pi}^\omega$. Then

$$\Delta_t \equiv \mathrm{var}_i(\log P_t(i)) = \int_0^1 [\log P_t(i) - \overline{P}_t]^2 \, di = \underbrace{\int_0^{1-\lambda} [\log P_t(i) - \overline{P}_t]^2 \, di}_{price\ change} + \underbrace{\int_{1-\lambda}^1 [\log P_t(i) - \overline{P}_t]^2 \, di}_{indexed\ price}$$

$$= \int_0^{1-\lambda} [\log P_t(i) - \log P_t]^2 \, di + 2\int_0^{1-\lambda} [\log P_t(i) - \log P_t][\log P_t - \overline{P}_t] \, di + \int_0^{1-\lambda} [\log P_t - \overline{P}_t]^2 \, di$$

$$+ \int_{1-\lambda}^1 [\log P_{t-1}(i) - \overline{P}_t + \omega\overline{\pi}]^2 \, di$$

$$= \int_0^{1-\lambda} [\log P_t(i) - \log P_t]^2 \, di + 2\int_0^{1-\lambda} [\log P_t(i) - \log P_t][\log P_t - \overline{P}_t] \, di + \int_0^{1-\lambda} [\log P_t - \overline{P}_t]^2 \, di$$

$$+ \int_{1-\lambda}^1 [\log P_{t-1}(i) - \overline{P}_{t-1}]^2 \, di + 2\int_{1-\lambda}^1 [\log P_{t-1}(i) - \overline{P}_{t-1}][\overline{P}_{t-1} - \overline{P}_t + \omega\overline{\pi}] \, di + \int_{1-\lambda}^1 [\overline{P}_{t-1} - \overline{P}_t + \omega\overline{\pi}]^2 \, di$$

$$= (1-\lambda)b_t^2 + 2(1-\lambda)b_t[\log P_t - \overline{P}_t] + (1-\lambda)[\log P_t - \overline{P}_t]^2$$

$$+ \lambda\Delta_{t-1} + 2[\overline{P}_{t-1} - \overline{P}_t + \omega\overline{\pi}] \underbrace{\int_{1-\lambda}^1 [\log P_{t-1}(i) - \overline{P}_{t-1}] \, di}_{=0} + \lambda[\overline{P}_t - \overline{P}_{t-1} + \omega\overline{\pi}]^2$$

So that

$$\Delta_t = (1-\lambda)b_t^2 + 2(1-\lambda)b_t[\log P_t - \overline{P}_t] + (1-\lambda)[\log P_t - \overline{P}_t]^2 + \lambda\Delta_{t-1} + \lambda[\overline{P}_t - \overline{P}_{t-1} - \omega\overline{\pi}]^2 \qquad (A.10)$$

From **Lemma 3** we have

$$\log P_t - \overline{P}_t \approx Q_p^0 + \tfrac{1-\theta}{2}Q_p^1(\Delta_t - \overline{\Delta}). \qquad (A.11)$$

Also from equation (21) in the paper we have that

$$\pi_t - \overline{\pi} = \frac{1 - \lambda\overline{\Pi}^{(\theta-1)(1-\omega)}}{\lambda\overline{\Pi}^{(\theta-1)(1-\omega)}}(b_t - \overline{b}) \Rightarrow b_t = \overline{b} + \frac{\lambda\overline{\Pi}^{(\theta-1)(1-\omega)}}{1 - \lambda\overline{\Pi}^{(\theta-1)(1-\omega)}}(\pi_t - \overline{\pi}) = \overline{b} + M(\pi_t - \overline{\pi}). \qquad (A.12)$$

We use a guess and verify procedure to determine the dispersion of prices. In particular, we posit that if the deviation of cross-sectional price dispersion from its non-stochastic steady state level $\overline{\Delta}$ is $\Xi_t \equiv \Delta_t - \overline{\Delta}$ then

$$\Xi_t = K_Q(\pi_t - \overline{\pi}) + Z_{1,Q}(\pi_t - \overline{\pi})^2 + F_Q\Xi_{t-1} \qquad (A.13)$$

Where we assume that K_Q is of the same order as the shock processes, so that the first term is of second order. Assuming that the initial degree of price dispersion is also of second order then the degree of price dispersion is of second order at all times.

Plug (A.12) and (A.11) into (A.10) and using the guess (A.13) yields,

$$\Delta_t = (1-\lambda)[\overline{b} + M(\pi_t - \overline{\pi})]^2 + 2(1-\lambda)[\overline{b} + M(\pi_t - \overline{\pi})][Q_p^0 + \tfrac{1-\theta}{2}Q_p^1\Xi_t]$$

$$+ (1-\lambda)[Q_p^0 + \tfrac{1-\theta}{2}Q_p^1\Xi_t]^2 + \lambda\Delta_{t-1}$$

$$+ \lambda[(\ln P_t - \ln P_{t-1} - \omega\overline{\pi}) - (\ln P_t - \overline{P}_t) + (\ln P_{t-1} - \overline{P}_{t-1})]^2$$

$$= (1-\lambda)\overline{b}^2 + (1-\lambda)M^2(\pi_t - \overline{\pi})^2 + 2(1-\lambda)\overline{b}M(\pi_t - \overline{\pi})$$

$$+ 2(1-\lambda)\overline{b}Q_p^0 + (1-\lambda)(1-\theta)\overline{b}Q_p^1\Xi_t + 2(1-\lambda)Q_p^0 M(\pi_t - \overline{\pi})$$

$$+ (1-\lambda)(1-\theta)MQ_p^1 K_Q(\pi_t - \overline{\pi})^2$$

$$+(1-\lambda)[Q_p^0]^2 + (1-\lambda)(1-\theta)Q_p^0 Q_p^1 \Xi_t$$

$$+(1-\lambda)\tfrac{(1-\theta)^2}{4}[Q_p^1]^2 K_Q^{~2}(\pi_t - \overline{\pi})^2 + \lambda\Delta_{t-1}$$

$$+\lambda(\pi_t - \overline{\pi})^2 + \lambda(1-\omega)^2\overline{\pi}^2 + 2\lambda(1-\omega)\overline{\pi}(\pi_t - \overline{\pi})$$

$$+\lambda\tfrac{(1-\theta)^2}{4}[Q_p^1]^2 K_Q^{~2}(\pi_t - \overline{\pi})^2 + \lambda[Q_p^0]^2 + \lambda(1-\theta)Q_p^0 Q_p^1 \Xi_t$$

$$+\lambda\tfrac{(1-\theta)^2}{4}[Q_p^1]^2 \Xi_{t-1}^2 + \lambda[Q_p^0]^2 + \lambda(1-\theta)Q_p^0 Q_p^1 \Xi_{t-1}$$

$$-\lambda(1-\theta)Q_p^1 K_Q(\pi_t - \overline{\pi})^2 - 2\lambda Q_p^0(\pi_t - \overline{\pi})$$

$$-2\lambda(1-\omega)\overline{\pi}Q_p^0 - \lambda(1-\theta)(1-\omega)\overline{\pi}Q_p^1 \Xi_t$$

$$+2\lambda Q_p^0(\pi_t - \overline{\pi}) + 2\lambda(1-\omega)\overline{\pi}Q_p^0 + \lambda(1-\theta)(1-\omega)\overline{\pi}Q_p^1 \Xi_{t-1}$$

$$-2\lambda[Q_p^0]^2 - \lambda(1-\theta)Q_p^0 Q_p^1 \Xi_{t-1} - \lambda(1-\theta)Q_p^0 Q_p^1 \Xi_t + hot.$$

The non-stochastic steady state value is given by

$$\overline{\Delta} = (1-\lambda)\overline{b}^2 + 2(1-\lambda)\overline{b}Q_0^p + (1-\lambda)[Q_0^p]^2 + \lambda\overline{\Delta} + \lambda(1-\omega)^2\overline{\pi}^2 = (\overline{b} + Q_0^p)^2 + \tfrac{\lambda(1-\omega)^2}{(1-\lambda)}\overline{\pi}^2. \tag{A.14}$$

We can solve for deviations from steady state dispersion $\Xi_t \equiv \Delta_t - \overline{\Delta}$

$$\Xi_t = (\pi_t - \overline{\pi})\left\{2(1-\lambda)\overline{b}M + 2(1-\lambda)Q_p^0 M + 2\lambda(1-\omega)\overline{\pi}\right\}$$

$$+(\pi_t - \overline{\pi})^2\left\{(1-\lambda)M^2 + \lambda + \tfrac{(1-\theta)^2}{4}[Q_p^1]^2 K_Q^2 + (1-\lambda)(1-\theta)MQ_p^1 K_Q - \lambda(1-\theta)Q_p^1 K_Q\right\}$$

$$+\Xi_t\left\{(1-\lambda)(1-\theta)\overline{b}Q_p^1 + (1-\lambda)(1-\theta)Q_p^0 Q_p^1 - \lambda(1-\theta)(1-\omega)\overline{\pi}Q_p^1\right\} \tag{A.15}$$

$$+\Xi_{t-1}\left\{\lambda + \lambda(1-\theta)(1-\omega)\overline{\pi}Q_p^1\right\} + hot.$$

Note that the form of (A.15) is the same as the form of our guess in (A.13). Let $\Gamma_0 = \{1 + (\theta-1)Q_1^P[(1-\lambda)(\overline{b} + Q_0^P) - \lambda(1-\omega)\overline{\pi}]\}^{-1}$ and $\Gamma_1 = [1 - (\theta-1)(1-\omega)Q_1^P\overline{\pi}]\Gamma_0$. Then matching coefficients with the guess and verify equation (A.13), we have $\Gamma_2 = 2\{(1-\lambda)M(\overline{b} + Q_p^0) + \lambda(1-\omega)\overline{\pi}\}\Gamma_0$ which confirms the guess for the functional form of $\Delta_t - \overline{\Delta}$ with $F_Q = \lambda\Gamma_1$. Hence

$$\Xi_t = \Gamma_2(\pi_t - \overline{\pi}) + \lambda\Gamma_1\Xi_{t-1} +$$

$$+(\pi_t - \overline{\pi})^2\Gamma_0\left\{(1-\lambda)M^2 + \lambda + \tfrac{(1-\theta)^2}{4}[Q_p^1]^2\Gamma_2^2 + (1-\lambda)(1-\theta)MQ_p^1\Gamma_2 - \lambda(1-\theta)Q_p^1\Gamma_2\right\} + h.o.t.$$

which completes the proof. ∎

Proposition 1.

From **Lemma 5**, we have $E_i\tilde{y}_t(i) = \tilde{y}_t - Q_y^0 - \tfrac{1}{2}\tfrac{\theta-1}{\theta}Q_y^1\left(\mathrm{var}_i\,\hat{y}_t(i) - \overline{\Upsilon}\right) + hot$. Therefore, we can express the approximation from the disutility of labor supply given in **Lemma 2** as follows:

$$\int_0^1 \tilde{\upsilon}(Y_t(i))di \quad \approx \overline{Y}_t^F \tilde{\upsilon}'_{\overline{Y}_t^F} \left\{ E_i \tilde{y}_t(i) + \tfrac{1}{2}(1+\eta^{-1}) \int_0^1 \tilde{y}_t^2(i)di \right\}$$

$$= \overline{Y}_t^F \tilde{\upsilon}'_{\overline{Y}_t^F} \left\{ E_i \tilde{y}_t(i) + \tfrac{1}{2}(1+\eta^{-1}) \Big(\text{var}_i\, \hat{y}_t(i) + [E_i \tilde{y}_t(i)]^2 \Big) \right\}$$

$$= \overline{Y}_t^F \tilde{\upsilon}'_{\overline{Y}_t^F} \left\{ (1 - (1+\eta^{-1})Q_y^0)\tilde{y}_t \right. \tag{A.16}$$

$$+ \tfrac{1}{2}[Q_y^1 \theta^{-1} - Q_y^1 + (1+\eta^{-1})(1 + \tfrac{\theta-1}{\theta} Q_y^0 Q_y^1)]\,\text{var}_i\, \hat{y}_t(i) +$$

$$+ \tfrac{1}{2}(1+\eta^{-1})(\tfrac{1}{4}(\tfrac{\theta-1}{\theta})^2 [Q_y^1(\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon})]^2 - \tfrac{\theta-1}{\theta} \tilde{y}_t Q_y^1[\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon}])$$

$$\left. + \tfrac{1}{2}(1+\eta^{-1})\tilde{y}_t^2 + (1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta} Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon} \right\} + hot + tip$$

As Woodford (2003) shows, $\tilde{\upsilon}'_{\overline{Y}_t^F} = u'_{\overline{C}_t^F}(1-\Phi)$, where (with log utility) $\Phi = -\ln(\tfrac{\theta-1}{\theta})$ which follows from the intra-temporal condition for labor supply. Hence,

$$\int_0^1 \tilde{\upsilon}(Y_t(i))di = (1-\overline{g}_y)^{-1}\hat{C}_t^F u'_{\hat{C}_t^F}(1-\Phi)\left\{(1 - (1+\eta^{-1})Q_y^0)\tilde{y}_t + \tfrac{1}{2}(1+\eta^{-1})\tilde{y}_t^2 + \right. \tag{A.17}$$

$$+ \tfrac{1}{2}[Q_y^1 \theta^{-1} - Q_y^1 + (1+\eta^{-1})(1 + \tfrac{\theta-1}{\theta} Q_y^0 Q_y^1)]\,\text{var}_i\, \hat{y}_t(i)$$

$$+ \tfrac{1}{2}(1+\eta^{-1})(\tfrac{1}{4}(\tfrac{\theta-1}{\theta})^2 [Q_y^1(\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon})]^2 - \tfrac{\theta-1}{\theta} \tilde{y}_t Q_y^1[\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon}])$$

$$\left. + (1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta} Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon} \right\} + hot + tip$$

As in **Lemma 1**, $\hat{C}_t^F u'_{\hat{C}_t^F} = 1$. Using equation (35) from the derivations in the paper and **Lemma 4**, we replace the cross-section dispersion of output in (A.17) with the cross-sectional dispersion of prices.

Given the low levels of inflation we consider in our analysis, Γ_2 is close to zero and $\Gamma_2(\pi_t - \overline{\pi})$ has (in expectation) negligible effects on Ξ_t so that

$$E(\Xi_t) = \Gamma_3 E(\hat{\pi}_t^2) \tag{A.18}$$

where

$$\Gamma_3 = \Gamma_0 (1-\lambda\Gamma_1)^{-1}\left\{(1-\lambda)M^2 + \lambda\right\}. \tag{A.19}$$

Note that from the definition of Ξ_t, we have

$$E(\Delta_t) = E(\Xi_t) + \overline{\Delta}. \tag{A.20}$$

Since all variables are stationary and $\hat{y}_t = \tilde{y}_t - \log(\overline{X})$, we can compute the expected per period utility as follows

$$EU_t \quad \approx E\left\{\tilde{c}_t - (1-g)^{-1}(1-\Phi)(1 - (1+\eta^{-1})Q_y^0)\tilde{y}_t - \tfrac{1}{2}(1-g)^{-1}(1-\Phi)(1+\eta^{-1})\tilde{y}_t^2 \right.$$

$$- \tfrac{1}{2}(1-g)^{-1}(1-\Phi)[Q_y^1\theta^{-1} - Q_y^1 + (1+\eta^{-1})(1 + \tfrac{\theta-1}{\theta} Q_y^0 Q_y^1)]\theta^2 \Delta_t + \tfrac{1}{2}(1-g)^{-1}(1-\Phi)(1+\eta^{-1})\tfrac{\theta-1}{\theta} Q_y^1 \theta^2 \tilde{y}_t \Xi_t$$

$$\left. - (1-g)^{-1}(1-\Phi)\left[(1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta} Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon}\right]\right\} + hot + tip$$

$$= [1 - \tfrac{(1-\Phi)}{(1-g)}(1 - (1+\eta^{-1})Q_y^0)]\log(\overline{X}) - \tfrac{(1-\Phi)}{(1-g)}\left[(1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta} Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon}\right]$$

$$- \tfrac{1}{2}(1-g)^{-1}(1+\eta^{-1})\{(1-\Phi)\log(\overline{X})^2 + E(\hat{y}_t^2)\} - \tfrac{1}{2}\tfrac{(1-\Phi)}{(1-g)}[Q_y^1\theta^{-1} - Q_y^1 + (1+\eta^{-1})(1 + \tfrac{\theta-1}{\theta} Q_y^0 Q_y^1)]\theta^2 E(\Delta_t)$$

$$+ hot + tip$$

After using (A.18) and (A.20) and collecting terms, one arrives at equation (37). ∎

Appendix B: Model with Capital

The representative consumer aims to maximize the present discount value of the utility stream from consumption and leisure

$$\max E_t \sum_{j=0}^{\infty} \beta^j \{ \ln C_{t+j} - \frac{\eta}{\eta+1} \int_0^1 N_{t+j}(i)^{1+\eta^{-1}} di \}$$ (B.1)

where C is consumption of the final good, $N_t(i)$ is labor supplied to individual industry i, h is the degree of external habit formation, η is the Frisch labor supply elasticity and β_t is the time-varying discount factor. The budget constraint each period is given by

$$\xi_{t+j} : C_{t+j} + \frac{S_{t+j}}{P_{t+j}} + I_{t+j} + \frac{\psi}{2} \left(\frac{I_{t+j}}{K_{t+j-1}} - \delta - \overline{GY} + 1 \right)^2 K_{t+j-1}$$

$$\leq \int_0^1 \frac{n_{t+j}(i) W_{t+j}(i)}{P_{t+j}} di + \frac{S_{t+j-1} R_{t+j-1} q_{t+j-1}}{P_{t+j}} + \frac{K_{t+j-1} R_{t+j}^k}{P_{t+j}} + T_{t+j}$$ (B.2)

where S is the stock of one-period bonds held by the consumer, R is the gross nominal interest rate, P is the price of the final good, $W(i)$ is the nominal wage earned from labor in industry i, K is capital owned by the representative consumer, R^k is the nominal rental rate on capital, T is transfers and profits from ownership of firms, q_t is the risk premium shock, and ξ is the shadow value of wealth. The fourth term on the left hand side is a quadratic adjustment cost to the stock of capital held by the consumer, where ψ measures the strength of the adjustment cost. δ is the rate of depreciation on capital and $\overline{GY}$ the steady state growth rate of output. Capital can be accumulated according to the law of motion

$$\mu_{t+j} : K_{t+j} = (1-\delta) K_{t+j-1} + I_{t+j}$$ (B.3)

The first order conditions from this utility-maximization problem are then:

$$C_{t+j}^{-1} = \xi_{t+j}$$ (B.4)

$$N_{t+j}(i)^{1/\eta} = \xi_{t+j} \frac{W_{t+j}(i)}{P_{t+j}}$$ (B.5)

$$\frac{\xi_{t+j}}{P_{t+j}} = E_{t+j} \beta \xi_{t+j+1} \frac{R_{t+j} q_{t+j}}{P_{t+j+1}}$$ (B.6)

$$q_{t+j} = 1 + \psi \left(\frac{I_{t+j}}{K_{t+j-1}} - \delta - \overline{GY} + 1 \right)$$ (B.7)

$$0 = E_{t+j} \beta \lambda_{t+j+1} \left[\tilde{R}_{t+j+1}^k - \frac{\psi}{2} \left(\frac{I_{t+j+1}}{K_{t+j}} - \delta - \overline{GY} + 1 \right)^2 + \psi \frac{I_{t+j+1}}{K_{t+j}} \left(\frac{I_{t+j+1}}{K_{t+j}} - \delta - \overline{GY} + 1 \right) \right]$$

$$+ (1-\delta) E_{t+j} \beta \mu_{t+j+1} + \mu_{t+j}$$ (B.8)

where $\lambda_{t+j} / \mu_{t+j}$ is Tobin's q and $\tilde{R}_{t+j}^k = R_{t+j}^k / P_{t+j}$ is the real rental rate of capital.

Production of the final good is done by a perfectly competitive sector which combines a continuum of intermediate goods into a final good per the following aggregator

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}$$ (B.9)

where Y_t is the final good and $Y(i)$ is intermediate good i, while θ denotes the elasticity of substitution across intermediate goods, yielding the following demand curve for goods of intermediate sector i

$$Y_t(i) = Y_t(P_t(i)/P_t)^{-\theta}$$ (B.10)

and the following expression for the aggregate price level

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}$$ (B.11)

The production of each intermediate good is done by a monopolist facing a Cobb-Douglas production function with capital share α

$$Y_t(i) = K_{t+j-1}(i)^\alpha (A_{t+j} N_{t+j}(i))^{1-\alpha}$$ (B.12)

where A denotes the level of technology, common across firms. We assume that capital is perfectly mobile across firms,

$$K_{t+j-1} = \int_0^1 K_{t+j-1}(i) di$$ (B.13)

Each intermediate good producer has sticky prices, modeled as in Calvo (1983) where 1-λ is the probability that each firm will be able to reoptimize its price each period. We allow for indexation of prices to steady-state inflation by firms who do not reoptimize their prices each period, with ω representing the degree of indexation (0 for no indexation to 1 for full indexation). Denoting the optimal reset price of firm i by $B(i)$, re-optimizing firms solve the following profit-maximization problem

$$\max E_t \sum_{j=0}^\infty \lambda^j Q_{t,t+j} y_{t+j}(i) [B_t(i)\overline{\Pi}^{\omega j} - W_{t+j}(i) N_{t+j}(i) - R_{t+j}^k K_{t+j-1}(i)]$$ (B.14)

where Q is the stochastic discount factor and $\overline{\Pi}$ is the gross steady-state level of inflation. The optimal relative reset price is then given by

$$\frac{B_t(i)}{P_t} = \frac{\theta}{\theta-1} \frac{E_t \sum_{j=0}^\infty \lambda^j Q_{t,t+j} Y_{t+j} (P_{t+j}/P_t)^{\theta+1} \overline{\Pi}^{-j\omega\theta} (MC(i)_{t+j}/P_{t+j})}{E_t \sum_{j=0}^\infty \lambda^j Q_{t,t+j} Y_{t+j} (P_{t+j}/P_t)^{\theta} \overline{\Pi}^{-j\omega(\theta-1)}}$$ (B.15)

where firm-specific marginal costs can be related to aggregate variables using

$$\frac{MC_{t+j}(i)}{P_{t+j}} = \left(\frac{C_{t+j}}{A_{t+j}}\right)^{\frac{(1-\alpha)\eta}{\eta+\alpha}} \left(\frac{Y_{t+j}}{A_{t+j}}\right)^{\frac{1-\alpha}{\eta+\alpha}} \left(\frac{R_{t+j}^k}{P_{t+j}}\right)^{\frac{\alpha(\eta+1)}{\eta+\alpha}} \times$$
$$\times \left(\frac{B_t(i)}{P_t}\right)^{-\frac{\theta(1-\alpha)}{\eta+\alpha}} \left(\frac{P_{t+j}}{\overline{\Pi}^{j\omega} P_t}\right)^{\frac{\theta(1-\alpha)}{\eta+\alpha}} [\alpha^\alpha (1-\alpha)^{1-\alpha}]^{-\frac{\eta+1}{\eta+\alpha}}.$$ (B.16)

Firm-specific marginal cost remain a function of aggregate variables and the reset price only because full capital mobility implies that all firms have the same (adjusted) capital labor ratio. The first order condition for the firms' input usage is

$$\frac{\alpha}{1-\alpha} \frac{N_t(i)}{K_{t-1}(i)} = \frac{R_t^k}{W_t(i)}$$ (B.17)

which we combine with the first order condition for labor supply to get

$$\frac{\alpha}{1-\alpha} \frac{N_t(i)^{1+\eta^{-1}}}{K_{t-1}(i)} = \frac{R_t^k}{P_t} \xi_t$$ (B.18)

Since the right hand side consists of aggregate variables only, the left hand side must be identical across firms.

Give these price-setting assumptions, the dynamics of the price level are governed by

$$P_t^{1-\theta} = (1-\lambda) B_t^{1-\theta} + \lambda P_{t-1}^{1-\theta} \overline{\Pi}^{\omega(1-\theta)}$$ (B.19)

We allow for government consumption of final goods (G_t), so the goods market clearing condition for the economy is

$$Y_t = C_t + G_t + I_t. \tag{B.20}$$

We define the aggregate labor input as

$$N_{t+j} = \left[\int_0^1 N_{t+j}(i)^{1+\eta^{-1}} di \right]^{1/(1+\eta^{-1})}. \tag{B.21}$$

Note that this definition of labor input differs from the labor-only model. We make different assumptions to simplify our aggregation problems.

Log-linearization
In the model with capital the goods market clearing condition becomes

$$\hat{y}_t = \overline{c}_y \hat{c}_t + \overline{g}_y \hat{g}_t + \overline{i}_y \hat{i}_t \tag{B.22}$$

where $\overline{c}_y$, $\overline{g}_y$, and $\overline{i}_y$ are the steady-state ratios of consumption, government and investment to output respectively.

Lemma 6. Let $\overline{c}_y$, $\overline{g}_y$, and $\overline{i}_y$ be the steady-state ratios of consumption, government and investment to output respectively. Then

$$\overline{c}_y = (1 - \overline{g}_y) - \frac{\theta - 1}{\theta} \frac{\alpha \beta (\delta + \overline{GY} - 1)}{\overline{GY} - \beta(1 - \delta)}, \tag{B.23}$$

$$\overline{i}_y = \frac{\theta - 1}{\theta} \frac{\alpha \beta (\delta + \overline{GY} - 1)}{\overline{GY} - \beta(1 - \delta)}, \tag{B.24}$$

Proof: See Appendix C.

Corollary 1. Since $\overline{c}_y$, $\overline{g}_y$, and $\overline{i}_y$ are independent of the degree of price stickiness λ, they are equal to their flexible price level counterparts, $\overline{c}_y = \overline{c}_y^F$, $\overline{g}_y = \overline{g}_y^F$, and $\overline{i}_y = \overline{i}_y^F$.

Also, integrating over firm-specific production functions and log-linearizing yields the aggregate production function:

Lemma 7. Let lower case letters denote the deviations from steady state values. Then the aggregate production function up to a first order approximation is given by

$$\hat{y}_t = \alpha \hat{k}_t + (1 - \alpha) \hat{n}_t, \tag{B.25}$$

Proof: See Appendix C.

Allowing for capital also changes the steady state effects of positive trend inflation. For example, the steady-state level of the output gap (which is defined as the deviation of output from its flexible price level counterpart) is given by

$$\overline{X}^{\frac{(1-\alpha)(\eta+1)}{\eta+\alpha}} = \frac{1-[1-\lambda\beta^{-1}\overline{\Pi}^{(1-\omega)\theta}]\left(\dfrac{1-\lambda\overline{\Pi}^{(1-\omega)(\theta-1)}}{1-\lambda}\right)^{\frac{\eta+\alpha+\theta(1-\alpha)}{(\eta+\alpha)(\theta-1)}}}{\lambda\beta^{-1}\overline{\Pi}^{\frac{(1-\omega)\theta(\eta+1)}{\eta+\alpha}}} \tag{B.26}$$

Note that the steady-state level of the gap is again equal to one when steady-state inflation is zero (i.e., $\overline{\Pi}=1$) or when the degree of price indexation is exactly equal to one.

Lemma 8. The expansion of the utility part that corresponds to disutility from labor supply in the capital model is given by

$$\int_0^1 \upsilon(n_t(i))di = \overline{N}_t^F \upsilon'_{\overline{N}_t^F} \left\{ (1-(\tfrac{\eta+1}{\eta})Q_n^0)\hat{n}_t + \tfrac{1}{2}[(\tfrac{\eta+1}{\eta})(1-Q_n^1+\tfrac{\eta+1}{\eta}Q_n^0Q_n^1)]\operatorname{var}_i \hat{n}_t(i) + \right.$$

$$+ \tfrac{1}{8}(\tfrac{\eta+1}{\eta})^3[Q_n^1(\operatorname{var}_i \hat{n}_t(i)-\overline{\Upsilon})]^2 - \tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2\hat{n}_tQ_n^1(\operatorname{var}_i \hat{n}_t(i)-\overline{\Upsilon}) +$$

$$\left. + \tfrac{1}{2}(\tfrac{\eta+1}{\eta})\hat{n}_t^2 - Q_n^0 + (\tfrac{\eta+1}{\eta})[Q_n^0]^2 + \tfrac{\eta+1}{\eta}Q_n^1[\tfrac{1}{2}-(\tfrac{\eta+1}{\eta})Q_n^0]\overline{\Upsilon} \right\} + hot + tip$$

where $Q_n^1 = \dfrac{1-\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2\overline{\Upsilon}}{\left[1+\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2\overline{\Upsilon}\right]^3} \leq 1$ and $Q_n^0 = \tfrac{1}{2}\tfrac{\eta+1}{\eta}\dfrac{\overline{\Upsilon}}{\left[1+\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2\overline{\Upsilon}\right]^2}$ and $\overline{\Upsilon}$ is the steady state dispersion of labor supply.

Proof: See Appendix C.

Lemma 9. The cross sectional variations in labor input and prices are proportional,

$$\operatorname{var}_i \hat{n}_t(i) = \left(\frac{\theta}{1+\alpha\eta^{-1}}\right)^2 \operatorname{var}_i \ln p_t(i)$$

Proof: See Appendix C.

Before we proceed with Proposition 2 we write the inter-temporal condition for labor supply as,

$$\frac{\upsilon'_{\overline{N}_t^F}\overline{N}_t^F}{u'_{\overline{C}_t^F}} = \overline{N}_t^F \frac{W_t}{P_t}$$

The right hand side of this expression is simply total labor income. Because of the distortion due to monopolistic competition it is equal to

$$\frac{\upsilon'_{\overline{N}_t^F}\overline{N}_t^F}{u'_{\overline{C}_t^F}} = (1-\Phi)(1-\alpha)\overline{Y}_t^F$$

where $1-\Phi = (\theta-1)/\theta$. Since we assume log utility this expression further simplifies to

$$\upsilon'_{\overline{N}_t^F}\overline{N}_t^F = (1-\Phi)(1-\alpha)(\overline{c}_y^F)^{-1}$$

where $\overline{c}_y^F$ is the share of consumption in output at the flexible price steady state.

$$\int_0^1 \upsilon(n_t(i))di = (1-\alpha)(c_y^F)^{-1}(1-\Phi)\left\{(1-(\tfrac{\eta+1}{\eta})Q_n^0)\hat{n}_t + \tfrac{1}{2}[(\tfrac{\eta+1}{\eta})(1-Q_n^1 + \tfrac{\eta+1}{\eta}Q_n^0 Q_n^1)]\operatorname{var}_i \hat{n}_t(i) + \right.$$

$$+ \tfrac{1}{8}(\tfrac{\eta+1}{\eta})^3[Q_n^1(\operatorname{var}_i \hat{n}_t(i) - \overline{\Upsilon})]^2 - \tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \hat{n}_t Q_n^1(\operatorname{var}_i \hat{n}_t(i) - \overline{\Upsilon}) + \tag{B.27}$$

$$\left. + \tfrac{1}{2}(\tfrac{\eta+1}{\eta})\hat{n}_t^2 - Q_n^0 + (\tfrac{\eta+1}{\eta})[Q_n^0]^2 + \tfrac{\eta+1}{\eta}Q_n^1[\tfrac{1}{2} - (\tfrac{\eta+1}{\eta})Q_n^0]\overline{\Upsilon}\right\} + hot + tip$$

Proposition 2. In the model for capital, the second order approximation to utility (1) is

$$\Omega_0 + \Omega_1 E(\hat{y}_t^2) + \Omega_2 E(\hat{\pi}_t^2)$$

where parameters $\Omega_i, i = 0,...,2$ depend on the steady state inflation $\overline{\pi}$ and are given by

$$\Omega_0 = \log(\overline{X}) - \tfrac{1}{2}(1-\alpha)(\overline{c}_y)^{-1}(1-\Phi)\left[\tfrac{\eta+1}{\eta}[Q_n^0]^2 - Q_n^0 + Q_n^1 \tfrac{\eta+1}{\eta}[\tfrac{1}{2} - (\tfrac{\eta+1}{\eta})Q_n^0]\tfrac{\theta^2}{(1+\alpha\eta^{-1})^2}\overline{\Delta}\right]$$

$$- \tfrac{1}{2}(1-\alpha)(\overline{c}_y)^{-1}(1-\Phi)[(\tfrac{\eta+1}{\eta})(1-Q_n^1 + \tfrac{\eta+1}{\eta}Q_n^0 Q_n^1)]\tfrac{\theta^2}{(1+\alpha\eta^{-1})^2}\overline{\Delta},$$

$$\Omega_1 = -\tfrac{1}{2}(\overline{c}_y)^{-1}(1+\tfrac{1}{\eta}),$$

$$\Omega_2 = -\tfrac{1}{2}(1-\alpha)(\overline{c}_y)^{-1}[(\tfrac{\eta+1}{\eta})(1-Q_n^1 + \tfrac{\eta+1}{\eta}Q_n^0 Q_n^1)]\tfrac{\theta^2}{(1+\alpha\eta^{-1})^2}\Gamma_3,$$

$$Q_n^1 = \frac{1 - \tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}}{\left[1 + \tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}\right]^3}$$

$$Q_n^0 = \tfrac{1}{2}\tfrac{\eta+1}{\eta}\frac{\overline{\Upsilon}}{\left[1 + \tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}\right]^2}$$

$$\overline{\Upsilon} = (\tfrac{\theta}{1+\alpha\eta})^2 \overline{\Delta}$$

$$\Phi = -\theta^{-1}.$$

and Γ_3 is a defined in Proposition 1.
Proof: See Appendix C.

Appendix C: Proofs for model with capital

Lemma 6

From the first order condition for the use of capital we know that capital for a firm that reset its price at time t evolves as follows

$$k_{t+j-1}(i) = GY_t^j \overline{\Pi}^{j(1-\omega)(\theta-1)} \underbrace{\frac{\theta-1}{\theta} Y_t \alpha \frac{1}{\tilde{R}_t^k} \left(\frac{\overline{B}}{P}\right)^{1-\theta}}_{k_{t-1}^R} \tag{C.1}$$

where k_t^R is the optimal reset stock of capital at time t. Due to Calvo pricing the aggregate capital stock today is an aggregate of an infinite sum of past capital stocks, adjusted for their growth rate in the interim

$$K_{t-1} = \int_0^1 k_{t-1}(i) di = \sum_{j=0}^{\infty} (1-\lambda)\lambda^j k_{t-j-1}^R GY_t^j \overline{\Pi}^{j(1-\omega)(\theta-1)}$$

$$= \sum_{j=0}^{\infty} (1-\lambda)\lambda^j \frac{\theta-1}{\theta} Y_{t-j} \alpha \frac{1}{\tilde{R}^k} \left(\frac{\overline{B}}{P}\right)^{1-\theta} GY_t^j \overline{\Pi}^{j(1-\omega)(\theta-1)} \tag{C.2}$$

$$= Y_t \frac{\theta-1}{\theta} \alpha \frac{1}{\tilde{R}^k} \frac{1-\lambda}{1-\lambda\overline{\Pi}^{(1-\omega)(\theta-1)}} \left(\frac{\overline{B}}{P}\right)^{1-\theta} = Y_t \frac{\theta-1}{\theta} \frac{\alpha\beta}{\overline{GY} - \beta(1-\delta)}$$

$$= Y_t \overline{k}_y$$

Thus, the ratio of capital to output $\overline{k}_y$. From the capital accumulation equation we know that in the steady state the share of output going to investment is equal to

$$\overline{i}_y = \frac{\theta-1}{\theta} \frac{\alpha\beta(\delta + \overline{GY} - 1)}{\overline{GY} - \beta(1-\delta)}, \tag{C.3}$$

The conclusion for $\overline{c}_y$ then follows directly from the resource constraint. ∎

Lemma 7

We first rewrite the individual production function as

$$y_t(i) = A_t^{1-\alpha} n_t(i)^{1+\alpha\eta^{-1}} \left(\frac{k_{t-1}(i)}{n_t(i)^{1+\eta^{-1}}}\right)^{\alpha} \tag{C.4}$$

From the first order condition for labor supply and the optimal capital labor ratio, we know that the term in parentheses is the same across firms. Integrating this equation then yields,

$$Y_t = A_t^{1-\alpha} \left[\int_0^1 n_t(i)^{(1+\alpha\eta^{-1})(\theta-1)/\theta} di\right]^{\frac{\theta}{\theta-1}} \left(\frac{K_{t-1}}{N_t^{1+\eta^{-1}}}\right)^{\alpha} \tag{C.5}$$

We then take logs of this expression and consider a first order approximation to the integral,

$$\frac{\theta}{\theta-1} \ln\left[\int_0^1 n_t(i)^{(1+\alpha\eta^{-1})(\theta-1)/\theta} di\right] \approx$$

$$\approx \frac{\eta+\alpha}{\eta+1} \left[\int_0^1 \ln n_t(i)^{(1+\eta^{-1})} di\right] + \frac{1}{2} \frac{\theta}{\theta-1} \frac{[\frac{\eta+\alpha}{\eta}\frac{\theta-1}{\theta}]^2 \, \text{var}_i(\ln n_t(i))}{\{1 + \frac{1}{2}[\frac{\eta+\alpha}{\eta}\frac{\theta-1}{\theta}]^2 \, \text{var}_i(\ln n_t(i))\}^2}$$

$$\approx \ln N_t^{1+\alpha\eta^{-1}} + \underbrace{\frac{1}{2}\frac{\theta}{\theta-1}\frac{[\frac{\eta+\alpha}{\eta}\frac{\theta-1}{\theta}]^2\,\overline{\Upsilon}}{\{1+\frac{1}{2}[\frac{\eta+\alpha}{\eta}\frac{\theta-1}{\theta}]^2\,\overline{\Upsilon}\}^2}}_{Z_n^0} + \underbrace{\frac{1}{2}\frac{\theta}{\theta-1}\frac{[\frac{\eta+\alpha}{\eta}\frac{\theta-1}{\theta}]^2\{1-\frac{1}{2}[\frac{\eta+\alpha}{\eta}\frac{\theta-1}{\theta}]^2\,\overline{\Upsilon}\}}{\{1+\frac{1}{2}[\frac{\eta+\alpha}{\eta}\frac{\theta-1}{\theta}]^2\,\overline{\Upsilon}\}^3}}_{Z_n^1}(\Upsilon_t - \overline{\Upsilon})$$

$$= \ln N_t^{1+\alpha\eta^{-1}} + Z_n^0 + Z_n^1 \frac{\theta^2}{(1+\alpha\eta^{-1})^2}\underbrace{(\overline{\Delta}_t - \overline{\Delta})}_{hot} + hot \tag{C.6}$$

$$= \ln N_t^{1+\alpha\eta^{-1}} + Z_n^0 + hot$$

where the last line follows from price dispersion being of second order (Lemma 4, Appendix A). ∎

Lemma 8

Note that

$$\upsilon(n_t(i))di \approx \overline{N}_t^F \upsilon'_{\overline{N}_t^F}\left\{\hat{n}_t(i) + \tfrac{1}{2}(1+\eta^{-1})\hat{n}_t^2(i)\right\} + tip \tag{C.7}$$

where $\hat{n}_t(i) = \log(n_t(i)/\overline{N}_t^F)$ is the deviation of firm i's labor input from flexible price level of labor input $\overline{N}_t^F$ and η is the Frisch elasticity of labor supply. Because of our definition of labor input in the capital model the flexible price steady state labor input and the Calvo-pricing steady state labor input are the same, $\overline{N}_t = \overline{N}_t^F \Rightarrow \hat{n}_t = \tilde{n}_t$. Compute the average across firms

$$\int_0^1 \upsilon(n_t(i))di = \overline{N}_t^F \upsilon'_{\overline{N}_t^F}\left\{E_i\hat{n}_t(i) + \tfrac{1}{2}(1+\eta^{-1})\int_0^1 \hat{n}_t^2(i)di\right\} + hot \tag{C.8}$$

Consider the first term $E_i\hat{n}_t(i)$. We define aggregate labor input as

$$N_t = \left[\int_0^1 n_t(i)^{1+\eta^{-1}}di\right]^{\frac{1}{1+\eta^{-1}}} \Rightarrow N_t/\overline{N}_t^F = \left[\int_0^1 \left(n_t(i)/\overline{N}_t^F\right)^{1+\eta^{-1}}di\right]^{\frac{1}{1+\eta^{-1}}} \tag{C.9}$$

Note that

$$\ln N_t/\overline{N}_t^F = \frac{\eta}{\eta+1}\ln(\int_0^1\left(\frac{n_t(i)}{\overline{N}_t^F}\right)^{\frac{\eta+1}{\eta}}di) = \frac{\eta}{\eta+1}\int_0^1 \frac{\eta+1}{\eta}\ln(\frac{n_t(i)}{\overline{N}_t^F})di + \frac{1}{2}\frac{\eta}{\eta+1}\frac{1}{\left[E(\frac{n_t(i)}{\overline{N}_t^F})^{\frac{\eta+1}{\eta}}\right]^2}\,\mathrm{var}_i((\frac{n_t(i)}{\overline{N}_t^F})^{\frac{\eta+1}{\eta}}) + hot$$

$$= E_i\hat{n}_t(i) + \frac{1}{2}\frac{\eta}{\eta+1}\frac{1}{\left[E(\frac{n_t(i)}{\overline{N}_t^F})^{\frac{\eta+1}{\eta}}\right]^2}\,\mathrm{var}_i((\frac{n_t(i)}{\overline{N}_t^F})^{\frac{\eta+1}{\eta}}) + hot \tag{C.10}$$

Using the delta method, we have

$$\mathrm{var}_i((n_t(i)/\overline{N}_t^F)^{\frac{\eta+1}{\eta}}) = \mathrm{var}_i(\exp\{\tfrac{\eta+1}{\eta}\ln(\tfrac{n_t(i)}{\overline{N}_t^F})\}) \approx (\tfrac{\eta+1}{\eta})^2[\exp\{\tfrac{\eta+1}{\eta}E\ln(\tfrac{n_t(i)}{\overline{N}_t^F})\}]^2\,\mathrm{var}_i(\ln(\tfrac{n_t(i)}{\overline{N}_t^F})) \tag{C.11}$$

Also observe that

$$E(n_t(i)/\overline{N}_t^F)^{\frac{\eta+1}{\eta}} = E[\exp\{\tfrac{\eta+1}{\eta}\ln(n_t(i)/\overline{N}_t^F)\}]$$

$$\approx \exp\{\tfrac{\eta+1}{\eta}E\ln(n_t(i)/\overline{N}_t^F)\} + \tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2\exp\{\tfrac{\eta+1}{\eta}E\ln(n_t(i)/\overline{N}_t^F)\}\,\mathrm{var}_i(\ln(n_t(i)/\overline{N}_t^F))$$

$$= \exp\{\tfrac{\eta+1}{\eta}E\ln(n_t(i)/\overline{N}_t^F)\}\left[1+\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2\,\mathrm{var}_i(\ln(n_t(i)/\overline{N}_t^F))\right] \tag{C.12}$$

Combine (C.11) and (C.12) to re-write (C.10) as follows

$$\ln N_t/\overline{N}_t^F = E_i\hat{n}_t(i) + \frac{1}{2}\frac{\eta}{\eta+1}\frac{(\frac{\eta+1}{\eta})^2\,\mathrm{var}_i(\ln(\frac{n_t(i)}{\overline{N}_t^F}))}{\left[1+\frac{1}{2}(\frac{\eta+1}{\eta})^2\,\mathrm{var}_i(\ln(\frac{n_t(i)}{\overline{N}_t^F}))\right]^2} + hot \tag{C.13}$$

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We use Taylor series expansion of the second term,

$$\ln N_t / \bar{N}_t^F = E_i \hat{n}_t(i) + \tfrac{1}{2}\tfrac{\eta+1}{\eta}\frac{\overline{\Upsilon}}{\left[1+\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}\right]^2} + \tfrac{1}{2}\tfrac{\eta+1}{\eta}\frac{1-\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}}{\left[1+\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}\right]^3}\left[\Upsilon_t - \overline{\Upsilon}\right] + hot$$

where $\Upsilon_t = \mathrm{var}_i(\ln(n_t(i)/\bar{N}_t^F))$ and $\overline{\Upsilon} = E\Upsilon_t$. We can consider a modification where

$$\ln N_t / \bar{N}_t^F = E_i \hat{n}_t(i) + Q_n^0 + \tfrac{1}{2}\tfrac{\eta+1}{\eta} Q_n^1(\Upsilon_t - \overline{\Upsilon}) + hot \tag{C.14}$$

where $Q_n^1 = \dfrac{1-\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}}{\left[1+\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}\right]^3} \le 1$ and $Q_n^0 = \tfrac{1}{2}\tfrac{\eta+1}{\eta}\dfrac{\overline{\Upsilon}}{\left[1+\tfrac{1}{2}(\tfrac{\eta+1}{\eta})^2 \overline{\Upsilon}\right]^2}$.

It then follows from (C.14),

$$\hat{n}_t = \ln N_t / \bar{N}_t = \ln N_t / \bar{N}_t^F = E_i \hat{n}_t(i) + Q_n^0 + \tfrac{1}{2}\tfrac{\eta+1}{\eta} Q_n^1 \left[\mathrm{var}_i(\ln(n_t(i)/\bar{N}_t^F)) - \overline{\Upsilon}\right] + hot \tag{C.15}$$

which implies that the first order term in (C.8) is equal to

$$E_i \hat{n}_t(i) = \hat{n}_t - Q_n^0 - \tfrac{1}{2}\tfrac{\eta+1}{\eta} Q_n^1 \left[\mathrm{var}_i(\ln(\tfrac{n_t(i)}{\bar{N}_t^F})) - \overline{\Upsilon}\right] + hot = \hat{n}_t - Q_n^0 - \tfrac{1}{2}\tfrac{\eta+1}{\eta} Q_n^1 \left[\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon}\right] + hot \tag{C.16}$$

Now consider the second term in (C.8):

$$\int_0^1 \hat{n}_t^2(i)di = \int_0^1 [\hat{n}_t(i) - E_i\hat{n}_t(i) + E_i\hat{n}_t(i)]^2 di = \mathrm{var}_i \hat{n}_t(i) + [E_i\hat{n}_t(i)]^2 \tag{C.17}$$

Using (C.16), we can re-write (C.17) as follows

$$\int_0^1 \hat{n}_t^2(i)di = \mathrm{var}_i \hat{n}_t(i) + [\hat{n}_t - Q_n^0 - \tfrac{1}{2}\tfrac{\eta+1}{\eta} Q_n^1(\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon})]^2$$

$$= \mathrm{var}_i \hat{n}_t(i) + \hat{n}_t^2 + \tfrac{1}{4}(\tfrac{\eta+1}{\eta})^2[Q_n^1(\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon})]^2 - \tfrac{\eta+1}{\eta}\hat{n}_t Q_n^1(\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon}) + \tag{C.18}$$

$$+ [Q_n^0]^2 - 2Q_n^0\hat{n}_t + \tfrac{\eta+1}{\eta} Q_n^0 Q_n^1(\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon})$$

Now we combine (C.18) and (C.16) to finally get the part which corresponds to the second-order approximation of utility due to the disutility of labor supply (C.8):

$$\int_0^1 \upsilon(n_t(i))di = \bar{N}_t^F \upsilon'_{\bar{N}_t^F}\left\{E_i\hat{n}_t(i) + \tfrac{1}{2}(1+\eta^{-1})\int_0^1 \hat{n}_t^2(i)di\right\} + hot$$

$$= \bar{N}_t^F \upsilon'_{\bar{N}_t^F}\left\{\hat{n}_t - Q_n^0 - \tfrac{1}{2}\tfrac{\eta+1}{\eta} Q_n^1[\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon}]\right.$$

$$+ \tfrac{1}{2}(1+\eta^{-1})\left([1+\tfrac{\eta+1}{\eta}Q_n^0 Q_n^1]\mathrm{var}_i \hat{n}_t(i) + \hat{n}_t^2 - 2Q_n^0\hat{n}_t\right)$$

$$+ \tfrac{1}{8}(1+\eta^{-1})(\tfrac{\eta+1}{\eta})^2[Q_n^1(\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon})]^2$$

$$- \tfrac{1}{2}(1+\eta^{-1})\tfrac{\eta+1}{\eta}\hat{n}_t Q_n^1(\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon}) + (1+\eta^{-1})[Q_n^0]^2$$

$$\left. - (1+\eta^{-1})\tfrac{\eta+1}{\eta} Q_n^0 Q_n^1 \overline{\Upsilon}\right\} + hot + tip \tag{C.19}$$

$$= \bar{N}_t^F \upsilon'_{\bar{N}_t^F}\left\{(1 - (\tfrac{\eta+1}{\eta})Q_n^0)\hat{n}_t + \tfrac{1}{2}[(\tfrac{\eta+1}{\eta})(1 - Q_n^1 + \tfrac{\eta+1}{\eta}Q_n^0 Q_n^1)]\mathrm{var}_i \hat{n}_t(i) + \right.$$

$$+ \tfrac{1}{8}(\tfrac{\eta+1}{\eta})^3[Q_n^1(\mathrm{var}_i \hat{n}_t(i) - \overline{\Upsilon})]^2 +$$

$$\left. + \tfrac{1}{2}(\tfrac{\eta+1}{\eta})\hat{n}_t^2 - Q_n^0 + (\tfrac{\eta+1}{\eta})[Q_n^0]^2 + \tfrac{\eta+1}{\eta}Q_n^1[\tfrac{1}{2} - (\tfrac{\eta+1}{\eta})Q_n^0]\overline{\Upsilon}\right\} + hot + tip. \blacksquare$$

Lemma 9
We first note that,

$$\mathrm{var}_i(\hat{n}_t(i)) = \mathrm{var}_i(\ln n_t(i) - \ln N_t^F) = \mathrm{var}_i(\ln n_t(i))$$

To replace variability of log labor with variability of log prices we will use the market firms' first order conditions for inputs and combine it with the agents first order condition for labor supply,

$$\frac{(1-\alpha)}{\alpha}\frac{k_t(i)}{n_t(i)} = \frac{W_t(i)}{R_t^k} = \frac{P_t C_t n_t(i)^{1/\eta}}{R_t^k}$$

This implies that the idiosyncrasy in the capital-labor ratio is proportional to log labor input across firms,

$$\ln\frac{k_t(i)}{n_t(i)} = \frac{1}{\eta}\ln n_t(i) + \text{aggregate terms}$$

This then allows us to calculate the cross sectional variation in output,

$$\text{var}_i\left(\ln y_t(i)\right) = \text{var}_i\left(\ln n_t(i)^{1+\alpha\eta^{-1}} + \alpha\ln\frac{k_t(i)}{n_t(i)^{1+\alpha\eta^{-1}}} + \ln A_t\right)$$

$$= \text{var}_i\left[\left(1+\alpha\eta^{-1}\right)\ln n_t(i) + \text{aggregate terms}\right] = \left(1+\alpha\eta^{-1}\right)^2\text{var}_i\left(\ln n_t(i)\right)$$

Finally, using the demand curve for each variety, we have

$$\text{var}_i\left(\ln y_t(i)\right) = \text{var}_i\left(\ln Y_t - \theta(\ln p_t(i) - \ln P_t)\right) = \theta^2\,\text{var}_i\left(\ln p_t(i)\right) \tag{C.20}$$

and it follows that the cross-sectional variation in labor input is proportional to the cross-sectional variation in prices.

$$\text{var}_i\,\hat{n}_t(i) = \left(\tfrac{\theta}{1+\alpha/\eta}\right)^2\text{var}_i\,\ln p_t(i).\ \blacksquare \tag{C.21}$$

Proposition 2.
Combine **Lemmas 8-9** and **Corollary 1** with **Lemma 1** and **Lemma 4** to get

$$EU_t = \log(\overline{X}) - \tfrac{1}{2}(1-\alpha)(c_y^F)^{-1}(\tfrac{\eta+1}{\eta})E(\hat{n}_t^2)$$

$$-\tfrac{1}{2}(1-\alpha)(c_y^F)^{-1}[(\tfrac{\eta+1}{\eta})(1 - Q_n^1 + \tfrac{\eta+1}{\eta}Q_n^0 Q_n^1)]\frac{\theta^2}{(1+\alpha\eta^{-1})^2}\Gamma_3\,\text{var}(\hat{\pi}_t^2)$$

$$-\tfrac{1}{2}(1-\alpha)(c_y^F)^{-1}(1-\Phi)\left[\tfrac{\eta+1}{\eta}[Q_n^0]^2 - Q_n^0 + Q_n^1\tfrac{\eta+1}{\eta}[\tfrac{1}{2} - (\tfrac{\eta+1}{\eta})Q_n^0]\frac{\theta^2}{(1+\alpha\eta^{-1})^2}\overline{\Delta}\right]$$

$$-\tfrac{1}{2}(1-\alpha)(c_y^F)^{-1}(1-\Phi)[(\tfrac{\eta+1}{\eta})(1 - Q_n^1 + \tfrac{\eta+1}{\eta}Q_n^0 Q_n^1)]\frac{\theta^2}{(1+\alpha\eta^{-1})^2}\overline{\Delta}$$

$$+hot + tip.\ \blacksquare$$

Appendix D: Taylor-pricing model with 3-quarter length contracts

In the Taylor-pricing model with 3-quarter length contracts, in each period one third of all firms can change their price. This price will then have to last for 3 quarters before it can be reset. The model is otherwise identical to the baseline model.

The firms maximization problem is now given by

$$\max E_t \sum_{j=0}^{2} Q_{t,t+j} y_{t+j}(i)[B_t(i)\bar{\Pi}^{\omega j} - W_{t+j}(i)N_{t+j}(i)] \tag{D.1}$$

since the price is only fixed for 3 quarters. The first order condition from this optimization problem is

$$\frac{B_t(i)}{P_t} = \frac{\theta}{\theta-1} \frac{E_t \sum_{j=0}^{2} Q_{t,t+j} Y_{t+j} (P_{t+j}/P_t)^{\theta+1} \bar{\Pi}^{-\omega\theta} (MC(i)_{t+j}/P_{t+j})}{E_t \sum_{j=0}^{2} Q_{t,t+j} Y_{t+j} (P_{t+j}/P_t)^{\theta} \bar{\Pi}^{-\omega(\theta-1)}} \tag{D.2}$$

Substituting the equations for the discount factor and the firm-specific marginal cost from the baseline model yields,

$$\left(\frac{B_t}{P_t}\right)^{1+\theta\eta^{-1}} = \frac{\theta}{\theta-1} \frac{E_t \sum_{j=0}^{2}\left[\prod_{i=1}^{j} R_{t+i-1}^{-1} GY_{t+i}\Pi_{t+i}^{1+\theta(1+\eta^{-1})}\right]\left(\frac{C_{t+j}}{A_{t+j}}\right)\left(\frac{Y_{t+j}}{A_{t+j}}\right)^{\eta^{-1}}}{E_t \sum_{j=0}^{2}\left[\prod_{i=1}^{j} R_{t+i-1}^{-1} GY_{t+i}\Pi_{t+i}^{\theta-1}\right]} \tag{D.3}$$

Given these price-setting assumptions, the dynamics of the price level are governed by

$$P_t^{1-\theta} = \frac{1}{3}[B_t^{1-\theta} + B_{t-1}^{1-\theta} + B_{t-2}^{1-\theta}] \tag{D.4}$$

The remaining equations are as in the baseline model.

Steady State:

The steady state real reset price is then equal to

$$\left(\frac{\bar{B}}{P}\right) = \left[\frac{1 + \bar{\Pi}^{\theta-1} + \bar{\Pi}^{2(\theta-1)}}{3}\right]^{\frac{1}{\theta-1}} \tag{D.5}$$

which is greater than one for positive trend inflation. This implies that the output gap is equal to

$$\bar{X} = \left\{ \left(\frac{\bar{B}}{P}\right)^{1+\theta\eta^{-1}} \frac{\sum_{j=0}^{2}\left[\beta^{-1}\bar{\Pi}^{\theta-1}\right]^{j}}{\sum_{j=0}^{2}\left[\beta^{-1}\bar{\Pi}^{\theta(1+\eta^{-1})}\right]^{j}} \right\}^{\frac{1}{1+\eta^{-1}}} \tag{D.6}$$

In the steady state, the cross sectional mean of prices is given by

$$\bar{P}_t(i) = E_i \log P_t(i) = \frac{1}{3}[B_t + B_{t-1} + B_{t-2}] = \bar{b} + \log P_t - \log(\bar{\Pi}) \tag{D.7}$$

This implies that the cross sectional variance of prices in the steady state is equal to

$$\overline{\Delta} = \mathrm{var}_i (\log P_t(i)) = \frac{1}{3}\overline{\pi}^2 + 0 + \frac{1}{3}\overline{\pi}^2 = \frac{2}{3}\overline{\pi}^2 \tag{D.8}$$

Dynamic Equations:

With staggered pricing the log-linearized equations are now as follows. For equation (D.4),

$$\hat{b}_t = \left(\overline{\Pi}^{\theta-1}\right)\left(\hat{\pi}_t - \hat{b}_{t-1}\right) + \left(\overline{\Pi}^{2(\theta-1)}\right)\left(\hat{\pi}_t + \hat{\pi}_{t-1} - \hat{b}_{t-2}\right) \tag{D.9}$$

and for equation (D.3)

$$
\begin{aligned}
(1+\theta/\eta)\hat{b}_t &= G_2^{-1}(\hat{c}_t + \gamma_2\hat{c}_{t+1} + \gamma_2\hat{c}_{t+2}) + G_2^{-1}\eta^{-1}(\hat{y}_t + \gamma_2\hat{y}_{t+1} + \gamma_2\hat{y}_{t+2}) \\
&\quad + (\hat{g}y_{t+1} - \hat{r}_t)[G_2^{-1}(\gamma_2 + \gamma_2^2) - G_1^{1}(\gamma_1 + \gamma_1^2)] + (\hat{g}y_{t+2} - \hat{r}_{t+1})(G_2^{-1}\gamma_2^2 - G_1^{-1}\gamma_1^2) \\
&\quad + \hat{\pi}_{t+1}[(1+\theta(1+\theta/\eta))(\gamma_2 + \gamma_2^2)G_2^{-1} - \theta G_1^{-1}(\gamma_1 + \gamma_1^2)] \\
&\quad + \hat{\pi}_{t+2}[(1+\theta(1+\theta/\eta))\gamma_2^2 G_2^{-1} - \theta\gamma_1^2 G_1^{-1}]
\end{aligned}
\tag{D.10}
$$

Where $\gamma_1 = \lambda\beta^{-1}\overline{\Pi}^{(1-\omega)(\theta-1)}$ and $\gamma_2 = \gamma_1\overline{\Pi}^{(1-\omega)(1+\theta/\eta)}$, and $G_2 = 1 + \gamma_2 + \gamma_2^2$, $G_1 = 1 + \gamma_1 + \gamma_1^2$.

Lemma 4 in Taylor model:
In the baseline model, we derived the cross-sectional price dispersion up to a second order approximation. Because, the pricing contracts are different in the Taylor model, this section derives an equivalent Lemma for the Taylor model.

Lemma 4 (Taylor):
Let $\Xi_t \equiv \Delta_t - \overline{\Delta}$ be the deviation of cross-section price dispersion from its non-stochastic steady state level $\overline{\Delta}$. Then

$$
\begin{aligned}
\Delta_t - \overline{\Delta} &= \frac{1}{3}\Gamma_1[(b_t - \overline{b})^2 + 2\overline{b}(b_t - \overline{b}) \\
&\quad + (b_{t-1} - \overline{b})^2 + (\pi_t - \overline{\pi})^2 - 2(b_{t-1} - \overline{b})(\pi_t - \overline{\pi}) + 2(b_{t-1} - \overline{b})(\overline{b} - \overline{\pi}) - 2(\pi_t - \overline{\pi})(\overline{b} - \overline{\pi}) \\
&\quad + (b_{t-2} - \overline{b})^2 + (\pi_t - \overline{\pi})^2 + (\pi_{t-1} - \overline{\pi})^2 - 2(b_{t-2} - \overline{b})(\pi_t - \overline{\pi}) - 2(b_{t-2} - \overline{b})(\pi_{t-1} - \overline{\pi}) \\
&\quad + 2(\pi_t - \overline{\pi})(\pi_{t-1} - \overline{\pi}) + 2(b_{t-2} - \overline{b})(\overline{b} - 2\overline{\pi}) - 2(\pi_t - \overline{\pi})(\overline{b} - 2\overline{\pi}) - 2(\pi_{t-1} - \overline{\pi})(\overline{b} - 2\overline{\pi})] \\
&\quad + \frac{2}{3}\Gamma_1[(b_t - \overline{b}) + (b_{t-1} - \overline{b}) + (b_{t-2} - \overline{b}) - 2(\pi_t - \overline{\pi}) - (\pi_{t-1} - \overline{\pi})][Q_p^0] + h.o.t
\end{aligned}
\tag{D.11}
$$

where

$$\Gamma_1 = [1 + (\theta-1)Q_p^1(\overline{b} - \overline{\pi} + Q_p^0)]^{-1}. \tag{D.12}$$

Proof:
At any given time the cross sectional dispersion is given by

$$
\begin{aligned}
\Delta_t \quad &= \mathrm{var}_i (\log P_t(i)) = \int_0^1 [\log P_t(i) - \overline{P}_t]di \\
&= \frac{1}{3}[b_t + \log P_t - \overline{P}_t]^2 + \frac{1}{3}[b_{t-1} + \log P_{t-1} - \overline{P}_t]^2 + \frac{1}{3}[b_{t-2} + \log P_{t-2} - \overline{P}_t]^2 \\
&= \frac{1}{3}[b_t + (\log P_t - \overline{P}_t)]^2 + \frac{1}{3}[b_{t-1} - \pi_t + (\log P_t - \overline{P}_t)]^2 + \frac{1}{3}[b_{t-2} - \pi_t - \pi_{t-1} + (\log P_t - \overline{P}_t)]^2
\end{aligned}
$$

$$= \frac{1}{3}[b_t^2 + (b_{t-1} - \pi_t)^2 + (b_{t-2} - \pi_t - \pi_{t-1})^2] + \frac{2}{3}[b_t + (b_{t-1} - \pi_t) + (b_{t-2} - \pi_t - \pi_{t-1})](\log P_t - \overline{P}_t)$$

$$+ (\log P_t - \overline{P}_t)^2$$

$$= \frac{1}{3}[b_t^2 + (b_{t-1} - \pi_t)^2 + (b_{t-2} - \pi_t - \pi_{t-1})^2]$$

$$+ \frac{2}{3}[b_t + (b_{t-1} - \pi_t) + (b_{t-2} - \pi_t - \pi_{t-1})][Q_p^0 + \tfrac{1-\theta}{2} Q_p^1 (\Delta_t - \overline{\Delta})]$$

$$+ [Q_p^0]^2 + (1-\theta)Q_p^0 Q_p^1 (\Delta_t - \overline{\Delta}) + \tfrac{(1-\theta)^2}{4}[Q_p^1]^2 \underbrace{(\Delta_t - \overline{\Delta})^2}_{hot}$$

$$= \frac{1}{3}[(b_t - \overline{b})^2 + 2\overline{b}b_t - \overline{b}^2$$

$$+ (b_{t-1} - \overline{b})^2 + (\pi_t - \overline{\pi})^2 - 2(b_{t-1} - \overline{b})(\pi_t - \overline{\pi}) + 2(b_{t-1} - \overline{b})(\overline{b} - \overline{\pi}) - 2(\pi_t - \overline{\pi})(\overline{b} - \overline{\pi}) + (\overline{b} - \overline{\pi})^2$$

$$+ (b_{t-2} - \overline{b})^2 + (\pi_t - \overline{\pi})^2 + (\pi_{t-1} - \overline{\pi})^2 - 2(b_{t-2} - \overline{b})(\pi_t - \overline{\pi}) - 2(b_{t-2} - \overline{b})(\pi_{t-1} - \overline{\pi})$$

$$+ 2(\pi_t - \overline{\pi})(\pi_{t-1} - \overline{\pi}) + 2(b_{t-2} - \overline{b})(\overline{b} - 2\overline{\pi}) - 2(\pi_t - \overline{\pi})(\overline{b} - 2\overline{\pi}) - 2(\pi_{t-1} - \overline{\pi})(\overline{b} - 2\overline{\pi}) + (\overline{b} - 2\overline{\pi})^2]$$

$$+ \frac{2}{3}[(b_t - \overline{b}) + (b_{t-1} - \overline{b}) + (b_{t-2} - \overline{b}) - 2(\pi_t - \overline{\pi}) - (\pi_{t-1} - \overline{\pi}) + 3\overline{b} - 3\overline{\pi}][Q_p^0 + \tfrac{1-\theta}{2} Q_p^1 (\Delta_t - \overline{\Delta})]$$

$$+ [Q_p^0]^2 + (1-\theta)Q_p^0 Q_p^1 (\Delta_t - \overline{\Delta}) + h.o.t \tag{D.13}$$

The deviation from the steady state is then equal to

$$\Delta_t - \overline{\Delta} = \frac{1}{3}\Gamma_1[(b_t - \overline{b})^2 + 2\overline{b}(b_t - \overline{b})$$

$$+ (b_{t-1} - \overline{b})^2 + (\pi_t - \overline{\pi})^2 - 2(b_{t-1} - \overline{b})(\pi_t - \overline{\pi}) + 2(b_{t-1} - \overline{b})(\overline{b} - \overline{\pi}) - 2(\pi_t - \overline{\pi})(\overline{b} - \overline{\pi})$$

$$+ (b_{t-2} - \overline{b})^2 + (\pi_t - \overline{\pi})^2 + (\pi_{t-1} - \overline{\pi})^2 - 2(b_{t-2} - \overline{b})(\pi_t - \overline{\pi}) - 2(b_{t-2} - \overline{b})(\pi_{t-1} - \overline{\pi})$$

$$+ 2(\pi_t - \overline{\pi})(\pi_{t-1} - \overline{\pi}) + 2(b_{t-2} - \overline{b})(\overline{b} - 2\overline{\pi}) - 2(\pi_t - \overline{\pi})(\overline{b} - 2\overline{\pi}) - 2(\pi_{t-1} - \overline{\pi})(\overline{b} - 2\overline{\pi})] \tag{D.14}$$

$$+ \frac{2}{3}\Gamma_1[(b_t - \overline{b}) + (b_{t-1} - \overline{b}) + (b_{t-2} - \overline{b}) - 2(\pi_t - \overline{\pi}) - (\pi_{t-1} - \overline{\pi})][Q_p^0]$$

$$+ h.o.t$$

where

$$\Gamma_1 = [1 + (\theta - 1)Q_p^1 (\overline{b} - \overline{\pi} + Q_p^0)]^{-1}. \blacksquare \tag{D.15}$$

Proposition 2 (Taylor) in the Taylor model

Given Lemmas 1-5 for the Taylor model, the second order approximation to utility is

$$\Theta_0 + \Theta_1 E(\hat{y}_t^2) + \Theta_2 E(\hat{\pi}_t^2) + \Theta_3 E(\hat{b}_t^2) + \Theta_4 E(\hat{b}_{t-1}\hat{\pi}_t) + \Theta_5 E(\hat{b}_{t-2}\hat{\pi}_t) + \Theta_6 E(\hat{\pi}_t \hat{\pi}_{t-1}) \tag{D.16}$$

where parameters $\Theta_i, i = 0,...,6$ depend on the steady state inflation $\overline{\pi}$ and are given by

$$\Theta_0 = [1-(1-\Phi)(1-(1+\tfrac{1}{\eta})Q_y^0)]\log(\overline{X}) - \tfrac{(1-\Phi)}{(1-\overline{g}_y)}\Big[(1+\tfrac{1}{\eta})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta}Q_y^1[\tfrac{1}{2} - (1+\tfrac{1}{\eta})Q_y^0]\overline{\Upsilon}\Big]$$
$$-\tfrac{1}{2}\tfrac{(1+\tfrac{1}{\eta})(1-\Phi)}{(1-\overline{g}_y)}[\log(\overline{X})]^2 - \tfrac{1}{2}\tfrac{1}{(1-\overline{g}_y)}[Q_y^1(\tfrac{1}{\theta}-1) - Q_y^1 + (1+\tfrac{1}{\eta})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)]\theta^2\overline{\Delta}, \tag{D.17}$$

$$\Theta_1 = -\tfrac{1}{2}\tfrac{(1+\tfrac{1}{\eta})}{(1-\overline{g}_y)}, \tag{D.18}$$

$$\Theta_2 = -\tfrac{1}{2}\tfrac{1}{(1-\overline{g}_y)}\theta^2\Big\{[Q_y^1(\tfrac{1}{\theta}-1) + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)] - (1+\tfrac{1}{\eta})\tfrac{\theta-1}{\theta}Q_y^1\log(\overline{X})\Big\}\Gamma_1, \tag{D.19}$$

$$\Theta_3 = -\tfrac{1}{2}\tfrac{1}{(1-\overline{g}_y)}\theta^2\Big\{[Q_y^1(\tfrac{1}{\theta}-1) + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)] - (1+\tfrac{1}{\eta})\tfrac{\theta-1}{\theta}Q_y^1\log(\overline{X})\Big\}\Gamma_1, \tag{D.20}$$

$$\Theta_4 = \tfrac{2}{3}\tfrac{1}{(1-\overline{g}_y)}\theta^2\Big\{[Q_y^1(\tfrac{1}{\theta}-1) + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)] - (1+\tfrac{1}{\eta})\tfrac{\theta-1}{\theta}Q_y^1\log(\overline{X})\Big\}\Gamma_1, \tag{D.21}$$

$$\Theta_5 = \tfrac{1}{3}\tfrac{1}{(1-\overline{g}_y)}\theta^2\Big\{[Q_y^1(\tfrac{1}{\theta}-1) + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)] - (1+\tfrac{1}{\eta})\tfrac{\theta-1}{\theta}Q_y^1\log(\overline{X})\Big\}\Gamma_1, \tag{D.22}$$

$$\Theta_6 = -\tfrac{1}{3}\tfrac{1}{(1-\overline{g}_y)}\theta^2\Big\{[Q_y^1(\tfrac{1}{\theta}-1) + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)] - (1+\tfrac{1}{\eta})\tfrac{\theta-1}{\theta}Q_y^1\log(\overline{X})\Big\}\Gamma_1, \tag{D.23}$$

$$\Phi = -\log(\tfrac{\theta-1}{\theta}). \tag{D.24}$$

Proof:

From **Lemma 5**, we have $E_i \tilde{y}_t(i) = \tilde{y}_t - Q_y^0 - \tfrac{1}{2}\tfrac{\theta-1}{\theta}Q_y^1\big(\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon}\big) + hot$. Therefore, we can express the approximation from the disutility of labor supply given in **Lemma 2** as follows:

$$\int_0^1 \tilde{\upsilon}(Y_t(i))di \approx \overline{Y}_t^F \tilde{\upsilon}'_{\overline{Y}_t^F}\left\{ E_i\tilde{y}_t(i) + \tfrac{1}{2}(1+\eta^{-1})\int_0^1 \tilde{y}_t^2(i)di\right\}$$
$$= \overline{Y}_t^F \tilde{\upsilon}'_{\overline{Y}_t^F}\left\{ E_i\tilde{y}_t(i) + \tfrac{1}{2}(1+\eta^{-1})\big(\text{var}_i\, \hat{y}_t(i) + [E_i\tilde{y}_t(i)]^2\big)\right\}$$
$$= \overline{Y}_t^F \tilde{\upsilon}'_{\overline{Y}_t^F}\Big\{(1-(1+\eta^{-1})Q_y^0)\tilde{y}_t \tag{D.25}$$
$$+ \tfrac{1}{2}[Q_y^1\theta^{-1} - Q_y^1 + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)]\,\text{var}_i\, \hat{y}_t(i) +$$
$$+ \tfrac{1}{2}(1+\eta^{-1})(\tfrac{1}{4}(\tfrac{\theta-1}{\theta})^2[Q_y^1(\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon})]^2 - \tfrac{\theta-1}{\theta}\tilde{y}_t Q_y^1[\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon}])$$
$$+ \tfrac{1}{2}(1+\eta^{-1})\tilde{y}_t^2 + (1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta}Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon}\Big\} + hot + tip$$

As Woodford (2003) shows, $\tilde{\upsilon}'_{\overline{Y}_t^F} = u'_{\overline{C}_t^F}(1-\Phi)$, where (with log utility) $\Phi = -\ln(\tfrac{\theta-1}{\theta})$ which follows from the intra-temporal condition for labor supply. Hence,

$$\int_0^1 \tilde{\upsilon}(Y_t(i))di = (1-\overline{g}_y)^{-1}\hat{C}_t^F u'_{\hat{C}_t^F}(1-\Phi)\Big\{(1-(1+\eta^{-1})Q_y^0)\tilde{y}_t + \tfrac{1}{2}(1+\eta^{-1})\tilde{y}_t^2 +$$
$$+ \tfrac{1}{2}[Q_y^1\theta^{-1} - Q_y^1 + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)]\,\text{var}_i\, \hat{y}_t(i) \tag{D.26}$$
$$+ \tfrac{1}{2}(1+\eta^{-1})(\tfrac{1}{4}(\tfrac{\theta-1}{\theta})^2[Q_y^1(\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon})]^2 - \tfrac{\theta-1}{\theta}\tilde{y}_t Q_y^1[\text{var}_i\, \hat{y}_t(i) - \overline{\Upsilon}])$$
$$+ (1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta}Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon}\Big\} + hot + tip$$

As in **Lemma 1**, $\overline{C}_t^F u'_{\overline{C}_t^F} = 1$. Using equation (42) and **Lemma 4 (Taylor)** we replace the cross-section dispersion of output in (A.17) with the cross-sectional dispersion of prices. Using $E(\pi_t - \overline{\pi}) = 0$, we compute the expected value of Ξ_t:

$$E(\Xi_t) = \Gamma_1\{E(\hat{b}_t^2) + E(\hat{\pi}_t^2) - \tfrac{4}{3}E(\hat{b}_{t-1}\hat{\pi}_t) - \tfrac{2}{3}E(\hat{b}_{t-2}\hat{\pi}_t) + \tfrac{2}{3}E(\hat{\pi}_t\hat{\pi}_{t-1})]$$

$$+\underbrace{\Gamma_1[2(Q_p^0 + \overline{b} - \overline{\pi})E(\hat{b}_t) - 2(Q_p^0 + \overline{b} - \frac{5}{3}\overline{\pi})E(\hat{\pi}_t)]}_{hot} \tag{D.27}$$

where

$$\Gamma_1 = [1 + (\theta - 1)Q_p^1(\overline{b} - \overline{\pi} + Q_p^0)]^{-1}. \tag{D.28}$$

Note that from the definition of Ξ_t, we have

$$E(\Delta_t) = E(\Xi_t) + \overline{\Delta}. \tag{D.29}$$

Since all variables are stationary and $\hat{y}_t = \tilde{y}_t - \log(\overline{X})$, we can compute the expected per period utility as follows

$$
\begin{aligned}
EU_t \quad &\approx E\Big\{\tilde{c}_t \\
&-(1-g)^{-1}(1-\Phi)(1-(1+\eta^{-1})Q_y^0)\tilde{y}_t \\
&-\tfrac{1}{2}(1-g)^{-1}(1-\Phi)(1+\eta^{-1})\tilde{y}_t^2 \\
&-\tfrac{1}{2}(1-g)^{-1}[Q_y^1\theta^{-1} - Q_y^1 + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)]\theta^2\Delta_t \\
&+\tfrac{1}{2}(1-g)^{-1}(1+\eta^{-1})\tfrac{\theta-1}{\theta}Q_y^1\theta^2\tilde{y}_t\Xi_t \\
&-(1-g)^{-1}(1-\Phi)\Big[(1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta}Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon}\Big]\Big\} + hot + tip \\
&= [1-(1-g)^{-1}(1-\Phi)(1-(1+\eta^{-1})Q_y^0)]\log(\overline{X}) \\
&\quad -(1-g)^{-1}(1-\Phi)\Big[(1+\eta^{-1})[Q_y^0]^2 - Q_y^0 + \tfrac{\theta-1}{\theta}Q_y^1[\tfrac{1}{2} - (1+\eta^{-1})Q_y^0]\overline{\Upsilon}\Big] \\
&\quad -\tfrac{1}{2}(1-g)^{-1}(1+\eta^{-1})\{(1-\Phi)[\log(\overline{X})]^2 + E(\hat{y}_t^2)\} \\
&\quad -\tfrac{1}{2}(1-g)^{-1}[Q_y^1\theta^{-1} - Q_y^1 + (1+\eta^{-1})(1+\tfrac{\theta-1}{\theta}Q_y^0 Q_y^1)]\theta^2 E(\Delta_t) \\
&\quad + hot + tip
\end{aligned}
$$

After using (A.18) and (A.20) and collecting terms, one arrives at equation (D.16).∎