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Adverse Selection in Competitive Search Equilibrium  
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**ABSTRACT**

We extend the concept of competitive search equilibrium to environments with private information, and in particular adverse selection. Principals (e.g. employers or agents who want to buy assets) post contracts, which we model as revelation mechanisms. Agents (e.g. workers, or asset holders) have private information about the potential gains from trade. Agents observe the posted contracts and decide where to apply, trading off the contracts' terms of trade against the probability of matching, which depends in general on the principals' capacity constraints and market search frictions. We characterize equilibrium as the solution to a constrained optimization problem, and prove that principals offer separating contracts to attract different types of agents. We then present a series of applications, including models of signaling, insurance, and lemons. These illustrate the usefulness and generality of the approach, and serve to contrast our findings with standard results in both the contract and search literatures.

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# 1 Introduction

This paper studies equilibrium and efficiency in economies with adverse selection as well as the frictions modeled in competitive search theory. In our framework, a large number of uninformed principals compete to attract agents. For example, firms may compete to attract workers who have private information about their productivity or preferences, or buyers of assets may compete to attract sellers who have private information concerning the quality of their holdings. Principals post incentive-compatible contracts that specify actions if they match with particular agent types. Agents observe the posted contracts and direct their search, or apply, to the most attractive ones. Matching is limited by capacity constraints (each principal can match with at most one agent) and possibly by search frictions (it may be more difficult to contact a principal when more agents apply). Principals and agents form rational expectations about the market tightness for each contract—the ratio of the measure of principals posting that contract to the measure of agents who apply—as well as the types of agents who apply.

Part of our contribution is technical: we extend the standard competitive search model (Montgomery, 1991; Peters, 1991; Moen, 1997; Shimer, 1996; Acemoglu and Shimer, 1999; Burdett, Shi and Wright, 2001; Mortensen and Wright, 2002) to allow ex-ante heterogeneous agents with private information about their type. We prove that, under mild assumptions, including a single-crossing condition, equilibrium exists where principals offer separating contracts: each contract posted attracts only one type of agent, and different types direct their search to different contracts. The expected utility of each type of agent is uniquely determined in equilibrium. Moreover, equilibrium is easily found by sequentially solving a constrained optimization problem for each type. We also present a series of examples and applications to illustrate the generality and usefulness of our approach. These examples also show that some well-known results in contract theory and search theory change when we combine elements of both in the same model, and they allow us to explore the role of several key assumptions.

The first application is a classic signaling problem (Spence, 1973; Akerlof, 1976). Suppose that workers are heterogeneous with respect to both their productivity and their cost of working longer hours, where more productive workers find long hours less costly, and firms care about productivity but not the length of the workday. In equilibrium, firms require that more productive workers accept longer hours than they would under full information—a version of the rat race. In this example, although hours are distorted, the probability that any worker gets a job is not. The equilibrium is not generally efficient: a Pareto improvement is possible if there are sufficiently few low-productivity agents or the difference in the cost of

working is small relative to the difference in productivity. We also consider the case where long workdays are less costly for low-productivity workers; in this case there is an equilibrium with pooling and no distortion in the workday (although other equilibria exist).

Our second example is a version of the Rothschild and Stiglitz (1976) insurance model. Consider a labor market interpretation, rather than pure insurance, as in the original model.<sup>1</sup> Risk-averse workers and risk-neutral firms match in pairs to produce output, but only some pairs are productive. Workers differ in the probability that they can form a productive match, and if a match is unproductive the worker is let go. In equilibrium, firms separate workers by only partially insuring them against the probability the match will be unproductive. This means that workers can be worse off if they lose their job than if they had never found a one. We interpret this as an explanation for the observation that firms do not fully insure workers against layoff risk: if they did, they would attract low-productivity applicants. In this example, we show that a partial-pooling allocation—pooling only some types of workers—may Pareto dominate the equilibrium.

Competitive search also offers a novel resolution to the famous nonexistence problem in Rothschild and Stiglitz (1976). When there are relatively few low productivity workers, equilibrium may not exist in the original model, for the following reason: given any separating contract, profit for an individual firm can be increased by a deviation to a pooling contract that cross subsidizes low-productivity workers. Here, such a pooling contract will not increase profit. The key difference is that, in our model, because firms are capacity constrained, a deviation cannot serve the entire population. Suppose a firm posts a contract designed to attract a representative cross section of agents. Because of capacity constraints, the more workers that apply for this contract the less likely it is that any one will match. This discourages some workers from applying. Critically, it is the most productive workers who are the first to withdraw their applications, because their outside option—trying to obtain a separating contract—is more attractive. Hence, only undesirable workers are attracted by such a deviation, making it unprofitable.

In the above applications, asymmetric information affects the contracts offered in equilibrium, but not market tightness. Our third example reverses this: adverse selection makes it harder for agents to find a partner, but it does not affect the terms of trade conditional on a match. Consider a market where agents want to sell a heterogeneous object with uncertain quality (Akerlof, 1970). Imagine these objects are apples, meant to represent assets, that could be high or low quality.<sup>2</sup> Principals want to buy apples from agents, but some apples

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<sup>1</sup>We present the results in a labor market context because the assumption that a firm can only hire a fraction of the available workforce may seem more reasonable than the assumption that an insurance company can only serve a fraction of the potential customers.

<sup>2</sup>In our static environment, apples stand in for the claims to trees bearing fruit as dividends in the

are bad—they are lemons. To make the case stark, assume there are no fundamental search frictions, so that everyone on the short side of the market can match. But since the short side of the market is endogenous, in equilibrium, agents with good apples only match probabilistically. This serves to screen out low quality. This is different from related results in the literature (e.g. Nosal and Wallace 2007), where lotteries are used to similar effect. Interestingly, screening through matching saves resources, since posting contracts and trading with lotteries is wasteful. Still, equilibrium here can be Pareto dominated by a pooling allocation if there are few bad apples. We also find that in some cases adverse selection completely shuts down the market—an extreme lemons problem that may have something to do with the recent collapse in credit markets.

Our paper is related to a growing literature exploring competitive search theory with informational frictions, including Faig and Jerez (2005), Guerrieri (2008), and Moen and Rosén (2006), who propose different extensions of the standard model to allow one-sided private information. However, in all those papers, agents are ex ante homogeneous, and heterogeneity is match-specific. Inderst and Müller (1999) provide a model that extends competitive search to an environment with ex ante heterogeneous agents; their model is a special case of our first example, where the single-crossing condition is satisfied. Inderst and Wambach (2001) explore a version of the Rothschild and Stiglitz (1976) model with a finite number of principals and agents, and capacity constraints; this model is related to our second example. Our paper is the first to develop a general framework for analyzing competitive search with adverse selection and present a variety of applications.

Many papers study adverse selection without search, of course. Driven by the nonexistence issue in Rothschild and Stiglitz (1976), Miyazaki (1977), Wilson (1977), and Riley (1979) propose alternative notions of equilibrium. In contrast to those papers, we do not get cross subsidization in equilibrium. As we have mentioned above, the key difference with Rothschild and Stiglitz (1976) in our model is this: since no principal can serve all the agents, each must deduce which agents are most attracted to pooling contracts, and this eliminates the incentive to pool. Prescott and Townsend (1984) study adverse selection in competitive economies, concluding that “there do seem to be fundamental problems for the operation of competitive markets for economies or situations which suffer from adverse selection” (p. 44). This is consistent with our substantive findings, even though the models are very different. More recently, Bisin and Gottardi (2006) study adverse selection with competitive markets, where agents are restricted to trade incentive-compatible contracts, and also show that there

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standard asset-pricing model of Lucas (1978). Lester, Postlewaite and Wright (2007) review some related asset models with search and private information, but that literature does not analyze adverse selection and competitive search.

exists a separating equilibrium. Although our equilibrium concept is more strategic, some features are similar, and the incentive-compatibility condition they impose is analogous to one that we generate endogenously.

The rest of the paper is organized as follows. In Section 2, we develop the general environment, define equilibrium, and discuss some critical assumptions. In Section 3, we show how to find equilibrium by solving a constrained optimization problem, prove that a separating equilibrium always exists, and show that equilibrium payoffs are unique. In Section 4, we define a class of incentive-feasible allocations in order to discuss whether equilibrium outcomes are efficient within this class. In Sections 5–7, we present the series of applications discussed above, in each case characterizing equilibria and discussing efficiency. Section 8 concludes. All proofs are relegated to the Appendix.

## 2 The Model

There is a measure 1 of agents, a fraction  $\pi_i > 0$  of whom are of type  $i \in \mathbb{I} \equiv \{1, 2, \dots, I\}$ . Type is an agent’s private information. There is a large set of ex ante homogeneous principals. A principal may post a contract, at cost  $k > 0$ , that provides an opportunity to match with an agent (we discuss the nature of a contract below). To keep the analysis focused, we consider a static environment, where principals and agents have a single opportunity to match.

There is a compact and nonempty set of feasible actions for principals and agents who are matched,  $\mathbb{Y}$ , contained in a metric space with metric  $d(y, y')$  for  $y, y' \in \mathbb{Y}$ . A typical element  $y \in \mathbb{Y}$  may specify actions by the principal, actions by the agent, transfers between them, and other possibilities; as we make explicit in the applications, it can include lotteries. A principal who matches with a type  $i$  agent gets a payoff  $v_i(y) - k$  if they undertake action  $y$ . A principal who does not post a contract gets a payoff normalized to 0, while one who posts a contract but fails to match gets  $-k$ . A type  $i$  agent matched with a principal gets a payoff  $u_i(y)$  if they undertake  $y$ , while an unmatched agent gets a payoff also normalized to 0. For all  $i$ ,  $u_i : \mathbb{Y} \mapsto \mathbb{R}$  and  $v_i : \mathbb{Y} \mapsto \mathbb{R}$  are continuous.

By the revelation principle, without loss of generality we assume that contracts are revelation mechanisms. Precisely, a contract is a vector of actions,  $C \equiv \{y_1, \dots, y_I\} \in \mathbb{Y}^I$ , specifying that if a principal and agent match, the latter (truthfully) announces a type  $i$  and they implement  $y_i$ . Contract  $C$  is incentive compatible if  $u_i(y_i) \geq u_i(y_j)$  for all  $i, j$ .<sup>3</sup> Let  $\mathbb{C} \subset \mathbb{Y}^I$  denote the set of incentive compatible contracts. Principals only post contracts in

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<sup>3</sup>Since we are not concerned with moral hazard, here, we assume that  $y$  can be implemented by any principal and agent.

$\mathbb{C}$ .

We turn now to the matching process. All agents observe the set of posted contracts and apply, or direct their search to the most attractive ones. Let  $\Theta(C)$  denote the principal-agent ratio associated with contract  $C$ , defined as the measure of principals offering  $C$  over the measure of agents who apply to that contract,  $\Theta : \mathbb{C} \mapsto [0, \infty]$ . Let  $\gamma_i(C)$  denote the share of agents that apply to  $C$  whose type is  $i$ , with  $\Gamma(C) \equiv \{\gamma_1(C), \dots, \gamma_i(C), \dots, \gamma_I(C)\} \in \Delta^I$ , the  $I$ -dimensional unit simplex. That is,  $\Gamma(C)$  satisfies  $\gamma_i(C) \geq 0$  for all  $i$  and  $\sum_i \gamma_i(C) = 1$ , and  $\Gamma : \mathbb{C} \mapsto \Delta^I$ . The functions  $\Theta$  and  $\Gamma$  are determined endogenously in equilibrium, and are defined for all incentive compatible contracts, not only the ones that are posted in equilibrium.

An agent who applies to contract  $C$  matches with a principal with probability  $\mu(\Theta(C))$ , independent of type, where  $\mu : [0, \infty] \mapsto [0, 1]$  is nondecreasing. A principal offering  $C$  matches with a type  $i$  agent with probability  $\eta(\Theta(C))\gamma_i(C)$ , where  $\eta : [0, \infty] \mapsto [0, 1]$  is nonincreasing. We impose  $\mu(\theta) = \theta\eta(\theta)$  for all  $\theta$ , since the left hand side is the matching probability of an agent and the right hand side is the matching probability of a principal times the principal-agent ratio. Together with the monotonicity of  $\mu$  and  $\eta$ , this implies both functions are continuous. It is convenient to let  $\bar{\eta} \equiv \eta(0) > 0$  denote the highest probability that a principal can match with an agent, obtained when the principal-agent ratio is 0. Similarly let  $\bar{\mu} \equiv \mu(\infty) > 0$  denote the highest probability that an agent can match with a principal. Conversely,  $\mu(\theta) = \theta\eta(\theta)$  ensures that  $\eta(\infty) = \mu(0) = 0$ .

The expected utility of principals who post  $C = \{y_1, \dots, y_I\}$  is

$$\eta(\Theta(C)) \sum_{i=1}^I \gamma_i(C) v_i(y_i) - k.$$

The expected utility of type  $i$  agents who apply to  $C$  and report type  $j$  if they match is

$$\mu(\Theta(C)) u_i(y_j).$$

We now have the following definition of equilibrium.

**Definition 1** *A competitive search equilibrium is a vector  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}} \in \mathbb{R}_+^I$ , a measure  $\lambda$  on  $\mathbb{C}$  with support  $\bar{\mathbb{C}}$ , a function  $\Theta(C) : \mathbb{C} \mapsto [0, \infty]$ , and a function  $\Gamma(C) : \mathbb{C} \mapsto \Delta^I$  satisfying:*

(i) *principals' profit maximization and free-entry: for any  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$ ,*

$$\eta(\Theta(C)) \sum_{i=1}^I \gamma_i(C) v_i(y_i) \leq k,$$

with equality if  $C \in \bar{\mathbb{C}}$ ;

(ii) *agents' optimal search*: let

$$\bar{U}_i = \max \left\{ 0, \max_{C' = \{y'_1, \dots, y'_I\} \in \bar{\mathbb{C}}} \mu(\Theta(C')) u_i(y'_i) \right\}$$

and  $\bar{U}_i = 0$  if  $\bar{\mathbb{C}} = \emptyset$ ; then for any  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$  and  $i$ ,

$$\bar{U}_i \geq \mu(\Theta(C)) u_i(y_i),$$

with equality if  $\Theta(C) < \infty$  and  $\gamma_i(C) > 0$ ; moreover, if  $u_i(y_i) < 0$ , either  $\Theta(C) = \infty$  or  $\gamma_i(C) = 0$ ;

(iii) *market clearing*:

$$\int_{\bar{\mathbb{C}}} \frac{\gamma_i(C)}{\Theta(C)} d\lambda(\{C\}) \leq \pi_i \text{ for any } i,$$

with equality if  $\bar{U}_i > 0$ .

To understand our notion of equilibrium, first consider contracts that are actually posted in equilibrium,  $C \in \bar{\mathbb{C}}$ . Part (i) of the definition implies that principals earn zero profits from any such contract. Since  $\eta(\infty) = 0 < k$ , it must be that  $\Theta(C) < \infty$ . Part (ii) then implies that if type  $i$  agents apply for a contract,  $\gamma_i(C) > 0$ , they cannot earn a higher level of utility from any other posted contract. And part (iii) guarantees that all type  $i$  agents apply to some contract, unless they are indifferent about participating in the market, which gives them the outside option  $\bar{U}_i = 0$ .

Equilibrium also imposes restrictions on contracts that are not posted in equilibrium. Each principal anticipates that it cannot affect the utility of any type of agent, and so takes  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$  as given. Intuitively, when a principal considers posting a new contract  $C \in \mathbb{C}$ , he initially imagines an infinite principal-agent ratio. Some contracts will not be able to attract agents even at that ratio,  $\bar{U}_i \geq \bar{\mu} u_i(y_i)$  for all  $i$ , in which case  $\Theta(C) = \infty$  and the choice of  $\Gamma(C)$  is arbitrary and immaterial. Otherwise, agents would be attracted to the contract, pulling down the principal-agent ratio. This process would stop at the value of  $\Theta(C)$  such that  $\bar{U}_i = \mu(\Theta(C)) u_i(y_i)$  for some  $i$  and  $\bar{U}_j \geq \mu(\Theta(C)) u_j(y_j)$  for all  $j$ . Moreover, only type  $i$  agents with  $\bar{U}_i = \mu(\Theta(C)) u_i(y_i)$  would be attracted to the contract, pinning down  $\Gamma(C)$ . This is all captured by equilibrium condition (ii).<sup>4</sup> Equilibrium condition (i)

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<sup>4</sup>The requirement that if  $u_i(y_i) < 0$  then either  $\Theta(C) = \infty$  or  $\gamma_i(C) = 0$  rules out the possibility that type  $i$  agents earn zero utility, but apply for contract  $C = \{y_1, \dots, y_I\}$  with the expectation they will not be able to get it,  $\Theta(C) = 0$ . Other agents with  $\bar{U}_j > 0$  would then not apply for the contract. Such a belief might



then imposes that, for principals not to post  $C$ , in equilibrium, they must not earn positive profit from it, given  $\Theta(C)$  and  $\Gamma(C)$ .

Let

$$\bar{\mathbb{Y}}_i \equiv \{y \in \mathbb{Y} \mid \bar{\eta}v_i(y) \geq k \text{ and } u_i(y) \geq 0\}$$

be the set of actions for a type  $i$  agent that deliver nonnegative utility while permitting the principal to make nonnegative profits if the principal-agent ratio is 0, and

$$\bar{\mathbb{Y}} \equiv \bigcup_i \bar{\mathbb{Y}}_i.$$

In equilibrium, actions that are not in  $\bar{\mathbb{Y}}$  are not implemented.

For the benchmark analysis, we make three assumptions on preferences over  $y \in \bar{\mathbb{Y}}$ , as we now discuss.

**Assumption A1** *Monotonicity: for all  $y \in \bar{\mathbb{Y}}$ ,*

$$v_1(y) \leq v_2(y) \leq \dots \leq v_I(y).$$

This mild assumption says that for any fixed action, principals weakly prefer higher types.

For the next assumption, let  $B_\varepsilon(y) \equiv \{y' \in \mathbb{Y} \mid d(y, y') < \varepsilon\}$  be a ball of radius  $\varepsilon$ .

**Assumption A2** *Local non-satiation: for all  $i \in \mathbb{I}$ ,  $j < i$ ,  $y \in \bar{\mathbb{Y}}_i$ , and  $\varepsilon > 0$ , there exists a  $y' \in B_\varepsilon(y)$  such that  $v_i(y') > v_i(y)$  and  $u_j(y') \leq u_j(y)$ .*

Another mild assumption, A2 is satisfied in any application where actions allow transfers.<sup>5</sup>

The next assumption guarantees that it is possible to design contracts that attract some agents without attracting less desirable agents.

**Assumption A3** *Sorting: for all  $i \in \mathbb{I}$ ,  $y \in \bar{\mathbb{Y}}_i$ , and  $\varepsilon > 0$ , there exists a  $y' \in B_\varepsilon(y)$  such that*

$$u_j(y') > u_j(y) \text{ for all } j \geq i \text{ and } u_j(y') < u_j(y) \text{ for all } j < i.$$

One should interpret A3 as a generalized version of a standard single-crossing condition. Assume for illustration that:  $y \equiv (y_1, y_2) \in \mathbb{R}^2$ ;  $\bar{\mathbb{Y}}$  is open; and the functions  $u_i(y_1, y_2)$  are

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make a deviation unprofitable, if  $\bar{\eta}v_i(y_i) < k$ , but we find it implausible. In particular, it is inconsistent with the adjustment process described in the text.

<sup>5</sup>Moreover, we use A2 only in the proof of Proposition 2 below, and nowhere else. We use it to establish that it is possible to make a principal better off while not improving the well-being of agents. If  $\eta$  is strictly decreasing, we could do this by adjusting market tightness, but since for some examples it is interesting to have  $\eta$  only weakly decreasing, we introduce A2.

differentiable. Then A3 holds if the marginal rate of substitution between  $y_1$  and  $y_2$  is higher for higher types—i.e., if

$$\frac{\partial u_i(y_1, y_2)/\partial y_1}{\partial u_i(y_1, y_2)/\partial y_2}$$

is monotone in  $i$ .<sup>6</sup> Assumption A3 is important for the results: in Section 5.4 we show how the nature of equilibrium changes when A3 fails.

### 3 Equilibrium Characterization

We now show that equilibrium solves a set of optimization problems. For any type  $i$ , consider the following problem:

$$\begin{aligned} & \max_{\theta \in [0, \infty], y \in \mathbb{Y}} \mu(\theta) u_i(y) && \text{(P-}i\text{)} \\ \text{s.t. } & \eta(\theta) v_i(y) \geq k, \\ & \text{and } \mu(\theta) u_j(y) \leq \bar{U}_j \text{ for all } j < i. \end{aligned}$$

In terms of economics, (P- $i$ ) chooses market tightness  $\theta$  and an action  $y$  to maximize the expected utility of type  $i$  subject to a principal at least breaking even when only type  $i$  agents apply, and subject to types lower than  $i$  not wanting to apply.

Now consider the larger problem (P) of solving (P- $i$ ) for all  $i$ . More precisely, we say that a set  $\mathbb{I}^* \subset \mathbb{I}$  and three vectors  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  solve (P) if:

1.  $\mathbb{I}^*$  denotes the set of  $i$  such that the constraint set of (P- $i$ ) is non empty and the maximized value is strictly positive, given  $(\bar{U}_1, \dots, \bar{U}_{i-1})$ ;
2. for any  $i \in \mathbb{I}^*$ , the pair  $(\theta_i, y_i)$  solves problem (P- $i$ ) given  $(\bar{U}_1, \dots, \bar{U}_{i-1})$ , and  $\bar{U}_i = \mu(\theta_i) u_i(y_i)$
3. for any  $i \notin \mathbb{I}^*$ ,  $\bar{U}_i = 0$ .

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<sup>6</sup>Actually, A3 is more general. For example, suppose

$$u_i(y_1, y_2) = \left(\frac{1}{2}y_1^{\rho_i} + \frac{1}{2}y_2^{\rho_i}\right)^{1/\rho_i} - \frac{y_1^2 + y_2^2}{2} - \frac{1}{8}.$$

for  $\rho_i < 1$ , and  $\rho_i$  is higher for higher types. Then the elasticity of substitution between  $y_1$  and  $y_2$ ,  $1/(1 - \rho_i)$ , is increasing in  $i$ . Let  $\mathbb{Y} = [0, 1]^2$ , a superset of the points where  $u_i(y_1, y_2) > 0$ . Any point  $(y_1, y_2)$  on the boundary of  $\mathbb{Y}$  is not in  $\bar{\mathbb{Y}}$  since  $u_i(y_1, y_2) < 0$ . This example fails the single-crossing condition when  $y_1 = y_2$ , but satisfies A3, since it is possible to increase the spread between  $y_1$  and  $y_2$  and attract higher types while repelling lower ones.

Proposition 1 below says that we can find any equilibrium by solving (P); conversely, Proposition 2 says that any solution to (P) generates an equilibrium. As a preliminary step, we prove that (P) has a solution and provide a partial characterization, by showing that the zero profit condition binds and that higher types are not attracted by  $(\theta, y)$ . This implies that only downward incentive constraints are relevant in equilibrium (while, in fact, in the primitive definition incentive compatibility requires no  $j > i$  or  $j < i$  would want to apply and pretend to be type  $i$ ).

**Lemma 1** *Assume A1-A3. There exists  $\mathbb{I}^*$ ,  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  that solve (P). At any solution,*

$$\begin{aligned} \eta(\theta_i)v_i(y_i) &= k \text{ for all } i \in \mathbb{I}^*, \\ \mu(\theta_i)u_j(y_i) &\leq \bar{U}_j \text{ for all } j \in \mathbb{I} \text{ and } i \in \mathbb{I}^*. \end{aligned}$$

All formal proofs are in the Appendix, but the idea is to notice that (P) has a recursive structure. As a first step, (P-1) depends only on exogenous variables and thus determines  $\bar{U}_1$ . In general, at step  $i$ , (P- $i$ ) depends on the previously determined values of  $\bar{U}_j$  for  $j < i$  and determines  $\bar{U}_i$ . Thus, we can solve (P) in  $I$  iterative steps.

We now show that a solution to (P) can be used to construct an equilibrium in which some principals offer a contract to attract type  $i \in \mathbb{I}^*$ , while keeping out other types. The relevant contract is suggested by the solution to (P), but we must also show that no other contract gives positive profit.

**Proposition 1** *Assume A1-A3. Suppose  $\mathbb{I}^*$ ,  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  solve (P). Then there exists a competitive search equilibrium  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, \Gamma\}$  with  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\bar{\mathbb{C}} = \{C_i\}_{i \in \mathbb{I}^*}$  where  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ , and  $\gamma_i(C_i) = 1$ .*

The next result establishes that any equilibrium can be characterized using (P). The proof is based on a variational argument, showing that if  $(\theta_i, y_i)$  does not solve (P), it cannot be part of an equilibrium.

**Proposition 2** *Assume A1-A3. Let  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, \Gamma\}$  be a competitive search equilibrium. Let  $\{\bar{U}_i\}_{i \in \mathbb{I}} = \bar{U}$  and  $\mathbb{I}^* = \{i \in I | \bar{U}_i > 0\}$ . For each  $i \in \mathbb{I}^*$ , there exists a contract  $C \in \bar{\mathbb{C}}$  with  $\Theta(C) < \infty$  and  $\gamma_i(C) > 0$ . Moreover, take any  $\{\theta_i\}_{i \in \mathbb{I}^*}$  and  $\{y_i\}_{i \in \mathbb{I}^*}$  such that for each  $i \in \mathbb{I}^*$ , there exists a contract  $C_i = \{y_1, \dots, y_i, \dots, y_I\} \in \bar{\mathbb{C}}$  (so the  $i^{\text{th}}$  element of  $C_i$  is  $y_i$ ) with  $\theta_i = \Theta(C_i) < \infty$  and  $\gamma_i(C_i) > 0$ . Then  $\mathbb{I}^*$ ,  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  solve (P).*

The above results imply that in equilibrium any contract  $C = \{y_1, \dots, y_i, \dots, y_I\}$  that attracts type  $i$  solves (P- $i$ ), in the sense that the solution to the problem has  $\theta = \Theta(C)$  and

$y = y_i$ . This does not necessarily mean that each contract attracts only one type of agent. In general, a contract could attract more than one type, or the same type could apply to more than one type of contract. But if a contract  $C = \{y_1, \dots, y_I\}$  attracts two types, say  $i$  and  $j$ , then  $\Theta(C)$  and  $y_i$  solve (P- $i$ ) and  $\Theta(C)$  and  $y_j$  also solve (P- $j$ ).

The existence of equilibrium and uniqueness of equilibrium payoffs follow immediately from the above results.

**Proposition 3** *Assume A1-A3. Then competitive search equilibrium exists, and the equilibrium  $\bar{U}$  is unique.*

The last result of this section shows that, when there are strict gains from trade for all types, all agents get strictly positive utility.

**Proposition 4** *Assume A1-A3, and that for all  $i$  there exists  $y \in \mathbb{Y}$  with  $\bar{\eta}v_i(y) > k$  and  $u_i(y) > 0$ . Then in any competitive search equilibrium,  $\bar{U}_i > 0$  for all  $i$ , and in particular there exists a contract  $C \in \mathbb{C}$  with  $\Theta(C) < \infty$  and  $\gamma_i(C) > 0$ .*

The proof follows by showing that the maximized value of any (P- $i$ ) is positive as long as  $\bar{U}_j > 0$  for  $j < i$ . One might imagine a stronger claim, that if there are strict gains from trade for any type  $i$  then  $\bar{U}_i > 0$ , but Section 7 shows that this may not be the case. In particular, if there are no gains from trade for some type  $j < i$  and  $\bar{U}_j = 0$ , it may be that  $\bar{U}_i = 0$  even though there would be gains from trade for type  $i$  with full information.

## 4 Incentive Feasible Allocations

To set the stage for studying efficiency, we define an incentive feasible allocation. We begin by defining an allocation, by which we basically mean a description of the posted contracts together with the implied search behavior and payoffs of agents.

**Definition 2** *An allocation is a vector  $\bar{U}$  of expected utilities for the agents, a measure  $\lambda$  over the set of incentive-compatible contracts  $\mathbb{C}$  with support  $\bar{\mathbb{C}}$ , a function  $\tilde{\Theta} : \bar{\mathbb{C}} \mapsto [0, \infty]$ , and a function  $\tilde{\Gamma} : \bar{\mathbb{C}} \mapsto \Delta^I$ .*

Note that  $\tilde{\Theta}$  and  $\tilde{\Gamma}$  are different from the  $\Theta$  and  $\Gamma$  in the definition of equilibrium, because the former are defined only over the set of posted contracts, while the latter are defined for all incentive compatible contracts.

An allocation is incentive feasible whenever: (1) each posted contract offers the maximal expected utility to agents who direct their search for that contract and no more to those

who do not; (2) the economy's resource constraint is satisfied; and (3) markets clear.<sup>7</sup> More formally, we have:

**Definition 3** *An allocation  $\{\bar{U}, \lambda, \bar{C}, \tilde{\Theta}, \tilde{\Gamma}\}$  is incentive feasible if*

1. *for any  $C \in \bar{C}$  and  $i \in \{1, \dots, I\}$  such that  $\tilde{\gamma}_i(C) > 0$  and  $\tilde{\Theta}(C) < \infty$ ,*

$$\bar{U}_i = \mu(\tilde{\Theta}(C))u_i(y_i),$$

*and*

$$\bar{U}_i \equiv \max_{C' \in \bar{C}} \mu(\tilde{\Theta}(C'))u_i(y'_i)$$

*where  $C' = \{y'_1, \dots, y'_I\}$ ;*

- 2.

$$\int \left( \eta(\tilde{\Theta}(C)) \sum_{i=1}^I \tilde{\gamma}_i v_i(y_i) - k \right) d\lambda(C) = 0;$$

3. *for all  $i \in \{1, \dots, I\}$ ,*

$$\int \frac{\tilde{\gamma}_i(C)}{\tilde{\Theta}(C)} d\lambda(C) \leq \pi_i,$$

*with equality if  $\bar{U}_i > 0$ .*

The set of incentive feasible allocations provides a benchmark for what the economy could conceivably achieve, perhaps through legal or other restrictions on the contracts that can be offered.

## 5 Application I: The Rat Race

We now proceed with the first of our three main applications, a version of a classic signaling model (Akerlof, 1976). For concreteness, think of agents here as workers who are heterogeneous in terms of both their productivity and their cost of working long hours, and principals as firms that are willing to pay more for high-productivity workers and can observe hours but not productivity. The main substantive result in this example is this: if the cost of a long workday is lower for more productive workers, equilibrium will separate types by distorting hours relative to the first best, but will not distort market tightness.<sup>8</sup>

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<sup>7</sup>The resource constraint in the formal definition to follow can be read as saying that principals' profits are 0 on average, but this is not the only interpretation, since there may be different ways to implement the same outcome—e.g. we can imagine a planner posting the contracts directly.

<sup>8</sup>This example is essentially a static version of Inderst and Müller (1999), although we also extend it to consider cases where the sorting assumption A3 fails.

## 5.1 Setup

An action here is  $y = \{t, x\}$ , where  $t$  is a transfer from the firm to the worker and  $x \geq 0$  is the length of the workday. The payoff of a type  $i$  worker who undertakes  $\{t, x\}$  is

$$u_i(t, x) = t - \frac{x}{a_i},$$

where higher values of  $a_i$  imply that  $x$  is less costly. The payoff of a firm matched with type  $i$  who undertakes  $\{t, x\}$  is

$$v_i(t, x) = b_i - t,$$

where  $b_i$  is the productivity the worker. We assume  $I = 2$  and  $b_2 > b_1$ . We restrict the set of feasible actions to  $\mathbb{Y} = \{(t, x) | t \in [-\varepsilon, b_2] \text{ and } x \in [0, b_2 \max\{a_1, a_2\}]\}$  for some  $\varepsilon > 0$ . This ensures  $\mathbb{Y}$  is compact but is not otherwise important (although it is convenient to allow for the possibility of a small negative transfer).<sup>9</sup>

The space of actions that provide nonnegative utility to type  $i$  and nonnegative profit when the firm-worker ratio is 0 is

$$\bar{\mathbb{Y}}_i = \{(t, x) \in \mathbb{Y} | x/a_i \leq t \leq b_i - k/\bar{\eta}\}.$$

The fact that  $b_2 > b_1$  implies A1, and A2 holds because  $(t, x) \in \bar{\mathbb{Y}}_i$  implies  $t \geq 0$ , so  $t$  can be reduced to raise  $v_i(t, x)$  and lower  $u_j(t, x)$  (this is where it is convenient to allow negative transfers). In terms of A3, consider  $(t, 0) \in \bar{\mathbb{Y}}$ . There are nearby points  $(t', x')$  with  $u_1(t', x') < u_1(t, 0)$  and  $u_2(t', x') > u_2(t, 0)$  if and only if  $a_2 > a_1$ , so more productive workers find it less costly to have higher  $x$ . At any other  $\{t, x\} \in \bar{\mathbb{Y}}$ , A3 holds for all  $a_1$  and  $a_2$ . We impose  $a_2 > a_1$ , for now, and discuss later what happens if this is violated. Finally, assume  $\bar{\eta}b_1 > k$ , so that there are gains from trade for both types, and hence Proposition 4 implies  $\bar{U}_1 > 0$  and  $\bar{U}_2 > 0$ .

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<sup>9</sup>A firm would never offer  $t > b_2$  and the worker would never accept  $t < 0$ . Also, a worker would never provide  $x > b_2 \max\{a_1, a_2\}$ , given that  $b_2 \leq t$ .

## 5.2 Equilibrium

Based on Section 3, we characterize equilibrium by solving (P). In this example, (P-i) is

$$\begin{aligned} \bar{U}_i &= \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_i} \right) \\ \text{s.t. } &\eta(\theta)(b_i - t) \geq k, \\ \text{and } &\mu(\theta) \left( t - \frac{x}{a_j} \right) \leq \bar{U}_j \text{ for } j \leq i. \end{aligned}$$

Let us assume  $\mu$  is strictly concave and continuously differentiable. Then we have the following (again, all proofs are in the Appendix):

**Result 1** *There exists a unique competitive search equilibrium with  $\mu'(\theta_i)b_i = k$ , for  $i = 1, 2$ , and so  $\theta_1 < \theta_2$ . Moreover, we have*

$$x_1 = 0; x_2 = \frac{a_1}{\mu(\theta_2)} \left[ \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right) - \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) \right] k > 0;$$

$$t_i = \left( \frac{1}{\mu'(\theta_i)} - \frac{\theta_i}{\mu(\theta_i)} \right) k, \text{ for } i = 1, 2;$$

$$\bar{U}_1 = \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k \text{ and } \bar{U}_2 = \left[ \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) + \left( 1 - \frac{a_1}{a_2} \right) \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right) \right] k.$$

In equilibrium, some firms post contracts to attract only type 1 workers, while others post contracts to attract only type 2 workers. To achieve separation, the contracts of type 1 are undistorted, while those of type 2 are distorted by having them work (unproductively) long hours. Notice that market tightness is equal to the first-best or full-information level for both types, so matching and hence employment are not distorted in this example. Also, note that without additional assumptions we cannot rule out the possibility that  $t_2 < t_1$ , so type 2 workers may actually receive less compensation in addition to working longer hours; they are compensated for this by a higher probability of being hired,  $\theta_2 > \theta_1$ .

## 5.3 Efficiency

Equilibrium is not necessarily efficient. Consider an allocation that treats the two types identically,  $\bar{C} = \{C\}$ , where  $C = \{(t, 0), (t, 0)\}$ . Then let  $\tilde{\Theta}(C) = \theta^*$ , with  $\theta^*$  solving  $\mu'(\theta^*)(\pi_1 b_1 + \pi_2 b_2) = k$ ,  $\tilde{\gamma}_i(C) = \pi_i$ , and  $\lambda(\{C\}) = 1/\theta^*$ . Note that  $\theta_1 < \theta^* < \theta_2$ , so market tightness now is in between the levels for the different types in equilibrium. Then choose  $t$

so profits are 0:

$$t = \left( \frac{1}{\mu'(\theta^*)} - \frac{\theta^*}{\mu(\theta^*)} \right) k$$

This contract is obviously incentive compatible, since types are treated identically. All workers want to apply, and get the same expected utility, so condition (1) of feasibility is satisfied. And the resource and market clearing conditions are satisfied by the choice of  $t$  and  $\lambda$ . Hence, this constitutes an incentive feasible allocation.

The expected utility for all workers is

$$\bar{U} = \left( \frac{\mu(\theta^*)}{\mu'(\theta^*)} - \theta^* \right) k.$$

Comparing this with equilibrium,  $\bar{U} > \bar{U}_1$  since  $\theta^* > \theta_1$ , and  $\bar{U} \geq \bar{U}_2$  if and only if

$$\frac{\mu(\theta^*)}{\mu'(\theta^*)} - \theta^* \geq \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) + \left( 1 - \frac{a_1}{a_2} \right) \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right).$$

This inequality holds if  $a_1/a_2$  is close to 1 (screening is very costly) or if  $\pi_1$  is close to zero (there are very few type 1 agents). The reason is that, in equilibrium, firms who want to attract type 2 need to screen out type 1, or they would be swamped by applications from low-productivity workers. Screening may not be socially optimal, however, if there are few type 1 workers or screening is costly. Thus it may be socially preferable to subsidize type 1 workers and eliminate screening, but this is not consistent with equilibrium, any individual firm has incentive to deviate and try to screen.

## 5.4 Pooling

The sorting condition A3 plays a critical role. To understand this, consider a variant where signaling is cheaper for less productive workers,  $a_1 \geq a_2$  while  $b_1 < b_2$ .<sup>10</sup> Now firms would like to screen out low-productivity workers, but cannot: if they try to attract high-productivity types, the low-productive types would apply and claim to be high-productivity. In this case, we now show that there is a class of equilibria in which firms attract both types, and ask the low-productivity agents to work  $x \geq 0$  unproductive hours.

When A3 is violated, we cannot use the analysis in Section 3 to characterize the outcome, and we must go back to the primitive definition of equilibrium. Given this, we describe a

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<sup>10</sup>We claim this violates A3. Fix  $t$  and set  $x = 0$ . For any nearby contract  $(t', x')$ ,

$$u_1(t', x') - u_1(t, 0) = t' - t - x'/a_1 \geq t' - t - x'/a_2 = u_2(t', x') - u_2(t, 0),$$

since  $x' \geq 0$ . Hence, there is no  $(t', x')$  with  $u_1(t', x') < u_1(t, 0)$  and  $u_2(t', x') > u_2(t, 0)$ .



class of equilibria indexed by  $x_1 \in [0, a_1(b_2 - b_1)(1 - \pi_1)/\pi_1]$ . All firms post the same contract  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$ , where  $t$  is chosen to make profit 0, all workers apply, and  $\gamma_i(C) = \pi_i$ . Given this, suppose a firm considers offering a contract that attracts only one type. If it tries to attract type 1, it loses the benefit of cross-subsidization, and is unable to attract them while earning positive profits. If it tries to attract type 2 workers, it is unable to devise a contract that will exclude type 1, again making the deviation unprofitable.

**Result 2** *Suppose  $a_1 \geq a_2$ . For any  $x_1 \in [0, a_1(b_2 - b_1)(1 - \pi_1)/\pi_1]$ , there exists a competitive search equilibrium where  $\bar{C} = \{C\}$  with  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$  and*

$$t = \pi_1 \left( b_1 - \frac{x_1}{a_1} \right) + \pi_2 b_2 - \frac{\theta}{\mu(\theta)} k,$$

where  $\theta$  solves

$$\mu'(\theta) \left( \pi_1 \left( b_1 - \frac{x_1}{a_1} \right) + \pi_2 b_2 \right) = k.$$

The expected utility for all workers is  $\bar{U} = \mu(\theta)t$ .

There are equilibria where  $x > 0$ , because if firms did not set  $x > 0$  they would be stuck exclusively with type 1 workers. Clearly,  $x > 0$  is socially wasteful, and these equilibria can be Pareto ranked, with  $x_1 = 0$  being the best. Moreover, if  $a_1 > a_2$ , then not only do there exist the equilibria in Result 2, we can show that any equilibrium where all firms post the same contract must be in that class.

**Result 3** *Suppose  $a_1 > a_2$ . If there exists a competitive search equilibrium with  $\bar{C} = \{C\}$  where  $C = \{(t_1, x_1), (t_2, x_2)\}$ ,  $\gamma_i(C) = \pi_i$ , and  $\Theta(C) < \infty$ , then  $x_2 = 0$ , and  $t_1 = t_2 + x_1/a_1$ .*

## 6 Application II: Insurance

Our next application is based on Rothschild and Stiglitz (1976), where risk neutral principals offer insurance to risk averse agents who are heterogeneous in their probability of a loss. This illustrates several features. First, although we had linear utility in the previous section, this is not necessary in the general framework. Second, we show here that even if a pooling allocation does not Pareto dominate the equilibrium, a partial-pooling allocation may. Finally, to illustrate that traditional search frictions are not necessary, in this example we allow the short side of the market to match for sure:  $\mu(\theta) = \min\{\theta, 1\}$ .

## 6.1 Setup

We again specify the model in terms of worker-firm matching.<sup>11</sup> Now the productivity of a match is initially unknown by both the worker and firm. Some workers are more likely than others to generate productive matches, but firms cannot observe this: type  $i$  produce 1 unit of output with probability  $p_i$  and 0 otherwise, and  $p_i$  is the agent's private information. A contract specifies a transfer to the worker conditional on realized productivity and reported type. Workers are risk averse and firms are risk neutral. In the absence of adverse selection, full insurance equates the marginal utility of agents across states. We show that firms here do not provide full insurance, because incomplete insurance helps keep undesirable workers from applying.

An action consists of a pair of consumption levels, conditional on employment or unemployment after match productivity has been realized,  $y = \{c_e, c_u\}$ . The payoff of a matched type  $i$  worker given  $y$  is

$$u_i(c_e, c_u) = p_i U(c_e) + (1 - p_i) U(c_u),$$

where  $p_1 < p_2 < \dots < p_I < 1$  and  $U : [\underline{c}, \infty) \rightarrow \mathbb{R}$  is increasing and strictly concave with  $\lim_{c \rightarrow \underline{c}} U(c) = -\infty$  for some  $\underline{c} < 0$  and  $U(0) = 0$ . The payoff of a firm matched with type  $i$  given  $y$  is

$$v_i(c_e, c_u) = p_i(1 - c_e) - (1 - p_i)c_u.$$

To ensure A1 is satisfied, we restrict the set of feasible actions to  $\mathbb{Y} = \{(c_e, c_u) | c_u + 1 \geq c_e \geq \underline{c} \text{ and } c_u \geq \underline{c}\}$ . The assumption  $\lim_{c \rightarrow \underline{c}} U(c) = -\infty$  ensures that actions of the form  $\{c_e, \underline{c}\}$  yield negative utility for all types and so are not in  $\bar{\mathbb{Y}}$ . Then, since a reduction in  $c_u$  raises  $v_i(y)$  and lowers  $u_i(y)$ , and is feasible for all  $y \in \bar{\mathbb{Y}}$ , A2 is satisfied. To verify A3, consider an incremental increase in  $c_e$  to  $c_e + dc_e$  and an incremental reduction in  $c_u$  to  $c_u - dc_u$  for some  $dc_e > 0$  and  $dc_u > 0$ . For a type  $i$  worker, this raises utility by approximately  $p_i u'(c_e) dc_e - (1 - p_i) u'(c_u) dc_u$ , which is positive if and only if

$$\frac{dc_e}{dc_u} > \frac{1 - p_i}{p_i} \frac{u'(c_u)}{u'(c_e)}.$$

Since  $(1 - p_i)/p_i$  is decreasing in  $i$ , an appropriate choice of  $dc_e/dc_u$  yields an increase in utility if and only if  $j \geq i$ , which verifies A3.

Finally, assume  $p_1 \leq k < p_I$ , which ensures that there are no gains from employing the

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<sup>11</sup>Again, we frame the discussion in terms of labor markets, rather than general insurance, because it seems more reasonable to assume an employer can only hire a fraction of the available workforce than to assume an insurance company can only serve a fraction of its potential customers.

lowest type, even in the absence of asymmetric information, but there may be gains from trade for higher types, say by setting  $c_e = c_u = p_I - k > 0$ . Let  $i^*$  denote the lowest type without gains from trade, so  $p_{i^*} \leq k < p_{i^*+1}$  (the results extend to the case  $k < p_1$  by defining  $i^* = 0$ ).

## 6.2 Equilibrium

We can again characterize equilibrium using (P), leading to:

**Result 4** *There exists a competitive search equilibrium where for all  $i \leq i^*$ ,  $\bar{U}_i = 0$ ; and for all  $i > i^*$ ,  $\theta_i = 1$ ,  $\bar{U}_i > 0$ , and  $c_{e,i} > c_{e,i-1}$  and  $c_{u,i} < c_{u,i-1}$  are the unique solution to*

$$p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} = k$$

$$\text{and } p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) = p_{i-1}U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1}),$$

where  $c_{e,i^*} = c_{u,i^*} = 0$ .

One interesting feature of equilibrium here is that  $c_{u,i} < 0$  for all  $i > i^*$ , and therefore a worker is worse off when he matches and turns out to be unproductive than when he does not match in the first place. If one interprets a bad match as a layoff, contracts give laid-off workers less utility than those who never match, because this keeps inferior workers from applying for the job.<sup>12</sup>

## 6.3 Efficiency

Again we show equilibrium may not be efficient. First note that a worker with  $p_i$  close to 1 suffers little from the distortions introduced by the information problem. At the extreme, if  $p_I = 1$ ,  $c_{u,I} = \underline{c}$  excludes other workers without distorting the type  $I$  contract at all. Generally, adverse selection has the biggest impact on the utility of workers with an intermediate value of  $p_i$ . We show that a Pareto improvement may result from partial pooling. Consider  $p_1 = 1/4$ ,  $p_2 = 1/2$ , and  $p_3 = 3/4$  and suppose there are equal numbers of type 1 and 3, so that half of all matches are productive. Set  $U(c) = \log(1 + c)$  and  $k = 3/8$ . Then in equilibrium,  $\bar{U}_1 = 0$ ;  $c_{e,2} = 0.344$ ,  $c_{u,2} = -0.094$ , and  $\bar{U}_2 = U(0.104)$ ; and  $c_{e,3} = 0.576$ ,  $c_{u,3} = -0.227$ , and  $\bar{U}_3 = U(0.319)$ .

Pooling all three types, the best incentive-feasible allocation involves  $c_e = c_u = 1/8$  and  $\bar{U}_i = U(1/8)$ . Compared to the equilibrium, this raises the utility of type 1 and type 2

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<sup>12</sup>To interpret a bad match as a layoff, it might help to imagine that an agent actually must work for some time before productivity is realized, as in Jovanovic (1979).

workers, but reduces the utility of type 3 workers. Now consider an allocation that pools types 1 and 2. If there are sufficiently few type 1 workers, it is feasible to set  $c_e = c_u > 0.104$ , delivering utility greater to types 1 and 2. For example, suppose  $\pi_1 = \pi_3 = 0.01$  and  $\pi_2 = 0.98$ . Then the utility of types 1 and 2 rises to  $U(0.122)$ . By raising the utility of type 2, it is easier to exclude them from type 3 contracts, reducing the requisite inefficiency of those contracts. This raises the utility of type 3, in this case, to  $U(0.325)$ .

## 6.4 Relationship to Rothschild-Stiglitz

Rothschild and Stiglitz (1976, p. 630) “consider an individual who will have income of size  $W$  if he is lucky enough to avoid accident. In the event an accident occurs, his income will be only  $W - d$ . The individual can insure against this accident by paying to an insurance company a premium  $\alpha_1$  in return for which he will be paid  $\hat{\alpha}_2$  if an accident occurs. Without insurance his income in the two states, ‘accident,’ ‘no accident,’ was  $(W, W - d)$ ; with insurance it is now  $(W - \alpha_1, W - d + \alpha_2)$  where  $\alpha_2 = \hat{\alpha}_2 - \alpha_1$ .”

We can normalize the utility of an uninsured individual to zero and express the utility of one who anticipates an accident with probability  $p_i$  as

$$u_i(\alpha_1, \alpha_2) = p_i U(W - \alpha_1) + (1 - p_i) U(W - d + \alpha_2) - \kappa_i,$$

where  $\kappa_i \equiv p_i U(W) + (1 - p_i) U(W - d)$ . Setting  $W = d = 1$  and defining  $c_e = 1 - \alpha_1$  and  $c_u = \alpha_2$ , this is equivalent to our example. Our results apply to their setup, with one wrinkle: our fixed cost of posting contracts.

Rothschild and Stiglitz (1976) show that in any equilibrium, principals who attract type  $i$  agents,  $i > 1$ , offer incomplete insurance to deter type  $i - 1$  agents, which is of course very similar to our finding. Under some conditions, however, their equilibrium does not exist. Starting from a configuration of separating contracts, suppose one principal deviates by offering a pooling contract to attract multiple types. In their setup, this is profitable if the least-cost separating contract is Pareto inefficient. But such a deviation is never profitable in our environment.

The key difference is that in the original model a deviating principal can capture all the agents in the economy, or at least a representative cross-section, while in our model a principal cannot serve all the agents who are potentially attracted to a contract. Instead, agents are rationed thorough the endogenous movement in market tightness  $\theta$ . Whether such a deviation is profitable depends on which agents are most willing to accept a decline in  $\theta$ . What we find is that high types are the first to give up on the pooling contract when applications get crowded by low types. A lower type, with an inferior outside option

$\bar{U}_{i-1} < \bar{U}_i$ , is more inclined to stick it out. Hence, a principal who tries to offer a pooling contract will end up with a long queue of type 1 agents—the worst possible outcome. For this reason, a deviation to a pooling contract is not profitable, and equilibrium with separating contracts always exists.

## 7 Application III: Asset Markets

A feature of the previous two examples is that market tightness is not distorted:  $\theta_i$  is at its full-information level for all  $i$ . We now present a model where tightness can be used to screen bad types. Although the results hold more generally, to stress the point, we again abstract from traditional search frictions and assume  $\mu(\theta) = \min\{\theta, 1\}$ , so that matching is determined by the short side of the market and  $\bar{\eta} = 1$ .

### 7.1 Setup

Consider an asset market with lemons, in the sense of Akerlof (1970). Buyers (principals) always value an asset more than sellers (agents) value it, but some assets are better than others and their values are private information to the seller. Market tightness, or probabilistic trading, seems in principle a good way to screen out low quality asset holders, since sellers with more valuable assets are more willing to accept a low probability of trade at any given price. This model shows how an illiquid asset market may have a useful role as a screening device.

Each type  $i$  seller is endowed with one indivisible asset, which call an apple, of type  $i$ , with value  $a_i^S > 0$  to the seller and  $a_i^B > 0$  to the buyer, both expressed in units of a numeraire good. An action for type  $i$  sellers is a pair  $\{\alpha_i, t_i\}$ , where  $\alpha_i$  is the probability that the seller gives the buyer the apple and  $t_i$  is the transfer of the numeraire to the seller.<sup>13</sup> The payoff of a matched type  $i$  seller who reports type  $j$  is

$$u_i(\alpha_j, t_j) = t_j - \alpha_j a_i^S,$$

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<sup>13</sup>Given that apples are indivisible, it may be efficient to use lotteries, with  $\alpha$  the probability apples change hands, as e.g. in Prescott and Townsend (1984) and Rogerson (1988). Nosal and Wallace (2007) provide a related model of an asset (money) market where probabilistic trade is useful due to private information, but they use random and not directed search, leading to quite different results. It would actually be equivalent here to assume apples are perfectly divisible and preferences are linear, with  $\alpha$  reinterpreted as the fraction traded, but we like the indivisibility since it allows us to contrast our results with models that use lotteries.

while the payoff of a buyer matched with a type  $i$  seller who reports truthfully is

$$v_i(\alpha_i, t_i) = \alpha_i a_i^B - t_i.$$

Note that we have normalized the no-trade payoff to 0.

We set  $I = 2$  and impose a number of restrictions on payoffs. First, both buyers and sellers prefer type 2 apples and both types of agents like apples:

$$a_2^S > a_1^S > 0 \text{ and } a_2^B > a_1^B > 0.$$

Second, there would gains from trade, including the cost  $k$  of posting, if the buyer were sure to trade:

$$a_i^S + k < a_i^B \text{ for } i = 1, 2.$$

The available actions are  $\mathbb{Y} = [0, 1] \times [0, 1]$ , with  $\bar{\mathbb{Y}}_i = \{(\alpha, t) \in \mathbb{Y} \mid \alpha a_i^S \leq t \leq \alpha a_i^B - k\}$ . Using these restrictions, we verify our three assumptions. As a preliminary step, note that  $(\alpha, t) \in \bar{\mathbb{Y}}_i$  implies  $\alpha \geq k/(a_i^B - a_i^S) > 0$  and  $t \geq k a_i^S/(a_i^B - a_i^S) > 0$ , so in any equilibrium contract, trades are bounded away from zero.

Since  $\alpha > 0$  whenever  $(\alpha, t) \in \bar{\mathbb{Y}}_i$ , the restriction  $a_1^B < a_2^B$  implies A1. Also, A2 holds because for any  $(\alpha, t) \in \bar{\mathbb{Y}}_i$ , a movement to  $(\alpha, t - \varepsilon)$  with  $\varepsilon > 0$  is feasible and raises buyer utility. The important assumption is again A3, which is here guaranteed by  $a_1^S < a_2^S$ . Fix  $(\alpha, t) \in \bar{\mathbb{Y}}$  and  $\bar{a} \in (a_1^S, a_2^S)$ . For arbitrary  $\delta > 0$ , consider  $(\alpha', t') = (\alpha - \delta, t - \bar{a}\delta)$ . This is feasible for small  $\delta$  because  $(\alpha, t) \in \bar{\mathbb{Y}}$  guarantees that  $\alpha > 0$  and  $t > 0$ . Then

$$u_2(\alpha', t') - u_2(\alpha, t) = \delta(a_2^S - \bar{a}) > 0$$

$$u_1(\alpha', t') - u_1(\alpha, t) = \delta(a_1^S - \bar{a}) < 0.$$

Now for fixed  $\varepsilon > 0$ , choose  $\delta \leq \varepsilon/\sqrt{1 + \bar{a}^2}$ . This ensures  $(\alpha', t') \in B_\varepsilon(\alpha, t)$ , so A3 holds.

## 7.2 Equilibrium

We again use problem (P) to characterize the equilibrium.

**Result 5** *There exists a unique competitive search equilibrium with  $\alpha_i = 1$ ,  $t_i = a_i^B - k$ ,  $\theta_1 = 1$ ,  $\bar{U}_1 = a_1^B - a_1^S - k$ ,  $\theta_2 = \frac{a_1^B - a_1^S - k}{a_2^B - a_1^S - k} < 1$ , and  $\bar{U}_2 = \theta_2(a_2^B - a_2^S - k)$ .*

With full information,  $\theta_2 = 1$  and  $\bar{U}_2 = a_2^B - a_2^S - k$ . Relative to this benchmark, buyers post too few contracts designed to attract type 2 sellers, and hence too many of them fail to

trade. Since type 2 sellers have better apples than type 1, they are more willing to accept this in return for a better price when they do trade. Agents with inferior assets are less willing to accept a low probability of trade because they do not want to be stuck with their own apple, which is in fact a lemon. Note that the alternative of setting  $\theta_2 = 1$  but rationing through the probability of trade in a match,  $\alpha_2 < 1$ , wastes resources, because it involves posting more contracts at cost  $k$ . In other words, reducing the matching rate is a cost-effective screening device compared to lotteries.

### 7.3 Efficiency

Consider a pooling contract, with  $\alpha_1 = \alpha_2 = 1$  and  $t_1 = t_2 = t$ . That is,  $\bar{Y} = \{C\}$ , where  $C = \{(1, t), (1, t)\}$ . Moreover,  $\tilde{\Theta}(C) = 1$ ,  $\tilde{\gamma}_i(C) = \pi_i$ , and  $\lambda(\{C\}) = 1$ . Finally, set  $t = \pi_1 a_1^B + \pi_2 a_2^B - k$ . The choice of  $t$  ensures that the resource constraint holds and the choice of  $\lambda$  ensures that markets clear. All sellers apply and the contract is trivially incentive compatible. The expected payoff for type  $i$  sellers is

$$\bar{U}_i = \pi_1 a_1^B + \pi_2 a_2^B - a_i^S - k.$$

With this pooling allocation, type 1 sellers are always better off than they were in equilibrium since  $a_1^B < a_2^B$ , and type 2 sellers are better off if and only if

$$\pi_1 a_1^B + \pi_2 a_2^B - a_2^S - k > \frac{(a_2^B - a_2^S - k)(a_1^B - a_1^S - k)}{a_2^B - a_1^S - k}.$$

Since  $\pi_2 = 1 - \pi_1$ , this reduces to

$$\pi_1 < \frac{a_2^B - a_2^S - k}{a_2^B - a_1^S - k} = \frac{\bar{U}_2}{\bar{U}_1}.$$

Both the numerator and denominator are positive, but the numerator is smaller (gains from trade are smaller for type 2 sellers) because  $a_2^S > a_1^S$ . Thus type 2 sellers prefer the pooling allocation when there is not too much cross subsidization, or  $\pi_1$  is small, so that the cost of subsidizing type 1 sellers is worth the increased efficiency of trade.

### 7.4 No Trade

So far, we have assumed there are gains from trade for both types of sellers. Now suppose there are no gains from trade for type 1 apples,  $a_1^B \leq a_1^S + k$ . Then not only will type 1 seller fail to trade, in equilibrium, the entire market will shut down:

**Result 6** *If  $a_1^B \leq a_1^S + k$  then, in any equilibrium,  $\bar{U}_1 = \bar{U}_2 = 0$ .*

Notice that the market shuts down here even if there are still gains from trade in good apples,  $a_2^B > a_2^S + k$ . Intuitively, it is only possible to keep bad apples out of the market by reducing the probability of trade in good apples. If there is no market in bad apples, however, agents holding them would accept any probability of trade. Hence we cannot screen out bad apples, and this renders the good apple market inoperative. Whether this is related to the recent collapse in asset-backed securities markets seems worth further exploration.

## 8 Conclusion

We have developed a tractable general framework to analyze adverse selection in competitive search markets. Under our assumptions, there is a unique equilibrium, where principals post separating contracts. We characterized the equilibrium as the solution to a set of constrained optimization problems, and illustrated the use of the model through a series of examples.

We expect that one could extend the framework to dynamic situations, with repeated rounds of posting and search, as seems relevant in many applications—including labor and asset markets. It may also be interesting to study the case opposite to the one analyzed here, where the informed instead of the uninformed parties post contracts. In standard competitive search theory the outcome does not depend on who posts. With asymmetric information, contract posting by informed parties may introduce multiplicity of equilibrium through the usual signaling mechanism (see Delacroix and Shi (2007)). All of this is left for future work.

## APPENDIX

**Proof of Lemma 1.** In the first step, we prove that there exists a solution to (P). The second and third steps establish the stated properties of the solution.

Step 1: Consider  $i = 1$ . If the constraint set in (P-1) is empty, set  $\bar{U}_1 = 0$ . Otherwise, (P-1) is well-behaved, as the objective function is continuous and the constraint set compact. Hence, (P-1) has a solution and a unique maximum  $m_1$ . If  $m_1 \leq 0$  set  $\bar{U}_1 = 0$ ; otherwise set  $\bar{U}_1 = m_1$  and let  $(\theta_1, y_1)$  be one of the maximizers.

We now proceed by induction. Fix  $i > 1$  and assume that we have found  $\bar{U}_j$  for all  $j < i$  and  $(\theta_j, y_j)$  for all  $j \in \mathbb{I}^*$ ,  $j < i$ . Consider (P- $i$ ). If the constraint set is empty, set  $\bar{U}_i = 0$ . Otherwise, (P- $i$ ) again has a solution and a unique maximum  $m_i$ . If  $m_i \leq 0$ , set  $\bar{U}_i = 0$ ; otherwise let  $\bar{U}_i = m_i$  and let  $(\theta_i, y_i)$  be one of the maximizers.



Step 2: Suppose by way of contradiction that there exists  $i \in \mathbb{I}^*$  such that  $(\theta_i, y_i)$  solves (P- $i$ ) but  $\eta(\theta_i)v_i(y_i) > k$ . This together with  $\bar{U}_i = \mu(\theta_i)u_i(y_i) > 0$  implies that  $y_i \in \bar{\mathbb{Y}}_i$  and  $\mu(\theta_i) > 0$ . Fix  $\varepsilon > 0$  such that  $\eta(\theta_i)v_i(y) \geq k$  for all  $y \in B_\varepsilon(y_i)$ . Then A3 ensures there exists a  $y' \in B_\varepsilon(y_i)$  such that

$$\begin{aligned} u_j(y') &> u_j(y_i) \text{ for all } j \geq i, \\ u_j(y') &< u_j(y_i) \text{ for all } j < i. \end{aligned}$$

Then the pair  $(\theta_i, y')$  satisfies all the constraints of problem (P- $i$ ):

1.  $\eta(\theta_i)v_i(y') \geq k$  from the choice of  $\varepsilon$ ;
2.  $\mu(\theta_i)u_j(y') < \mu(\theta_i)u_j(y_i) \leq \bar{U}_j$  for all  $j < i$ , where the first inequality is by construction of  $y'$  and  $\mu(\theta_i) > 0$ , while the second holds since  $(\theta_i, y_i)$  solves (P- $i$ ).

Now  $(\theta_i, y')$  achieves a higher value than  $(\theta_i, y_i)$  for the objective function in (P- $i$ ), given  $\mu(\theta_i)u_i(y') > \mu(\theta_i)u_i(y_i)$ . Hence,  $(\theta_i, y_i)$  does not solve (P- $i$ ), a contradiction.

Step 3: Fix  $i \in \mathbb{I}^*$  and suppose by way of contradiction that there exists  $j > i$  such that  $\mu(\theta_i)u_j(y_i) > \bar{U}_j$ . Let  $h$  be the smallest such  $j$ . Since  $i \in \mathbb{I}^*$ ,  $\mu(\theta_i)u_i(y_i) = \bar{U}_i > 0$ , which implies  $\mu(\theta_i) > 0$  and  $u_i(y_i) > 0$ . Also, from the previous step  $(\theta_i, y_i)$  satisfies  $\eta(\theta_i)v_i(y_i) = k$ , which ensures  $\eta(\theta_i) > 0$  and  $v_i(y_i) > 0$ . In particular, this implies that  $y_i \in \bar{\mathbb{Y}}_i$ .

The pair  $(\theta_i, y_i)$  satisfies the constraints of (P- $h$ ) since

1.  $\eta(\theta_i)v_h(y_i) \geq \eta(\theta_i)v_i(y_i) = k$ , where the first inequality holds by A1 given  $h > i$  and  $y_i \in \bar{\mathbb{Y}}_i \subset \bar{\mathbb{Y}}$ , and the second comes from the previous step;
2.  $\mu(\theta_i)u_l(y_i) \leq \bar{U}_l$  for all  $l < h$ , which holds for
  - (a)  $l < i$  because  $(\theta_i, y_i)$  satisfy the constraints of (P- $i$ ),
  - (b)  $l = i$  because  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$  since  $i \in \mathbb{I}^*$ ,
  - (c)  $i < l < h$  by the choice of  $h$  as the smallest violation of  $\mu(\theta_i)u_j(y_i) > \bar{U}_j$ .

Since  $\mu(\theta_i)u_h(y_i) > \bar{U}_h \geq 0$ ,  $(\theta_i, y_i)$  is in the constraint set of (P- $h$ ) and delivers a strictly positive value for the objective function; hence  $h \in \mathbb{I}^*$ . But then the fact that  $\bar{U}_h$  is not the maximized value of (P- $h$ ) is a contradiction. ■

**Proof of Proposition 1.** We proceed by construction.

- The vector of expected utilities is  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$ .

- The set of posted contracts is  $\bar{\mathbb{C}} = \{C_i\}_{i \in \mathbb{I}^*}$  where  $C_i \equiv (y_i, \dots, y_i)$ .
- $\lambda$  is such that  $\lambda(\{C_i\}) = \pi_i \Theta(C_i)$  for any  $i \in \mathbb{I}^*$ .
- For  $i \in \mathbb{I}^*$  and  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ . Otherwise, for any incentive compatible  $C' = \{y'_1, \dots, y'_I\} \in \mathbb{C}$ , let  $J(C') = \{j | u_j(y'_j) > 0\}$  denote the types that attain positive utility from  $C'$ . If  $J(C') \neq \emptyset$  and  $\min_{j \in J(C')} \{\bar{U}_j / u_j(y'_j)\} < \bar{\mu}$  then

$$\mu(\Theta(C')) = \min_{j \in J(C')} \frac{\bar{U}_j}{u_j(y'_j)}.$$

If this equation is consistent with multiple values of  $\Theta(C')$ , pick the largest one. Otherwise, if  $J(C') = \emptyset$  or  $\min_{j \in J(C')} \{\bar{U}_j / u_j(y'_j)\} \geq \bar{\mu}$ , then  $\Theta(C') = \infty$ .

- For  $i \in \mathbb{I}^*$  and  $C_i = (y_i, \dots, y_i)$ , let  $\gamma_i(C_i) = 1$  and so  $\gamma_j(C_i) = 0$  for  $j \neq i$ . For any other  $C'$ , define  $\Gamma(C')$  such that  $\gamma_h(C') > 0$  only if  $h \in \arg \min_{j \in J(C')} \{\bar{U}_j / u_j(y'_j)\}$ . If there are multiple minimizers, let  $\gamma_h(C') = 1$  for the smallest such  $h$ . If  $J(C') = \emptyset$ , again choose  $\Gamma(C')$  arbitrarily, e.g.  $\gamma_1(C') = 1$ .

We now verify that all of the equilibrium conditions hold.

Condition (i): For any  $i \in \mathbb{I}^*$ ,  $(\theta_i, y_i)$  solves (P- $i$ ), and Lemma 1 implies  $\eta(\theta_i)v_i(y_i) = k$ . Thus, profit maximization and free entry hold for any  $\{C_i\}_{i \in \mathbb{I}^*}$ . Now consider an arbitrary incentive compatible contract; we claim that principals' profit maximization and free-entry condition is satisfied. Suppose, to the contrary, that there exists  $C' = (y'_1, \dots, y'_I) \in \mathbb{C}$  with  $\eta(\Theta(C')) \sum_i \gamma_i(C')v_i(y'_i) > k$ . This implies  $\eta(\Theta(C')) > 0$ , so  $\Theta(C') < \infty$ , and there exists some type  $j$  with  $\gamma_j(C') > 0$  and  $\eta(\Theta(C'))v_j(y'_j) > k$ . Since  $\gamma_j(C') > 0$  and  $\Theta(C') < \infty$ , our construction of  $\Theta(C')$  and  $\Gamma(C')$  implies that  $j$  is the smallest solution to  $\min_{h \in J(C')} \{\bar{U}_h / u_h(y'_h)\}$  and hence  $u_j(y'_j) > 0$  and  $\bar{U}_j = \mu(\Theta(C'))u_j(y'_j)$ . So for all  $h < j$  with  $u_h(y'_h) > 0$ ,

$$\bar{U}_h > \mu(\Theta(C'))u_h(y'_h) \geq \mu(\Theta(C'))u_h(y'_j),$$

where the first inequality is by construction and the second follows because  $C'$  is incentive compatible. Moreover, if  $u_h(y'_h) \leq 0$ ,  $\bar{U}_h \geq \mu(\Theta(C'))u_h(y'_h)$  since  $\bar{U}_h \geq 0$ . This proves that  $(\Theta(C'), y'_j)$  satisfies the constraints of (P- $j$ ). Then, since  $\eta(\Theta(C'))v_j(y'_j) > k$  and  $\bar{U}_h \geq \mu(\Theta(C'))u_h(y'_j)$  for all  $h < j$ , with strict inequality when  $u_h(y'_h) > 0$ , there exist a  $\theta > \Theta(C')$  such that  $(\theta, y'_j)$  satisfies all the constraints of (P- $j$ ) and achieves a higher value than  $(\Theta(C'), y'_j)$ . This implies  $\bar{U}_j$  is not the maximized value of (P- $j$ ), a contradiction.

Condition (ii): By construction,  $\Theta$  and  $\Gamma$  ensure that  $\bar{U}_i \geq \mu(\Theta(C'))u_i(y'_i)$  for all contracts  $C' = \{y'_1, \dots, y'_I\}$ , with equality if  $\Theta(C') < \infty$  and  $\gamma_i(C') > 0$ . Moreover, for any  $i \in \mathbb{I}^*$ ,

$\bar{U}_i = \mu(\theta_i)u_i(y_i) > 0$  where  $\theta_i = \Theta(C_i)$  and  $C_i = \{y_i, \dots, y_i\}$  is the equilibrium contract offered to  $i$ . Finally, if  $u_i(y_i) < 0$  so  $i \notin J(C')$ , either  $J(C') = \emptyset$ , in which case  $\Theta(C') = \infty$ , or  $J(C') \neq \emptyset$ , in which case  $\gamma_h(C') = 1$  for some  $h \in J(C')$  and so  $\gamma_i(C') = 0$ .

Condition (iii): Market clearing obviously holds given the way we construct  $\lambda$ . Since all the equilibrium conditions are satisfied the proof is complete. ■

**Proof of Proposition 2.** From equilibrium condition (i), any  $C \in \bar{\mathcal{C}}$  has  $\eta(\Theta(C)) > 0$ , hence  $\Theta(C) < \infty$ . From condition (iii),  $\bar{U}_i > 0$  implies  $\gamma_i(C) > 0$  for some  $C \in \bar{\mathcal{C}}$ . This proves that for each  $i \in \mathbb{I}^*$ , there exists a contract  $C \in \bar{\mathcal{C}}$  with  $\Theta(C) < \infty$  and  $\gamma_i(C) > 0$ .

The remainder of the proof proceeds in five steps. The first four steps show that for any  $i \in \mathbb{I}^*$  and  $C_i \in \bar{\mathcal{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $\gamma_i(C_i) > 0$ ,  $(\theta_i, y_i)$  solves (P- $i$ ). First, we prove that the constraint  $\eta(\theta_i)v_i(y_i) \geq k$  is satisfied. Second, we prove that the constraint  $\mu(\theta_i)u_j(y_i) \leq \bar{U}_j$  is satisfied for all  $j$ . Third, we prove that the pair  $(\theta_i, y_i)$  delivers  $\bar{U}_i$  to type  $i$ . Fourth, we prove that  $(\theta_i, y_i)$  solves (P- $i$ ). The fifth and final step shows that for any  $i \notin \mathbb{I}^*$ , either the constraint set of (P- $i$ ) is empty or the maximized value is nonpositive.

Step 1: Take  $i \in \mathbb{I}^*$  and  $C_i \in \bar{\mathcal{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $\gamma_i(C_i) > 0$ . We claim the constraint  $\eta(\theta_i)v_i(y_i) \geq k$  is satisfied in (P- $i$ ). Note first that  $i \in \mathbb{I}^*$  implies  $\bar{U}_i > 0$ . By equilibrium condition (ii),  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$ , and  $\mu(\theta_i) > 0$ . To derive a contradiction, assume  $\eta(\theta_i)v_i(y_i) < k$ . Equilibrium condition (i) implies  $\eta(\theta_i) \sum_j \gamma_j(C) v_j(y_j) = k$ , so there is an  $h$  with  $\gamma_h(C) > 0$  and  $\eta(\theta_i)v_h(y_h) > k$ . Since  $\eta(\theta_i) \leq \bar{\eta}$ ,  $\bar{\eta}v_h(y_h) > k$ . Moreover, because  $\theta_i = \Theta(C) < \infty$  and  $\gamma_h(C) > 0$ , optimal search implies  $u_h(y_h) > 0$ . This proves  $y_h \in \bar{Y}_h$ .

Next, fix  $\varepsilon > 0$  such that  $\eta(\theta_i)v_h(y) > k$  for all  $y \in B_\varepsilon(y_h)$ . Then A3 together with  $y_h \in \bar{Y}$  guarantees that there exists  $y' \in B_\varepsilon(y_h)$  such that

$$\begin{aligned} u_j(y') &> u_j(y_h) \text{ for all } j \geq h, \\ u_j(y') &< u_j(y_h) \text{ for all } j < h. \end{aligned}$$

Notice that  $y' \in \bar{Y}_h$  as well, given that  $u_h(y') > u_h(y_h) > 0$  and  $\bar{\eta}v_h(y') \geq \eta(\theta_i)v_h(y') > k$ .

Now consider  $C' = \{y', \dots, y'\}$  and  $\theta' \equiv \Theta(C')$ . Note that

$$\mu(\theta')u_h(y') \leq \bar{U}_h = \mu(\theta_i)u_h(y_h) < \mu(\theta_i)u_h(y'),$$

where the weak inequality follows from optimal search, the equality holds because  $\theta_i < \infty$  and  $\gamma_h(C) > 0$ , and the strict inequality holds by the construction of  $y'$ , since  $\mu(\theta_i) > 0$ . This implies  $\mu(\theta') < \mu(\theta_i)$ , and  $\theta' < \theta_i$ . Next observe that for all  $j < h$ , either  $u_j(y') < 0$ , in

which case  $\gamma_j(C') = 0$  by equilibrium condition (ii), or

$$\mu(\theta')u_j(y') < \mu(\theta_i)u_j(y_h) \leq \mu(\theta_i)u_j(u_j) \leq \bar{U}_j,$$

where the first inequality uses  $\mu(\theta') < \mu(\theta_i)$ , the second uses incentive compatibility, and the third follows from optimal search. Hence,  $\gamma_j(C') = 0$  for all  $j < h$ .

Finally, profits from posting  $C'$  are

$$\eta(\theta') \sum_{j=1}^I \gamma_j(C')v_j(y') \geq \eta(\theta')v_h(y') \geq \eta(\theta_i)v_h(y') > k.$$

The first inequality follows because  $\gamma_j(C') = 0$  if  $j < h$  and  $v_h(y')$  is nondecreasing in  $h$  by A1 together with  $y' \in \bar{Y}_h \subset \bar{Y}$ , the second follows because  $\theta' < \theta_i$ , and the last inequality uses the construction of  $\varepsilon$ . Posting  $C'$  is therefore strictly profitable, which is a contradiction, and this completes Step 1.

Step 2: Again take  $i \in \mathbb{I}^*$  and  $C_i \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $\gamma_i(C_i) > 0$ . Equilibrium condition (ii) implies  $\mu(\theta_i)u_j(y_j) \leq \bar{U}_j$  for all  $j$  while incentive compatibility implies  $u_j(y_i) \leq u_j(y_j)$ . Hence the constraint  $\mu(\theta_i)u_j(y_i) \leq \bar{U}_j$  in (P- $i$ ) is satisfied for all  $j$ .

Step 3: Again take  $i \in \mathbb{I}^*$  and  $C_i \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $\gamma_i(C_i) > 0$ . Equilibrium condition (ii) implies  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$ , since  $\theta_i < \infty$  and  $\gamma_i(C_i) > 0$ . Hence  $(\theta_i, y_i)$  delivers  $\bar{U}_i$  to type  $i$ .

Step 4: Again take  $i \in \mathbb{I}^*$  and  $C_i \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $\gamma_i(C_i) > 0$ . To find a contradiction, suppose there exists  $(\theta', y')$  that satisfies the constraints of (P- $i$ ) but delivers higher utility. That is,  $\eta(\theta')v_i(y') \geq k$ ,  $\mu(\theta')u_j(y') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y') > \bar{U}_i$ . We now use A2. Note that  $\mu(\theta')u_i(y') > \bar{U}_i > 0$  implies  $\mu(\theta') > 0$  and  $u_i(y') > 0$ , while  $\eta(\theta')v_i(y') \geq k$  implies  $v_i(y') > 0$  and so  $\bar{\eta}v_i(y') \geq k$ . In particular,  $y' \in \bar{Y}_i$ . We can therefore fix  $\varepsilon' > 0$  such that for all  $y \in B_{\varepsilon'}(y')$ ,  $\mu(\theta')u_i(y) > \bar{U}_i$ , and then choose  $y'' \in B_{\varepsilon'}(y')$  such that  $v_i(y'') > v_i(y')$  and  $u_j(y'') \leq u_j(y')$  for all  $j < i$ . This ensures  $\eta(\theta')v_i(y'') > k$ ,  $\mu(\theta')u_j(y'') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y'') > \bar{U}_i$ . Note that we still have  $y'' \in \bar{Y}_i$ .

We now use A3. Fix  $\varepsilon'' > 0$  such that for all  $y \in B_{\varepsilon''}(y'')$ ,  $\eta(\theta')v_i(y) > k$  and  $\mu(\theta')u_i(y) > \bar{U}_i$ . Choose  $y''' \in B_{\varepsilon''}(y'')$  such that

$$u_j(y''') > u_j(y'') \text{ for all } j \geq i,$$

$$u_j(y''') < u_j(y'') \text{ for all } j < i.$$

This ensures  $\eta(\theta')v_i(y''') > k$ ,  $\mu(\theta')u_j(y''') < \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y''') > \bar{U}_i$ . Note that we still have  $y''' \in \bar{Y}_i$ .

Now consider  $C''' \equiv \{y''', \dots, y'''\}$ . From equilibrium condition (ii),  $\mu(\theta')u_i(y''') > \bar{U}_i$  implies that  $\mu(\theta') > \mu(\Theta(C'''))$ , which guarantees  $\eta(\Theta(C'''))v_i(y''') > k$ . This also implies  $\Theta(C''') < \infty$ .

We next claim  $\gamma_j(y''') = 0$  for all  $j < i$ . Suppose  $\gamma_j(y''') > 0$  for  $j < i$ . Since  $\Theta(C''') < \infty$ , equilibrium condition (ii) implies  $u_j(y''') \geq 0$ . We have already shown that  $\mu(\theta') > \mu(\Theta(C'''))$ , and  $u_j(y'') > u_j(y''')$  by construction. Thus  $\mu(\theta')u_j(y'') > \mu(\Theta(C'''))u_j(y''')$ . The result follows from equilibrium condition (ii) and  $\bar{U}_j \geq \mu(\theta')u_j(y'')$ ; i.e.,  $\bar{U}_j > \mu(\Theta(C'''))u_j(y''')$  and  $\Theta(C''') < \infty$  implies  $\gamma_j(C''') = 0$ , a contradiction.

The profit from offering this contract is

$$\eta(\Theta(C''')) \sum_{j=1}^I \gamma_j(C''')v_j(y''') \geq \eta(\Theta(C'''))v_i(y''') > k,$$

where the first inequality uses  $\gamma_j(C''') = 0$  for  $j < i$  and A1. This contradicts the first condition in the definition of equilibrium, and proves  $(\theta_i, y_i)$  solves (P- $i$ ).

Step 5: Suppose there is an  $i \notin \mathbb{I}^*$  for which the constraint set of (P- $i$ ) is nonempty and the maximized value is positive. That is, suppose there exists  $(\theta', y')$  such that  $\eta(\theta')v_i(y') \geq k$ ,  $\mu(\theta')u_j(y') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y') > \bar{U}_i = 0$ . Replicating step 4, we can first find  $y''$  such that  $\eta(\theta')v_i(y'') > k$ ,  $\mu(\theta')u_j(y'') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y'') > 0$ . Then we find  $y'''$  such that  $\eta(\theta')v_i(y''') > k$ ,  $u_j(y'') > u_j(y''')$  for  $j < i$ , and  $u_j(y'') < u_j(y''')$  for  $j \geq i$ . Finally,  $C''' = \{y''', \dots, y'''\}$  only attracts type  $i$  or higher and hence must be profitable and deliver positive utility, a contradiction. ■

**Proof of Proposition 3.** By Lemma 1 there is a solution to (P). Proposition 1 shows that if  $\mathbb{I}^*$ ,  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  solve (P), there is an equilibrium  $\{\bar{U}, \lambda, \bar{C}, \Theta, \Gamma\}$  with the same  $\bar{U}$ ,  $\bar{C} = \{C_i\}_{i \in \mathbb{I}^*}$ , where  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ , and  $\gamma_i(C_i) = 1$ . This proves existence. Proposition 2 shows that in any equilibrium  $\{\bar{U}, \lambda, \bar{C}, \Theta, \Gamma\}$ ,  $\bar{U}_i$  is the maximum value of (P- $i$ ) for all  $i \in \mathbb{I}^*$ , and  $\bar{U}_i = 0$  otherwise. Lemma 1 shows there is a unique maximum value  $\bar{U}_i$  for (P- $i$ ) for all  $i \in \mathbb{I}^*$ . This proves that the equilibrium  $\bar{U}$  is unique. ■

**Proof of Proposition 4.** Consider  $i = 1$ . Fix  $y$  satisfying  $\bar{\eta}v_1(y) > k$  and  $u_1(y) > 0$ . Then fix  $\theta > 0$  satisfying  $\eta(\theta)v_1(y) = k$ . These points satisfy the constraints of (P-1) and deliver utility  $\mu(\theta)u_1(y) > 0$ . This proves  $\bar{U}_1 > 0$ . Now suppose  $\bar{U}_j > 0$  for all  $j < i$ . We claim  $\bar{U}_i > 0$ . Again fix  $y$  satisfying  $\bar{\eta}v_i(y) > k$  and  $u_i(y) > 0$ . Then fix  $\theta > 0$  satisfying  $\eta(\theta)v_i(y) \geq k$  and  $\mu(\theta)u_j(y) \leq \bar{U}_j$  for all  $j < i$ ; this is feasible since  $\bar{U}_j > 0$  and  $\mu(0) = 0$ . These points satisfy the constraints of (P- $i$ ) and deliver utility  $\mu(\theta)u_i(y) > 0$ , which proves

$\bar{U}_i > 0$ . By induction, the proof is complete. ■

**Proof of Result 1.** Using  $\eta(\theta) = \mu(\theta)/\theta$ , write (P-1) as

$$\bar{U}_1 = \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_1} \right) \text{ s.t. } \frac{\mu(\theta)}{\theta} (b_1 - t) \geq k.$$

By Lemma 1 the constraint is binding, so we can eliminate  $t$  and reduce the problem to

$$\bar{U}_1 = \max_{\theta \in [0, \infty], x \in [0, b_2 a_2]} \mu(\theta) \left( b_1 - \frac{x}{a_1} \right) - \theta k.$$

At the solution,  $x = 0$  and  $\theta = \theta_1$  solves  $\mu'(\theta_1)b_1 = k$ . Using this to eliminate  $b_1$  from the objective function delivers  $\bar{U}_1$ , and the constraint delivers  $t_1$ .

Next, solve (P-2) using the  $\bar{U}_1$  derived in the previous step:

$$\begin{aligned} \bar{U}_2 &= \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_2} \right) \\ \text{s.t. } &\frac{\mu(\theta)}{\theta} (b_2 - t) \geq k, \\ \text{and } &\mu(\theta) \left( t - \frac{x}{a_1} \right) \leq \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k. \end{aligned}$$

Again we can eliminate  $t$ . It is easy to see that the second constraint binds, so we can eliminate  $x$  and then check that at the solution  $x \geq 0$ . Thus, the problem reduces to

$$\bar{U}_2 = \max_{\theta \in [0, \infty]} \left[ \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k + \left( 1 - \frac{a_1}{a_2} \right) (\mu(\theta)b_2 - \theta k) \right].$$

Now  $\theta = \theta_2$  solves  $\mu'(\theta_2)b_2 = k$ , where the concavity of  $\mu$  implies  $\theta_1 < \theta_2$ . Substituting into the objective function gives  $\bar{U}_2$ . Then the constraints give  $t_2$  and  $x_2$ . Concavity of  $\mu$  ensures that  $\mu(\theta)/\mu'(\theta) - \theta$  is increasing in  $\theta$ , which implies  $x_2 > 0$ . ■

**Proof of Result 2.** Fix  $x_1 \leq a_1(b_2 - b_1)(1 - \pi_1)/\pi_1$ . We now construct an equilibrium. Assume  $\bar{\mathbb{C}} = \{C\}$  where  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$ , and  $\bar{U}_1 = \bar{U}_2 = \bar{U} = \mu(\theta)t$ , where  $t$  and  $\theta$  are defined above. Moreover,  $\Theta(C) = \theta$ ,  $\gamma_i(C) = \pi_i$  for  $i = 1, 2$ , and  $\lambda(\{C\}) = \theta$ . For any other incentive compatible contract  $C' = \{(t'_1, x'_1), (t'_2, x'_2)\} \in \mathbb{C}$ ,  $C' \neq C$ , suppose  $\Theta(C')$  solves

$$\bar{U} = \mu(\Theta(C')) \left( t'_1 - \frac{x'_1}{a_1} \right)$$

if this defines  $\Theta(C') < \infty$ ; if  $\bar{U} \geq \bar{\mu}(t'_1 - x'_1/a_1)$ ,  $\Theta(C') = \infty$ . And suppose  $\gamma_1(C') = 1$  and  $\gamma_2(C') = 0$  for all such contracts. By construction, profit maximization and free entry hold

for  $C$ , with  $t$  chosen so that firms break even.

For any other incentive-compatible  $C' \neq C$ , profit maximization and free entry reduce to

$$\eta(\Theta(C'))(b_1 - t'_1) \leq k.$$

Since  $\eta(\infty) = 0$ , this holds if  $\Theta(C') = \infty$ . Otherwise, use  $\Theta(C')$  to eliminate  $t'_1$ . We need to show that

$$\mu(\Theta(C')) \left( b_1 - \frac{x'_1}{a_1} \right) - \Theta(C')k \leq \bar{U}.$$

An upper bound on the left hand side is obtained setting  $x'_1 = 0$  and choosing  $\Theta(C')$  to maximize  $\mu'(\Theta(C'))b_1 = k$ . The restriction  $x_1 \leq a_1(b_2 - b_1)(1 - \pi_1)/\pi_1$  implies that  $b_1 \leq \pi_1(b_1 - \frac{x_1}{a_1}) + \pi_2 b_2$ , from which it follows that  $\Theta(C') \leq \theta$ . That is,

$$\mu(\Theta(C')) \left( b_1 - \frac{x'_1}{a_1} \right) - \Theta(C')k \leq \left( \frac{\mu(\Theta(C'))}{\mu'(\Theta(C'))} - \Theta(C') \right) k \leq \left( \frac{\mu(\theta)}{\mu'(\theta)} - \theta \right) k = \bar{U},$$

where the first inequality uses the preceding discussion, the second inequality holds because  $\Theta(C') \leq \theta$ , and the third holds from the construction of  $\bar{U}$ .

Next, optimal search holds by construction for type 1. For type 2, we need to verify

$$\bar{U} \geq \mu(\Theta(C')) \left( t'_2 - \frac{x'_2}{a_2} \right).$$

To this end, note that  $t'_1 - x'_1/a_1 \geq t'_2 - x'_2/a_1 \geq t'_2 - x'_2/a_2$ , where the first inequality comes from incentive compatibility of  $C'$  and the second from  $a_1 \geq a_2$  and  $x'_2 \geq 0$ . This implies the desired inequality:

$$\mu(\Theta(C')) \left( t'_2 - \frac{x'_2}{a_2} \right) \leq \mu(\Theta(C')) \left( t'_1 - \frac{x'_1}{a_1} \right) = \bar{U},$$

Finally, market clearing holds by construction. ■

**Proof of Result 3.** Throughout this proof, we suppose there is an equilibrium characterized by the single incentive compatible contract  $C = \{(t_1, x_1), (t_2, x_2)\}$ . In the first step we prove that  $x_2 = 0$  and in the second that  $t_1 = t_2 + x_1/a_1$ .

Step 1: Suppose  $x_2 > 0$ . Given that  $\gamma_i(C) = \pi_i > 0$  for  $i = 1, 2$  and  $\Theta(C) < \infty$ , optimal search requires

$$\bar{U}_i = \mu(\Theta(C)) \left( t_i - \frac{x_i}{a_i} \right),$$

for  $i = 1, 2$ . Now consider  $C' = \{(t_2 - x_2/a_2, 0), (t_2 - x_2/a_2, 0)\}$ . Optimal search by type 2

requires

$$\mu(\Theta(C')) \left( t_2 - \frac{x_2}{a_2} \right) \leq \bar{U}_2 = \mu(\Theta(C)) \left( t_2 - \frac{x_2}{a_2} \right),$$

which implies that  $\Theta(C') \leq \Theta(C)$ . Moreover, notice that

$$t_1 - \frac{x_1}{a_1} \geq t_2 - \frac{x_2}{a_1} > t_2 - \frac{x_2}{a_2},$$

where the first inequality follows from incentive compatibility and the second from  $a_1 > a_2$  and  $x_2 > 0$ .

Together with  $\Theta(C') \leq \Theta(C)$ , this implies

$$\bar{U}_1 = \mu(\Theta(C)) \left( t_1 - \frac{x_1}{a_1} \right) > \mu(\Theta(C')) \left( t_2 - \frac{x_2}{a_2} \right).$$

Thus,  $\gamma_1(C') = 0$ . Hence, given that  $\Theta(C') \leq \Theta(C) < \infty$ ,  $\gamma_2(C') = 1$  and  $\Theta(C') = \Theta(C)$ . Then expected profit for a firm posting  $C'$  is

$$\eta(\Theta(C')) \left( b_2 - t_2 + \frac{x_2}{a_2} \right) > \eta(\Theta(C)) (\pi_1(b_1 - t_1) + \pi_2(b_2 - t_2)) = k.$$

The first inequality follows from  $b_1 < b_2$ ,  $t_1 > t_2 - x_2/a_2 + x_1/a_1 \geq t_2 - x_2/a_2$ , and  $\Theta(C') = \Theta(C)$ , while the second follows from the fact that in the proposed equilibrium firms post  $C$  and break even. Hence,  $C'$  represents a profitable deviation, a contradiction.

Step 2: Incentive compatibility and  $x_2 = 0$  imply that  $t_1 - x_1/a_1 \geq t_2$ . By way of contradiction, suppose  $t_1 - x_1/a_1 > t_2$ . Consider  $C' = \{(t_2 + x_1/a_1, x_1), (t_2, 0)\}$ . Then

$$\mu(\Theta(C')) t_2 \leq \bar{U}_2 = \mu(\Theta(C)) t_2,$$

where the first inequality follows from optimal search by type 2 for  $C'$  and the second from optimal search for  $C$  together with  $\gamma_2(C) = \pi_2 > 0$ . This implies that  $\Theta(C') \leq \Theta(C)$ . Hence,

$$\bar{U}_1 = \mu(\Theta(C)) \left( t_1 - \frac{x_1}{a_1} \right) > \mu(\Theta(C')) t_2,$$

where the first equality follows from optimal search by type 1 for  $C$  and  $\gamma_1(C) = \pi_1 > 0$ , while the second comes from  $\Theta(C') \leq \Theta(C)$  and  $t_1 - x_1/a_1 > t_2$ . Hence,  $\gamma_1(C') = 0$ , and, given that  $\Theta(C') \leq \Theta(C) < \infty$ ,  $\gamma_2(C') = 1$  and  $\Theta(C') = \Theta(C)$ . So the expected profit from  $C'$  is

$$\eta(\Theta(C')) (b_2 - t_2) > \eta(\Theta(C)) (\pi_1(b_1 - t_1) + \pi_2(b_2 - t_2)) = k,$$



given that  $\Theta(C') = \Theta(C)$  and  $b_2 - t_2 > b_1 - (t_1 - x_1/a_1) > b_1 - t_1$ . Hence  $C'$  represents a profitable deviation, a contradiction. ■

**Proof of Result 4.** For  $i \leq i^*$ , consider (P- $i$ ) without the constraint of keeping out lower types. This relaxed problem should yield a higher payoff

$$\begin{aligned} \bar{U}_i &\leq \max_{\theta \in [0, \infty], (c_e, c_u) \in \mathbb{Y}} \min\{\theta, 1\} (p_i U(c_e) + (1 - p_i) U(c_u)) \\ \text{s.t. } &\min\{1, \theta^{-1}\} (p_i(1 - c_e) - (1 - p_i)c_u) \geq k. \end{aligned}$$

At the solution,  $c_{u,i} = c_{e,i} = c_i$ , so this reduces to

$$\begin{aligned} \bar{U}_i &\leq \max_{\theta \in [0, \infty], c \geq \underline{c}} \min\{\theta, 1\} U(c) \\ \text{s.t. } &\min\{1, \theta^{-1}\} (p_i - c) \geq k. \end{aligned}$$

Either the constraint set is empty (if  $p_i < k + \underline{c}$ ) or there are no points in the constraint set that give positive utility (given that  $p_i - k \leq 0$ ). In any case, this gives  $\bar{U}_i = 0$ .

Turn next to a typical problem (P- $i$ ),  $i > i^*$ :

$$\begin{aligned} \bar{U}_i &= \max_{\theta \in [0, \infty], (c_e, c_u) \in \mathbb{Y}} \min\{\theta, 1\} (p_i U(c_e) + (1 - p_i) U(c_u)) \\ \text{s.t. } &\min\{1, \theta^{-1}\} (p_i(1 - c_e) - (1 - p_i)c_u) \geq k \\ &\text{and } \min\{\theta, 1\} (p_j U(c_e) + (1 - p_j) U(c_u)) \leq \bar{U}_j \text{ for all } j < i. \end{aligned}$$

We claim the solution sets  $\theta_i \leq 1$  (if  $\theta_i > 1$ , reducing  $\theta_i$  to 1 relaxes the first constraint without otherwise affecting the problem). Hence, we can rewrite the problem as

$$\begin{aligned} \bar{U}_i &= \max_{\theta \leq 1, (c_e, c_u) \in \mathbb{Y}} \theta (p_i U(c_e) + (1 - p_i) U(c_u)) \\ \text{s.t. } &p_i(1 - c_e) - (1 - p_i)c_u \geq k \\ &\text{and } \theta (p_j U(c_e) + (1 - p_j) U(c_u)) \leq \bar{U}_j \text{ for all } j < i. \end{aligned}$$

Lemma 1 ensures the first constraint is binding, which proves  $p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} = k$ . It remains to prove that  $\theta_i = 1$ ,  $c_{e,i} > c_{e,i-1}$ ,  $c_{u,i} < c_{u,i-1}$ , and the constraint for  $j = i - 1$  binds.

We start by establishing these claims for  $i = i^* + 1$ . In this case,  $c_{e,i} = c_{u,i} = 0$  satisfies the constraints but leaves the first one slack. Lemma 1 implies that it is possible to do better, which proves  $\bar{U}_i > 0$ . On the other hand, consider any  $(c_{e,i}, c_{u,i})$  that delivers positive utility

and satisfies the last constraint, so

$$p_i U(c_{e,i}) + (1 - p_i) U(c_{u,i}) > 0 \text{ and } p_j U(c_{e,i}) + (1 - p_j) U(c_{u,i}) \leq 0$$

for all  $j < i$ . Subtracting inequalities gives

$$(p_i - p_j)(U(c_{e,i}) - U(c_{u,i})) > 0,$$

which proves  $c_{e,i} > c_{u,i}$ . Now if  $c_{e,i} > c_{u,i} \geq 0$ ,  $p_j U(c_{e,i}) + (1 - p_j) U(c_{u,i}) > 0$ , so this is infeasible. If  $0 \geq c_{e,i} > c_{u,i}$ ,  $p_i U(c_{e,i}) + (1 - p_i) U(c_{u,i}) < 0$ , so this is suboptimal. This proves  $c_{e,i} > 0 > c_{u,i}$  when  $i = i^* + 1$ . Finally, since  $p_j U(c_{e,i}) + (1 - p_j) U(c_{u,i}) \leq 0$  for all  $j < i$ , setting  $\theta_i = 1$  raises the value of the objective function without affecting the constraints and so is optimal.

We now proceed by induction. Fix  $i > i^* + 1$  and assume that for all  $j \in \{i^* + 1, \dots, i - 1\}$ , we have  $c_{e,j} > c_{e,j-1}$ ,  $c_{u,j} < c_{u,j-1}$ ,  $\theta_j = 1$ , and

$$p_{j-1} U(c_{e,j}) + (1 - p_{j-1}) U(c_{u,j}) = p_{j-1} U(c_{e,j-1}) + (1 - p_{j-1}) U(c_{u,j-1}) = \bar{U}_{j-1}.$$

We establish the result for  $i$ . Setting  $c_{e,i} = c_{e,i-1}$ ,  $c_{u,i} = c_{u,i-1}$ , and  $\theta_i = 1$  satisfies the constraints in (P- $i$ ). Since it leaves the first constraint slack, Lemma 1 implies it is possible to do better. Thus

$$\theta_i (p_i U(c_{e,i}) + (1 - p_i) U(c_{u,i})) > p_i U(c_{e,i-1}) + (1 - p_i) U(c_{u,i-1}).$$

On the other hand, incentive compatibility implies,

$$\theta_i (p_{i-1} U(c_{e,i}) + (1 - p_{i-1}) U(c_{u,i})) \leq \bar{U}_{i-1} = p_{i-1} U(c_{e,i-1}) + (1 - p_{i-1}) U(c_{u,i-1}).$$

Subtracting inequalities gives  $(p_i - p_{i-1})(\theta_i U(c_{e,i}) - U(c_{e,i-1}) - \theta_i U(c_{u,i}) + U(c_{u,i-1})) > 0$ . Using  $p_i > p_{i-1}$ , this implies  $\theta_i U(c_{e,i}) - U(c_{e,i-1}) > \theta_i U(c_{u,i}) - U(c_{u,i-1})$ . As before, we can rule out the possibility that  $\theta_i U(c_{u,i}) \geq U(c_{u,i-1})$ , because this is infeasible. We can rule out the possibility that  $\theta_i U(c_{e,i}) \leq U(c_{e,i-1})$ , because this is suboptimal. Hence  $\theta_i U(c_{e,i}) - U(c_{e,i-1}) > 0 > \theta_i U(c_{u,i}) - U(c_{u,i-1})$ . Now since  $U(c_{e,i-1}) > 0$  and  $\theta_i \in [0, 1]$ , the first inequality implies  $c_{e,i} > c_{e,i-1}$ . Since  $U(c_{u,i-1}) < 0$ , the second inequality implies  $c_{u,i} < c_{u,i-1}$ .

Next suppose  $\theta_i < 1$  and consider the following variation: raise  $\theta_i$  to 1 and increase  $c_e$  and reduce  $c_u$  while keeping both  $\theta(p_i U(c_e) + (1 - p_i) U(c_u))$  and  $p_i c_e + (1 - p_i) c_u$  unchanged;

i.e. set  $c_e > c_{e,i}$  and  $c_u < c_{u,i}$  so that

$$\begin{aligned}\theta_i(p_i U(c_{e,i}) + (1 - p_i)U(c_{u,i})) &= p_i U(c_e) + (1 - p_i)U(c_u) \\ \text{and } p_i c_{e,i} + (1 - p_i)c_{u,i} &= p_i c_e + (1 - p_i)c_u.\end{aligned}$$

For all  $j < i$ ,  $(p_j - p_i)(\theta_i U(c_{e,i}) - U(c_e) + U(c_u) - \theta_i U(c_{u,i})) > 0$  since  $p_j < p_i$ ,  $0 < c_{e,i} < c_e$ ,  $c_u < c_{u,i} < 0$ , and  $\theta_i < 1$ . Add this to  $\theta_i(p_i U(c_{e,i}) + (1 - p_i)U(c_{u,i})) = p_i U(c_e) + (1 - p_i)U(c_u)$  to obtain  $\theta_i(p_j U(c_{e,i}) + (1 - p_j)U(c_{u,i})) > p_j U(c_u) + (1 - p_j)U(c_e)$ . This implies the perturbation relaxes the remaining constraints and so is feasible. This proves  $\theta_i = 1$ .

Finally, we prove that the constraints for  $j < i - 1$  are slack. If not then

$$p_j U(c_{e,i}) + (1 - p_j)U(c_{u,i}) = \bar{U}_j \geq p_j U(c_{e,i-1}) + (1 - p_j)U(c_{u,i-1}),$$

where the inequality uses the constraints in problem (P-( $i - 1$ )). On the other hand,

$$p_{i-1} U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) \leq \bar{U}_{i-1} = p_{i-1} U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1}),$$

where the inequality is a constraint in (P- $i$ ) and the equality is the definition of  $\bar{U}_{i-1}$ . Subtracting these equations,  $(p_{i-1} - p_j)(U(c_{e,i-1}) - U(c_{e,i}) + U(c_{u,i}) - U(c_{u,i-1})) \geq 0$ . Since  $p_{i-1} > p_j$ ,  $c_{e,i} > c_{e,i-1}$ , and  $c_{u,i} < c_{u,i-1}$ , we have a contradiction. The constraints for all  $j < i - 1$  are slack, while the constraint for  $i - 1$  binds, otherwise the solution to (P- $i$ ) would have  $c_{e,i} = c_{u,i} = p_i - k$ . ■

**Proof of Result 5.** Write (P-1) as

$$\begin{aligned}\bar{U}_1 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\}(t - \alpha a_1^S) \\ \text{s.t. } \min\{1, \theta^{-1}\}(\alpha a_1^B - t) &\geq k.\end{aligned}$$

By Lemma 1 we can rewrite the problem as

$$\bar{U}_1 = \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\}\alpha(a_1^B - a_1^S) - \theta k.$$

Since  $a_1^B > a_1^S + k$ , it is optimal to set  $\alpha = \theta = 1$ . It follows that  $\bar{U}_1 = a_1^B - a_1^S - k$ .

Now consider (P-2)

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\}(t - \alpha a_2^S) \\ \text{s.t. } & \min\{1, \theta^{-1}\}(\alpha a_2^B - t) \geq k \\ & \min\{\theta, 1\}(t - \alpha a_1^S) \leq a_1^B - a_1^S - k.\end{aligned}$$

Again, we can eliminate  $t$  using the first constraint to write

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\}\alpha(a_2^B - a_2^S) - \theta k \\ \text{s.t. } & \min\{\theta, 1\}\alpha(a_2^B - a_1^S) - \theta k = a_1^B - a_1^S - k.\end{aligned}$$

Then use the last constraint to eliminate  $\alpha$  and write

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty]} \frac{a_1^B - a_1^S - (1 - \theta)k}{a_2^B - a_1^S} (a_2^B - a_2^S) - \theta k \\ \text{s.t. } & \frac{a_1^B - a_1^S - (1 - \theta)k}{\min\{\theta, 1\}(a_2^B - a_1^S)} \in [0, 1],\end{aligned}$$

where the constraint here ensures that  $\alpha$  is a probability. Since  $a_1^S < a_2^S < a_2^B$ , the objective function is decreasing in  $\theta$ , and we set it to the smallest value consistent with the constraints:

$$\theta_2 = \frac{a_1^B - a_1^S - k}{a_2^B - a_1^S - k} < 1.$$

This implies  $\alpha_2 = 1$ , so the constraint binds. Then  $\bar{U}_2$  is easy to compute. ■

**Proof of Result 6.** Write problem (P-1) as

$$\begin{aligned}\bar{U}_1 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\}(t - \alpha a_1^S) \\ \text{s.t. } & \min\{1, \theta^{-1}\}(\alpha a_1^B - t) \geq k.\end{aligned}$$

Again we can eliminate  $t$  and rewrite this as

$$\bar{U}_1 = \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\}\alpha(a_1^B - a_1^S) - \theta k.$$

Since  $a_1^B \leq a_1^S + k$ ,  $\bar{U}_1 = 0$ , which is attained by  $\theta = 0$ .

Now consider (P-2):

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\} (t - \alpha a_2^S) \\ \text{s.t. } & \min\{1, \theta^{-1}\} (\alpha a_2^B - t) \geq k \\ & \min\{\theta, 1\} (t - \alpha a_1^S) \leq 0.\end{aligned}$$

Eliminating  $t$  using the first constraint gives

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\} \alpha (a_2^B - a_2^S) - \theta k \\ \text{s.t. } & \min\{\theta, 1\} \alpha (a_2^B - a_1^S) - \theta k = 0.\end{aligned}$$

Eliminating  $\alpha$  using the last constraint gives

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty]} \frac{a_1^S - a_2^S}{a_2^B - a_1^S} \theta k \\ \text{s.t. } & \frac{\theta k}{\min\{\theta, 1\} (a_2^B - a_1^S)} \in [0, 1].\end{aligned}$$

Since  $a_2^B > a_2^S > a_1^S$ , the fraction in the objective function is negative. Hence  $\theta = 0$  and  $\bar{U}_2 = 0$ . ■

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