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A NOTE ON REGIME SWITCHING, MONETARY POLICY, AND MULTIPLE EQUILIBRIA

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ABSTRACT

When monetary policy is subject to regime switches conditions for determinacy become more complex. Davig and Leeper (2007) and Farmer, Waggoner and Zha (2009a) have studied such conditons. Using some new results from stochastic processes, we characterize the moments of the stationary distribution of inflation under regime switching to obtain conditions for indeterminacy that can be easily checked and interpreted in terms of expected values of Taylor coefficients. In the last section, we outline methods to compute the moments of stationary distributions in regime switching models of higher dimensions.

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1 Introduction

In simple settings the conditions under which monetary policy can lead indeterminacy are well understood: active Taylor rules generate determinacy and passive rules generate indeterminacy.[1] When monetary policy is subject to regime switches, presumably because monetary policy has to shift randomly with changes in some underlying economic conditions, the situation becomes more complex, especially if policy is active in some regimes and passive in others. It is natural then to expect that some average over the regimes, possibly weighted by transition probabilities, would allow the characterization of determinacy vs. indeterminacy, once indeterminacy is appropriately defined. The question has been studied by Davig and Leeper (2006, 2007), Chung, Davig and Leeper (2007), and by Farmer, Waggoner and Zha (2009a, 2009b). We hope to further clarify the conditions for indeterminacy by characterizing the moments of the stationary distribution of inflation when monetary policy can switch across active and passive regimes according to a Markov process. In the last section, we outline methods to compute the moments of stationary distributions in regime switching models of higher dimensions.

2 A simple model

We start with the simplest possible model, and leave the extensions for later. The simplest model has flexible prices where π_t is the inflation rate, r_t is the real rate, and R_t is the nominal rate at time t. The Fisher equation is satisfied, that is

$$R_t = E\left(\pi_{t+1}\right) + r_t \tag{1}$$

and the monetary authority sets the nominal rate according to the Taylor rule:

$$R_t = \tilde{R} + \phi_t\left(\pi_t - \tilde{\pi}\right) \tag{2}$$

We assume that $\{r_t\}_t$ is a bounded *iid* random variable over the state space $\varrho = \left(r^1, ...r^n\right)$ with mean $\tilde{r}$, that $\{\phi_t\}_t$ is an irreducible, aperiodic, stationary Markov chain over state space $\Phi = \left(\bar{\phi}_1, ...\bar{\phi}_s\right)$ with transition matrix P and stationary distribution $\nu = \left(\nu_1, ...\nu_s\right)$, and that the target inflation rate is $\tilde{\pi} = \tilde{R} - \tilde{r}$. Then, substituting (2) into (1) and subtracting $\tilde{r}$ from both sides, we have:

$$\tilde{R} - \tilde{r} + \phi_t\left(\pi_t - \tilde{\pi}\right) = E\left(\pi_{t+1}\right) + r_t - \tilde{r}$$

$$\phi_t\left(\pi_t - \tilde{\pi}\right) = E\left(\pi_{t+1}\right) - \left(\left(\tilde{R} - \tilde{r}\right) - \left(r_t - \tilde{r}\right)\right)$$

$$\phi_t\left(\pi_t - \tilde{\pi}\right) = E\left(\pi_{t+1}\right) - \left(\tilde{\pi} - \left(r_t - \tilde{r}\right)\right)$$

$$\phi_t\left(\pi_t - \tilde{\pi}\right) = E\left(\pi_{t+1} - \tilde{\pi}\right) + \left(r_t - \tilde{r}\right)$$

[1] We have in mind simple Taylor rules in simple settings where the a policy is active if the central bank changes the nominal rate by more than the change in the inflation rate, and passive otherwise.

If we set $q_t = \pi_t - \tilde{\pi}$, and we define the *iid* random variable $\varepsilon_t = r_t - \tilde{r}$ so that $E(\varepsilon_t) = 0$, we get:

$$\phi_t q_t = E(q_{t+1}) + \varepsilon_t \tag{3}$$

We can then explore solutions of (3) that satisfy

$$q_{t+1} = \phi_t q_t - \varepsilon_t + \gamma_{t+1} \tag{4}$$

provided has $E_t(\gamma_{t+1}) = 0$ for the *iid* process $\{\gamma_t\}_t$. By repeated substitution we obtain

$$q_{t+N} = \left(\prod_{l=0}^{N-1} \phi_{t+l}\right) q_t + \sum_{l=0}^{N-1} (\gamma_{t+1} - \varepsilon_t) \prod_{m=l+1}^{N-1} \phi_{t+m} \tag{5}$$

It is clear that if $\bar{\phi}_i > 1$ for $i = 1, ...s$, the only solution satisfying (3) that is bounded or that has finite moments is the Minimum State Variable solution (see McCallum (1983)

$$q_t = \frac{\varepsilon_t}{\phi_t} \tag{6}$$

When $\bar{\phi}_s < 1$ for one or more values of s, indeterminacy can become an issue and additional solutions of (3) may emerge. For any initial q_0 and finite state *iid* sunspot process $\{\gamma_t\}_t$ with $E_t(\gamma_{t+1}) = 0$ for all t, there may be other solutions of (3) satisfying (4) that are bounded, or that have finite moments. It may therefore be useful to consider what the set of admissible solutions to (3) are.

Typically, transversality conditions associated with underlying optimization problems are given in terms of the expected discounted value of assets in the limit as time goes to infinity. If for example the supply of nominal bonds or nominal balances are fixed, under apropriate assumptions fast unbounded deflations may generate real asset levels that go to infinity, violating transversality conditions. Fast unbounded inflations that drive the real value of bonds or money to zero may also be inefficient or infeasible, so it is indeed reasonable, from the perspective of microfoundations, to impose conditions assuring that at least the mean of the stationary distribution of $\{q_t\}_t$ exists. Other more stringent criteria may only require the existence of second or even higher moments.

3 Indeterminacy

Let us start with the existence of stationary solutions of (4). Since the state spaces for $\{\phi_t\}_t, \{\varepsilon_t\}_t$ and $\{\gamma_t\}_t$ are finite with $\{\varepsilon_t\}_t$ and $\{\gamma_t\}_t$ zero mean *iid* processes, we can immediately apply a theorem of Brandt (1986). Recall that ν is the stationary probability induced by the transition matrix P. Brandt (1986) shows that if the condition $\nu \ln |\Phi'| < 0$ holds, that is if the expected value of ln $|\phi|$ taken with respect to the stationary probabilities induced by the transition matrix P is negative, then (4) has a unique ergodic stationary solution. Thus we see that the existence of stationary solutions requires not that $|\bar{\phi}_i| < 1$ for every i, but that the average over $\ln|\Phi'|$ computed using stationary probabilities over the state space of the Taylor coefficient is negative. Clearly, the condition $\nu \ln|\Phi| < 0$ cannot be satisfied if $|\bar{\phi}_i| > 1$ for all i.

But this is not much help since a stationary distribution need not have finite moments, let alone be bounded. In fact it is precisely the finiteness of moments that will be the focus next. For this we invoke a recent Theorem of Saporta (2005).[2] Let Q be the diagonal matrix with diagonal entries $\bar{\phi}_i$.

Theorem 1 *(Saporta (2005), Thm 2) Let*

$$q_{t+1} = \phi_t q_t - \left(\varepsilon_t - \gamma_{t+1}\right)$$

Assume: (i) $\nu \ln|\Phi| < 0$,[3] and (ii) $\ln \phi_i$ $i = 1, ..s$ are not integral multiples of the same number.[4] Then for $x = \{-1, 1\}$, the tails of the stationary distribution of q_n, $P_{.>}(q_n > q)$, are asymptotic to a power law:

$$\Pr{}_{>}(x q_n > q) \sim L(x)\, q^{-\mu}$$

with $L(1) + L(-1) > 0$, where $\mu > 0$ satisfies

$$\lambda\left(Q^{\mu} P'\right) = 1$$

and where $\lambda\left(Q^{\mu} P'\right)$ is the dominant root of $Q^{\mu} P'$.

Remark 2 *The stationary distribution of $\{q_t\}_t$ is two-tailed because realizations of ε_t and γ_t as well as $\bar{\phi}_i$ may be positive or negative.*[5]

Remark 3 *Note that the i'th the column sum of the matrix QP' gives the expected value of the Taylor coefficient conditional on starting at state i.*

Remark 4 *Most importantly, it follows from power law tails that if the solution of $\mu = \hat{\mu}$, then the stationary distribution has only moments $m < \hat{\mu}$.*

The above result is still not sharp enough because it does not sufficiently restrict the range of μ. Suppose for example, on grounds of microfoundations, we wanted to make sure that $\hat{\mu} > m$ for some m. To assure that the first moment of the stationary distribution of $\{q_t\}_t$ exists, we would want $\hat{\mu} > 1$, or if we wanted the variance to exist (mean square stability) would want $\hat{\mu} > 2$.

[2] In a very different context Benhabib and Bisin (2008) use similar techniques to study wealth distribution with stochastic returns to capital as well as stochastic earnings.

[3] Condition (i) may be viewed as a passive logarithmic Taylor rule in expectation. We will also use an expected passive Taylor rule in Assumption 1 and Proposition 1 but not in logarithms.

[4] Condition (ii) is a non-degeneracy condition often used to avoid lattice disatributions in renewal theory, and that will hold generically.

[5] The distribution would only have a right tail if we had $-\varepsilon_t + \gamma_{t+1} > 0$, and $\bar{\phi}_i > 0$ for all i, that is we would have $L(-1) = 0$. See Saporta (2005), Thm 1.

The assumptions to guarantee this however are easy to obtain and trivial to check, given the transition matrix P and the state space Φ.

Define $\Phi^m = \left((\phi_1)^m, ... (\phi_1)^m\right)$ for some positive integer m that we choose.

Assumption 1 *(a) Let the column sums of $Q^m P'$ be less than unity, that is $P(\Phi^m)' < \mathbf{1}$, where $\mathbf{1}$ is a vector with elements equal to 1, (b) Let $P_{ii} > 0$ for all i, and (c) Assume that there exists some i for which $\bar{\phi}_i > 1$.*

Remark 5 *In Assumption 1, (a) implies, for $m = 1$, that the expected value of the Taylor coefficient ϕ_t conditional on any realization of ϕ_{t-1}, is less than 1, that is that the policy is passive in expectation. (b) implies that there is a positive probability that the Taylor coefficient does not change from one period to the next, and (c) implies that there exists a state in which the Taylor rule is active.*

We now turn to our result on the conditions for indeterminacy.

Proposition 1 *Let assumption 1 hold. The stationary distribution of inflation exists and has moments of order m or lower.*

Proof. We have to show that there exists a solution $\hat{\mu} > m$ of $\lambda(Q^\mu P') = \mathbf{1}$. Saporta shows that $\mu = 0$ is a solution for $\lambda(Q^\mu P') = \mathbf{1}$, or equivalently for $\ln(\lambda(Q^\mu P')) = 0$. This follows because $Q^0 = I$ and P is a stochastic matrix with a unit dominant root. Let $E \ln q$ denote the expected value of $\ln q$ evaluated at its stationary distribution. Saporta, under the assumption $E \ln q < 0$, shows that $\frac{d \ln \lambda(A^\mu P')}{\delta \mu} < 0$ at $\mu = 0$, and that $ln(\lambda(A^\mu P'))$ is a convex function of μ.[6] Therefore, if there exists another solution $\mu > 0$ for $\ln(\lambda(A^\mu P')) = 0$, it is positive and unique. To assure that $\hat{\mu} > m$ we replace the condition $E \ln q < 0$ with $P(\Phi^m)' < \mathbf{1}$. . Since $Q^m P'$ is positive and irreducible, its dominant root is smaller than the maximum column sum. Therefore for $\mu = m$, $\lambda(Q^\mu P') < 1$. Now note that if $P_{ii} > 0$ and $\bar{\phi}_i > 1$ for some i, the trace of $Q^\mu P'$ goes to infinity if μ does (see also Saporta (2004) Proposition 2.7). But the trace is the sum of the roots so that the dominant root of $Q^\mu P'$, $\lambda(Q^\mu P')$, goes to infinity with μ. It follows that the solution of $ln(\lambda(Q^\mu P')) = 0$, $\hat{\mu} > m$. ■

Remark 6 *It follows from the Proposition for example, that if admissible solutions of (4) require the mean of the stationary distribution of q to exist, we can apply the assumptions of the Proposition with $m = 1$; if we require both the mean and the variance to exist, we invoke the assumptions with $m = 2$. If on the other hand $P(\Phi)' > \mathbf{1}$ then from the proof of Proposition 1 the stationary solutions to (4), which exist if the assumptions of Theorem 1 are satisfied, will not have a first moment[7], and therefore such solutions to (4) may be considered inadmissible.*

[6] This follows because $\lim_{n\to\infty} \frac{1}{n} \ln E(q_0 q_{-1} ... q_{n-1})^\mu = \ln(\lambda(Q^\mu P'))$ and the log-convexity of the moments of non-negative random variables (see Loeve(1977), p. 158).

[7] This is because if a positive μ exists it will have to be less than 1.

Remark 7 *Note also that the condition for the existence of a mean for the stationary distribution of inflation,* $P(\Phi)' < \mathbf{1}$*, implies that the expected value of* ϕ *from every state is less than* 1*. Since the dominant root* $\lambda(QP')$ *lies between the maximum and minimum column sums of* QP'*, it represents an average value for* ϕ *with probabilities weighted appropriately over states, required to be less than unity; an average Taylor Principle for indeterminate solutions.*

The following Corollary follows immediately since it implies $\lambda(Q^m P') > 1$.

Corollary 1 *If* $P(\Phi^m)' \geq \mathbf{1}$*, then the stationary distribution of inflation, which exists if* $\nu \ln |\Phi| < 0$*, has no moments of order* m *or higher.*

Remark 8 *If we have a Markov chain for* ϕ_t *and we want it to be iid, then the rows of* P *must be identical: transition probabilities must be independent of the state. The dominant root* $\lambda(Q^\mu P')$ *is simply the trace of* $Q^\mu P'$ *since the other roots are zero, and column sums* $\Sigma_i \left(\bar{\phi}_i\right)^\mu P_{ji}$ *are identical for any* j*.*

Remark 9 *Comparative statics for* μ *can be obtained easily since the dominant root is an increasing function of the elements of* $Q^\mu P'$*. Since* $\lambda(Q^\mu P')$ *is a log-convex function of* μ*, the effect of mean preserving spreads the random variable* $\lim_{N\to\infty} \left(\prod_{n=0}^{N-1} \left(\phi_{-n}\right)^\mu \right)^{\frac{1}{N}}$ *can be studied though second order dominance to show that they will decrease* μ*.*

The results above are also consistent with Proposition 1 of Davig and Leeper (2007). First note that as long as there is a state of the Taylor coefficient, $\bar{\phi}_i > 1$ with $P_{ii} > 0$, and $\gamma_{t+1} - \varepsilon_t$ is *iid* with zero mean, then a stationary distribution of inflation that solves (4) will unbounded: there will always be a positive probability of a sufficiently long run of $\bar{\phi}_i > 1$ coupled with non-negative shocks, to reach any level of inflation. Therefore we may seek to obtain bounded solutions of (4) with $0 < \bar{\phi}_i < 1$, all i. In that case however, the matrix given by Davig and Leeper (2007), $M = Q^{-1}P$ will have elements larger than those of P. But the dominant root of P, larger in modulus than other roots, is 1, and as is well known, an increasing function of its elements. So M must have a root larger than 1, and bounded solutions would be consistent with the results of Davig and Leeper (2007)..Conversely, if $\bar{\phi}_i > 1$ for all i, the dominant root, as well as other roots of $M = Q^{-1}P$ will be within the unit circle and satisfy the condition given by Davig and Leeper (2007) to rule out bounded solutions to (4).

However, as shown by Farmer, Waggoner and Zha (2009b) in an example with a two state Markov chain, bounded sunspot solutions that satisfy (3) may still exist. With regime-switching we may allow the sunspot variable γ_{t+1} to be proportional to $\phi_t q_t$ for all transitions to the active regime, and to thereby dampen the multiplicative effect of the Taylor coefficient, effectively transforming the system into one that behaves as if the policies were passive. The reason that this is compatible with a zero mean sunspot variable is that the dampening of the active policy can be offset by a value of γ_{t+1} for all transitions to the passive regime, again proportional to the value of $\phi_t q_t$, and thereby preserve the zero mean of γ. Therefore given transition probabilities, the random switching model makes it possible maintain the zero mean of the sunspot variable, as long as we allow a correlation between the sunspot variable and the contemporaneous realization of the Taylor coefficient ϕ. Boundedness follows because this scheme effectively delivers a stochastic difference equation with random switching between Taylor coefficients that are below one in each regime.

Even more generally, in a New Keynesian model, Farmer, Waggoner and Zha (2009a) construct examples of bounded solutions without sunspots that depend not only on the fundamental shocks of the Minimum State Variable solution, but also on additional autoregressive shocks driven by fundamental shocks. The coefficients of the autoregressive structure have to depend on the transitions between the regimes as well as on the transition probabilities in order to satisfy the analogue of (3). Markov switching across regimes permits the construction of such solutions, but the autoregressive structure thus constructed must also be non-explosive to allow the solutions to be bounded. Farmer, Waggoner and Zha (2009a) show that this can be accomplished if at least one of the regimes is passive so that it would permit indeterminacy operating on its own. A key element of the construction is the dependence of the additional non-fundamental shocks on the transitions between states and transition probabilities.

For an example of a MSV sunspot solution when ε_t and ϕ_t in equation (3) are correlated, take the case where ϕ_t is *iid* over $\boldsymbol{\phi} = \left\{\phi^1, ...\phi^n\right\}$ with probabilities $\boldsymbol{p} = (p_1, ...p^n)$, and set $\varepsilon_t = \phi_t - 1$. We now have a stationary solution that solves (3) irrespective of the state space $\boldsymbol{\phi}$ and probabilities $\boldsymbol{p}$, given by $q_t = 1$. Note however that with this specification we have $\Pr(\phi_t q + (1-\phi_t) = q) = 1$ for $q = 1$. (For a discussion of such solutions see Bougerol and Picard (1992), and also footnote 12 below.) We may now also introduce a "regime switching" sunspot process with $\gamma_{t+1} = \phi_t \gamma_t + \omega_t$, where ω_t is *iid* with zero mean, whose distribution can be characterized by the methods of Theorem 1, provided $E \ln \phi_t < 0$. Now consider the solution $q_t = 1 + \gamma_t$. Since $E\left(q_{t+1}\right) = 1 + E\left(\gamma_{t+1}\right) = 1 + \phi_t \gamma_t$, this solution also satisifies (3).

4 Extensions

1. The results can be extended to the case where $\{\phi_t, \varepsilon_t\}_t$ is not necessarily *iid*. We can define a Markov modulated process where we have a Markov chain on $\left\{\phi_t, \varepsilon_t, \gamma_{t+1}\right\}_t$ with the restriction that

$$\Pr\left(\phi_t, \varepsilon_t, \gamma_{t+1} | \phi_{t-1}, \varepsilon_{t-1}, \gamma_t\right) = \Pr\left(\phi_t, \varepsilon_t, \gamma_{t+1} | \phi_{t-1}\right)$$

The idea is that a single Markov process, here for simplicity $\{\phi_t\}_t$, drives the distributions of ε_t and γ_t, so that the parameters of the distribution of ε_t and γ_t depend on ϕ_{t-1} but not on past realizations of ε and γ. (See Saporta (2005) in remarks following Thm. 2). A pertinent example of such conditional independence is where the expectations of interest rate deviations ε_t and the sunspot variable γ_t remain at zero, irrespective of the realizations of ϕ_{t-1}, but other parameters of their distribution may be affected by ϕ_{t-1}. With an additional technical assumption the results of the previous sections go through unchanged.[8] Furthermore, the finite state discrete Markov chain assumptions can also be relaxed. (See Roitershtein (2007).)

2. To study economic models with regime switching in more than one dimension, the results of the previous sections have to be generalized. For example New Keynesian models with sticky prices require two dimensions, inflation and output, while $AR(q)$ models with random coefficients can be transformed into a first order system. We may therefore want to study higher order systems of the type $q_{t+1} = A_t q_t + b_t$, where A_t and b_t are random d-dimensional square matrices and vectors, respectively. The theory to study such multidimensional processes has been developed by Kesten (1973) and recently extended by others. (For an overview see Saporta (2004), sections 4 and 5.) While the results concerning power tails in the one-dimensional case generalize at least for the case of *iid* transitions, the technical conditions that must be verified, although similar to the ones for the one dimensional case, are more complex. So we consider the simplest cases sufficient for our purposes based on Kesten (1973), and refer the reader to Saporta (2004a, sections 4 and 5) and Saporta, Guivarc'h, and Le Page (2004b)) for generalizations.

Let $\{A_t\}_t$ be an *iid* process on $\mathbf{A}= \left(A^1, ...A^n\right)$ with associated probability distribution $p = (p_1...p_n) > 0$, where each $d \times d$ square matrix $A^i \geq 0$ that has no zero rows. We let $\delta(A_i)$ denote the dominant root of A_i, z_i denote the maximum column sum of A^i where $z = (z_1, ...z_n)$, and y_i denote the minimum column sum of A_i, where $y, (y_1, ...y_n)$. Similarly let $\{b_t\}_t$ be an *iid* process on $\mathbf{b}= \left(b^1, ...b^m\right)$, with associated probability distribution $\nu = \left(\nu^1, ...\nu^m\right) > 0$ where, for all i, b^i is not a vector with all zero elements[9].

The two assumptions we need, easily checked, are (i) $\alpha = p \ln z < 0$,[10] and (ii) $py^{\sigma} > d^{\frac{\sigma}{2}}$ where d is the dimension of the system. These two assumptions play a role analogous to (i) in Theorem 1 and $P(\Phi^m)' > \mathbf{1}$ in Corollary 1.

In addition we need two assumptions that rule out exceptional cases: (a) the elements $\delta(A_i)$ are not integral multiples of the same number, and (b) for some $\delta(A_i)$, b^j and b^k, $u^i\left(b^j - b^k\right) \neq 0$, where u^i is the characteristic vector of

[8] The technical assumption is

$$\Pr\left(\phi_i q + \varepsilon_i + \gamma_{i+1} = q\right) < 1 \text{ for any } i \text{ and } q.$$

This prevents special cases where the stochastic process gets stuck at a particular value of q. See the last paragraph of the previous section.

[9] We could easily allow a continous distribution for b, but we are trying to keepp notation and exposition to a minimum.

[10] To assure that $\alpha = p \ln z < 0$ we may use a stronger condition, that the expected value of the dominant root of $\left(A_1 \otimes^t A_1\right)$, $\delta\left(A_1 \otimes^t A_1\right) < 1$, where $\otimes$ is the Kronecker product.This condition guarantees not only that $\alpha < 0$ and therefore, the existence of a stationary solution of of $\{q_t\}_t$, but also the existence of the first and second moments. For a proof see Klüppelberg and Pergamenchtchiko (2004), Saporta (2004) Proposition 4.1, and its proof. A similar and related condition is used by Farmer, Waggoner and Zha (2009c) to assure the existence of second moments to yield the desirable "mean square stability" results.

$\delta(A_i)$. Assumption (a) is the analogue of (ii) in Theorem 1. Assumption (b) would not be needed if > 0. (See remark following Theorem 4 in Kesten (1973).) It is clear that both the Assumptions $(a) - (b)$ exclude exceptional cases and can be perturbed away if needed.

The above assumptions assure that there exists $0 < \mu \leq \sigma$ such that the power law and moment results in the one dimensional case generalize: the power law will apply to xq, with x any normalized unit row vector of the same dimension as q. The power law is given by $\lim_{t\to\infty} \Pr(xq \geq t) = C(x)\, t^{-\mu}$ with $C(x)$ strictly positive for some but not necessarily for all x, because some or all elements of the vector q may have only a left rather than a right power tail.[11] Note that with $C(x) > 0$, if $\sigma < 1$ the stationary distribution of $\{xq_t\}_t$ has no mean, if $\sigma < 2$, it has no variance, and more generally no moment $m > \sigma$. If σ is not finite, all moments of $\{q_t\}_t$ will exist. We could however study whether a particular moment fails to exist by computing the value of $\sigma = \hat{\sigma}$ that solves $py^{\hat{\sigma}} = d^{\frac{\hat{\sigma}}{2}}$.[12]

3. To simplify matters, with some additional assumptions we can introduce a Phillips curve in a simplified model while still remaining in one dimension. Let the simple Phillips curve be given by $q_t = kx_t$ where x_t is output and q_t is inflation, and let the IS curve be $x_t = -m\left[R_t - Eq_{t+1}\right] + E_t x_{t+1}$ where R_t is the nominal interest rate. Let the Taylor rule be given by $i_t = \phi_t q_t$. Then after substitutions the system can be written as

$$Eq_{t+1} = \left(\frac{\phi_t mk + 1}{mk + 1}\right) q_t = \chi_t q_t$$

where $\chi_t = \chi > 1$ (< 1) if $\phi_t = \phi > 1$ (< 1). There is always a bounded solution given by $q_t = 0$ where inflation is always at its target steady state.. However, if ϕ_t is generated by a Markov chain, there may also be sunspot solutions given by

$$q_{t+1} = \chi_t q_t + \gamma_{t+1}$$

where γ_{t+1} is a sunspot variable. This equation may then be analyzed by the

[11] Note that is not restricted to be positive. Then for some but not all unit vectors x we may have $C = 0$. In some cases, for example for $\mathbf{b} > 0$, the distribution of xq only has a right tail for all $x > 0$ so that $\lim_{t\to\infty} t^{\mu} \Pr(xq \geq t) = C(x) > 0$, (see Kesten (1973), Thm 4 and the remark that follows), but does not have a left tail. Then $\lim_{t\to\infty} t^{\mu} \Pr(-xq \geq t) = \lim_{t\to\infty} t^{\mu} \Pr(xq \leq -t) = 0$. Conversely for example for $\mathbf{b} < 0$ only the left but not the right power tail exists. For more general statements that guarantee $C > 0$ for all unit vectors x, see Kesten (1973), thm 6 or Saporta (2004), thms 10-13 and especially 5.1 in sections 4 and 5. The additional assumptions guarantee sufficient mixing so that both left and right power tails exist for all elements of q.

[12] We may also inquire as to whether $\alpha = p \ln z > 0$ rules out the existence of a stationary distribution for the solution of $q_{t+1} = A_t q_t + b_t$. Bougerol and Picard (1992) prove that this is indeed the case where $\{A_s, b_s\}_s$ is *iid* under the assumptions that (i) $\{A_s, b_s\}$ is independent of q_n for $s < n$ (so for example, the direction of time cannot be reversed), and (ii) if for some $\{A_0, b_0\}$ there exists an invariant affine subspace $H \in \mathbf{R}^d$ such that $\left\{A^i q + b^i | q \in H\right\}$ is contained in H, then H is $\mathbf{R}^d$. Condition (ii), which the authors call irreducibility, eliminates for example cases where $b_t = 0$ for all t, so that $q_t = 0$ is a stationary solution for all t irrespective of $\{A_s\}_s$, or where for some feasible A_0, b_0, q_0 we have $\Pr(A_0 q_0 + b_0 = q_0) = 1$.

same methods used above.

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