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ABSTRACT

This paper specifies a new convenient algorithm to construct policy projections conditional on alternative anticipated policy-rate paths in linearized dynamic stochastic general equilibrium (DSGE) models, such as Ramses, the Riksbank's main DSGE model. Such projections with anticipated policy-rate paths correspond to situations where the central bank transparently announces that it, conditional on current information, plans to implement a particular policy-rate path and where this announced plan for the policy rate is believed and then anticipated by the private sector. The main idea of the algorithm is to include among the predetermined variables (the "state" of the economy) the vector of nonzero means of future shocks to a given policy rule that is required to satisfy the given anticipated policy-rate path.

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1. Introduction

This paper specifies a new convenient way to construct policy projections conditional on alternative *anticipated* policy-rate paths in linearized dynamic stochastic general equilibrium (DSGE) models, such as Ramses, the Riksbank's main DSGE model.[1] Such projections with anticipated policy-rate paths correspond to situations where the central bank transparently announces that it, conditional on current information, plans to implement a particular policy-rate path and where this announced plan for the policy rate is believed and then anticipated by the private sector. Such projections are particularly relevant for central banks such as the Reserve Bank of New Zealand (RBNZ), Norges Bank, the Riksbank, and the Czech National Bank (CNB), where the policy announcement includes not only the current policy rate decision but also a forecast path for the future policy rate. They are also relevant in the discussion about the kind of "forward guidance" about the future policy rate that the Federal Reserve System and the Bank of Canada have recently given.

A common method to do policy simulations for alternative policy-rate paths is to add *unanticipated* shocks to a given instrument rule (a rule that specifies the policy rate as a function of observed variables), as in the method of modest interventions by Leeper and Zha [22] (see appendix D). That method is designed to deal with policy simulations that involve "modest" unanticipated deviations from a given instrument rule. Such policy simulations correspond to a situation when the central bank would nontransparently and secretly plan to surprise the private sector by deviations from an announced instrument rule (or, alternatively a situation when the central bank announces and follows a future path but the path is not believed by and each period surprises the private sector). Aside from corresponding to policy that is either non-transparent or lacks credibility, such deviations are in practical simulations often both serially correlated and large, which can be inconsistent with the assumption that they would remain unanticipated and interpreted as i.i.d. shocks by the private sector. In other words, they are in practice often not "modest" in the sense of Leeper and Zha. Projections with anticipated policy-rate paths would in many cases seem more relevant for the transparent flexible inflation targeting that central banks such as the RBNZ, Norges Bank, the Riksbank, and the CNB conduct.[2]

A standard way to incorporate anticipated shocks (that is, shocks with non-zero time-varying means) in an economic model with forward-looking variables is to use a deterministic, perfect-

[1] The policy rate (also called the instrument rate) is the short interest rate that the central bank uses as a (policy) instrument (control variable). For the Riksbank, the policy rate is the repo rate.

[2] However, as noted in Svensson [33], there are recent cases when the Riksbank's policy-rate path has been far from credible and when projections with unanticipated shocks may be more relevant.

foresight variant of the model where all future shocks are set equal to their means and are assumed to be known in the first period. Furthermore, a finite horizon is assumed, with a terminal condition where all variables equal their steady-state values. The problem can then be seen as a two-point boundary problem with an initial and a terminal condition. Stacking the model equations for the finite number of periods together with the initial and terminal condition gives rise to a finite-dimensional simultaneous equation system, nonlinear for a nonlinear model and linear for a linear model. The model can then be solved with the Fair-Taylor [13] algorithm or the so-called Stacked Time algorithm of Laffargue [21], Boucekkine [11], and Juillard [19]. The horizon is extended until it has a negligible effect on the solution.[3] The Dynare [12] collection of MatLab and Octave routines uses the Stacked Time algorithm for deterministic, perfect-foresight settings.

Assuming a linear model (a linearized DSGE model), we provide an alternative simple and convenient algorithm that allows a stochastic interpretation – more precisely a standard state-space representation of a stochastic linear model with forward-looking variables, the solution of which can be expressed in a recursive form and found with standard algorithms for the solution of linear rational-expectations systems, such as the Klein [20], Sims [28], or AIM algorithms (Anderson and Moore [9] and [10]). The main idea is to include among the predetermined variables (the "state" of the economy) the vector of nonzero means of future shocks to a given instrument rule. By modelling the shocks as a moving-average process – more precisely, the sum of zero-mean i.i.d. shocks – we allow a consistent stochastic interpretation of new information about the nonzero means. The policy-rate path can then be written as a function of the initial state of the economy, including the vector of anticipated shocks, and the vector of anticipated shocks can be chosen so as to result in any desired anticipated policy-rate path. This is a special case of the more general analysis of judgment in monetary policy in Svensson [31] and of optimal policy projections with judgment in Svensson and Tetlow [34].

Our algorithm thus adds an anticipated sequence of shocks to a general but constant policy rule, including targeting rules (conditions on the target variables, the variables that are the arguments of the loss function) and explicit or implicit instrument rules (instrument rules where the policy rate depends on predetermined variables only or also on forward-looking variables). It very conveniently allows the construction of policy projections for alternative arbitrary nominal and real policy-rate paths, whether or not these are optimal for a particular loss function or not.

We consider policy simulations where restrictions on the nominal or real policy-rate path are

[3] That is, one need only extend the horizon until such a point that the extension no longer affects the simulated results over the horizon of interest. This is a "type III iteration" in the parlance of Fair and Taylor [13].

eventually followed by an anticipated future switch to a given well-behaved policy rule, either optimal or arbitrary. With such a setup, there is a unique equilibrium for each specified set of restrictions on the nominal or real policy-rate path. The equilibrium will, in a model with forward-looking variables, depend on which future policy rule is implemented, but for any given such policy rule, the equilibrium is unique. It is well known since Sargent and Wallace [27] that an exogenous nominal policy-rate path will normally lead to indeterminacy in a model with forward-looking variables (and to an explosive development in a backward-looking model), so at some future time the nominal policy-rate must become endogenous for a well-behaved equilibrium to result (see also Gagnon and Henderson [14]). Such a setup with a switch to a well-behaved policy rule solves the problem with multiple equilibria for alternative policy-rate projections that Gali [15] has emphasized. On the other hand, consistent with Gali's results, the unique equilibrium depends on and is sensitive to both the time of the switch and the policy rule to which policy shifts.

We demonstrated our method for three different models, namely the small empirical backward-looking model of the U.S. economy of Rudebusch and Svensson [26], the small empirical forward-looking model of the U.S. economy of Lindé [23], and Ramses, the medium-sized model of the Swedish economy of Adolfson, Laséen, Lindé, and Villani [4].[4] From the examples examined in this paper, we see that, in a model without forward-looking variables such as the Rudebusch-Svensson model, there is no difference between policy simulations with anticipated and unanticipated restrictions on the policy-rate path. In a model with forward-looking variables, such as the Lindé model or Ramses, there is such a difference, and the impact of anticipated restrictions would generally be larger than that of unanticipated restrictions. In a model with forward-looking variables, exogenous restrictions on the policy-rate path are consistent with a unique equilibrium, if there is a switch to a well-behaved policy rule in the future. For given restrictions on the policy-rate path, the equilibrium depends on that policy rule.

If inflation is sufficiently sensitive to the real policy rate, "unusual" equilibria may result from restrictions for sufficiently many quarters on the nominal policy rate. Such cases have the property that a shift up of the real interest-rate path reduces inflation and inflation expectations so much that the nominal interest-rate path (which by the Fisher equation equals the real interest-rate path plus the path of inflation expectations) shifts down. Then, a shift up of the nominal interest-rate path requires an equilibrium where the path of inflation and inflation expectations shifts up more and the real policy-rate path shifts down. In the Rudebusch-Svensson model, which has no forward-

[4] See also Adolfson, Laséen, Lindé, and Villani [5] and [6].

looking variables, inflation is so sluggish and insensitive to changes in the real policy rate that there are only small differences between restrictions on the nominal and real policy rate. In the Lindé model, inflation is so sensitive to the real policy rate that restrictions for 5–6 quarters or more on the nominal policy rate result in unusual equilibria. In Ramses, unusual equilibria seem to require restrictions for 10 quarters or more. In order to avoid unusual equilibria, restrictions should be imposed for fewer quarters than that.

The paper is organized as follows: Section 2 presents the state-space representation of a linear(ized) DSGE model and shows how to do policy simulations with an arbitrary constant (that is, time-invariant) policy rule, such as an instrument rule or a targeting rule. Section 3 shows our convenient way of constructing policy projections that satisfy arbitrary anticipated restrictions on the nominal or real policy rate by introducing anticipated time-varying deviations in the policy rule. Section 4 provides examples of restrictions on nominal and real policy-rate paths for the Rudebusch-Svensson model, the Lindé model, and Ramses. Section 5 presents some conclusions.

A few appendices contain some technical details. Appendix A specifies the policy rule under optimal policy under commitment. Appendices B and C provide some details on the Rudebusch-Svensson and Lindé models, respectively. Appendix D demonstrates the Leeper and Zha [22] method of modest interventions in this framework.

2. The model

A linear model with forward-looking variables (such as a DSGE model like Ramses that is linearized around a steady state) can be written in the following practical state-space form,

$$\begin{bmatrix} X_{t+1} \\ Hx_{t+1|t} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + Bi_t + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{2.1}$$

for $t = ..., -1, 0, 1, ...$ Here, X_t is an n_X-vector of *predetermined* variables in period t (where the period is a quarter); x_t is an n_x-vector of *forward-looking* variables; i_t is generally an n_i-vector of (policy) *instruments* but in the cases examined here it is a scalar, the policy rate, in the Riksbank's case the repo rate, so $n_i = 1$; ε_t is an n_ε-vector of i.i.d. shocks with mean zero and covariance matrix I_{n_ε}; A, B, and C, and H are matrices of the appropriate dimension; and for any stochastic process y_t, $y_{t+\tau|t}$ denotes $\mathrm{E}_t y_{t+\tau}$, the rational expectation of $y_{t+\tau}$ conditional on information available in period t. The forward-looking variables and the instruments are the *nonpredetermined* variables.[5]

[5] A variable is predetermined if its one-period-ahead prediction error is an exogenous stochastic process (Klein [20]). For (2.1), the one-period-ahead prediction error of the predetermined variables is the stochastic vector $C\varepsilon_{t+1}$.

The variables can be measured as differences from steady-state values, in which case their unconditional means are zero. Alternatively, one of the components of X_t can be unity, so as to allow the variables to have nonzero means. The elements of the matrices A, B, C, and H are normally estimated with Bayesian methods. Here they are considered fixed and known for the policy simulations. More precisely, the matrices are considered structural, for instance, functions of the deep parameters in an underlying linearized DSGE model. Hence, with a linear model with additive uncertainty and a quadratic loss function as specified in appendix A, the conditions for certainty equivalence are satisfied, that is, mean forecasts are sufficient for policy decisions.

The upper block of (2.1) provides n_X equations determining the n_X-vector X_{t+1} in period $t+1$ for given X_t, x_t, i_t and ε_{t+1},

$$X_{t+1} = A_{11}X_t + A_{12}x_t + B_1 i_t + C\varepsilon_{t+1}, \tag{2.2}$$

where A and B are partitioned conformably with X_t and x_t as

$$A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{2.3}$$

The lower block provides n_x equations determining x_t in period t for given $x_{t+1|t}$, X_t, and i_t,

$$x_t = A_{22}^{-1}(Hx_{t+1|t} - A_{21}X_t - B_2 i_t). \tag{2.4}$$

Hence, we assume that the $n_x \times n_x$ submatrix A_{22} is nonsingular, which assumption must be satisfied by any reasonable model with forward-looking variables.[6]

In a backward-looking model, that is, a model without forward-looking variables, there is no vector x_t of forward-looking variables and no lower block of equations in (2.1).

With a constant (that is, time-invariant) arbitrary instrument rule, the policy rate satisfies

$$i_t = [f_X \quad f_x] \begin{bmatrix} X_t \\ x_t \end{bmatrix}, \tag{2.5}$$

where the $n_i \times (n_X + n_x)$ matrix $[f_X \quad f_x]$ is a given (linear) instrument rule and partitioned conformably with X_t and x_t. If $f_x \equiv 0$, the instrument rule is an *explicit* instrument rule; if $f_x \neq 0$, the instrument rule is an *implicit* instrument rule. In the latter case, the instrument rule is actually an equilibrium condition, in the sense that in a real-time analogue the policy rate in period t and the forward-looking variables in period t would be simultaneously determined.

[6] Without loss of generality, we assume that the shocks ε_t only enter in the upper block of (2.1), since any shocks in the lower block of (2.1) can be redefined as additional predetermined variables and introduced in the upper block.

The instrument rule that is estimated for Ramses is of the form (see the appendix of Adolfson, Laséen, Lindé, and Svensson [1] (ALLS1) for the notation)

$$\begin{aligned} i_t &= \rho_R i_{t-1} + (1-\rho_R)\left[\hat{\bar{\pi}}_t^c + r_\pi\left(\hat{\pi}_{t-1}^c - \hat{\bar{\pi}}_t^c\right) + r_y \hat{y}_{t-1} + r_x \hat{\tilde{x}}_{t-1}\right] \\ &\quad + r_{\Delta\pi}\left(\hat{\pi}_t^c - \hat{\pi}_{t-1}^c\right) + r_{\Delta y}\left(\hat{y}_t - \hat{y}_{t-1}\right) + \varepsilon_{Rt}. \end{aligned} \tag{2.6}$$

Since $\hat{\pi}_t^c$ and $\hat{y}_t$, the deviation of CPI inflation and output from trend, are forward-looking variables in Ramses, this is an implicit instrument rule.

An arbitrary more general (linear) policy rule (G, f) can be written as

$$G_x x_{t+1|t} + G_i i_{t+1|t} = f_X X_t + f_x x_t + f_i i_t, \tag{2.7}$$

where the $n_i \times (n_x + n_i)$ matrix $G \equiv [G_x \;\; G_i]$ is partitioned conformably with x_t and i_t and the $n_i \times (n_X + n_x + n_i)$ matrix $f \equiv [f_X \;\; f_x \;\; f_i]$ is partitioned conformably with X_t, x_t, and i_t. This general policy rule includes explicit, implicit, and forecast-based instrument rules (in the latter the policy rate depends on expectations of future forward-looking variables, $x_{t+1|t}$) as well as targeting rules (conditions on current, lagged, or expected future target variables).[7] When this general policy rule is an instrument rule, we require the $n_x \times n_i$ matrix f_i to be nonsingular, so (2.7) determines i_t for given X_t, x_t, $x_{t+1|t}$, and $i_{t+1|t}$.

The optimal instrument rule under commitment (see appendix A) can be written as

$$0 = F_{iX} X_t + F_{i\Xi} \Xi_{t-1} - i_t, \tag{2.8}$$

where the matrix F_i in (A.6) is partitioned conformably with X_t and Ξ_{t-1}. Here the n_x-vector of Lagrange multipliers Ξ_t in equilibrium follows

$$\Xi_t = M_{\Xi X} X_t + M_{\Xi\Xi} \Xi_{t-1}, \tag{2.9}$$

where the matrix M in (A.5) has been portioned conformably with X_t and Ξ_{t-1}. Thus, in order to include this optimal instrument rule in the set of policy rules (2.7) considered, the predetermined variables need to be augmented with Ξ_{t-1} and the equations for the predetermined variables with (2.9). For simplicity, the treatment below does not include this augmentation. Alternatively, below the vector of predetermined variables could consistently be augmented with the vector of Lagrange multipliers, so everywhere we would have $(X_t', \Xi_{t-1}')'$ instead of X_t, with corresponding augmentation of the relevant matrices.

[7] A targeting rule can be expressed in terms of expected leads, current values, and lags of the target variables (the arguments of the loss function); see Svensson [30], Svensson and Woodford [35], and Giannoni and Woodford [16].

The general policy rule can be added to the model equations (2.1) to form the new system to be solved. With the notation $\tilde{x}_t \equiv (x_t', i_t')'$, the new system can be written

$$\begin{bmatrix} X_{t+1} \\ \tilde{H}\tilde{x}_{t+1|t} \end{bmatrix} = \tilde{A} \begin{bmatrix} X_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} C \\ 0_{(n_x+n_i)\times n_\varepsilon} \end{bmatrix} \varepsilon_{t+1}, \tag{2.10}$$

for $t = ..., -1, 0, 1, ...$, where

$$\tilde{H} \equiv \begin{bmatrix} H & 0 \\ G_x & G_i \end{bmatrix}, \quad \tilde{A} \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ f_X & f_x & f_i \end{bmatrix},$$

and where $\tilde{H}$ is partitioned conformably with x_t and i_t and $\tilde{A}$ is partitioned conformably with X_t, x_t, and i_t.

Then, under the assumption that the policy rule gives rise to the saddlepoint property (that the number of eigenvalues with modulus greater than unity is equal to the number of non-predetermined variables), the system can be solved with the Klein [20] algorithm or the other algorithms for the solution of linear rational-expectations models mentioned in the introduction. The Klein algorithm generates the matrices M and F such that the resulting equilibrium satisfies

$$X_{t+1} = MX_t + C\varepsilon_{t+1}, \tag{2.11}$$

$$\tilde{x}_t \equiv \begin{bmatrix} x_t \\ i_t \end{bmatrix} = FX_t \equiv \begin{bmatrix} F_x \\ F_i \end{bmatrix} X_t \tag{2.12}$$

for $t = ..., -1, 0, 1, ...$, where the matrices M and F depend on $\tilde{A}$ and $\tilde{H}$, and thereby on A, B, H, G, and f.

In a backward-looking model, the time-invariant instrument rule depends on the vector of predetermined variables only, since there are no forward-looking variables, and the vector $\tilde{x}_t$ is identical to i_t.

Consider now *projections* in period t, that is, mean forecasts, conditional on information available in period t, of future realizations of the variables. For any stochastic vector process u_t, let $u^t \equiv \{u_{t+\tau,t}\}_{\tau=0}^{\infty}$ denote a *projection* in period t, where $u_{t+\tau,t}$ denotes the mean forecast of the realization of the vector in period $t+\tau$ conditional on information available in period t. We refer to τ as the horizon of the forecast $u_{t+\tau,t}$.

The projection (X^t, x^t, i^t) in period t is then given by (2.11) and (2.12) when we set the mean of future i.i.d. shocks equal to zero, $\varepsilon_{t+\tau,t} = \mathrm{E}_t\varepsilon_{t+\tau} = 0$ for $\tau > 0$. It then satisfies

$$X_{t+\tau,t} = M^\tau X_{t,t}, \tag{2.13}$$

$$\tilde{x}_{t+\tau,t} \equiv \begin{bmatrix} x_{t+\tau,t} \\ i_{t+\tau,t} \end{bmatrix} = FX_{t+\tau,t} \equiv \begin{bmatrix} F_x \\ F_i \end{bmatrix} X_{t+\tau,t} = \begin{bmatrix} F_x \\ F_i \end{bmatrix} M^\tau X_{t,t}, \tag{2.14}$$

$$X_{t,t} = X_{t|t}, \tag{2.15}$$

for $\tau \geq 0$, where $X_{t|t}$ is the estimate of predetermined variables in period t conditional on information available in the beginning of period t. Thus, "$,t$" and "$|t$" in subindices refer to projections (forecasting) and estimates ("nowcasting" and "backcasting") in the beginning of period t, respectively.

3. Projections with time-varying restrictions on the policy rate

The projection of the policy rate $i^t = \{i_{t+\tau,t}\}_{\tau=0}^{\infty}$ in period t is by (2.14) given by

$$i_{t+\tau,t} = F_i M^\tau X_{t+\tau,t}$$

for $\tau \geq 0$.[8]

Suppose now that we consider imposing restrictions on the policy-rate projection of the form

$$i_{t+\tau,t} = \bar{\imath}_{t+\tau,t}, \quad \tau = 0, ..., T, \tag{3.1}$$

where $\{\bar{\imath}_{t+\tau,t}\}_{\tau=0}^{T}$ is a sequence of $T+1$ given policy-rate levels. Alternatively, we can have restriction on the real policy-rate projection of the form

$$r_{t+\tau,t} = \bar{r}_{t+\tau,t}, \quad \tau = 0, ..., T, \tag{3.2}$$

where

$$r_t \equiv i_t - \pi_{t+1|t} \tag{3.3}$$

is the real policy rate and $\pi_{t+1|t}$ is expected inflation. With restrictions of this kind, the nominal or real policy rate is exogenous for period $t, t+1, ..., t+T$.

These restrictions are here assumed to be anticipated by both the central bank and the private sector, in contrast to Leeper and Zha [22] where they are anticipated and planned by the central bank but not anticipated by the private sector. Thus, our case corresponds to a situation where the restriction is announced to the private sector by the central bank and believed by the private sector, whereas the Leeper and Zha case corresponds to a situation where the central bank either

[8] The projection of the policy rate and the other variables will satisfy the policy rule,

$$G_x x_{t+\tau+1,t} + G_i i_{t+\tau+1,t} = f_X X_{t+\tau,t} + f_x x_{t+\tau,t} + f_i i_{t+\tau,t},$$

for $\tau \geq 0$.

makes secret plans to implement the restriction or the restriction is announced but not believed by the private sector.

The restrictions make the nominal or real policy-rate projection exogenous for the periods t, $t+1$, ..., $t+T$. We know from Sargent and Wallace [27] that exogenous interest rates may cause indeterminacy when there are forward-looking variables. In order to ensure determinacy, we assume that there is an anticipated switch in period $t+T+1$ to the policy rule (G, f). Then the restrictions can be implemented by augmenting a stochastic deviation, z_t, to the policy rule (2.7),

$$G_x x_{t+1|t} + G_i i_{t+1|t} = f_X X_t + f_x x_t + f_i i_t + z_t. \tag{3.4}$$

The projection $\{z_{t+\tau,t}\}_{\tau=0}^{T}$ of the future deviations is then chosen such that (3.1) or (3.2) is satisfied. The projection of the future deviation from the horizon $T+1$ and beyond is zero, corresponding to the anticipated shift then to the policy rule (G, f).

More precisely, we let the $(T+1)$-vector $z^t \equiv (z_{t,t}, z_{t+1,t}, ..., z_{t+T,t})'$ (where $z_{t,t} = z_t$) denote a projection of the stochastic variable $z_{t+\tau}$ for $\tau = 0, ..., T$. As in the treatment of central-bank judgment in Svensson [31], the stochastic variable z_t is called the deviation. In particular, we assume that the deviation is a moving-average process that satisfies

$$z_t = \eta_{t,t} + \sum_{s=1}^{T} \eta_{t,t-s}$$

for a given $T \geq 0$, where $\eta^t \equiv (\eta'_{t,t}, \eta'_{t+1,t}, ..., \eta'_{t+T,t})'$ is a zero-mean i.i.d. random $(T+1)$-vector realized in the beginning of period t and called the innovation in period t. For $T = 0$, we have $z_t = \eta_{t,t}$, and the deviation is a simple i.i.d. disturbance. For $T > 0$, the deviation instead follows a moving-average process. Then we have

$$\begin{aligned} z_{t+\tau,t+1} &= z_{t+\tau,t} + \eta_{t+\tau,t+1}, \quad \tau = 1, ..., T, \\ z_{t+T+1,t+1} &= \eta_{t+T+1,t+1}. \end{aligned}$$

It follows that the dynamics of the deviation and the projection z^t can be written

$$z^{t+1} = A_z z^t + \eta^{t+1}, \tag{3.5}$$

where the $(T+1) \times (T+1)$ matrix A_z is defined as

$$A_z \equiv \begin{bmatrix} 0_{T\times 1} & I_T \\ 0 & 0_{1\times T} \end{bmatrix}.$$

Hence, z^t is the central bank's mean projection of current and future deviations, and η^t can be interpreted as the new information the central bank receives in the beginning of period t about those deviations.[9]

Combining the model (2.1) with the augmented policy rule (3.4) gives the system

$$\begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{H}\tilde{x}_{t+1|t} \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{X}_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} \tilde{C} \\ 0_{(n_x+n_i)\times(n_\varepsilon+T+1)} \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1} \\ \eta^{t+1} \end{bmatrix}, \tag{3.6}$$

for $t = ..., -1, 0, 1, ...$, where

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ z^t \end{bmatrix}, \quad \tilde{x}_t \equiv \begin{bmatrix} x_t \\ i_t \end{bmatrix}, \quad \tilde{H} \equiv \begin{bmatrix} H & 0 \\ G_x & G_i \end{bmatrix},$$

$$\tilde{A} \equiv \begin{bmatrix} A_{11} & 0_{n_X\times 1} & 0_{n_X\times T} & A_{12} & B_1 \\ 0_{T\times n_X} & 0_{T\times 1} & I_T & 0_{T\times n_x} & 0_{T\times 1} \\ 0_{1\times n_X} & 0 & 0_{1\times T} & 0_{1\times n_x} & 0 \\ A_{21} & 0_{n_x\times 1} & 0_{n_x\times T} & A_{22} & B_2 \\ f_X & 1 & 0_{1\times T} & f_x & f_i \end{bmatrix}, \quad \tilde{C} \equiv \begin{bmatrix} C & 0_{n_X\times(T+1)} \\ 0_{(T+1)\times n_\varepsilon} & I_{T+1} \end{bmatrix}.$$

Under the assumption of the saddlepoint property, the system of difference equations (3.6) has a unique solution and there exist unique matrices M and F such that projection can be written

$$\begin{aligned} \tilde{X}_{t+\tau,t} &= M^\tau \tilde{X}_{t,t}, \\ \tilde{x}_{t+\tau,t} &= F\tilde{X}_{t+\tau,t} = FM^\tau \tilde{X}_{t,t} \end{aligned}$$

for $\tau \geq 0$, where $X_{t,t}$ in $\tilde{X}_{t,t} \equiv (X'_{t,t}, z^{t\prime})'$ is given but the $(T+1)$-vector z^t remains to be determined. Its elements are then determined by the restrictions (3.1) or (3.2).

In order to satisfy the restriction (3.1) on the nominal policy rate, we note that it can now be written

$$i_{t+\tau,t} = F_i M^\tau \begin{bmatrix} X_{t,t} \\ z^t \end{bmatrix} = \bar{\imath}_{t+\tau,t}, \quad \tau = 0, 1, ..., T.$$

This provides $T+1$ linear equations for the $T+1$ elements of z^t.

In order to instead satisfy the restriction (3.2) on the real policy rate, we note that inflation expectations in a DSGE model similar to Ramses generally satisfy

$$\pi_{t+1|t} \equiv \varphi\tilde{x}_{t+1|t} + \Phi \begin{bmatrix} \tilde{X}_t \\ \tilde{x}_t \end{bmatrix}. \tag{3.7}$$

for some vectors φ and Φ. These vectors φ and Φ are structural, not reduced-form expressions. For instance, if π_t is one of the elements of x_t, the corresponding element of φ is unity, all other

[9] In Svensson [31] the deviation z_t is an n_z-vector of terms entering the different equations in the model, and the projection z^t of future z_t deviation is identified with central-bank judgment. The graphs in Svensson [31] can be seen as impulse responses to η^t, the new information about future deviations. (The notation here is slightly different from Svensson [31] in that there the projection $z^t \equiv (z_{t+1,t}, ..., z_{t+T,t})'$ does not include the current deviation.)

elements of φ are zero, and $\Phi \equiv 0$. If $\pi_{t+1|t}$ is one of the elements of $\tilde{x}_t$, the corresponding element of Φ is unity, all other elements of Φ are zero, and $\varphi \equiv 0$. Then the restriction (3.2) can be written

$$r_{t+\tau,t} \equiv i_{t+\tau,t} - \pi_{t+\tau+1,t} = (F_i - \varphi FM - \Phi)M^\tau \begin{bmatrix} X_{t,t} \\ z^t \end{bmatrix} = \bar{r}_{t+\tau,t}, \quad \tau = 0, 1, ..., T.$$

This again provides $T+1$ linear equations for the $T+1$ elements of z^t.

When the restriction is on the nominal policy rate, we can think of the equilibrium as being implemented by the central bank announcing the nominal policy-rate path and the private sector incorporating this policy-rate projection in their expectations, with the understanding that the policy rate will be set according to the given policy rule (G, f) from period $t+T+1$. When the restriction is on the real policy rate, we need to consider the fact that in practice central banks set nominal policy rates, not real ones. The restriction on the real policy rate will result in an endogenously determined nominal policy-rate projection, which together with the endogenously determined inflation projection will be consistent with the real policy-rate path. We can then think of the equilibrium as being implemented by the central bank calculating that nominal policy-rate projection and then announce it to the private sector.

3.1. Backward-looking model

In a backward-looking model, the projection of the instrument rule with the time-varying constraints can be written

$$i_{t+\tau,t} = f_X X_{t+\tau,t} + z_{t+\tau,t}, \tag{3.8}$$

so it is trivial to determine the projection z^t recursively so as to satisfy the restriction (3.1) on the nominal policy-rate projection.

Inflation can be written

$$\pi_t = \Phi X_t$$

for some vector Φ, so expected inflation can be written

$$\pi_{t+1|t} = \Phi X_{t+1|t} = \Phi(AX_t + Bi_t). \tag{3.9}$$

By combining (3.8), (3.9) and (3.3), it is trivial to determine the projection z^t so as to satisfy the restriction (3.2) on the real policy-rate projection.

4. Examples

In this section we examine restrictions on the nominal and real policy-rate path for the backward-looking Rudebusch-Svensson model and the two forward-looking models, the Lindé model and Ramses. Appendices B and C provide some details on the Rudebusch-Svensson and Lindé models. We also show a simulation with Ramses with the method of modest interventions by Leeper and Zha. Appendix D provides some details on the Leeper-Zha method.

4.1. The Rudebusch-Svensson model

The backward-looking empirical Rudebusch-Svensson model [26] has two equations (with estimates rounded to two decimal points),

$$\pi_{t+1} = 0.70\,\pi_t - 0.10\,\pi_{t-1} + 0.28\,\pi_{t-2} + 0.12\,\pi_{t-3} + 0.14\,y_t + \varepsilon_{\pi,t+1}, \tag{4.1}$$

$$y_{t+1} = 1.16\,y_t - 0.25\,y_{t-1} - 0.10\left(\frac{1}{4}\Sigma_{j=0}^{3} i_{t-j} - \frac{1}{4}\Sigma_{j=0}^{3}\pi_{t-j}\right) + \varepsilon_{y,t+1}. \tag{4.2}$$

The period is a quarter, π_t is quarterly GDP inflation measured in percentage points at an annual rate, y_t is the output gap measured in percentage points, and i_t is the quarterly average of the federal funds rate, measured in percentage points at an annual rate. All variables are measured as differences from their means, their steady-state levels. The predetermined variables are $X_t \equiv (\pi_t, \pi_{t-1}, \pi_{t-2}, \pi_{t-2}, y_t, y_{t-1}, i_{t-1}, i_{t-2}, i_{t-3})'$. See appendix B for details.

The target variables are inflation, the output gap, and the first-difference of the federal funds rate. The period loss function is

$$L_t = \frac{1}{2}[\pi_t^2 + \lambda_y y_t^2 + \lambda_{\Delta i}(i_t - i_{t-1})^2], \tag{4.3}$$

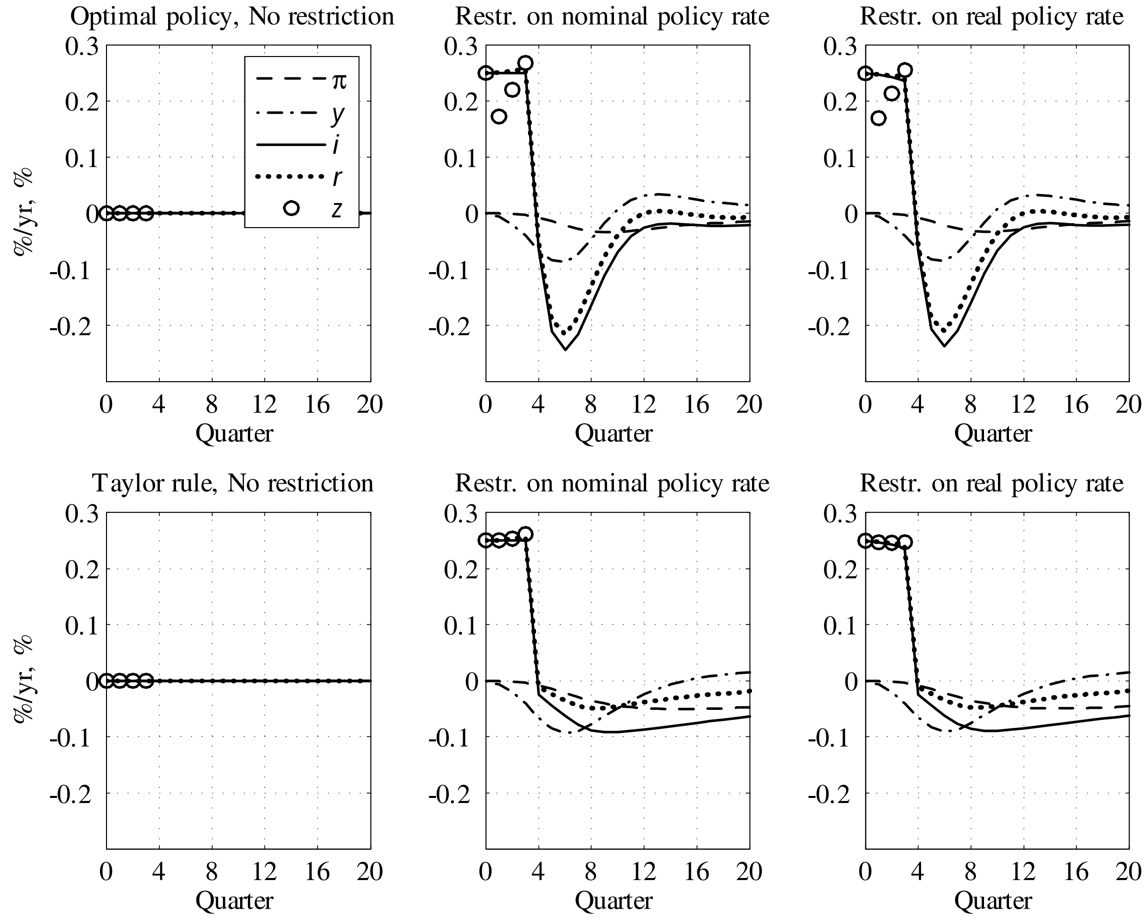
where π_t is measured as the difference from the inflation target, which is equal to the steady-state level. The discount factor, δ, and the relative weights on output-gap stabilization, λ_y, and interest-rate smoothing, $\lambda_{\Delta i}$, are set to satisfy $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$.

For the loss function (4.3) with the parameters $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$, and the case where ε_t is an i.i.d. shock with zero mean, the optimal instrument rule is (the coefficients are rounded to two decimal points)

$$i_t = 1.22\,\pi_t + 0.43\,\pi_{t-1} + 0.53\,\pi_{t-2} + 0.18\,\pi_{t-3} + 1.93\,y_t - 0.49\,y_{t-1} + 0.36\,i_{t-1} - 0.09\,i_{t-2} - 0.05\,i_{t-3}.$$

Figure 4.1 shows projections for the Rudebusch-Svensson model. The top row of panels show projections under the optimal policy, whereas the bottom row of panels show projections under a

Figure 4.1: Projections for Rudebusch-Svensson model with unrestricted and restricted nominal and real policy rate for optimal policy (top row) and Taylor rule (bottom row): 4-quarter restriction



Taylor rule,

$$i_t = 1.5\,\pi_t + 0.5\,y_t,$$

where the policy rate responds to the predetermined inflation and output gap with the standard coefficients 1.5 and 0.5, respectively.

The projections start in quarter 0 from the steady state, when all the predetermined variables are zero. The left column of panels show the projections when there is no restriction imposed on the nominal or real policy-rate path. This corresponds to zero projected deviations $z_{t+\tau,t}$ in the optimal instrument rule and the Taylor rule. These are denoted by circles for the first four quarters, quarters 0–3. The economy remains in the steady state, and inflation (denoted by a dashed curve), the output gap (denoted by a dashed-dotted curve), the nominal policy rate (denoted by a solid curve), and the real policy rate (denoted by a dotted curve) all remain at zero.

The middle column shows projections when the nominal policy-rate is restricted to equal 25 basis points for the first four quarters. For both the optimal policy and the Taylor rule, this requires positive and (except for quarter 1) increasing time-varying projected deviations in the instrument rule. The upward shift in quarters 0–3 in the nominal policy-rate path reduces inflation and expected inflation somewhat, and the real policy rate path shifts up a bit more than the nominal policy-rate path. The increased real policy rate also reduces the output gap. In the Rudebusch-Svensson model, inflation is very sluggish and the output gap responds more to the nominal and real policy rate than inflation. From quarter 4, there is no restriction on the policy-rate path, and according to both the optimal policy and the Taylor rule, the nominal and real policy rate are reduced substantially so as to bring the negative inflation and output gap eventually back to zero. The optimal policy is more effective in bringing back inflation and the output gap than the Taylor rule, which is natural since the Taylor rule is not optimal.

The right column shows projections when the real policy rate is restricted to equal unity during quarters 0–3. Since there is so little movement in inflation and expected inflation, the projections for these restrictions on the real and the nominal policy rate are very similar.

Since there are no forward-looking variables in the Rudebusch-Svensson model, there would be no difference between these projections with anticipated restrictions on the policy-rate path and simulations with unanticipated shocks as in Leeper and Zha [22].

4.2. The Lindé model

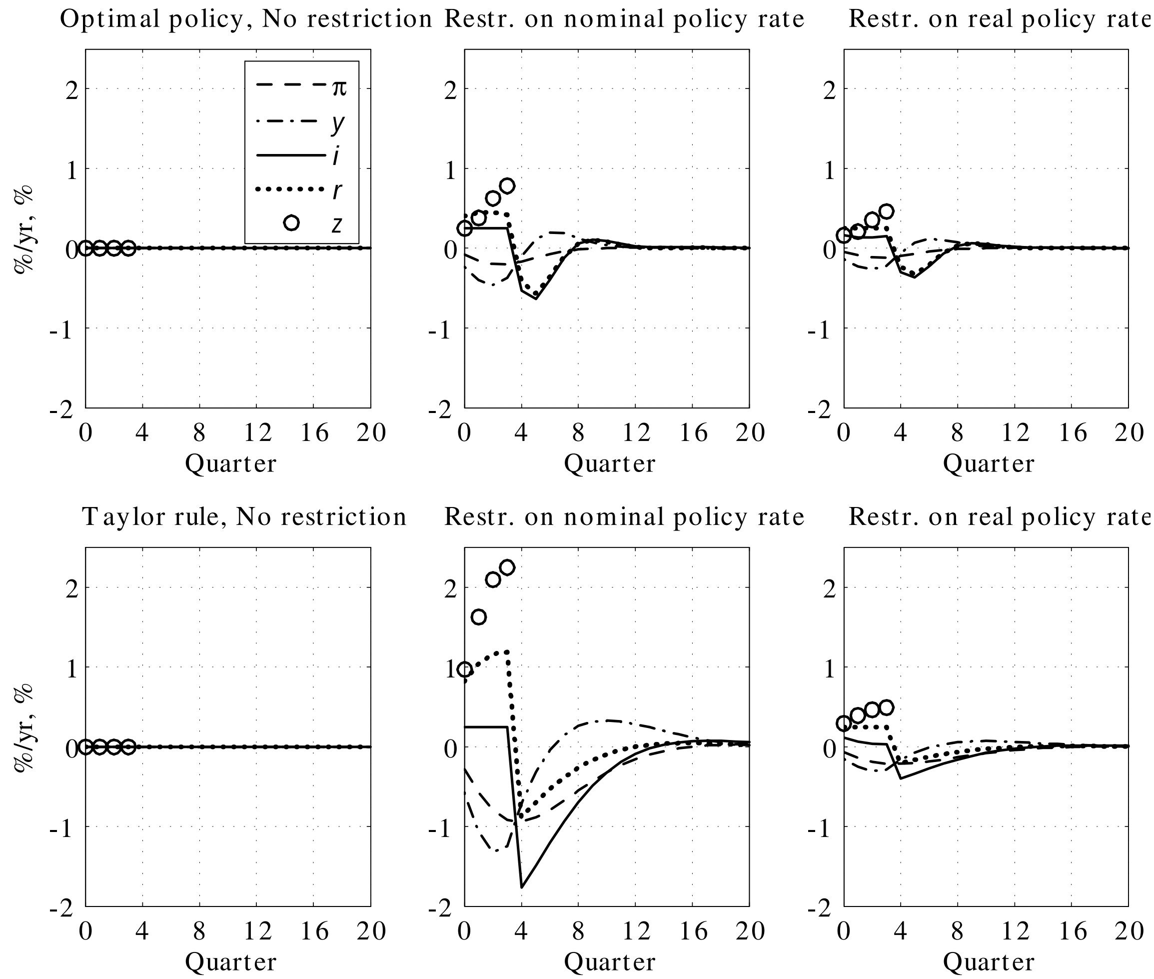
The empirical New Keynesian model of the US economy due to Lindé [23] also has two equations. We use the following parameter estimates,

$$\begin{aligned} \pi_t &= 0.457\, \pi_{t+1|t} + (1 - 0.457)\pi_{t-1} + 0.048 y_t + \varepsilon_{\pi t}, \\ y_t &= 0.425\, y_{t+1|t} + (1 - 0.425) y_{t-1} - 0.156(i_t - \pi_{t+1|t}) + \varepsilon_{yt}. \end{aligned}$$

The period is a quarter, and π_t is quarterly GDP inflation measured in percentage points at an annual rate, y_t is the output gap measured in percentage points, and i_t is the quarterly average of the federal funds rate, measured in percentage points at an annual rate. All variables are measured as differences from their means, their steady-state levels. The shock $\varepsilon_t \equiv (\varepsilon_{\pi t}, \varepsilon_{yt})'$ is i.i.d. with mean zero.

For the loss function (4.3), the predetermined variables are $X_t \equiv (\varepsilon_{\pi t}, \varepsilon_{yt}, \pi_{t-1}, y_{t-1}, i_{t-1})'$ (the lagged policy rate enters because it enters into the loss function, and the two shocks are included

Figure 4.2: Projections for the Lindé model with unrestricted and restricted nominal and real policy rate for optimal policy (top row) and Taylor rule (bottom row): 4-quarter restriction



among the predetermined variables in order to write the model on the form (2.1) with no shocks in the equations for the forward-looking variables). The forward-looking variables are $x_t \equiv (\pi_t, y_t)'$. See appendix C for details.[10]

For the loss function (4.3) with the parameters $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$, the optimal policy function (2.8) is (the coefficients are rounded to two decimal points),

$$i_t = 1.06\,\varepsilon_{\pi t} + 1.38\,\varepsilon_{yt} + 0.58\,\pi_{t-1} + 0.78\,y_{t-1} + 0.40\;i_{t-1} + 0.02\,\Xi_{\pi,t-1,t-1} + 0.20\,\Xi_{y,t-1,t-1},$$

where $\Xi_{\pi,t-1,t-1}$ and $\Xi_{y,t-1,t-1}$ are the Lagrange multipliers for the two equations for the forward-looking variables in the decision problem in period $t-1$ (see appendix A). The difference equation

[10] It is arguably unrealistic to consider inflation and output in the current quarter as forward-looking variables. Alternatively, current inflation and the output gap could be treated as predetermined, and one-quarter-ahead plans for inflation, the output gap, and the policy rate could be determined by the model above. Such a variant of the New Keynesian model is used in Svensson and Woodford [35].

(2.9) for the Lagrange multipliers is

$$\begin{bmatrix} \Xi_{\pi t} \\ \Xi_{yt} \end{bmatrix} = \begin{bmatrix} 10.20 & 0.74 & 5.54 & 0.43 & -0.21 \\ 0.74 & 1.48 & 0.40 & 0.85 & -0.28 \end{bmatrix} \begin{bmatrix} \varepsilon_{\pi t} \\ \varepsilon_{yt} \\ \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} 0.72 & 0.16 \\ 0.03 & 0.38 \end{bmatrix} \begin{bmatrix} \Xi_{\pi,t-1} \\ \Xi_{y,t-1} \end{bmatrix}.$$

We also examine the projections for a Taylor rule for which the policy rate responds to current inflation and the output gap,

$$i_t = 1.5\,\pi_t + 0.5\,y_t.$$

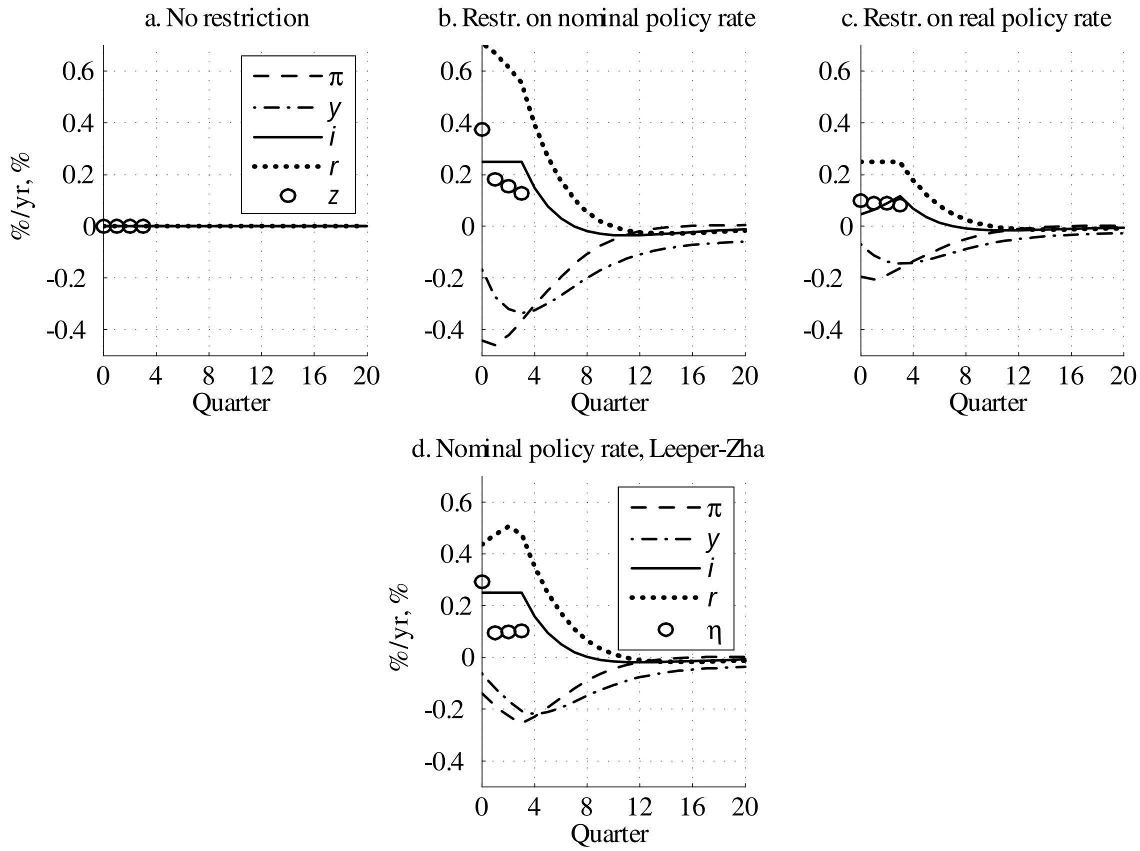
Figure 4.2 shows projections for the optimal policy (top row) and Taylor rule (bottom row) when there is a restriction to equal 25 basis points for quarters 0–3 for the nominal policy rate (middle column) and the real policy rate (right column). In the middle column, we see that a restriction to a 25 basis points higher nominal policy rate reduces inflation and inflation expectations so the projection of the real policy rate is above 25 basis points and higher than the policy rate for the first four quarters. In line with this, in the right column, the restriction on the real policy rate reduces inflation and inflation expectations so the corresponding nominal policy-rate projection is below 25 basis points. We note that these restrictions require positive and rising time-varying projected deviations (denoted by the circles). The magnitude of the projected deviations is larger than those in figure 4.1 for the Rudebusch-Svensson model. Using the magnitude of the projected deviations as indicating the severity of the restriction, we conclude that the restriction to nominal or real policy rates equal to unity is more severe in the Lindé model.

Because inflation is more sensitive to movements in the real policy rate in the Lindé model than in the Rudebusch-Svensson model, there is a greater difference between restrictions on the nominal and the real policy rate. Also, from quarter 4, when there is no restriction on the policy rate, a fall in the real and nominal policy rate, according to both the optimal policy and the Taylor rule, more easily stabilizes inflation and the output gap back to the steady state than in the Rudebusch-Svensson model.

4.3. Ramses

ALLS1 provides more details on Ramses, including the elements of the vectors X_t, x_t, i_t, and ε_t. Figure 4.3 shows projections with Ramses for the estimated instrument rule. The top row shows the result of restrictions on the nominal and real policy rate to equal 25 basis points for four quarters, quarters 0–3. We see that there is a substantial difference between restrictions on the nominal

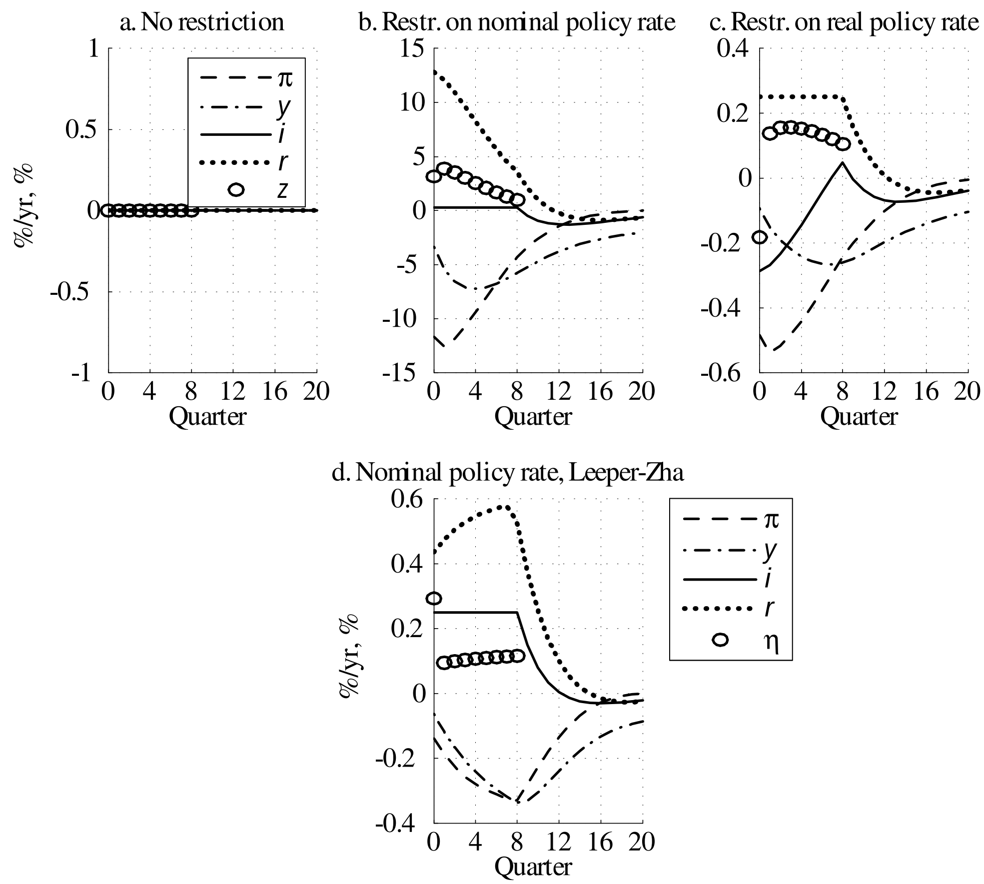
Figure 4.3: Projections for Ramses with anticipated unrestricted and restricted nominal and real policy rate (top row) and unanticipated restrictions on the nominal policy rate (bottom row): 4-quarter restriction



and the real policy rate, since inflation is quite sensitive to the real policy rate in Ramses. In the top middle panel, we see that a restriction on the nominal policy-rate projection to equal 25 basis points for quarters 0–3 corresponds to a very high and falling real policy-rate projection. In the top right panel we see that the restriction on the real policy rate to equal 25 basis points for quarters 0–3 corresponds to a nominal policy-rate projection quite a bit below the real policy rate.

The bottom panel of figure 4.3 shows the result of a projection with the Leeper-Zha method of modest interventions to implement a restriction on the nominal policy rate to equal 25 basis points for quarters 0–3. There, positive unanticipated shocks (denoted by circles) are added to the estimated instrument rule to achieve the restriction on the nominal policy rate. Comparing the bottom panel to the top right panel, we see that the impact on inflation, the output gap, and the real interest rate is smaller for the unanticipated shocks in the Leeper-Zha method than for the anticipated projected deviations in our method.

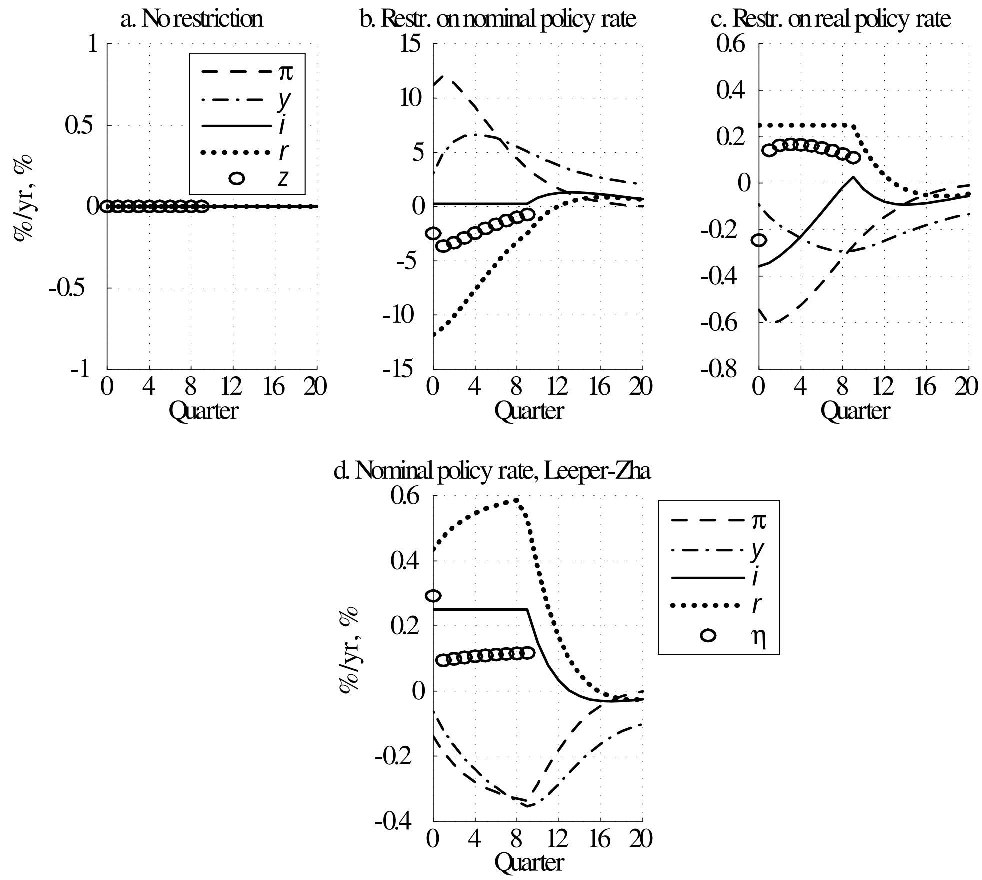
Figure 4.4: Projections for Ramses with anticipated unrestricted and restricted nominal and real policy rate (top row) and unanticipated restrictions on the nominal policy rate (bottom row): 9-quarter restriction



4.4. Unusual equilibria

If restrictions are imposed on the nominal policy rate for many periods, "unusual" equilibria can occur. We can illustrate this for Ramses in figure 4.4, where in panel b the nominal policy rate is restricted to equal 25 basis points for 9 quarters, quarters 0–8. This is a very contractionary policy, which shows in inflation and inflation expectations falling very much and the real policy rate becoming very high. (Note that the scale varies from panel to panel in figure 4.4.) If we look at panel c, where the real policy rate is restricted to equal 25 basis points for 9 quarters, we see that inflation and inflation expectations fall so much that the nominal policy rate becomes negative in quarter 0 (relative to when there is no restriction) and then rises to become positive only in quarter 7 and 8. We realize that, if inflation and inflation expectations respond so much that nominal and

Figure 4.5: Projections for Ramses with anticipated unrestricted and restricted nominal and real policy rate (top row) and unanticipated restrictions on the nominal policy rate (bottom row): 10-quarter restriction



real policy rates move in opposite directions, some unusual equilibria may arise. This is confirmed in figure 4.5, where in panel b the nominal policy rate is restricted at 25 basis points for one more quarter, quarter 9. We see that then there is no longer an equilibrium where the real policy rate is positive and high. Instead the equilibrium is such that the real policy rate is negative, policy is very expansionary, and inflation and inflation expectations are high.

This phenomenon of unusual equilibria clearly requires that inflation and inflation expectations are quite sensitive to the real policy rate so that for multiple-period restrictions the nominal and real policy rate moves in opposite directions. It requires as much as around 10-quarter restrictions to occur in Ramses. In the Lindé model, inflation is more sensitive to the real policy rate, so there it can occur already at 6-quarter restrictions. We have not observed the phenomenon in the

Rudebusch-Svensson model even for very long restrictions.

The phenomenon implies that restrictions for many quarters should be avoided in models where inflation and inflation expectations are sufficiently sensitive to the real policy rate.

5. Conclusions

We have presented a new convenient way to construct projections conditional on alternative *anticipated* policy-rate paths in linearized dynamic stochastic general equilibrium (DSGE) models, such as Ramses, the Riksbank's main DSGE model. The main idea is to include the anticipated future time-varying deviations from a policy rule in the vector of predetermined variables, the "state" of the economy. This allows the formulation of the linear(ized) model on a standard state-space form, the application of standard algorithms for the solution of linear rational-expectations models, and a recursive representation of the equilibrium projections. Projections for anticipated policy-rate paths are especially relevant for central banks, such as the Reserve Bank of New Zealand, Norges Bank, the Riksbank, and the Czech National Bank, that publish a policy-rate path, but they are also relevant for the discussion of the kind "forward guidance" recently given by the Fed and Bank of Canada.

From the examples in this paper, we have seen that, in a model without forward-looking variables such as the empirical model of the U.S. economy by Rudebusch and Svensson [26], there is no difference between policy simulations with anticipated and unanticipated restrictions on the policy-rate path. In a model with forward-looking variables, such as Ramses or the empirical New Keynesian model of the U.S. economy by Lindé [23], there is such a difference, and the impact of anticipated deviations from a policy rule will generally be larger than that of unanticipated deviations. In a model with forward-looking variables, exogenous restrictions on the policy-rate path are consistent with a unique equilibrium, if there is an anticipated switch to a well-behaved policy rule in the future. For given restrictions on the policy-rate path, the equilibrium depends on that policy rule.

Furthermore, our analysis shows that, if inflation is sufficiently sensitive to the real policy rate, "unusual" equilibria may result from restrictions on the nominal policy rate for sufficiently many periods. Such cases have the property that nominal and real policy rates move in opposite directions and nominal policy rates and inflation (expectations) move in the same direction. This phenomenon implies that restrictions on nominal policy rates for too many periods should be avoided.

Appendix

A. Optimal policy

Let Y_t be an n_Y-vector of *target* variables, measured as the difference from an n_Y-vector Y^* of *target levels*. This is not restrictive, as long as we keep the target levels time-invariant. If we would like to examine the consequences of different target levels, we can instead interpret Y_t as the absolute level of the target levels and replace Y_t by $Y_t - Y^*$ everywhere below. We assume that the target variables can be written as a linear function of the predetermined variables, the forward-looking variables, and the instruments,

$$Y_t = D \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} \equiv [D_X \quad D_x \quad D_i] \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}, \tag{A.1}$$

where D is an $n_Y \times (n_X + n_x + n_i)$ matrix and partitioned conformably with X_t, x_t, and i_t.

Let the intertemporal loss function in period t be

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau}, \tag{A.2}$$

where $0 < \delta < 1$ is a discount factor, L_t is the period loss given by

$$L_t \equiv Y_t' \Lambda Y_t, \tag{A.3}$$

and Λ is a symmetric positive semidefinite matrix containing the weights on the individual target variables.[11]

Optimization under commitment in a timeless perspective (Woodford [36]), which combined with the model equations (2.1) results in a system of difference equations (see Söderlind [29] and Svensson [32]). The system of difference equations can be solved with several alternative algorithms, for instance, those developed by Klein [20] and Sims [28] or the AIM algorithm of Anderson and Moore [9] and [10] (see Svensson [31] and [32] for details of the derivation and the application of the Klein algorithm). The equilibrium under optimal policy under commitment can be described by the following difference equation,

$$\begin{bmatrix} x_t \\ i_t \end{bmatrix} = \begin{bmatrix} F_x \\ F_i \end{bmatrix} \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \tag{A.4}$$

$$\begin{bmatrix} X_{t+1} \\ \Xi_t \end{bmatrix} = M \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1}. \tag{A.5}$$

[11] For plotting and other purposes, and to avoid unnecessary separate program code, it is convenient to expand the vector Y_t to include a number of variables of interest that are not necessary target variables or potential target variables. These will then have zero weight in the loss function.

The Klein algorithm returns the matrices F_x, F_i, and M. The submatrix F_i in (A.5) represents the optimal instrument rule,

$$i_t = F_i \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}. \tag{A.6}$$

These matrices depend on A, B, H, D, Λ, and δ, but they are independent of C. That they are independent of C demonstrates the certainty equivalence of optimal projections (the certainty equivalence that holds when the model is linear, the loss function is quadratic, and the shocks and the uncertainty are additive); only probability means of current and future variables are needed to determine optimal policy and the optimal projection. The n_x-vector Ξ_t consists of the Lagrange multipliers of the lower block of (2.1), the block determining the projection of the forward-looking variables. The initial value for Ξ_{t-1} is discussed in ALLS1.

In a backward-looking model, that is, a model without forward-looking variables, there is no vector x_t of forward-looking variables, no lower block of equations in (2.1), no Lagrange multiplier Ξ_t, and the vector of target variables Y_t only depends on the vector of predetermined variables X_t and the (vector of) instrument(s) i_t.

B. The Rudebusch-Svensson model: An empirical backward-looking model

The two equations of the model of Rudebusch and Svensson [26] are

$$\pi_{t+1} = \alpha_{\pi 1}\pi_t + \alpha_{\pi 2}\pi_{t-1} + \alpha_{\pi 3}\pi_{t-2} + \alpha_{\pi 4}\pi_{t-3} + \alpha_y y_t + z_{\pi,t+1} \tag{B.1}$$

$$y_{t+1} = \beta_{y1} y_t + \beta_{y2} y_{t-1} - \beta_r \left(\frac{1}{4}\Sigma_{j=0}^{3} i_{t-j} - \frac{1}{4}\Sigma_{j=0}^{3}\pi_{t-j} \right) + z_{y,t+1}, \tag{B.2}$$

where π_t is quarterly inflation in the GDP chain-weighted price index (P_t) in percentage points at an annual rate, i.e., $400(\ln P_t - \ln P_{t-1})$; i_t is the quarterly average federal funds rate in percentage points at an annual rate; y_t is the relative gap between actual real GDP (Q_t) and potential GDP (Q_t^*) in percentage points, i.e., $100(Q_t - Q_t^*)/Q_t^*$. These five variables were demeaned prior to estimation, so no constants appear in the equations.

The estimated parameters, using the sample period 1961:1 to 1996:2, are shown in table B.1.

Table B.1

$\alpha_{\pi 1}$	$\alpha_{\pi 2}$	$\alpha_{\pi 3}$	$\alpha_{\pi 4}$	α_y	β_{y1}	β_{y2}	β_r
0.70	−0.10	0.28	0.12	0.14	1.16	−0.25	0.10
(0.08)	(0.10)	(0.10)	(0.08)	(0.03)	(0.08)	(0.08)	(0.03)

The hypothesis that the sum of the lag coefficients of inflation equals one has a p-value of .16, so this restriction was imposed in the estimation.

The state-space form can be written

$$\begin{bmatrix} \pi_{t+1} \\ \pi_t \\ \pi_{t-1} \\ \pi_{t-2} \\ y_{t+1} \\ y_t \\ i_t \\ i_{t-1} \\ i_{t-2} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^4 \alpha_{\pi j} e_j + \alpha_y e_5 \\ e_1 \\ e_2 \\ e_3 \\ \beta_r e_{1:4} + \beta_{y1} e_5 + \beta_{y2} e_6 - \beta_r e_{7:9} \\ e_5 \\ e_0 \\ e_7 \\ e_8 \end{bmatrix} \begin{bmatrix} \pi_t \\ \pi_{t-1} \\ \pi_{t-2} \\ \pi_{t-3} \\ y_t \\ y_{t-1} \\ i_{t-1} \\ i_{t-2} \\ i_{t-3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\beta_r}{4} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} i_t + \begin{bmatrix} z_{\pi,t+1} \\ 0 \\ 0 \\ 0 \\ z_{y,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where e_j $(j = 0, 1, ..., 9)$ denotes a 1×9 row vector, for $j = 0$ with all elements equal to zero, for $j = 1, ..., 9$ with element j equal to unity and all other elements equal to zero; and where $e_{j:k}$ $(j < k)$ denotes a 1×9 row vector with elements $j, j+1, ..., k$ equal to $\frac{1}{4}$ and all other elements equal to zero. The predetermined variables are π_t, π_{t-1}, π_{t-2}, π_{t-3}, y_t, y_{t-1}, i_{t-1}, i_{t-2}, i_{t-2}, and i_{t-3}. There are no forward-looking variables.

For a loss function (4.3) with $\delta = 1$, $\lambda = 1$, and $\nu = 0.2$, and the case where z_t is an i.i.d. zero-mean shock; the optimal instrument rule is (the coefficients are rounded to two decimal points),

$$i_t = 1.22\,\pi_t + 0.43\,\pi_{t-1} + 0.53\,\pi_{t-2} + 0.18\,\pi_{t-3} + 1.93\,y_t - 0.49\,y_{t-1} + 0.36\,i_{t-1} - 0.09\,i_{t-2} - 0.05\,i_{t-3}.$$

C. The Lindé model: An empirical New Keynesian model

An empirical New Keynesian model estimated by Lindé [23] is

$$\begin{aligned} \pi_t &= \omega_f \pi_{t+1|t} + (1 - \omega_f)\pi_{t-1} + \gamma y_t + \varepsilon_{\pi t}, \\ y_t &= \beta_f y_{t+1|t} + (1 - \beta_f)(\beta_{y1} y_{t-1} + \beta_{y2} y_{t-2} + \beta_{y3} y_{t-3} + \beta_{y4} y_{t-4}) - \beta_r (i_t - \pi_{t+1|t}) + \varepsilon_{yt}, \end{aligned}$$

where the restriction $\sum_{j=1}^4 \beta_{yj} = 1$ is imposed and $\varepsilon_t \equiv (\varepsilon_{\pi t}, \varepsilon_{yt})'$ is an i.i.d. shock with mean zero. The estimated coefficients (Table 6a in Lindé [23], non-farm business output) are shown in table C.1.

Table C.1

ω_f	γ	β_f	β_r	β_{y1}	β_{y2}	β_{y3}
0.457	0.048	0.425	0.156	1.310	−0.229	−0.011
(0.065)	(0.007)	(0.027)	(0.016)	(0.174)	(0.279)	(0.037)

For simplicity, we set $\beta_{y1} = 1$, $\beta_{y2} = \beta_{y3} = \beta_{y4} = 0$. Then the state-space form can be written as

$$\begin{bmatrix} \varepsilon_{\pi,t+1} \\ \varepsilon_{y,t+1} \\ \pi_t \\ y_t \\ i_t \\ \omega_f \pi_{t+1|t} \\ \beta_r \pi_{t+1|t} + \beta_f y_{t+1|t} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -(1-\omega_f) & 0 & 0 & 1 & -\gamma \\ 0 & -1 & 0 & -(1-\beta_f) & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{\pi t} \\ \varepsilon_{yt} \\ \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \\ \pi_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \beta_r \end{bmatrix} i_t + \begin{bmatrix} \varepsilon_{\pi,t+1} \\ \varepsilon_{y,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The predetermined variables are $\varepsilon_{\pi t}$, ε_{yt}, π_{t-1}, y_{t-1}, and i_{t-1}, and the forward-looking variables are π_t and y_t.

For a loss function (4.3) with $\delta = 1$, $\lambda_y = 1$, and $\lambda_{\Delta i} = 0.2$, and the case where ε_t is an i.i.d. zero-mean shock; the optimal instrument rule is (the coefficients are rounded to two decimal points),

$$i_t = 1.06\,\varepsilon_{\pi t} + 1.38\,\varepsilon_{yt} + 0.58\,\pi_{t-1} + 0.78\,y_{t-1} + 0.40\;i_{t-1} + 0.02\,\Xi_{\pi,t-1,t-1} + 0.20\,\Xi_{y,t-1,t-1},$$

where $\Xi_{\pi,t-1,t-1}$ and $\Xi_{y,t-1,t-1}$ are the Lagrange multipliers for the two equations for the forward-looking variables in the decision problem in period $t-1$. The difference equation (2.9) for the Lagrange multipliers is

$$\begin{bmatrix} \Xi_{\pi t} \\ \Xi_{yt} \end{bmatrix} = \begin{bmatrix} 10.20 & 0.74 & 5.54 & 0.43 & -0.21 \\ 0.74 & 1.48 & 0.40 & 0.85 & -0.28 \end{bmatrix} \begin{bmatrix} \varepsilon_{\pi t} \\ \varepsilon_{yt} \\ \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} 0.72 & 0.16 \\ 0.03 & 0.38 \end{bmatrix} \begin{bmatrix} \Xi_{\pi,t-1} \\ \Xi_{y,t-1} \end{bmatrix}.$$

D. Unanticipated policy-rate shocks: "Modest interventions" as in Leeper and Zha [22]

The method of "modest interventions" of Leeper and Zha [22] can be interpreted as generating central-bank projections that satisfy the restriction on the policy rate by adding a sequence of additive shocks to the instrument rule. These planned shocks are unanticipated by the private sector.

In order to illustrate the Leeper and Zha [22] method of modest interventions, we set $T = 0$, in which case

$$z_t = \eta_{t,t}$$

and the deviation is a simple zero-mean i.i.d. disturbance. We can then write the projection model as perceived by the private sector as

$$\begin{bmatrix} \tilde{X}_{t+\tau+1,t} \\ \tilde{H}\tilde{x}_{t+\tau+1,t} \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{X}_{t+\tau,t} \\ \tilde{x}_{t+\tau,t} \end{bmatrix} \tag{D.1}$$

for $\tau \geq 0$, where

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ z_t \end{bmatrix}, \quad \tilde{x}_t \equiv \begin{bmatrix} x_t \\ i_t \end{bmatrix}, \quad \tilde{H} \equiv \begin{bmatrix} H & 0 \\ G_x & G_i \end{bmatrix},$$

$$\tilde{A} \equiv \begin{bmatrix} A_{11} & 0_{n_X \times 1} & A_{12} & B_1 \\ 0_{1\times n_X} & 0_{1\times 1} & 0_{1\times n_x} & 0_{1\times 1} \\ A_{21} & 0_{n_x\times 1} & A_{22} & B_2 \\ f_X & 1 & f_x & f_i \end{bmatrix}.$$

The solution to this system can be written

$$\begin{aligned} \begin{bmatrix} X^p_{t+\tau,t} \\ 0 \end{bmatrix} &= M^\tau \tilde{X}_{t,t}, \\ \tilde{x}^p_{t+\tau,t} &\equiv \begin{bmatrix} x^p_{t+\tau,t} \\ i^p_{t+\tau,t} \end{bmatrix} = F \begin{bmatrix} X^p_{t+\tau,t} \\ 0 \end{bmatrix} = \begin{bmatrix} F_x \\ F_i \end{bmatrix} M^\tau \tilde{X}_{t,t} \end{aligned}$$

for $\tau \geq 0$, where the superscript p denotes that this is the projection believed by the private sector in period t.

Let us demonstrate the method of modest interventions only for the restriction (3.1). The central bank plans to satisfy this restriction by a sequence of shocks $\{\tilde{\eta}_{t+\tau,t}\}_{\tau=0}^T$ that are unanticipated by the private sector. These shocks are chosen such that $\tilde{\eta}_{t,t}$ satisfies

$$i_{t,t} = F_i \begin{bmatrix} X_{t,t} \\ \tilde{\eta}_{t,t} \end{bmatrix} = \bar{\imath}_{t,t}.$$

Then the projection of the current forward-looking variables is given by

$$x_{t,t} = F_x \begin{bmatrix} X_{t,t} \\ \tilde{\eta}_{t,t} \end{bmatrix}.$$

For $\tau = 1, ..., T$, the projection of the predetermined variables is then given by

$$\begin{bmatrix} X_{t+\tau,t} \\ 0 \end{bmatrix} = M \begin{bmatrix} X_{t+\tau-1,t} \\ \tilde{\eta}_{t+\tau-1,t} \end{bmatrix},$$

the shock $\tilde{\eta}_{t+\tau,t}$ is chosen to satisfy

$$i_{t+\tau,t} = F_i \begin{bmatrix} X_{t+\tau,t} \\ \tilde{\eta}_{t+\tau,t} \end{bmatrix} = \bar{\imath}_{t+\tau,t},$$

and the projection of the forward-looking variables is given by

$$x_{t+\tau,t} = F_x \begin{bmatrix} X_{t+\tau,t} \\ \tilde{\eta}_{t+\tau,t} \end{bmatrix}.$$

There are some conceptual difficulties in a central bank announcing such a policy-rate path and projection to the private sector. The projection is only relevant if the private sector does not believe that the central bank will actually implement the path but instead follow the instrument rule with zero expected shocks to the instrument rule. The method of modest interventions is instead perhaps more appropriate for secret policy simulations and plans that are not announced to the private sector, or for a situation when the announced policy-rate path is not credible and the private sector is surprised each period when the path is implemented.

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