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INVENTORIES, STOCK-OUTS, AND PRODUCTION SMOOTHING

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ABSTRACT

If stock-outs are ignored and if demand shocks are additive, then optimal behavior requires that the marginal cost of production (MC) be equated with the expected marginal revenue of increasing expected sales by one unit (EMR). However, with more general demand shocks (and still ignoring stock-outs), the excess of MC over EMR has the same sign as the covariance of the slope of the demand curve and the marginal valuation of inventory. The equality of EMR and MC is also broken by taking account of stock-outs, even if demand shocks are additive.

If there is a production lag, then taking account of stock-outs implies that optimal behavior will be characterized by production smoothing even if the cost of production is linear. Two alternative definitions of production smoothing are presented and optimal behavior in the presence of stock-outs displays each type of smoothing.

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It is a well-known proposition that a firm producing a storable good under conditions of increasing marginal cost will tend to smooth the time profile of its production relative to the time profile of its sales. The incentive to smooth production arises from the fact that the cost function is a convex function of the level of production. For a given average level of production, average costs can be reduced by reducing the variation in production. Of course, if the cost function is linear in the level of production, then this incentive to smooth production disappears. However, we will demonstrate in this paper that if the possibility of stock-outs is explicitly incorporated into the firm's dynamic optimization problem (i.e., if we impose a non-negativity constraint on inventories), and if there is a lag in production, then optimal behavior can be characterized by production smoothing even if the cost function is linear.

Recently Blinder (1982) has shown that optimal behavior requires that the firm set its price and its level of production so as to equate the marginal cost of production (MC) with the expected marginal revenue (EMR) from increasing (expected) sales by one unit. Blinder's derivation of this result depends crucially on two assumptions of his model: (1) there are no stock-outs, i.e., if demand exceeds available inventory, the firm is allowed to sell short output and to cover the sale in a future period;[1] and (2) demand shocks are additive. Maintaining assumption (1) but relaxing the assumption of additive demand shocks, we show that this equality of MC and EMR does not hold in general. Indeed we demonstrate that the excess of MC over EMR has the same sign as the covariance of the marginal valuation of next period's inventory and the

1. Recently, Schutte (1983) has shown that there is a negative level of inventory in the stationary solution to Blinder's model. Schutte goes on to argue that accounting for stock-outs introduces substantial complications which are not readily handled in Blinder's model.

slope of the demand curve. For additive demand shocks, this covariance is zero and MC = EMR as in Blinder. We also show that if assumption (1) is relaxed but assumption (2) is maintained, we once again find that MC is not equal to EMR.

We present a simple stochastic model of the production and pricing behavior of a firm in Section I. This model explicitly incorporates stock-outs, but we show in Section I.A how this model can be used to analyze behavior if the firm is allowed to sell short its output, as in Blinder (1982). We then derive an expression for the excess of MC over EMR. In Section II we return to the model with stock-outs and show that even with additive demand shocks, MC > EMR.

In Sections III and IV we assume that the demand curve is perfectly inelastic and we examine two alternative definitions of production smoothing. In Section III we show that if beginning-of-period inventory is increased by one unit, the firm reduces its level of production by less than one unit. This is our first definition of production smoothing. In Section IV we present a second definition of production smoothing: the variance of sales is greater than the variance of production. We then show that if behavior is characterized by the first definition of production smoothing, then it is also characterized by the second definition. The concluding remarks in Section V include a brief discussion of the pitfalls of trying to infer the stock-out history of a firm from observations on the stock-out experience of an individual customer of that firm.

I. The Model

Consider a firm which produces a storable good.[2] Let x_t denote the physical stock of the good held in the firm's inventory at the beginning of period t. The demand for the firm's product, q_t, is a stochastic function of the price, p_t,

$$q_t = h(p_t, \varepsilon_t) \tag{1}$$

where ε_t is an i.i.d. random variable with continuous density function $f(\varepsilon_t)$. We assume that $h_p \leq 0$ and that $h_\varepsilon > 0$.

At the beginning of period t, before observing the realization of the random variable ε_t, the firm must decide how much output, y_t, to produce. This output, which costs $c(y_t)$, takes one period to produce and hence is not available for sale until period t + 1. We suppose that $c'(y) > 0$ and $c''(y) \geq 0$. The assumption that production does not take place instantaneously is important for the results concerning production smoothing in the presence of a linear cost function, and the implications of relaxing this assumption will be discussed in section III.A. The result that, in general, marginal cost is not equal to expected marginal revenue continues to hold whether or not there is a production lag. (See footnote 7 for further details.)

In addition to setting y_t the firm also sets the price p_t before observing the realization of ε_t. After the firm chooses values for y_t and p_t, the realization of ε_t is observed, and the firm sells goods out of inventory.[3] If

2. Gould (1978) analyzes inventories and stock-outs for a firm which produces a perishable good. In his model, "there are no intertemporal dynamic links in inventory planning by the firm." However, in this paper we analyze inventories of a storable good so that the firm faces an intertemporal optimization problem.

demand exceeds available inventory, the firm sells its entire inventory. Unsatisfied demand is not backlogged; it simply disappears.[4] Letting s_t denote the quantity of goods sold in period t, we have

$$s_t = \min(q_t, x_t) \tag{2}$$

The inventory accumulation equation is

$$x_{t+1} = x_t + y_t - s_t. \tag{3}$$

Substituting (1) into (2) and the result into (3), we have

$$x_{t+1} = y_t + \max(0, x_t - h(p_t, \varepsilon_t)) \tag{4}$$

Thus, if there is a stock-out in period t (i.e. $h(p_t, \varepsilon_t) > x_t$), the inventory carried into period t+1 is simply y_t. In the absence of a stock-out in period t, the inventory carried into period t+1 is $x_t + y_t - h(p_t, \varepsilon_t)$.

It will be convenient to let ω_t represent the maximum value of the demand shock ε_t which does not lead to a stock-out in period t. If $\varepsilon_t \leq \omega_t$, the firm is able to meet demand; if $\varepsilon_t > \omega_t$, there is a stock-out. More formally,

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3. This formulation differs from that in Reagan (1982) in which the firm observes the realization of ε_t before making its price and output decisions. In Reagan's model, the firm can raise price during periods of high demand shocks so that demand does not exceed available inventory at the market price.

4. Since we have assumed that ε_t is i.i.d., we are ignoring the fact that customers who are rationed in one period may have a higher demand in the following period. We might expect such an effect for a durable good. On the other hand, customers who are rationed might choose to take their business elsewhere next period. In this paper, we simply ignore these two opposing effects.

$$\omega_t = \omega(x_t, p_t) \qquad \text{s.t.} \qquad x_t = h(p_t, \omega_t) \tag{5}$$

From the properties of h(.,.), it is clear that $\omega_x > 0$ and $\omega_p \geq 0$.

The expected value of sales, s_t, is a function of both price, p_t, and available inventory, x_t. Letting $H(p_t, x_t)$ denote the expected value of s_t, we have

$$H(p_t, x_t) = \int_{-\infty}^{\omega_t} h(p_t, \varepsilon_t) f(\varepsilon_t) d\varepsilon_t + x_t[1-F(\omega_t)] \tag{6}$$

where F( ) is the cumulative distribution function of ε_t. Observe that

$$H_p(p_t, x_t) = \int_{-\infty}^{\omega_t} h_p(p_t, \varepsilon_t) f(\varepsilon_t) d\varepsilon_t \leq 0 \tag{7}$$

so that the expected value of s_t is a non-increasing function of the price p_t.

Now that we have described the economic environment in which the firm operates, we will discuss the firm's optimization problem. We assume that the firm is risk-neutral and maximizes the expected present value of its cash flow. Let β be the one-period discount factor where $0 < \beta < 1$. The value of the firm can be written as a function of its inventory level and satisfies the Bellman equation

$$V(x_t) = \max_{p_t, y_t} \{p_t H(p_t, x_t) - c(y_t) + \beta \int_{-\infty}^{\infty} V(x_{t+1}) f(\varepsilon_t) d\varepsilon_t\} \tag{8}$$

Substituting (4) into (8), and suppressing the time subscript, we obtain

$$V(x) = \max_{p,y}\{pH(p,x) - c(y) + \beta \int_{-\infty}^{\omega(x,p)} V(x+y-h(p,\varepsilon))f(\varepsilon)d\varepsilon$$

$$+ \beta[1 - F(\omega(x,p))]V(y)\} \tag{9}$$

Let D_y and D_p denote the derivatives, with respect to y and p, respectively, of the expression in curly brackets on the right hand side of (9). Optimality requires that D_y and D_p each be set equal to zero. Therefore,

$$D_y = -c'(y) + \beta\int_{-\infty}^{\omega} V'(x+y-h(p,\varepsilon))f(\varepsilon)d\varepsilon + \beta[1-F(\omega)]V'(y) = 0 \tag{10}$$

$$D_p = pH_p + H - \beta\int_{-\infty}^{\omega} V'(x+y-h(p,\varepsilon))h_p(p,\varepsilon)f(\varepsilon)d\varepsilon = 0 \tag{11}$$

According to (10), at the optimal level of production, the marginal cost, $c'(y)$, is equal to the expected present value of an additional unit of inventory at the beginning of period t+1. According to (11), the price is chosen so that the extra expected current revenue obtained from reducing the price is equal to the valuation of the reduction in next period's inventory due to reducing the price this year. Note that a reduction in the current price will increase current sales and reduce next period's inventory only if $\varepsilon_t < \omega_t$, i.e., only if there is not a stock-out in the current period.

Thus far we have ignored any inventory holding costs other than the opportunity cost of funds. It turns out that the results of this paper are not affected if we modify our model to include an inventory holding cost function $b(x_t)$ with $b' \geq 0$ and $b'' \geq 0$. Taking account of these holding costs, the Bellman equation in (8) must be rewritten as

$$V(x_t) = \max_{p_t, y_t} \{p_t H(p_t, x_t) - c(y_t) - b(x_t) + \beta \int_{-\infty}^{\infty} V(x_{t+1}) f(\varepsilon_t) d\varepsilon_t\}$$

However, the optimality conditions in (10) and (11) continue to hold without modification. More precisely, the value function $V(x_t)$ is affected by the introduction of holding costs, but relations involving the value function in (10) and (11) continue to hold as written. Henceforth, we will ignore any holding costs.

Before analyzing further the first-order conditions in (10) and (11), we digress to a discussion of the case in which stock-outs can be ignored.

I. A. Short Sales and Backlogs

In many inventory models, stock-outs are not explicitly modeled. Rather it is assumed (implicitly) that if demand exceeds available inventory, the firm in effect sells short and covers the sale in a future period. (See, for example, Blinder (1982)). Thus sales revenue is simply equal to price multiplied by demand. Formally, the first-order conditions derived above can be applied to this situation simply by setting ω equal to ∞. In effect, $\omega = \infty$ means that no values of ε lead to a stock-out.[5]

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5. If we allow short sales (by ignoring the non-negativity constraint on inventories), then we must introduce some feature into the model which prevents inventories from being run down to minus infinity. Typically, one introduces a convex cost of being away from some "target" level of inventories, as in Blinder (1982) or in Feldstein and Auerbach (1976). Letting $g(x)$ be a strictly convex cost of being away from some (implicitly defined) target level of inventories, the Bellman equation is $V(x) = \max_{p,y}\{\pi(p,y,x) + \beta E[V(x+y-h(p,\varepsilon))]\}$ where

Setting ω equal to ∞ in (10) and (11) yields the following first-order conditions for the situation in which stock-outs are ignored

$$c'(y) = \beta E[V'(x+y-h(p,\varepsilon))] \tag{12}$$

$$pH_p + H = \beta E[V'(x+y-h(p,\varepsilon))h_p(p,\varepsilon)] \tag{13}$$

where E[ ] denotes the expectation over ε. Now divide both sides of (13) by H_p and subtract the resulting equation from (12) to obtain

$$c'(y) - (p + \frac{H}{H_p}) = \frac{\beta}{H_p}\{H_p E[V'(x+y-h(p,\varepsilon))] - E[V'(x+y-h(p,\varepsilon))h_p(p,\varepsilon)]\} \tag{14}$$

Observe that if $\omega = \infty$ in (7), we obtain $H_p = E[h_p(p,\varepsilon)]$. Therefore, the term in curly brackets on the right-hand side of (14) is equal to $-\mathrm{Cov}[V'(x+y-h(p,\varepsilon)),h_p(p,\varepsilon)]$. Also note that $p + \frac{H}{H_p}$ is the change in expected revenue which accompanies a reduction in price which increases expected sales by one unit. Letting EMR denote this expected marginal revenue, we now rewrite (14) as

$$c'(y) - EMR = \frac{-\beta}{H_p}\mathrm{Cov}[V'(x+y-h(p,\varepsilon)),h_p(p,\varepsilon)] \tag{15}$$

Suppose that $h_p < 0$ for at least some interval of ε so that $H_p < 0$. Then $c'(y) - EMR$ has the same sign as $\mathrm{Cov}[V'(x+y-h(p,\varepsilon)),h_p(p,\varepsilon)]$.

$\pi(p,y,x) = pE[h(p,\varepsilon)] - c(y) - g(x)$. Note that the first-order conditions are identical to those in (12) and (13). Also, note that if $h_{pp} \equiv 0$, $c'' \geq 0$ and $g'' > 0$, then $\pi(p,y,x)$ is concave in p and y and strictly concave in x. Also, the transition equation $(x_{t+1} = x_t + y_t - h(p_t,\varepsilon_t))$ is linear x_t, y_t, and p_t. Therefore the value function is strictly concave. (See Lucas and Prescott (1971)).

We will consider two special cases of demand analyzed by Zabel (1972) in order to interpret (15): (1) additive demand shocks, and (2) multiplicative demand shocks. First suppose that demand shocks are additive, i.e., $h(p,\varepsilon)$ can be written as $h^*(p) + \varepsilon$. In this case $h_p(p,\varepsilon)$ is independent of ε, and the covariance in (15) is zero. Thus, if demand shocks are additive and if we ignore stock-outs, we obtain $c'(y) = EMR$ as in Blinder (1982).

Now suppose that demand shocks are multiplicative so that $h(p,\varepsilon)$ can be written as $h(p,\varepsilon) = h^*(p)\varepsilon$. In this case, $h_p(p,\varepsilon) = h^{*\prime}(p)\varepsilon < 0$ is a decreasing function of ε. Note that the inventory carried into next period, $x+y-h(p,\varepsilon)$ is a decreasing function of ε. Therefore, provided that $V(\ )$ is concave,[6] $V'(x+y-h(p,\varepsilon))$ is an increasing function of ε. Hence, the covariance in (15) is negative and we obtain $c'(y) < EMR$.

More generally, we find that if $h_{p\varepsilon} < 0$ for all ε, then $c'(y) < EMR$; if $h_{p\varepsilon} > 0$ for all ε, then $c'(y) > EMR$. If the demand shock is additive, we obtain the simple result that $EMR = c'(y)$.

II. Comparison of Marginal Revenue and Marginal Cost in the Presence of Stock-outs

We have shown in section I that if short sales are allowed, the equality of $c'(y)$ and EMR holds for additive demand shocks. We will show in this section that even with additive demand shocks, the equality of $c'(y)$ and EMR does not hold if we take account of stock-outs (i.e., do not implicitly allow short sales).

6. See footnote 5.

We proceed by rewriting (10) and (11) in more convenient forms. Equation (10) is clearly equivalent to

$$c'(y) = \beta E[V'(x+y-s)] = \beta E[V'(x+y-h(p,\varepsilon))|\varepsilon \le \omega]F(\omega) \qquad (16)$$

$$+ \beta V'(y)(1 - F(\omega))$$

Now assume that demand shocks are additive, $(h(p,\varepsilon) = h^*(p) + \varepsilon)$ and observe from (7) that $H_p = h^{*\prime}(p)F(\omega)$. Dividing both sides of (11) by H_p and recalling that the expected marginal revenue, EMR, is equal to $p + \frac{H}{H_p}$, we obtain

$$\text{EMR} = \beta E[V'(x+y-h(p,\varepsilon))|\varepsilon \le \omega] \qquad (17)$$

Comparing (16) and (17), we see that the marginal cost, $c'(y)$, is equated with the discounted unconditional expected marginal valuation of next period's inventory; however, expected marginal revenue, EMR, is equated with the discounted conditional expected marginal valuation of next period's inventory, where we condition on not stocking-out this period. The reason for this difference is that an increase in production will increase next period's inventory regardless of whether or not there is stock-out this period. However, reducing the current price to increase current expected sales by one unit will increase current sales and reduce next period's inventory only if there is no stock-out this period. Thus, we equate EMR with the conditional expected marginal valuation of next period's inventory.

To compare expected marginal revenue and marginal cost, we subtract (17) from (16) to obtain

$$c'(y) - EMR = \beta\,(1 - F(\omega))\{V'(y) - E[V'(x+y-h(p,\varepsilon))\,|\,\varepsilon \le \omega]\} \tag{18}$$

Provided that the value function is concave, $V'(y) > E[V'(x+y-h(p,\varepsilon))|\varepsilon \le \omega]$ (since $y < x+y-h(p,\varepsilon)$ if $\varepsilon < \omega$) so that marginal cost exceeds marginal revenue. The intuition for this result is that there are two ways to increase the expected value of next period's inventory by one unit: (1) increase production by one unit, or (2) raise price by $-H_p^{-1}$. Increasing production by a unit raises current costs by $c'(y)$ and increases next period's inventory for all realizations of the demand shock. Raising the price by $-H_p^{-1}$ imposes a current cost of EMR and raises next period's inventory only if the demand shock is small enough not to cause a stock-out. However, these are precisely the situations in which the extra unit of inventory has the least value. Therefore, the marginal benefit of increasing production by one unit exceeds the marginal benefit of raising price by $-H_p^{-1}$. Therefore, equating marginal benefits and marginal costs of each action requires that $c'(y)$ be greater than EMR.[7]

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7. Recall that we have assumed that production in period t is not available for sale until period t+1. Alternatively, if we assume that production in period t is available for sale in period t, then a stock-out will occur in period t if $\varepsilon_t > \omega^*_t$ where $\omega^*_t = \omega^*(x+y,p)$ is defined so that $x_t + y_t = h(p_t,\omega^*_t)$. In this case, the Bellman equation is

$$V(x) = \max_{p,y}\{pH(p,x+y) - c(y) + \beta\int_{-\infty}^{\omega^*} V(x+y-h(p,\varepsilon))f(\varepsilon)d\varepsilon + \beta[1 - F(\omega^*)]V(0)\} \tag{F7.1}$$

where $H(p,x+y) = \int_{-\infty}^{\omega^*} h(p,\varepsilon)f(\varepsilon)d\varepsilon + (x+y)[1 - F(\omega^*)]$. Observing that $H_y = 1 - F(\omega^*)$, the first-order conditions are

$$c'(y) = p[1 - F(\omega^*)] + \beta E[V'(x+y-h(p,\varepsilon))|\varepsilon \le \omega^*]F(\omega^*) \tag{F7.2}$$

$$pH_p + H = \beta E[V'(x+y-h(p,\varepsilon))h_p(p,\varepsilon)|\varepsilon \le \omega^*]F(\omega^*) \tag{F7.3}$$

III. Production Smoothing with Perfectly Inelastic Demand

In this section we present the first of two alternative definitions of production smoothing and show that if the demand curve is perfectly inelastic, then optimal behavior is characterized by production smoothing. Our first definition of production smoothing is that the optimal production rule as a function of initial inventory is such that $-1 < \frac{dy}{dx} < 0$. That is, an increase in initial inventory leads to a reduction in production. However, this decrease in production is smaller than the increase in initial inventory.

We assume that the demand curve is perfectly inelastic, $h_p \equiv 0$, and that the price of the good, p, is constant over time. The firm's only decision variable is the level of production, and the optimal level of production satisfies the first-order condition (10). Totally differentiating (10) with respect to x and y we obtain

$$\frac{dy}{dx} = -\frac{D_{yx}}{D_{yy}} \tag{19}$$

where

$$D_{yx} \equiv \frac{\partial D_y}{\partial x} = \beta \int_{-\infty}^{\omega} V''(x+y-h(p,\varepsilon))f(\varepsilon)d\varepsilon \tag{19a}$$

If we assume that demand shocks are additive, then $H_p = h_p F(\omega^*)$. Recalling that $EMR = p + \frac{H}{H_p}$, we can rewrite as (F7.3)

$$EMR = \beta E[V'(x+y-h(p,\varepsilon)) | \varepsilon \leq \omega^*] \tag{F7.4}$$

Substituting (F7.4) into (F7.2) yields

$$c'(y) - EMR = (1 - F(\omega^*))(p - EMR) \tag{F7.5}$$

This, if there is a positive probability of a stock $(1 - F(\omega^*))$, we see that c'(y) also exceeds EMR under the alternative assumption that production is immediately available for sale.

and

$$D_{yy} = \frac{\partial D_y}{\partial y} = -c''(y) + \beta \int_{-\infty}^{\omega} V''(x+y-h(p,\varepsilon))f(\varepsilon)d\varepsilon + \beta[1-F(\omega)]V''(y) \quad (19b)$$

Since, as shown in the next paragraph, the value function is concave, implying that $V'' < 0$, we see that $D_{yx} < 0$ and $D_{yy} < 0$. Therefore, from (19), $\frac{dy}{dx} < 0$. Now substitute (19a,b) into (19) to obtain

$$\frac{dy}{dx} = -\{1 + \frac{c''(y) - \beta V''(y)(1-F(\omega))}{D_{yy}}\} \quad (20)$$

The term in curly brackets in (20) is less than one if $c'' > 0$ or if there is a positive probability of a stock-out, $1-F(\omega)$. Therefore, provided that $1-F(\omega)$ is positive, we have

$$-1 < \frac{dy}{dx} < 0 \quad (21)$$

even if $c'' \equiv 0$. That is, even with a linear cost function, production smoothing will occur if we take account of stock-outs.[8] On the other hand, if short sales are permitted as in section I.A, then linearity of $c(y)$ implies $\frac{dy}{dx} = -1$ (Formally, this result is obtained by setting ω equal to ∞ in (20), as explained in section I).

We have used the fact that the value function is strictly concave, i.e., if $0 < \alpha < 1$ and if $x^A \neq x^B$, then

$$V(\alpha x^A + (1-\alpha)x^B) > \alpha V(x^A) + (1-\alpha)V(x^B) \quad (22)$$

To show that (22) holds, consider a firm with inventory $\alpha x^A + (1-\alpha)x^B$ and suppose that it operates two stores, A and B. Store A has an initial inventory

8. See Karlin and Scarf (1958) for a different derivation and presentation of this result.

of αX^A and store B has an initial inventory of $(1-\alpha)x^B$. In period t, the firm produces $\alpha y_t^A + (1-\alpha)y_t^B$ where y_t^i is the production in period t of a firm with initial inventory x^i, i = A,B. If, in each period, the firm ships αy_t^A to store A and $(1-\alpha)y_t^B$ to store B and if it directs a fraction α of its customers to store A and fraction $1-\alpha$ to store B, then the present value of net revenues from the two stores is greater than or equal to $\alpha V(x^A) + (1-\alpha)V(x^B)$, with strict inequality if $c'' > 0$. To establish the strict inequality in (22) in the case in which $c'' \equiv 0$, we note that if one store (say A) stocks out and the other store (B) does not stock out, the firm can increase the present value of its cash flow by transferring a unit of inventory from store B and selling it at store A. Thus, the value function is strictly concave.

This production smoothing result can be easily understood with the aid of Figure I. The upper panel of Figure I displays next period's inventory x_{t+1} as a function of the current demand shock ε_t, given the current value of inventory and the value of current production. For example, if the current inventory is x^*_t and the optimal value of current production is y^*_t, then the solid line shows next period's inventory as a function of ε_t. The bottom panel in Figure I shows the marginal valuation of next period's inventory as a function of ε_t. This relation is based on the fact that the marginal valuation of next period's inventory is a decreasing function of next period's inventory.

Now consider an increase in current inventory to $x^*_t + \delta$. If there is no production smoothing (i.e., $\frac{dy}{dx} = -1$), then current production falls to $y^*_t - \delta$ and the relation between x_{t+1} and ε_t is given by the piecewise linear function through points ABCD. The marginal valuation of next period's inventory is shown in the lower panel of Figure I as the curve through points A'B'C'D'.

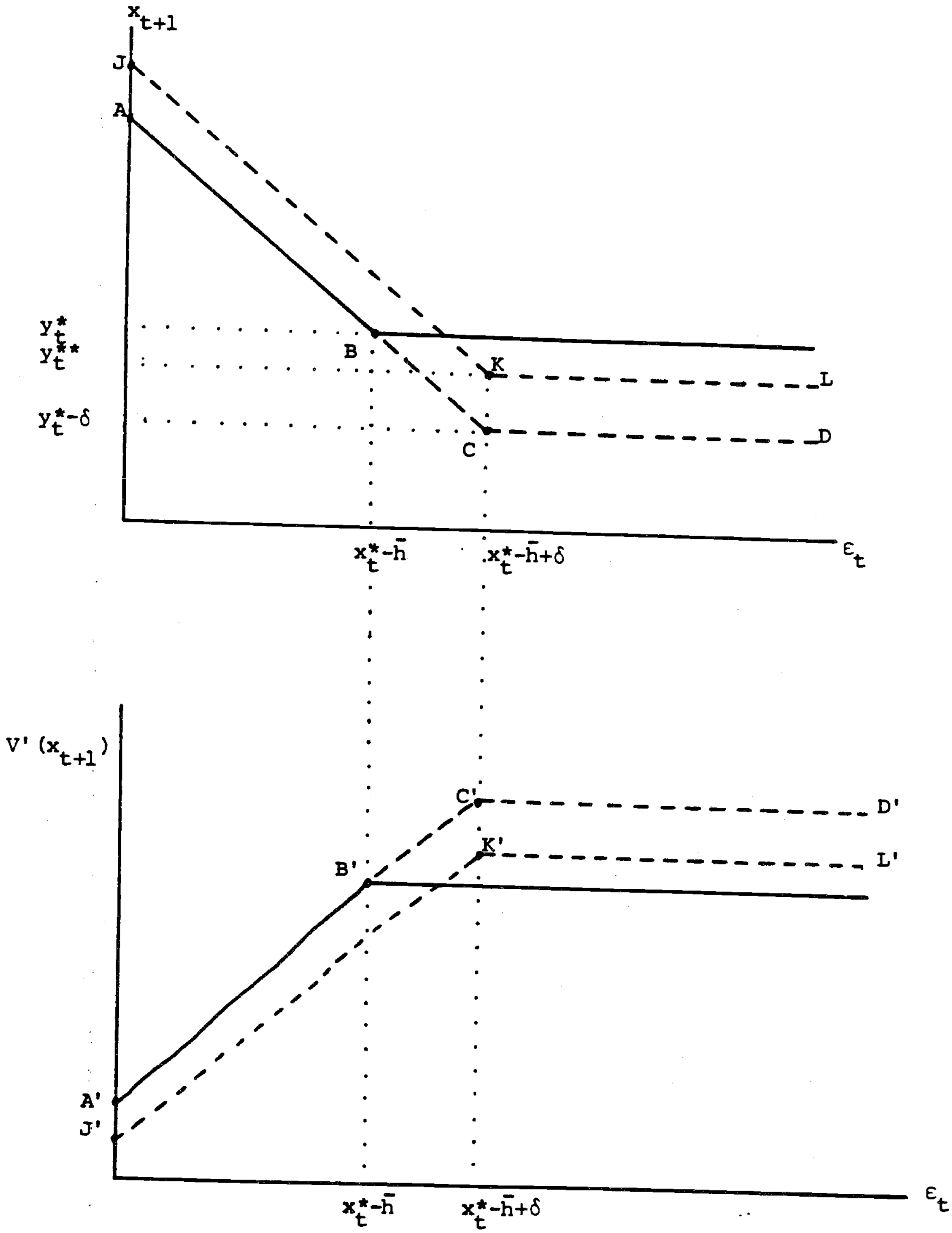

Figure I

$h(p_t, \varepsilon_t) = \bar{h} + \varepsilon_t$

Since for any given ε_t, the marginal valuation of next period's inventory along A'B'C'D' is greater than or equal to the marginal valuation along the solid line, the expected marginal valuation of inventory is greater along A'B'C'D' than along the solid line. However, if the cost function is linear, optimality requires that the new level of production be such that the expected marginal valuation of inventories remains unchanged (equal to the fixed value of $c'(y)$) when x_t changes. Therefore, $\frac{dy}{dx} = -1$ cannot characterize optimal behavior.

Now suppose that when inventory rises to $x^*_t + \delta$, the level of production falls to y^{**}_t, $y^*_t - \delta < y^{**}_t < y^*_t$. This situation is illustrated by the piecewise linear function through points JKL in the upper panel. For low realizations of ε_t, next period's inventory is higher along JKL than along the solid line; for high realizations of ε_t, next period's inventory is lower along JKL than along the solid line. For some appropriate value of y^{**}_t between $y^*_t - \delta$ and y^*_t, the expected marginal valuation of inventory is the same along J'K'L' as along the solid line in the bottom panel of Figure I. For such a y^{**}_t, the required equality of $c'(y)$ and expected marginal valuation of next period's inventory will be satisfied. Therefore, we obtain production smoothing even though the cost function is linear.

III.A <u>The Implications of Production Lags</u>

We have shown above that with a linear cost function and a one period production lag, optimal behavior is characterized $-1 < \frac{dy_t}{dx_t} < 0$. This finding of production smoothing in the presence of a linear cost function depends crucially on a lag in production. If production were instantaneous so that output y_t were available for sale in period t, then there would be no smoothing. In this case, optimal behavior would be characterized by an optimal level of inventory available for sale $x_t + y_t$.[9] This optimal level of inventory, say z, will be constant. Thus $y_t = z-x_t$. Furthermore since $x_{t+1} = z-s_t$ where s_t is sales in period t we obtain $y_{t+1} = s_t$. That is, with a linear cost function and instantaneous production, the optimal level of production in any period is equal to the sales of the previous period. Thus, sales and production have equal variance.

IV. Variances of Sales and Production

Up to this point we have defined production smoothing as $-1 < \frac{dy_t}{dx_t} < 0$ so that an increase in beginning-of-period inventory induces the firm to decrease its production by a smaller amount than the increase in inventory. An alternative definition of production smoothing is that the variance of production is less than the variance of sales. In this section we demonstrate that if $-1 < \frac{dy}{dx} < 0$, then the variance of sales is greater than the variance of production.

It will be convenient to define $z_t \equiv x_t+y_t$. Since production y_t can be written as a function of x_t, z_t can be expressed as a function of x_t,

9. In footnote 7, we derive the first-order conditions under the assumption that production is instantaneous. With a linear cost function, $c'(y)$ is a constant, say γ, so that the first-order condition (F7.2) is

$$\gamma = p[1-F(\omega^*)] + \beta E[V'(x+y-h(p,\varepsilon))|\varepsilon\leq\omega^*]F(\omega^*) \qquad (F9.1)$$

Recalling that with instantaneous production, $\omega^* = \omega^*(x+y,p)$, it is clear that x and y enter (F9.1) only as a sum x+y. Thus the optimal value of x+y is constant.

$$z_t = z(x_t) = x_t + y_t \tag{23a}$$

$$\text{where} \quad z'(x) \gtrless 0 \quad \text{as} \quad \frac{dy_t}{dx_t} \gtrless -1 \tag{23b}$$

According to (23b), if the optimal production rule exhibits production smoothing (defined as $-1 < \frac{dy}{dx} < 0$), then z_t is an increasing function of x_t. Observing that $x_{t+1} = z_{t+1} - y_{t+1}$, equation (3) can be rewritten as

$$z_{t+1} - y_{t+1} = z_t - s_t \tag{24}$$

Calculating the unconditional variance of each side of (24) yields

$$\text{var}(z) + \text{var}(y) - 2\text{Cov}(z_{t+1}, y_{t+1}) = \text{var}(z) + \text{var}(s) - 2\text{Cov}(z_t, s_t) \tag{25}$$

Rearranging (25) we obtain

$$\text{var}(s) = \text{var}(y) + 2\text{Cov}(z_t, s_t) - 2\text{Cov}(z_t, y_t) \tag{26}$$

Recall that we have shown that $\frac{dy_t}{dx_t} > -1$ so that $z'(x) > 0$. Thus, when x_t is high z_t is high, and y_t is low so that $\text{Cov}(z_t, y_t) < 0$. Also, when x_t is high, s_t is high so that $\text{Cov}(z_t, s_t) > 0$. Therefore, it follows from (26) that $\text{var}(s) > \text{var}(y)$.[10] Note that if $\frac{dy}{dx} \equiv -1$, then $z'(x) \equiv 0$ and

10. To calculate the covariances in (26), we let $g(x)$ be the steady state (unconditional) density function of x. Define $\bar{y} \equiv \int_0^\infty y(x)g(x)dx$ and $\bar{z} \equiv \int_0^\infty z(x)g(x)dx$ to be the unconditional expected values of y and z respectively. Observe that

$$\text{Cov}(y, z) = \int_0^\infty (y(x) - Y)(z(x) - \bar{z})g(x)dx \tag{F10.a}$$

where Y is some constant. Let $\hat{x}$ be defined by $\bar{z} = z(\hat{x})$. Then, replacing Y by the constant $y(\hat{x})$, we can rewrite (F10.a) as

$$\text{Cov}(y, z) = \int_0^{\hat{x}} (y(x) - y(\hat{x}))(z(x) - \bar{z})g(x)dx + \int_{\hat{x}}^\infty (y(x) - y(\hat{x}))(z(x) - \bar{z})g(x)dx \tag{F10.b}$$

$Cov(z_t, s_t) = 0 = Cov(z_t, y_t)$; thus $var(s_t) = var(y_t)$ as explained in section III.A.

Thus, we have shown that with a production lag, the variance of sales exceeds the variance of production even if the cost function c(y) is linear.

V. Concluding Remarks

We have examined the production and pricing behavior of an intertemporally optimizing firm. We demonstrated that marginal cost is equated with

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Observe that

$$y(x)-y(\hat{x}) \gtrless 0 \quad \text{as} \quad x \lessgtr \hat{x} \tag{F10.c}$$

$$z(x)-\bar{z} \lessgtr 0 \quad \text{as} \quad x \lessgtr \hat{x} \tag{F10.d}$$

From (F10.b)-(F10.d), it follows that $Cov(y,z) < 0$.

Now observe that $Cov(s,z) = E[s(z-\bar{z})]$ so that

$$Cov(s,z) = \int_0^\infty \{\int_{-\infty}^{\omega} h(p,\varepsilon)(z(x)-\bar{z})f(\varepsilon)d\varepsilon + \int_{\omega}^{\infty} x(z(x)-\bar{z})f(\varepsilon)d\varepsilon\} g(x)dx \tag{F10.e}$$

Using the definition of H(p,x) in (6) as expected sales, equation (F10.e) can be written as

$$Cov(s,z) = \int_0^\infty H(p,x)(z(x)-\bar{z})g(x)dx \tag{F10.f}$$

Observe that $H(p,\hat{x})$ is a constant and that

$$H(p,x) - H(p,\hat{x}) \lessgtr 0 \quad \text{as} \quad x \lessgtr \hat{x} \tag{F10.g}$$

Therefore,

$$Cov(s,z) = \int_0^\infty [H(p,x)-H(p,\hat{x})][z(x)-\bar{z}]g(x)dx > 0 \tag{F10.h}$$

expected marginal revenue if stock-outs are ignored and if demand shocks are additive. If either of these conditions is not met, then marginal cost is not equal to expected marginal revenue in general. We then showed that if demand is perfectly inelastic and if there is a production lag, then optimal behavior will be characterized by production smoothing even if the cost function is linear.

Casual empiricism or introspection may lead one to think that stock-outs are rather uncommon. However, we must be careful to distinguish between the probability that an individual customer will be rationed and the probability that the firm will stock out. As an extreme example, suppose that the firm always carries an inventory of 999 and that each of the firm's 1000 customers always demands one unit. In this case, the probability that any individual customer will be rationed is only 0.1%. However, the firm will stock-out in every period. Thus, the fact that an individual is only rarely rationed provides no evidence that, from the firm's point of view, stock-outs are unusual.

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