NBER WORKING PAPER SERIES

IMPOSSIBLE FRONTIERS

Thomas J. Brennan
Andrew W. Lo

Working Paper 14525
http://www.nber.org/papers/w14525

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
December 2008

The views and opinions expressed in this article are those of the authors only, and do not necessarily represent the views and opinions of AlphaSimplex Group, MIT, Northwestern University, any of their affiliates and employees, or the National Bureau of Economic Research. The authors make no representations or warranty, either expressed or implied, as to the accuracy or completeness of the information contained in this article, nor are they recommending that this article serve as the basis for any investment decision---this article is for information purposes only. Research support from AlphaSimplex Group and the MIT Laboratory for Financial Engineering is gratefully acknowledged. We thank Henry Cohn, Sanjiv Das, Arnout Eikeboom, Leonid Kogan, Tri-Dung Nguyen, and participants of the JOIM Fall 2008 Conference and the MIT Finance Lunch for helpful comments and discussions.

NBER working papers are circulated for discussion and comment purposes. They have not been peer-reviewed or been subject to the review by the NBER Board of Directors that accompanies official NBER publications.

© 2008 by Thomas J. Brennan and Andrew W. Lo. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Impossible Frontiers
Thomas J. Brennan and Andrew W. Lo
NBER Working Paper No. 14525
December 2008
JEL No. G1,G11,G12,G14,G23,G32

ABSTRACT

A key result of the Capital Asset Pricing Model (CAPM) is that the market portfolio---the portfolio of all assets in which each asset's weight is proportional to its total market capitalization---lies on the mean-variance efficient frontier, the set of portfolios having mean-variance characteristics that cannot be improved upon. Therefore, the CAPM cannot be consistent with efficient frontiers for which every frontier portfolio has at least one negative weight or short position. We call such efficient frontiers "impossible", and derive conditions on asset-return means, variances, and covariances that yield impossible frontiers. With the exception of the two-asset case, we show that impossible frontiers are difficult to avoid. Moreover, as the number of assets n grows, we prove that the probability that a generically chosen frontier is impossible tends to one at a geometric rate. In fact, for one natural class of distributions, nearly one-eighth of all assets on a frontier is expected to have negative weights for *every* portfolio on the frontier. We also show that the expected minimum amount of shortselling across frontier portfolios grows linearly with n, and even when shortsales are constrained to some finite level, an impossible frontier remains impossible. Using daily and monthly U.S. stock returns, we document the impossibility of efficient frontiers in the data.

Thomas J. Brennan
School of Law, Northwestern University
East Chicago Avenue
Chicago, IL 60611
t-brennan@law.northwestern.com

Andrew W. Lo
Sloan School of Management
MIT
50 Memorial Drive
Cambridge, MA 02142-1347
and NBER
alo@mit.edu

Contents

1 Introduction

A cornerstone of modern portfolio management is the "efficient frontier" of mean-variance analysis: the set of portfolios for which the lowest variance possible is attained for given levels of expected return, or the highest possible expected return is attained for a given level of variance. The main thrust of the Capital Asset Pricing Model (CAPM) is that the market portfolio—the portfolio of all assets where each asset's weight is proportional to its total market capitalization—must lie somewhere on the efficient frontier. Since, by definition, every component of the market portfolio has a positive weight (because its market capitalization must be positive), we would expect at least one portfolio on the efficient frontier to have this property. If, for a given a set of asset-return parameters (means, variances, and covariances), the corresponding efficient frontier does not have any such portfolio, we call this an "impossible frontier" for obvious reasons.

In this paper, we show that, as the number of assets grows large, nearly all efficient frontiers are impossible.

Specifically, for any arbitrary set of expected returns, and for a randomly chosen covariance matrix, we show that the probability that the resulting frontier is impossible approaches one as the number of assets increases without bound. This result depends, of course, on the specific distribution from which we draw the covariance matrix, and we consider two classes: the uniform distribution (Haar measure), and distributions centered around linear-factor models such as the CAPM and Ross's (1976) Arbitrage Pricing Theory (APT). For both classes of distributions, mean-variance efficient frontiers are almost surely impossible.

This remarkable result is not an artifact of pathological parameters, except in the two-asset case, but is apparently a generic property of mean-variance efficient portfolios. For typical parameter values, *every* portfolio on the efficient frontier will contain at least one short position, i.e., a negative weight. This implies that such an efficient frontier cannot be consistent with a CAPM equilibrium in which every investor holds the tangency portfolio, for such an equilibrium requires all weights to be positive for that portfolio. Alternatively, our impossibility result implies that the set of expected return vectors and covariance matrices $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ that is consistent with a CAPM equilibrium is extremely small—in fact, measure-zero in the limit—hence we should not expect typical empirical estimates of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ to yield plausible portfolios from the CAPM perspective unless the CAPM is literally true and estimation error is negligible.

Our results provide one explanation for the skepticism that most long-only portfolio managers have for standard mean-variance optimization—from their perspective, an impossible frontier is truly impossible for them to implement. Moreover, it is well known that the output of standard portfolio optimizers yield weights that must be constrained, but until now, the non-negativity restriction that has become second nature to practitioners was thought to be a consequence of estimation error. The results in our paper show that even in the ideal case where the means and covariance matrix of asset returns are known with perfect certainty, the efficient frontier will almost always contain negative weights. To the extent that estimation error generates means and covariances that deviate from the CAPM, such sampling variation will only exacerbate the problem, making it more likely that the sample efficient frontier is impossible. Our impossibility results may also provide a partial explanation for the recent popularity of so-called "active extension" strategies such as 130/30 portfolios in which a limited amount of shortselling is permitted.

We begin in Section 2 with a brief review of the literature, and in Section 3 we derive analytical results for the two- and three-asset cases to build intuition and motivate our more general results. The main results of the paper are contained in Section 4, where we propose two classes of probability measures for covariance matrices and show that under both these classes of measures, impossible frontiers become the rule, not the exception, as the number of assets increases without bound. We also show that the expected minimum amount of shortselling across frontier portfolios grows linearly with n, and even when shortsales are constrained to some finite level, an impossible frontier remains impossible. Given the importance of the CAPM, in Section 5 we examine the linear one-factor return-generating model in more detail, and show how to construct a covariance matrix that does not yield an impossible tangency portfolio. In Section 6, we provide an empirical illustration of our theoretical findings using daily and monthly returns for a subset of S&P 500 stocks, and show that the usual sample estimators of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ do yield impossible frontiers. We conclude in Section 7 with a discussion of the theoretical and practical significance of our results.

2 Literature Review

Any review of the mean-variance portfolio selection literature must begin with Markowitz (1952) who first introduced this powerful framework to the economics literature. Building on the Markowitz mean-variance framework, Tobin (1958), Sharpe (1964), and Lintner (1965) derived the equilibrium implications under the assumption that all investors held mean-variance-optimal or "efficient" portfolios, culminating in the "Capital Market Line", the line in mean-standard deviation space connecting the riskfree rate on the expected-return axis with the tangency portfolio on the efficient frontier in mean-standard deviation space.

The role of shortsales in mean-variance analysis has also been considered by several authors. In fact, Markowitz (1959, p. 132) recognized the importance of implementing constraints on portfolio weights, one of which was a non-negativity or shortsales constraint. However, Lintner (1965) was perhaps the first to study the impact of shortsales on capital market equilibrium, deriving alternative equilibria under shortsales prohibitions as well as shortsales constraints. Lintner concluded that investors would not engage in shortsales in equilibrium because of the Tobin separation theorem, i.e., all investors are indifferent between holding portfolios of all assets versus portfolios of just two funds—the riskless asset and the tangency portfolio. None of these authors studied the prevalence of short positions in the tangency portfolio, essentially ruling out such "impossible frontiers" in equilibrium.

A number of papers have been written about the impact of shortsales constraints on asset prices in various settings, including Pogue (1970), Brito (1978), Jarrow (1980), Diamond and Verrecchia (1987), Heaton and Lucas (1996), Detemple and Shashidhar (1997), Duffie, Garleanu, and Pedersen (2002), and Sun and Wang (2006). However, these studies are focused on the equilibrium effects of credit constraints, not on quantifying the frequency or amount of shorting in the generic mean-variance-efficient portfolio, which we propose to do in this paper.

More recently, Markowitz (2005) has argued that empirical deviations from the CAPM are not surprising in light of the counterfactual assumptions on which the CAPM is based. In particular, he observes that "When one clearly unrealistic assumption of the capital asset pricing model is replaced by a real-world version, some of the dramatic CAPM conclusions no longer follow". An example is the fact that unlimited borrowing and lending at identical yields is not possible in practice, and this limitation implies that the market portfolio need not be mean-variance-efficient in equilibrium.

Markowitz's (2005) caveats are well taken, but the results of our paper are considerably stronger. We argue that even if all the assumptions of the CAPM are true, the market portfolio need not be mean-variance efficient. Specifically, Markowitz (2005) states the following assumptions:

(A1) Transaction costs and other illiquidities can be ignored.

(A2) All investors hold mean-variance efficient portfolios.

(A3) All investors hold the same (correct) beliefs about means, variances, and covariances of securities.

(A4) Every investor can lend all she or he has or can borrow all she or he wants at the risk-free rate.

and argues that Conclusion 1 follows:

(C1) The market portfolio is a mean-variance efficient portfolio.

The results of Sections 3–5 below show that there exist certain combinations of means, variances, and covariances for which *every* mean-variance efficient portfolio contains short positions, implying that none can be the market portfolio. And as the number of assets grows without bound, the likelihood of coming across a set of parameter values with this characteristic is almost certain.

3 Some Examples of Impossible Frontiers

We begin with some notation. Let $\boldsymbol{\mu}$ be the vector of expected returns for n assets, and let $\boldsymbol{\Sigma}$ be the covariance matrix of those returns.[1] For a given level of expected return μ_o, the corresponding portfolio on the efficient frontier is the vector $\boldsymbol{\omega}$ which minimizes the value of

$$\boldsymbol{\omega}^t \boldsymbol{\Sigma} \boldsymbol{\omega} \quad \text{subject to} \quad \boldsymbol{\omega}^t \boldsymbol{\iota} = 1 \ , \ \boldsymbol{\omega}^t \boldsymbol{\mu} = \mu_o \ . \tag{1}$$

where $\boldsymbol{\iota}$ is a column vector of ones of the appropriate length. The set of optimal $\boldsymbol{\omega}$ can be found using the method of Lagrange multipliers (see, for example, Merton, 1972):

$$\mathcal{F} \ = \ \left\{ \boldsymbol{\omega} : \boldsymbol{\omega} = \frac{BC}{D}\left(\mu_o - \frac{B}{C}\right)(\boldsymbol{\omega}_{\boldsymbol{\mu}} - \boldsymbol{\omega}_g) + \boldsymbol{\omega}_g \ , \ \text{for } \mu_o \geq \frac{B}{C} \right\} \tag{2}$$

where

$$A \ \equiv \ \boldsymbol{\mu}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \ , \ B \ \equiv \ \boldsymbol{\mu}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota} \ , \ C \ \equiv \ \boldsymbol{\iota}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota} \ , \ D \ \equiv \ AC - B^2 \tag{3}$$

and

$$\boldsymbol{\omega}_g \ \equiv \ \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota} / C \ , \ \boldsymbol{\omega}_{\boldsymbol{\mu}} \ \equiv \ \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} / B \ . \tag{4}$$

[1] Throughout this paper, we maintain the following notational conventions: (1) all vectors are column vectors unless otherwise indicated; (2) matrix transposes are indicated by t superscripts, hence $\boldsymbol{\omega}^t$ is the transpose of $\boldsymbol{\omega}$; and (3) vectors and matrices are always typeset in boldface, i.e., X and μ are scalars and $\mathbf{X}$ and $\boldsymbol{\mu}$ are vectors or matrices.

Note that $\boldsymbol{\omega}_g$ is the global minimum-variance portfolio, and $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is the vector that maximizes the Sharpe ratio relative to the risk-free rate of zero, i.e., $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ maximizes the function $\boldsymbol{\mu}^t\boldsymbol{\omega}/\sqrt{\boldsymbol{\omega}^t\boldsymbol{\Sigma}\boldsymbol{\omega}}$.

The frontier starts at the expected return level $\mu_o = B/C$. In fact, we can compute minimum-variance portfolios for values of μ_o less than B/C, but these portfolios would lie on the "inefficient" branch of the portfolio frontier, i.e., the portion of the frontier for which return is not maximized for a given level of risk.

We call a frontier "impossible" with respect to the i-th component if the weight of the i-th component at each point on the frontier is negative. Clearly, a sufficient condition for a frontier to be impossible is that it be impossible for the i-th asset, $1 \leq i \leq n$. From (2), we see that every point on an efficient frontier can be written in the form

$$\boldsymbol{\omega} \;=\; \frac{C}{D}\left(\mu_o - \frac{B}{C}\right)\boldsymbol{\omega}_P \;+\; \boldsymbol{\omega}_g \tag{5}$$

where $\boldsymbol{\omega}_P \equiv B\boldsymbol{\omega}_{\boldsymbol{\mu}} - B\boldsymbol{\omega}_g$. The values of C and D are non-negative by the Cauchy-Schwartz inequality, so a frontier will be impossible with respect to the i-th asset exactly when $\boldsymbol{\omega}_g$ and $\boldsymbol{\omega}_P$ both have negative i-th components.

Our technique for proving that an efficient frontier is impossible is to show that the i-th elements of both $\boldsymbol{\omega}_g$ and $\boldsymbol{\omega}_P$ are negative for some i. Using this method, we can calculate a lower bound for the probability that a generically chosen efficient frontier is impossible, as well as lower bounds for the expected number of coordinates on a frontier with respect to which the frontier is impossible, and also lower bounds on the expected amount of total shortsales at each point on the frontier.

In Section 3.1, we investigate the special case of $n=2$ and find that certain frontiers are impossible, but only under some rather unnatural conditions. However, in Section 3.2, we show that when $n=3$, a variety of frontiers become impossible without any unnatural conditions.

3.1 The Two-Asset Case

For the case of $n=2$ assets, we can characterize all situations in which a frontier will be impossible (proofs are included in the Appendix):

Proposition 1 *For $n=2$, let the assets be ordered so that $\mu_1 < \mu_2$, let σ_i denote the risk of the i-th asset, and let ρ denote the correlation between the assets. The efficient frontier is* *impossible if and only if*

$$\frac{\sigma_2}{\sigma_1} \quad < \quad \rho \ .$$

Because $\rho \leq 1$, the proposition implies that a necessary condition for a frontier with two assets to be impossible is that $\sigma_2 < \sigma_1$. Also, since the volatilities are both non-negative, it is also necessary that $\rho > 0$. Thus, for a frontier to be impossible, the asset with higher expected return must also have lower risk, and the two assets must be positively correlated. In such a circumstance, it will be optimal to have a short position in the low-return/high-risk asset at every point on the efficient frontier.

This condition is unnatural because the lower expected-return asset is strictly dominated by the higher expected-return asset given that the latter is less risky than the former. Therefore, on purely economic grounds, it is possible to rule out impossible frontiers in the two-asset case. However, we show in the next section that with just one more asset, there is no natural way to avoid impossible frontiers.

3.2 The Three-Asset Case

For $n = 3$ assets, we provide a characterization of 3×3 covariance matrices that gives rise to a minimum-variance portfolio with negative weights when the volatilities of all assets are equal. This result will allow us to specify a class of covariance matrices that imply arbitrarily large amounts of shortselling in the minimum-variance portfolio in the equal-volatilities case. Finally, we use our results to illustrate an example of a situation in which three assets can give rise to an impossible frontier without any unnatural restrictions on the risks and returns of the assets.

For the moment, we assume that the volatilities of all three assets are the same, which we normalize to 1 without loss of generality, hence the covariance matrix has the form:

$$\boldsymbol{\Sigma} \quad = \quad \begin{bmatrix} 1 & a & c \\ a & 1 & b \\ c & b & 1 \end{bmatrix} \ . \tag{6}$$

The range of possible values for a, b, and c for which $\boldsymbol{\Sigma}$ is positive definite is given by the following subset of $\mathbf{R}^3$:

$$\left\{ \ (a,b,c) \ : \ c = ab + \widetilde{c}\sqrt{1-a^2}\sqrt{1-b^2} \ , \ \text{with } a,b,\widetilde{c} \in (-1,1) \ \right\} \ . \tag{7}$$

We can then completely characterize the values of a, b, and c that yield negative weights in the minimum-variance portfolio:

Proposition 2 *If $\mathbf{\Sigma}$ is of the form (6), the first component of $\boldsymbol{\omega}_g$ has a negative value when*

$$c \ > \ 1 \ - \ a \ + \ b \ .$$

The second component has a negative value when

$$c \ < \ a \ + \ b \ - \ 1 \ .$$

The third component has a negative value when

$$c \ > \ a \ - \ b \ + \ 1 \ .$$

These three conditions are all mutually exclusive since $a,b,c < 1$, so at most one component of $\boldsymbol{\omega}_g$ may be negative.

Corollary 1 *Let $\mathbf{\Sigma}$ be of the form (6). If we let $d = 1-\varepsilon$, for $\varepsilon \to 0^+$, and we set $a = d$, $b = d$, and $c = ab - d\sqrt{1-a^2}\sqrt{1-b^2}$, then $\mathbf{\Sigma}$ is a non-degenerate covariance matrix and the short position required in the minimum-variance portfolio $\boldsymbol{\omega}_g$ is*

$$\omega_{m2} \ \ = \ \ -\frac{1}{2\varepsilon} \ + \ \mathrm{O}(1) \ .$$

Corollary 1 implies that, by allowing ε to tend toward zero, arbitrarily large short positions in the minimum-variance portfolio can be generated. The proof of Proposition 2 follows directly from the calculation of a formula for the value of $\boldsymbol{\omega}_g$ in the three-asset case, and the proof of the corollary follows from that same explicit formula and an additional calculation (see the Appendix for further details).

Armed with these results, we can now construct a non-trivial three-asset example with impossible frontiers. We begin with a correlation matrix $\mathcal{C}$ of the same form as the covariance matrix in Corollary 1, with $\varepsilon = 0.35$:

$$\mathcal{C} = \begin{bmatrix} 1.0000 & 0.6500 & 0.0471 \\ 0.6500 & 1.0000 & 0.6500 \\ 0.0471 & 0.6500 & 1.0000 \end{bmatrix} \tag{8}$$

Let the expected returns for the three assets be 10%, 14%, and 18%, respectively, and let their volatilities be 15%, 20%, and 25%, respectively, hence $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ are:

$$\boldsymbol{\mu} = \begin{bmatrix} 0.10 \\ 0.14 \\ 0.18 \end{bmatrix} \quad , \quad \boldsymbol{\Sigma} = \begin{bmatrix} 0.0225 & 0.0195 & 0.0018 \\ 0.0195 & 0.0400 & 0.0325 \\ 0.0018 & 0.0325 & 0.0625 \end{bmatrix} . \tag{9}$$

Note that unlike the condition in Proposition 1, these parameters do not imply any type of dominance relation among the three assets—higher expected-return assets have higher volatilities. These parameters yield the following values for $\boldsymbol{\omega}_P$ and $\boldsymbol{\omega}_g$:

$$\boldsymbol{\omega}_P = B(\boldsymbol{\omega}_{\boldsymbol{\mu}} - \boldsymbol{\omega}_g) = \begin{bmatrix} -1.4046 \\ 0.7212 \\ 0.6834 \end{bmatrix} \quad , \quad \boldsymbol{\omega}_g = \begin{bmatrix} 1.0888 \\ -0.5859 \\ 0.4971 \end{bmatrix} . \tag{10}$$

Given the expression (5) for the efficient frontier, it is apparent that every frontier portfolio close to $\boldsymbol{\omega}_g$ has a short position in asset 2, and every frontier portfolio close to a multiple of $\boldsymbol{\omega}_P$ has a short position in asset 1. A calculation shows that in fact there is a short position in asset 2 for each frontier portfolio with an expected return less than 18.42%. At this point, there is also a short position of −5.23% in asset 1, and the short position in asset 1 increases for in portfolios with higher expected returns. Thus, all portfolios on the efficient frontier have an aggregate short position of at least −5.23%.

In fact, the efficient frontier (10) will continue to be impossible if the values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are allowed to vary within a small neighborhood of (9). Many other three-asset examples of impossible frontiers can be constructed, and with empirically plausible parameters. By increasing the number of assets from two to three, the set of impossible frontiers seems to have grown significantly. In Section 4, we show that this is no coincidence, and that as n increases without bound, an arbitrarily chosen frontier is almost surely impossible.

4 The General Case

In this section, we consider the general case of an arbitrary number of n assets. Unfortunately, simple analytical results like those for the two- and three-asset cases of Section 3 are not available for an arbitrary number of assets. However, we propose to conduct the following thought experiment: for a given vector $\boldsymbol{\mu}$ of expected returns, and a randomly selected covariance matrix $\boldsymbol{\Sigma}$, what is the likelihood that the resulting frontier is impossible? To compute such a probability, we must, of course, propose a probability distribution for a covariance matrix, which is not a straightforward exercise. Although distributions of covariance matrices have been developed in the statistics literature, e.g., the Wishart distribution, they are sampling distributions of covariance-matrix *estimators* applied to independently and identically distributed multivariate normal data (see Anderson, 1984, chapter 7). Such distributions are highly parametric—if multivariate normality does not hold, then neither does the Wishart—and also do not necessarily capture the randomness that we seek, i.e., the random drawing of an arbitrary population covariance matrix from the space of all possible covariance matrices. In particular, Wishart distributions are typically "centered" at the estimated sample covariance matrix with multivariate tails that decline exponentially fast. This may be a reasonable model of the randomness associated with sampling error, but seems less compelling as a mechanism for drawing an arbitrary covariance matrix at random.

Instead, we seek a more general distribution, such as a uniform distribution over the space of all possible covariance matrices, i.e., the space of all $(n \times n)$ symmetric positive-definite matrices with real elements. However, because this space is not compact, a uniform distribution over this space will have infinite mass. Nevertheless, in the same way that an "improper prior" can be specified in Bayesian inference,[2] we can construct an "uninformative" distribution as a proxy for the uniform. We provide such a distribution for covariance matrices in Section 4.1 using the concept of Haar measure, which will allow us to gauge the probability that a randomly selected covariance matrix gives rise to an impossible frontier, yielding the conclusion that impossible frontiers are almost certain to arise as the number of assets increases without bound.

However, it may be argued that an uninformative distribution of covariance matrices will not yield economically relevant draws because the resulting covariance matrices lack the factor structure hypothesized in the most popular asset-pricing models. To address this concern, in Section 4.2 we introduce another class of probability distributions centered around the covariance matrices generated by linear factor-pricing models such as the CAPM and APT, and derive lower bounds on the probability that a frontier is impossible if it is

[2] See, for example, Jeffreys (1961, pp. 180–181) and Box and Tiao (1973, p. 426).

chosen randomly with respect to one of the distributions in this class. We show that this lower bound also approaches unity as n grows without bound.

In Section 4.3, we calculate lower bounds on the expected number of assets with respect to which a frontier will be impossible, as well as estimates for the expected minimum size of short positions across frontier portfolios. We also find that an impossible frontier will remain impossible even if constraints are placed on the total amount of shortselling allowed in any portfolio.

4.1 Haar Measure and Covariance Matrices

Haar measure is the unique measure (up to a constant) that is invariant under the natural action of the group $\mathbf{GL}_n$ of invertible linear transformations on $\mathbf{R}^n$ on the space of covariance matrices. For $\mathbf{G} \in \mathbf{GL}_n$, this action is defined by $\mathbf{\Sigma} \mapsto \mathbf{G\Sigma G}^t$ for $\mathbf{\Sigma} \in \mathcal{P}_n$ where $\mathcal{P}_n$ is the symmetric space of all positive-definite matrices on $\mathbf{R}^n$, and any covariance matrix $\mathbf{\Sigma}$ can be mapped to any other covariance matrix under some such action. Thus, any such action takes a neighborhood around a specified covariance matrix to a corresponding neighborhood around any other covariance matrix, and Haar measure assigns the same volume to every such image of the original neighborhood. In this sense, Haar measure behaves uniformly on all of $\mathcal{P}_n$ and represents an "uninformative" prior distribution over all possible $(n \times n)$ covariance matrices. The following definition summarizes Haar measure on $\mathbf{GL}_n$ (see Jorgenson and Lang, 2005 for further discussion).

Definition 1 *Haar measure on $\mathcal{P}_n$ is the measure, ν_n, that is invariant under transformations of the form $\mathbf{\Sigma} \mapsto \mathbf{G\Sigma G}^t$, for $\mathbf{G} \in \mathbf{GL}_n$. Thus, for any region $\mathbf{S} \subseteq \mathcal{P}_n$, Haar measure has the property that*

$$\nu_n(\mathbf{S}) \quad = \quad \nu_n\left(\mathbf{GSG}^t\right) \tag{11}$$

for all $\mathbf{G} \in \mathbf{GL}_n$. This measure is unique up to multiplication by a positive constant, and in terms of the elements of the matrix $\mathbf{\Sigma} = [\Sigma_{i,j}]$, we have

$$d\nu_n(\mathbf{\Sigma}) \quad = \quad \frac{1}{(\det(\mathbf{\Sigma}))^{(n+1)/2}} \prod_{i \le j} d\Sigma_{i,j} \tag{12}$$

where $d\Sigma_{i,j}$ is the element of Euclidean measure.

Under Haar measure, the entire space $\mathcal{P}_n$ has infinite volume so we cannot scale by a constant to transform Haar measure into a proper probability density. Instead, we calculate the probability that a selected frontier is impossible on cross sections of $\mathcal{P}_n$ using the probability density induced by Haar measure on those cross sections. We need first to introduce a useful system of coordinates on $\mathcal{P}_n$ with respect to which we can easily define our cross sections.

Definition 2 *Each matrix $\mathbf{M} \in \mathcal{P}_n$ can be uniquely expressed in terms of (partial) Iwasawa coordinates as $(\mathbf{X}, \mathbf{W}, \mathbf{V})$, where $\mathbf{W} \in \mathcal{P}_2$, $\mathbf{V} \in \mathcal{P}_{n-2}$, and $\mathbf{X} \equiv [\mathbf{x}_1, \mathbf{x}_2]$, with $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^{n-2}$. The relationship between $\mathbf{M}$ and $(\mathbf{X}, \mathbf{W}, \mathbf{V})$ is defined by the formula*

$$\mathbf{M} \quad = \quad \begin{pmatrix} \mathbf{I}_2 & \mathbf{X}^t \\ \mathbf{0} & \mathbf{I}_{n-2} \end{pmatrix} \begin{pmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{pmatrix} \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{X} & \mathbf{I}_{n-2} \end{pmatrix} \quad = \quad \begin{pmatrix} \mathbf{W} + \mathbf{X}^t\mathbf{V}\mathbf{X} & \mathbf{X}^t\mathbf{V} \\ \mathbf{V}\mathbf{X} & \mathbf{V} \end{pmatrix} . \tag{13}$$

Moreover, each matrix $\mathbf{W}$ can be uniquely expressed in terms of Iwasawa coordinates as (y, u, v), where $u, v \in \mathbf{R}_+$ and $y \in \mathbf{R}$, according to the relationship

$$\mathbf{W} \quad = \quad \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \quad = \quad \begin{pmatrix} u + y^2 v & yv \\ yv & v \end{pmatrix} . \tag{14}$$

Finally, we can also express each matrix $\mathbf{X}$ in terms of polar coordinates $(r_1, \ldots, r_{n-2}, \theta_1, \ldots, \theta_{n-2})$, where $r_i \in \mathbf{R}_+$ and $\theta_i \in S^1$, using the relationships $x_{1,i} = r_i \cos\theta_i$ and $x_{2,i} = r_i \sin\theta_i$. Therefore, each $\mathbf{M} \in \mathcal{P}_n$ can be written in terms of coordinates

$$\mathbf{M} \quad = \quad (r_1, \ldots, r_{n-2}, \theta_1, \ldots, \theta_{n-2}, y, u, v, \mathbf{V}) \tag{15}$$

so that the space $\mathcal{P}_n$ can be viewed as the product

$$\mathcal{P}_n \quad = \quad (\mathbf{R}_+)^{n-2} \times \left(S^1\right)^{n-2} \times \mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathcal{P}_{n-2} . \tag{16}$$

Using the coordinate system of Definition 2, we consider cross sections of $\mathcal{P}_n$ that have fixed values of all coordinates except the θ_i. We write $Z = Z\,(r_1, \ldots, r_{n-2}, y, u, v, \mathbf{V})$ for such a cross section with specified fixed values of the coordinates $r_1, \ldots, r_{n-2}$, y, u, v and $\mathbf{V}$. This cross section is thus a product of $(n-2)$ copies of S^1, and the measure on this cross section induced by Haar measure on $\mathcal{P}_n$ is

$$d\nu_Z \quad = \quad \frac{1}{(2\pi)^{n-2}}\, d\theta_1 \cdots d\theta_{n-2}\ . \tag{17}$$

The measure ν_Z is therefore a proper probability distribution on the cross-sectional space Z; although probabilities cannot be computed with respect to Haar measure on all of $\mathcal{P}_n$, they *can* be computed with respect to ν_Z on each cross section Z of $\mathcal{P}_n$.

To calculate the probability that a covariance matrix, $\mathbf{\Sigma}$, gives rise to an impossible frontier, it is convenient first to change variables from $\mathbf{\Sigma}$ to $\mathbf{M}$ using the correspondence

$$\mathbf{\Sigma} \quad = \quad \mathbf{AMA}^t \tag{18}$$

where $\mathbf{A} = \mathbf{A}\,(c_1, \ldots, c_n)$ is the unique matrix in $\mathbf{GL}_n$ with columns defined by

$$\mathbf{Ae}_1 \quad = \quad c_1\boldsymbol{\iota} \quad , \quad \mathbf{Ae}_2 \quad = \quad c_2\boldsymbol{\mu} \quad , \quad \text{and} \quad \mathbf{Ae}_j \quad = \quad c_j\mathbf{e}_j \quad \text{for } 3 \le j \le n \tag{19}$$

for specified values of $c_i > 0$. Haar measure is invariant under this change of variables, and so we can replace $\mathbf{\Sigma}$ with $\mathbf{M}$ and use Haar measure on $\mathbf{M}$ as the basis for our probability calculations. We calculate the probability that a matrix $\mathbf{\Sigma} = \mathbf{AMA}^t$ gives rise to an impossible frontier for a matrix $\mathbf{M}$ in a cross section Z, where the probability is calculated with respect to the distribution ν_Z. In Theorem 1, we obtain a lower bound for the probability of impossibility, but first we need a lemma specifying a useful test for impossibility.

Lemma 1 *For a frontier to be impossible with respect to the i-th coordinate, it is necessary and sufficient that*

$$\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\boldsymbol{\iota} \quad < \quad 0 \quad \textit{and} \quad \mathbf{e}_i^t\mathbf{\Sigma}^{-1}\boldsymbol{\mu} \;-\; \left(\frac{\boldsymbol{\mu}^t\mathbf{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}^t\mathbf{\Sigma}^{-1}\boldsymbol{\iota}}\right)\mathbf{e}_i^t\mathbf{\Sigma}^{-1}\boldsymbol{\iota} \quad < \quad 0\ . \tag{20}$$

If $\mathbf{\Sigma} = \mathbf{AMA}^t$ *and* $i > 2$, *the conditions in (20) are equivalent to*

$$\cos\theta_{i-2} \quad > \quad y\sin\theta_{i-2} \quad \textit{and} \quad \sin\theta_{i-2} \quad > \quad 0 \tag{21}$$

where $\mathbf{M}$ *has coordinates as in Definition 2.*

Theorem 1 *Let* $\mathbf{M} \in Z = Z(r_1, \ldots, r_{n-2}, y, u, v, \mathbf{V})$ *be chosen randomly with respect to the distribution* ν_Z*. The probability,* p_Z*, that the covariance matrix* $\mathbf{\Sigma} = \mathbf{AMA}^t$ *gives rise to an impossible frontier is bounded below as*

$$p_Z \;\; \geq \;\; 1 - \left(1 - \frac{1}{2\pi}\cot^{-1} y\right)^{n-2} \;\; \geq \;\; 1 \; - \; \left(1 - \frac{1}{2\pi\left(1 + \max(0, y)\right)}\right)^{n-2} \; . \tag{22}$$

This theorem shows that, for any fixed value of y, the probability, p_Z, that a covariance matrix in a cross section Z gives rise to an impossible frontier tends to 1 geometrically as n grows. Moreover, if y is bounded above by y_+, the probability for any cross section Z with a such a y coordinate tends to 1 at least as quickly as

$$p_Z \;\; \geq \;\; 1 \; - \; \left(1 - \frac{1}{2\pi\left(1 + \max(0, y_+)\right)}\right)^{n-2} \; .$$

The following corollary extends the previous results to yield a lower bound on the probability of impossibility for probability densities on the entire space $\mathcal{P}_n$.

Corollary 2 *Let* φ *be any probability density on* $\mathcal{P}_n$ *which factors into a product of densities*

$$\varphi \;\; = \;\; \left(\prod_{i=1}^{n-2} \varphi_{r_i}\right) \times \left(\prod_{i=1}^{n-2} \varphi_{\theta_i}\right) \times \varphi_y \times \varphi_u \times \varphi_v \times \varphi_{\mathbf{V}} \tag{23}$$

where the φ_{θ_i} *are uniform probability densities on* S^1 *and the other distributions are aribtrary distributions on the respective spaces* $r_i \in \mathbf{R}$*,* $y \in \mathbf{R}$*,* $u \in \mathbf{R}_+$*,* $v \in \mathbf{R}_+$*, and* $\mathbf{V} \in \mathcal{P}_{n-2}$*. Let* $\mathbf{\Sigma} = \mathbf{AMA}^t$ *be an arbitrary covariance matrix, with* $\mathbf{A}$ *as defined in (19), and with* $\mathbf{M}$ *chosen randomly in accordance with the distribution* φ*. The probability,* p*, that* $\mathbf{\Sigma}$ *gives rise to a frontier which is impossible is bounded below by*

$$p \;\; \geq \;\; 1 \; - \; \int_{\mathbf{R}} \left(1 - \frac{1}{2\pi\left(1 + \max(0, y)\right)}\right)^{n-2} \varphi_y(y) \; . \tag{24}$$

4.2 Linear-Factor Models and Impossibility

Although the generality of Haar measure in representing the selection of an arbitrary covariance matrix is compelling, some may consider it too general because it does not differentiate among outcomes according to their economic plausibility. In particular, Haar measure places the same probabilistic weight on covariance matrices arising from quantum mechanics as it does on covariance matrices from economic models—there is nothing intrinsic to Haar measure in which economic structure is incorporated. Accordingly, one could argue that Haar measure places too much weight on financially irrelevant covariance matrices. This argument is debatable, not in the least because we do not usually develop economic theories to yield specific implications for covariance matrices, hence it is not clear what "financially relevant" covariance matrices look like.

However, there does exist an important class of financial models that places restrictions on asset-return covariance matrices, and that is the set of linear factor models such as the CAPM and APT. If a linear factor-pricing model holds, then a typical covariance matrix drawn randomly from this economy will have a different distribution than Haar measure.

In this section, we introduce a class of probability distributions based upon the covariance matrix implied by linear factor models such as the CAPM and APT, and we calculate probabilities of impossibility with respect to distributions in the class. The construction of this class uses many of the techniques and notations developed in connection with our analysis of Haar measure in Section 4.1, so our exposition will be less detailed.

We start with $\mathbf{T}_0 = \mathbf{T}_0(\boldsymbol{\mu}, \mu_m, \sigma_m, r_f)$, the covariance matrix implied by a linear one-factor model for a chosen value of the expected return vector $\boldsymbol{\mu}$ and for arbitrarily specified values of the expected return on the market, μ_m, the market volatility, σ_m, and the riskfree rate, r_f, assuming for the moment that there are no idiosyncratic components to asset returns. The matrix $\mathbf{T}_0$ can be written

$$\mathbf{T}_0 \quad = \quad \sigma_m^2 \boldsymbol{\beta}\boldsymbol{\beta}^t$$

where $\boldsymbol{\beta}$ is the vector of "beta" values, $\boldsymbol{\beta} = (\boldsymbol{\mu} - \boldsymbol{\iota} r_f)/(\mu_m - r_f)$ (recall that we have assumed no idiosyncratic shocks for the moment).

To incorporate independent idiosyncratic risks, non-negative amounts can be added to diagonal elements of $\mathbf{T}_0$, and the elements of the matrix may be additionally adjusted to reflect deviations from the CAPM. We define a family of such matrices,

$$\mathcal{T} \quad = \quad \mathcal{T}(\boldsymbol{\mu}, \mu_m, \sigma_m, r_f) \quad = \quad \left\{ \mathbf{T} = \sigma_m^2 \boldsymbol{\beta}\boldsymbol{\beta}^t + \boldsymbol{\delta}\mathbf{I}_n + \varepsilon \boldsymbol{\iota}\boldsymbol{\iota}^t \colon \varepsilon \geq 0, \delta_i \geq 0 \right\}$$

and we define the subfamily $\mathcal{T}_2$ to be those matrices in $\mathcal{T}$ with $\delta_1 = 0$, $\delta_2 = 0$, and $\delta_i > 0$ for $3 \leq i \leq n$.

For any $\mathbf{T} \in \mathcal{T}_2$, we can write $\mathbf{T} = \mathbf{A}\mathbf{A}^t$, where $\mathbf{A} = \mathbf{A}(c_1, \ldots, c_n)$ is defined in (19), with

$c_1 = \varepsilon^{1/2}$, $c_2 = \sigma_m/\mu_m$, and $c_i = \delta_i^{1/2}$, for $3 \leq i \leq n$. We write covariance matrices $\mathbf{\Sigma}$ in the form $\mathbf{\Sigma} = \mathbf{AMA}^t$ for some $\mathbf{M} \in \mathcal{P}_n$, and we consider probability distributions on $\mathbf{\Sigma}$ defined in terms of probability distributions on $\mathbf{M}$. Since every $\mathbf{\Sigma}$ corresponds to a unique $\mathbf{M}$ under this relationship, every probability distribution for $\mathbf{\Sigma}$ can be realized in this way. Also, when $\mathbf{M} = \mathbf{I}_n$, we have $\mathbf{\Sigma} = \mathbf{T}$, and so distributions for $\mathbf{\Sigma}$ are "centered" on the CAPM-based matrix $\mathbf{T}$ to the same extent the distributions for $\mathbf{M}$ are centered on $\mathbf{I}_n$. We can now define a broad class of probability distributions for $\mathbf{\Sigma}$ and "centered" on CAPM-based matrices $\mathbf{T} \in \mathcal{T}_2$.

Definition 3 *For $c > 0$, a distribution φ on $\mathbf{\Sigma} \in \mathcal{P}_n$ is in the class $\mathcal{D}(\mathcal{T}_2; c)$ if the corresponding distribution $\varphi_{\mathbf{M}}$ on $\mathbf{M} \in \mathcal{P}_n$ can be factored into a product of distributions*

$$\varphi \quad = \quad \left(\prod_{i=1}^{n-2} \varphi_{r_i}\right) \times \left(\prod_{i=1}^{n-2} \varphi_{\theta_i}\right) \times \varphi_y \times \varphi_u \times \varphi_v \times \varphi_{\mathbf{V}} \tag{25}$$

where the φ_{θ_i} are uniform probability densities on S^1, where φ_y is bounded above by $c\, e^{-y^2}$ for $y \geq 0$, and where the other distributions are arbitrary distributions on the respective spaces $r_i \in \mathbf{R}_+$, $u \in \mathbf{R}_+$, $v \in \mathbf{R}_+$, and $\mathbf{V} \in \mathcal{P}_{n-2}$. Here we use the notation of Definition 2 for the coordinates for $\mathbf{M}$, and we use the correspondence $\mathbf{\Sigma} = \mathbf{AMA}^t$ for the relationship between $\mathbf{\Sigma}$ and $\mathbf{M}$.

We now turn to the central result of this section: a lower bound for the probability of impossibility which is uniform across all distributions in the class $\mathcal{D}(\mathcal{T}_2; c)$.

Theorem 2 *For any given expected-return vector $\boldsymbol{\mu}$, expected return on the market μ_m, market volatility σ_m, and riskfree rate r_f, let φ be a probability distribution in $\mathcal{D}(\mathcal{T}_2; c) = \mathcal{D}\left(\mathcal{T}_2(\boldsymbol{\mu}, \mu_m, \sigma_m, r_f); c\right)$, for a specified $c > 0$. With respect to this distribution, the probability that a random choice of $\mathbf{\Sigma}$ gives rise to an efficient frontier that is impossible is bounded below by:*

$$P_I \quad \geq \quad 1 \;-\; \left(\frac{6}{7}\right)^{n-2} \;-\; 4c \exp\left(-\left(\frac{n-2}{3\pi}\right)^{2/3}\right) \,. \tag{26}$$

This lower bound holds uniformly across all $\varphi \in \mathcal{D}(\mathcal{T}_2; c)$, as well as across all choices of $\boldsymbol{\mu}$, μ_m, σ_m, and r_f. As n increases without bound, the probability that a generically chosen frontier is impossible tends to unity.

The remarkable generality of Theorem 2 raises the question of how tight the lower bound can be, especially given the fact that we have placed no restrictions on the expected-return vector $\boldsymbol{\mu}$. Table 1 shows that even for the 50-asset case—a relatively small number of assets for most financial applications—the likelihood of an impossible frontier is nearly certain.

n	Lower Bound
25	0.9059
50	0.9787
75	0.9920
100	0.9966

Table 1: Lower bound for the probability that a randomly chosen n-asset covariance matrix yields an impossible frontier under any measure $\varphi \in \mathcal{D}(\mathcal{T}_2; 0.1)$ over the space of all $n \times n$ symmetric positive-definite covariance matrices with real elements.

It should come as no surprise that Theorem 2 can easily be extended to the case where returns satisfy any linear k-factor model, $k \ll n$. In this case, the factor $(n-2)$ in (26) is replaced by $(n-k-2)$ and some of the constants are slightly different, but the asymptotic implications of the bound are identical. As n increases without bound, the probability of an impossible frontier approaches unity.

4.3 Additional Impossibility Results

In this section we derive several additional results about impossible frontiers. We determine the expected number of assets with respect to which a generic frontier will be impossible, and derive a lower bound for the expected sizes of short positions across a generic frontier. We also generalize Theorem 2 to the case in which a constraint is placed on the total size of short positions at each point on the frontier.

Theorem 3 *For any given expected-return vector $\boldsymbol{\mu}$, expected return on the market μ_m, market volatility σ_m, and riskfree rate r_f, let φ be an arbitrary probability distribution in $\mathcal{D}(\mathcal{T}_2; c) = \mathcal{D}\left(\mathcal{T}_2(\boldsymbol{\mu}, \mu_m, \sigma_m, r_f); c\right)$. With respect to this probability distribution, the expected number of assets with respect to which the frontier corresponding to a random choice of $\boldsymbol{\Sigma}$ gives rise to an efficient frontier that is impossible is bounded below by:*

$$E_n \quad \geq \quad c'(n-2) \ , \tag{27}$$

for a positive constant c' defined as

$$c' \quad \equiv \quad \int_{\mathbf{R}} \left(\frac{1}{2\pi \left(1 + \max(0, y)\right)} \right) \varphi_y(y) \tag{28}$$

which depends only on the factor φ_y of the probability distribution φ. If φ_y is a normal distribution with unit variance, a numerical lower bound for E_n is $(n-2)/8$.

This result follows from an estimate of the integral defining the expected value (see the Appendix), and shows that the number of assets requiring short positions on a typical frontier grows linearly with the number of assets.

We can also determine lower bounds for the aggregate size of the short positions among efficient-frontier portfolios. The following definition makes this notion precise:

Definition 4 *For $1 \leq i \leq n$, let S_i denote the infimum of the short position in the i-th asset, measured as a fraction of the portfolio's net asset value, where the infimum is taken over all points on a given efficient frontier. Let S denote the infimum of the aggregate amount of shortselling, where the infimum is also taken over all portfolios on a given efficient frontier.*

With this definition, we are able to derive a lower bound on the magnitude of shorting among efficient-frontier portfolios:

Theorem 4 *For any given expected-return vector $\boldsymbol{\mu}$, expected return on the market μ_m, market volatility σ_m, and riskfree rate r_f, let φ be an arbitrary probability distribution in $\mathcal{D}(\mathcal{T}_2; c) = \mathcal{D}\left(\mathcal{T}_2\left(\boldsymbol{\mu}, \mu_m, \sigma_m, r_f\right); c\right)$. With respect to φ, for $3 \leq i \leq n$, the expected value of S_i satisfies*

$$\mathrm{E}[S_i] \quad \geq \quad c_i \tag{29}$$

where

$$c_i \quad \equiv \quad \frac{1}{2\pi} \left(\int_0^{\infty} r_{i-2} \varphi_{r_{i-2}} \right) \left(\int_{-\infty}^{0} (1-y) \varphi_y \right) \tag{30}$$

and $\varphi_{r_{i-2}}$ and φ_y are as in Definition 3. Note that if all the functions $\varphi_{r_{i-2}}$ are identical, so that all the c_i have a common value c_*, then the expected value of S has the following lower *bound:*

$$\mathrm{E}[S] \quad \geq \quad c_*(n-2) \ . \tag{31}$$

Finally, we consider the effect of imposing shortsales constraints by first defining the concept of a constrained efficient frontier:

Definition 5 *For $b \geq 0$, a constrained efficient frontier $\mathcal{F}_b$ is the set of portfolio weight vectors that provide maximum returns for given levels of volatility, subject to the condition that the total size of the short positions in such weight vectors be no more than a fraction b of the portfolio's net asset value. Such a constrained frontier $\mathcal{F}_b$ is an impossible frontier if every point on $\mathcal{F}_b$ has a negative weight for at least one asset.*

Remarkably, imposing shortsales constraints does not decrease the probability that a frontier is impossible, as the next result shows:

Theorem 5 *Let $\mathcal{F}$ be an unconstrained efficient frontier and let $\mathcal{F}_b$ be the corresponding constrained efficient frontier for some $b > 0$. If $\mathcal{F}$ is an impossible frontier, then $\mathcal{F}_b$ is an impossible frontier as well. Thus, the probability that a constrained efficient frontier, with $b > 0$, is impossible is at least as large as the probability that an unconstrained efficient frontier is impossible.*

5 The One-Factor Model

Given the overwhelming importance of the CAPM to financial theory and practice, we consider the special case of the linear one-factor model that underlies the CAPM. In particular, let the $(n\times1)$-vector of returns of n assets be given by the following linear one-factor model:

$$\mathbf{r} \quad = \quad \boldsymbol{\iota}\, r_f \ + \ \boldsymbol{\beta}(r_m - r_f) \ + \ \boldsymbol{\epsilon} \tag{32}$$

where r_m is the stochastic market return, $\boldsymbol{\beta}$ is an $(n\times1)$ constant vector, and $\boldsymbol{\epsilon}$ is an $(n\times1)$ stochastic vector of idiosyncratic shocks. We assume that the expected value of $\boldsymbol{\epsilon}$ is zero, and we write $\boldsymbol{\Omega}$ for its covariance matrix.

Let μ_m and σ_m denote the expected return and standard deviation of r_m, respectively. According to the CAPM relation (32), the mean vector and covariance matrix for asset returns, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, can be written in terms of μ_m, σ_m, $\boldsymbol{\beta}$, r_f, and $\boldsymbol{\Omega}$ as

$$\boldsymbol{\mu} \;=\; \boldsymbol{\iota}\, r_f \;+\; \boldsymbol{\beta}(\mu_m - r_f) \quad \text{and} \quad \boldsymbol{\Sigma} \;=\; \boldsymbol{\beta}\boldsymbol{\beta}^t \sigma_m^2 \;+\; \boldsymbol{\Omega}\,. \tag{33}$$

The tangency portfolio implied by the CAPM is $\boldsymbol{\omega}_{\boldsymbol{\mu}}$, defined in (4) as $\boldsymbol{\omega}_{\boldsymbol{\mu}} \equiv \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/B$. And under the assumption that $\boldsymbol{\Omega}$ is diagonal and $\boldsymbol{\mu}$ contains all positive elements, it can be shown that the tangency portfolio is, in fact, not impossible, i.e., it contains no negative weights and may, therefore, be consistent with capital market equilibrium in which the weights are proportional to the market capitalizations of the securities. In this section, we explore the impossibility of the tangency portfolio for more general residual covariance matrices $\boldsymbol{\Omega}$ and with no constraints on $\boldsymbol{\mu}$, and find that as before, impossibility is the rule, not the exception as n increases without bound.

In Section 5.1, we introduce the techniques needed to characterize impossible tangency portfolios, and in Section 5.2 we derive a lower bound on the probability that a randomly selected tangency portfolio is impossible. In Section 5.3, we show how to construct the unique covariance matrix that is consistent with a given vector of means $\boldsymbol{\mu}$, the riskfree rate r_f, a set of market-capitalization weights $\boldsymbol{\omega}_m$, and CAPM equilibrium (i.e., where those market weights correspond to those of the tangency portfolio), and which is as "close" as possible to a given covariance matrix $\boldsymbol{\Sigma}$. In other words, we derive the covariance matrix that is as close as possible to $\boldsymbol{\Sigma}$ but which is consistent with the CAPM.

5.1 Characterizing Impossible Tangency Portfolios

As in Section 4, the key to characterizing impossible tangency portfolios is the choice of coordinates in which to express the covariance matrix, which will allow us to focus on the portion of the matrix that is relevant for impossibility. Any covariance matrix $\boldsymbol{\Sigma}$ can be written in the form $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{M}\mathbf{A}^t$, where $\mathbf{M}$ is a positive-definite symmetric matrix, and where $\mathbf{A}$ is the unique matrix that takes $\mathbf{e}_1$ to $\boldsymbol{\mu}$ and $\mathbf{e}_i$ to $\mathbf{e}_i$ for $2 \le i \le n$.[3] Also, as we showed in Section 4.1, $\mathbf{M}$ can be expressed in terms of partial Iwasawa coordinates as

$$\mathbf{M} \;=\; \begin{pmatrix} w & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{pmatrix} \left[\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{x} & \mathbf{I}_{n-1} \end{pmatrix} \right] \;=\; \begin{pmatrix} w + \mathbf{x}^t\mathbf{V}\mathbf{x} & \mathbf{x}^t\mathbf{V} \\ \mathbf{V}\mathbf{x} & \mathbf{V} \end{pmatrix} \tag{34}$$

[3] Note that the definitions of $\mathbf{A}$ and $\mathbf{M}$ are slightly different here than in Section 4.1, but we keep the same notation because these matrices play the same role as before.

where $w \in \mathbf{R}_+$, $\mathbf{x} \in \mathbf{R}^{n-1}$, and $\mathbf{V}$ is a covariance matrix of dimension $(n-1)\times(n-1)$. Here we have used the notation $\mathbf{G}[\mathbf{H}]$ for $\mathbf{H}^t\mathbf{G}\mathbf{H}$.

In these coordinates, the portfolio $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ can be expressed simply as

$$\boldsymbol{\omega}_{\boldsymbol{\mu},i} = \begin{cases} (1+\mu_2 x_1 + \cdots + \mu_n x_{n-1})/d & \text{for } i=1, \\ -\mu_1 x_{i-1}/d & \text{for } 2 \le i \le n. \end{cases} \tag{35}$$

$$d \equiv 1 + (\mu_2-\mu_1)x_1 + \cdots + (\mu_n-\mu_1)x_{n-1} .$$

Therefore, $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is completely determined by $\mathbf{x}$ and $\boldsymbol{\mu}$. This allows us to characterize the impossibility of the tangency portfolio via the following proposition:

Proposition 3 *The tangency portfolio, $\boldsymbol{\omega}_{\boldsymbol{\mu}}$, implied by the CAPM is impossible if and only if any one of the following three conditions holds: (i) two elements of $\mathbf{x}$ have different signs; (ii) all elements of $\mathbf{x}$ have the same sign as μ_1/d; or (iii) the quantity $(1+\mu_2 x_1+\cdots+\mu_n x_{n-1})/d$ is negative, where $d \equiv 1+(\mu_2-\mu_1)x_1+\cdots+(\mu_n-\mu_1)x_{n-1}$.*

We will make the most use out of the first condition for impossibility in Proposition 3, since it describes the bulk of the cases in which the tangency portfolio is impossible.

5.2 The Probability of Impossible Tangency Portfolios

To determine the probability that the CAPM tangency portfolio is impossible, we need to choose a probability distribution on the underlying variables $\boldsymbol{\beta}$, μ_m, σ_m, r_f, and $\boldsymbol{\Omega}$. Once these variables are determined, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are determined as well, and Proposition 3 will allow us to assess the impossibility of the corresponding tangency portfolio.

For our probability distribution, we allow $\boldsymbol{\beta}$, μ_m, σ_m, and r_f to be specified arbitrarily—our results will hold uniformly across any choice of these variables. With respect to $\boldsymbol{\Omega}$, we decompose the matrix into components and allow all but one of those components to be specified arbitrarily. Specifically, we write $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \begin{pmatrix} \Omega_{11} & \mathbf{0} \\ \mathbf{0} & \widetilde{\boldsymbol{\Omega}} \end{pmatrix} \left[\begin{pmatrix} 1 & \boldsymbol{\gamma}^t \\ \mathbf{0} & \mathbf{I}_{n-1} \end{pmatrix} \right] \begin{pmatrix} \Omega_{11} & \Omega_{11}\boldsymbol{\gamma}^t \\ \Omega_{11}\boldsymbol{\gamma} & \widetilde{\boldsymbol{\Omega}} + \Omega_{11}\boldsymbol{\gamma}\boldsymbol{\gamma}^t \end{pmatrix} \tag{36}$$

where $\boldsymbol{\gamma} \in \mathbf{R}^{n-1}$, $\widetilde{\boldsymbol{\Omega}}$ is an $(n-1)\times(n-1)$ positive-definite matrix, and $\Omega_{11} > 0$. The values of $\widetilde{\boldsymbol{\Omega}}$ and Ω_{11} can be specified arbitrarily. With respect to $\boldsymbol{\gamma}$, we impose a probability distribution

$\varphi_{\boldsymbol{\gamma}}(\boldsymbol{\gamma})$ on $\mathbf{R}^{n-1}$, and in our probability calculations we consider several possible choices of $\varphi_{\boldsymbol{\gamma}}$. Thus, for our probability distribution on the underlying variables, we allow completely arbitrary specification of all terms except $\boldsymbol{\gamma}$, and with respect to $\boldsymbol{\gamma}$ we focus on a number of different choices of probability distributions on $\mathbf{R}^{n-1}$.

The characterization of impossibility in Proposition 3 relies on an expression of $\boldsymbol{\Sigma} = \mathbf{AMA}^t$ in terms of coordinates $\mathbf{x}$, w and $\mathbf{V}$ for $\mathbf{M}$. Our probability distribution, however, is expressed in terms of another set of coordinates for $\boldsymbol{\Sigma}$, namely $\boldsymbol{\gamma}$, Ω_{11}, $\widetilde{\boldsymbol{\Omega}}$, σ_m, and $\boldsymbol{\beta}$. Thus, we need to calculate the relationship between these choices of coordinates to apply the characterization of impossibility to draws from our distribution. The relationship of primary importance will be the expression of $\mathbf{x}$ in terms of the coordinates for the probability distribution, so we now turn to this calculation.

Multiplying on the right by $(\mathbf{A}^t)^{-1}$ and on the left by $\mathbf{A}^{-1}$ in the expression for $\boldsymbol{\Sigma}$ in (33), and using the definition of $\mathbf{A}$, we see that

$$\begin{aligned} \mathbf{M} &= \mathbf{e}_1\mathbf{e}_1^t\,(\sigma_m/\mu_m)^2 + \begin{pmatrix} \Omega_{11}/\mu_1^2 & (\Omega_{11}/\mu_1)\mathbf{z}^t \\ (\Omega_{11}/\mu_1)\mathbf{z} & \widetilde{\boldsymbol{\Omega}} + \Omega_{11}\mathbf{z}\mathbf{z}^t \end{pmatrix} \\ &= \begin{pmatrix} \Omega_{11}/\mu_1^2 + (\sigma_m/\mu_m)^2 & (\Omega_{11}/\mu_1)\mathbf{z}^t \\ (\Omega_{11}/\mu_1)\mathbf{z} & \widetilde{\boldsymbol{\Omega}} + \Omega_{11}\mathbf{z}\mathbf{z}^t \end{pmatrix} \end{aligned} \tag{37}$$

where $\mathbf{z} = \boldsymbol{\gamma} - \widetilde{\boldsymbol{\mu}}/\mu_1$, with $\widetilde{\boldsymbol{\mu}} = (\mu_2, \ldots, \mu_n)^t$. In light of the expression for $\mathbf{M}$ in (34), we see that

$$\mathbf{x} = \mathbf{V}^{-1}(\mathbf{V}\mathbf{x}) = (\Omega_{11}/\mu_1)\left(\widetilde{\boldsymbol{\Omega}} + \Omega_{11}\mathbf{z}\mathbf{z}^t\right)^{-1}\mathbf{z}\ . \tag{38}$$

Since $\boldsymbol{\mu} = \boldsymbol{\iota} r_f + \boldsymbol{\beta}(\mu_m - r_f)$, and since $\mathbf{z}$ is determined by $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$, we see that (38) expresses $\mathbf{x}$ in terms of the coordinates for our probability distribution, as desired. After some algebraic manipulation, we can also write this expression for $\mathbf{x}$ as

$$\mathbf{x} = \left(\frac{\Omega_{11}/\mu_1}{1 + \Omega_{11}\left\|\widetilde{\boldsymbol{\Omega}}^{-1/2}\mathbf{z}\right\|^2}\right)\widetilde{\boldsymbol{\Omega}}^{-1}\ . \tag{39}$$

This is a more useful formula for $\mathbf{x}$ since we are primarily interested in the signs of the elements of $\mathbf{x}$ and this expression shows these are the same as the signs of the elements of

$\widetilde{\mathbf{\Omega}}^{-1}\mathbf{z}/\mu_1$, since the remaining multiplicative factor is always positive.

Theorem 6 *Let p be the probability that the tangency portfolio implied by the CAPM is impossible when the probability distribution on the term $\boldsymbol{\gamma}$ underlying the covariance matrix $\mathbf{\Omega}$ has a distribution given by $\varphi_{\boldsymbol{\gamma}}$. A lower bound for p is*

$$p \;\; \geq \;\; \det\left(\widetilde{\mathbf{\Omega}}\right) \int_{\mathbf{R}^{n-1}} F(\boldsymbol{\gamma}) \varphi_{\boldsymbol{\gamma}} \left(\widetilde{\mathbf{\Omega}}\boldsymbol{\gamma} + \widetilde{\boldsymbol{\mu}}/\mu_1\right) \tag{40}$$

where F is equal to 1 whenever $\boldsymbol{\gamma}$ has two elements with different signs and equal to 0 otherwise.

We now make the result more concrete by applying the theorem to a specific choices for the distribution $\varphi_{\boldsymbol{\gamma}}$.

Corollary 3 *If $\varphi_{\boldsymbol{\gamma}}$ has an $(n-1)$-dimensional multivariate normal distribution with mean $\widetilde{\boldsymbol{\mu}}/\mu_1$ and covariance matrix $s\widetilde{\mathbf{\Omega}}^2$, for some $s > 0$, then the probability that $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is impossible satisfies*

$$p \;\; \geq \;\; 1 \; - \; 2^{2-n} \tag{41}$$

and this result is independent of the choice of s.

Note that choices of $\varphi_{\boldsymbol{\gamma}}$ not centered at $\widetilde{\boldsymbol{\mu}}/\mu_1$ will generally have a lower probability of impossibility. However, for choices of $\varphi_{\boldsymbol{\gamma}}$ that are close to the uniform distribution, choices with large variance, for example, the probability of impossibility will have a lower bound similar to that in the corollary.

5.3 A Non-Impossible Covariance Matrix

Given the simple structure of the linear one-factor model (32), it should be possible to find some covariance matrix $\widetilde{\mathbf{\Sigma}}$ "close" to $\mathbf{\Sigma}$ in some sense that yields a non-impossible tangency portfolio, i.e., a tangency portfolio that has strictly positive market-capitalization weights $\boldsymbol{\omega}_m$, and is consistent with $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, and r_f. Using the techniques developed in Section 5.1, we construct such a "non-impossible" covariance matrix in this section and show how it is related to Black and Litterman's (1992) approach to asset allocation with prior information.

Suppose that a mean return vector, $\boldsymbol{\mu}$, and a market-capitalization weight vector, $\boldsymbol{\omega}_m$ are given, and consider a covariance matrix, $\mathbf{\Sigma}$, that is derived either empirically or from prior information, but which is not necessarily compatible with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in the sense that $\boldsymbol{\omega}_m \neq \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, as required by the CAPM. The matrix most compatible with the observed $\boldsymbol{\Sigma}$ but still conforming to the known values of $\boldsymbol{\mu}$ and $\boldsymbol{\omega}_m$ can be determined in the following manner. Write $\boldsymbol{\Sigma} = \mathbf{AMA}^t$ and write $\mathbf{M}$ in terms of w, $\mathbf{x}$, and $\mathbf{V}$, as in (34). Replace $\mathbf{x}$ by $\widetilde{\mathbf{x}}$, where $\widetilde{\mathbf{x}}$ is defined by

$$\tilde{x}_i \quad = \quad \frac{-\boldsymbol{\omega}_{m,i+1}}{\mu_1 + (\mu_2 - \mu_1)\boldsymbol{\omega}_{m,2} + \cdots + (\mu_n - \mu_1)\boldsymbol{\omega}_{m,n}} \tag{42}$$

for $1 \leq i \leq n-1$. The formula in (42) inverts the relationship between $\boldsymbol{\omega}_m$ and $\mathbf{x}$ from (35), so the value of $\widetilde{\mathbf{x}}$ is the unique value compatible with the market weight vector $\boldsymbol{\omega}_m$ and the expected return vector $\boldsymbol{\mu}$.

The change from $\mathbf{x}$ to $\widetilde{\mathbf{x}}$ described in the last paragraph corresponds to a change in the overall covariance matrix. Replace $\boldsymbol{\Sigma}$ by $\widetilde{\boldsymbol{\Sigma}}$ where

$$\widetilde{\boldsymbol{\Sigma}} \quad \equiv \quad \mathbf{A}\widetilde{\mathbf{M}}\mathbf{A}^t \quad , \quad \widetilde{\mathbf{M}} \quad \equiv \quad \begin{pmatrix} w + \widetilde{\mathbf{x}}^t\mathbf{V}\widetilde{\mathbf{x}} & \widetilde{\mathbf{x}}^t\mathbf{V} \\ \mathbf{V}\widetilde{\mathbf{x}} & \mathbf{V} \end{pmatrix} . \tag{43}$$

This new covariance matrix, $\widetilde{\boldsymbol{\Sigma}}$, is then compatible with $\boldsymbol{\omega}_m$ and $\boldsymbol{\mu}$ in that $\boldsymbol{\omega}_m$ is the tangency portfolio resulting from this mean and covariance. In addition, $\widetilde{\boldsymbol{\Sigma}}$ is the covariance matrix most compatible with the specified values of $\boldsymbol{\mu}$ and $\boldsymbol{\omega}_m$ and the observed value of $\boldsymbol{\Sigma}$ in that it requires precisely the amount of alteration to $\boldsymbol{\Sigma}$ needed to make the three sets of parameters compatible.

Therefore, for those who have strong conviction that the CAPM must hold and that $\boldsymbol{\mu}$ and $\boldsymbol{\omega}_m$ are, in fact, the correct expected returns and market weights, and $\boldsymbol{\Sigma}$ is their best estimate of the covariance matrix, the covariance matrix they should adopt is $\widetilde{\boldsymbol{\Sigma}}$ given in (43).

6 Empirical Analysis

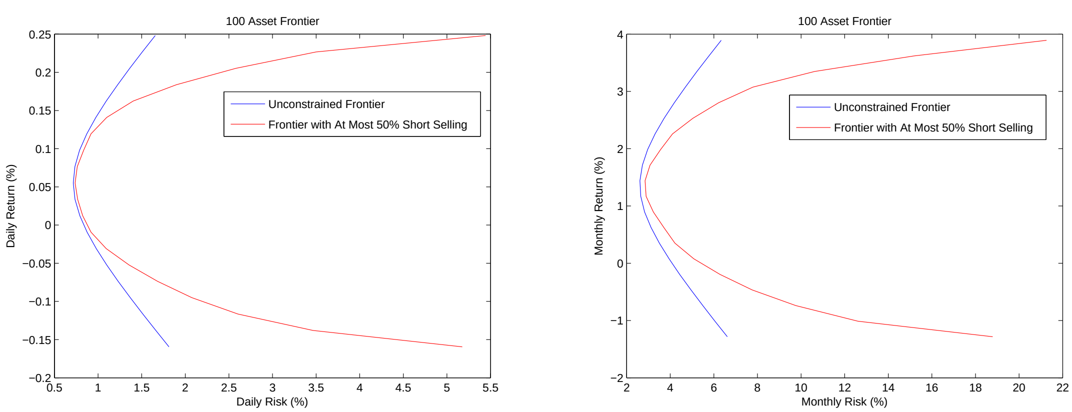
To gauge the empirical relevance of our impossibility results, we use daily and monthly returns for stocks in the S&P 500 index to estimate portfolio parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and show that the realizations of impossible frontiers in the historical record are nontrivial.

6.1 The Data

The monthly data consists of returns for stocks listed on the S&P 500 in December of 1995 for which monthly return data was available for the period from January 1980 through December 2005. The daily data consists of returns for stocks listed on the S&P 500 in December of 1995 for which daily return data was available for the period from January 1, 1995 through December 31, 2005. There are a total of 271 stocks in the monthly data set and 326 stocks in the daily data set.

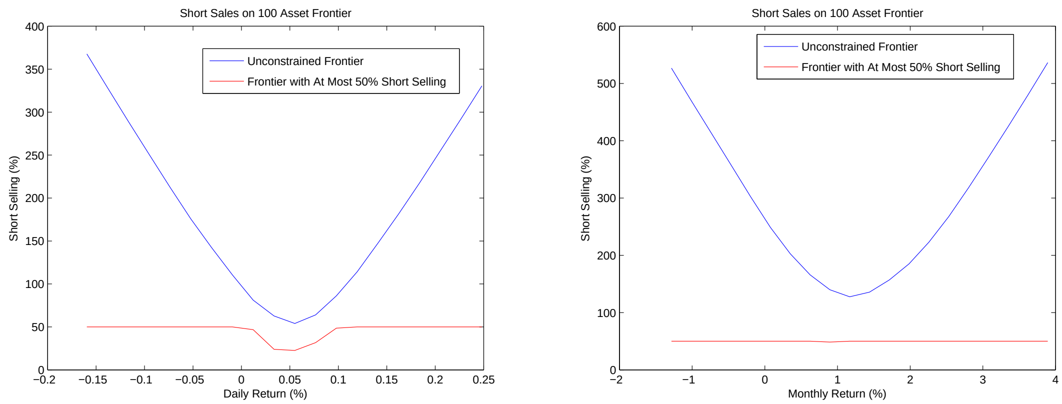
6.2 A 100-Stock Empirical Efficient Frontier

For concreteness, we construct the efficient frontier for the first 100 assets for both daily and monthly returns using standard estimators for the means and covariance matrices. The two frontiers are plotted in Figure 1, and we find that both are impossible. The blue lines indicate the unconstrained frontiers, and the red lines indicate the frontiers constrained to allow only 50% shortselling. Figure 2 shows the amount of shortselling for points on both of these frontiers. Clearly the shortsales constraints do not eliminate the problem of impossible frontiers, and have a significant impact on the characteristics of the constrained optimal portfolio.



(a) Daily Returns (b) Monthly Returns

Figure 1: Unconstrained efficient frontier for 100 stocks in the S&P 500 index, as well as the frontier constrained to allow no more than 50% shortselling, based on (a) daily returns from January 1, 1996 to December 31, 2005; and (b) monthly returns from January 1980 to December 2005.



(a) Daily Returns (b) Monthly Returns

Figure 2: Magnitude of short positions for points on the unconstrained efficient frontier for 100 stocks in the S&P 500 index, as well as the frontier constrained to allow no more than 50% shortselling, based on (a) daily returns from January 1, 1996 to December 31, 2005; and (b) monthly returns from January 1980 to December 2005.

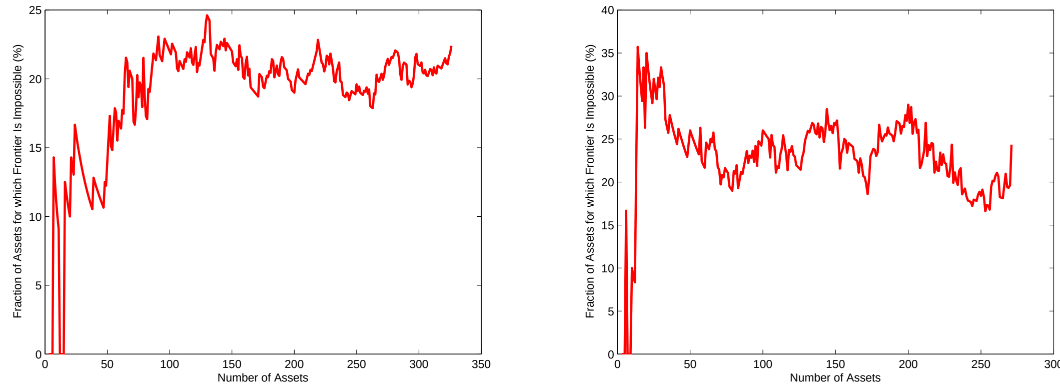
6.3 More Impossible Frontiers

Applying the usual sample mean and covariance-matrix estimators to daily and monthly returns, we compute estimates $(\hat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}})$ and construct efficient frontiers for each of 2 through 326 assets for daily returns, and 2 through 271 assets for monthly returns. Figure 3 shows the fraction of assets with respect to which each frontier is impossible. Figure 4 shows the size of the short positions in the portfolios $\boldsymbol{\omega}_g$ and $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ for each of these frontiers. These results show that negative holdings are the rule rather than the exception for empirical efficient frontiers, and non-negativity constraints are likely to have a major impact on the characteristics of mean-variance-optimized portfolios.

6.4 Estimation Error

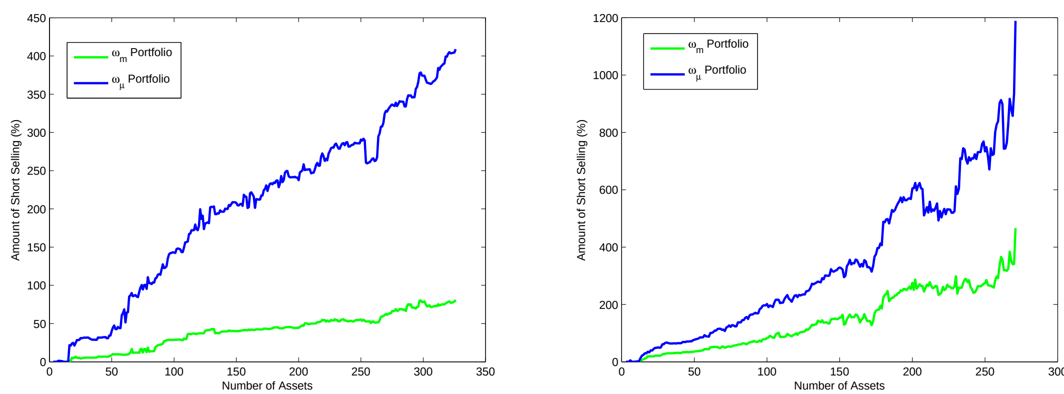
One possible critique of our empirical analysis is that estimation error is likely to yield sample means and covariances that are inconsistent with the CAPM, so it is not surprising that we find impossible frontiers in the data. But this observation only underscores the ubiquity of impossible frontiers in practice. Since the population means and covariance matrix must always be estimated in financial applications, estimation error is an unavoidable aspect of practical portfolio management. While a number of authors have explored the impact of estimation error on portfolio optimization,[4] and alternatives such as Bayesian

[4]See, for example, Brown (1976), Bawa, Brown, and Klein (1979), Frost and Savarino (1986), Jorion (1986), Tu and Zhou (2004, 2007, 2008), Wang (2005), DeMiguel, Garlappi, and Uppal (2007), Garlappi,



(a) Daily Returns (b) Monthly Returns

Figure 3: The fraction of assets with respect to which the empirical frontiers are impossible, as a function of the number of assets underlying the frontiers, based on a subset of S&P 500 stocks using (a) daily returns from January 1, 1996 to December 31, 2005, with the number of stocks n ranging from 2 to 326; and (b) monthly returns from January 1980 to December 2005, with the number of stocks n ranging from 2 to 271.



(a) Daily Returns (b) Monthly Returns

Figure 4: Magnitude of short positions in the portfolios $\boldsymbol{\omega}_g$ and $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ for the empirical frontiers, based on a subset of S&P 500 stocks using (a) daily returns from January 1, 1996 to December 31, 2005, with the number of stocks n ranging from 2 to 326; and (b) monthly returns from January 1980 to December 2005, with the number of stocks n ranging from 2 to 271.

inference (Brown, 1976), robust portfolio optimization (Fabozzi et al., 2007), and resampling (Michaud, 1998) have been developed in response, none of these methods addresses the impossibility of the *population* mean-variance efficient frontier.

In particular, Theorems 1 and 2 show that impossible frontiers are almost certain to occur even in the absence of estimation error. To the extent that estimation error can be viewed as random perturbations of population parameters (as opposed to perturbations that yield parameters closer to those satisfying a CAPM/APT relation), it is even more likely that estimated means and covariances will yield impossible frontiers. In other words, if a frontier is impossible for a set of population parameters, adding random noise to those parameters is unlikely to yield frontiers that are consistent with the CAPM.

7 Conclusion

In this paper, we have shown that mean-variance efficient frontiers almost always contain short positions, implying a fundamental inconsistency between efficiency and economic equilibrium as described by the CAPM. This result is distinct from earlier concerns in the literature regarding the mean-variance efficiency of the market portfolio. Those concerns involved the observability of the total market portfolio, the existence of non-traded assets such as human capital, estimation errors in the sample means and covariance matrix, non-stationarities, asymmetric information, and other capital-market imperfections. Even in a frictionless world where all parameters are fixed and known, and where all of the other perfect-markets assumptions of the CAPM hold, mean-variance efficient frontiers are almost always impossible.

This surprisingly general result provides a potential explanation for the near universal disdain with which long-only portfolio managers regard standard mean-variance optimization techniques. These investment professionals—who comprise the majority of end-users of commercial portfolio construction software such as the BARRA Optimizer and the Northfield Portfolio Optimizer—have railed against mindless optimization for years, arguing that portfolio weights obtained in this manner are ill-behaved and must be constrained or otherwise post-processed. However, the typical rationale for these complaints is that the weights of frontier portfolios are too unstable and too sensitive to estimation error to be of practical value. We have identified a distinctly different rationale, which is the ubiquity of short positions in frontier portfolios even in the absence of estimation error. An impossible frontier is, in fact, literally impossible for the long-only portfolio manager. The surging popular-

Uppal, and Wang (2007), and Kan and Zhou (2007).

ity of 130/30 strategies among such managers and their investors may well be a practical manifestation and an unintended consequence of the impossibility of mean-variance-optimal portfolios.

The virtual certainty of impossible frontiers also has implications for the interpretation of economic equilibrium. The converse of our impossibility theorem is that the set of parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ that are "possible", i.e., that are consistent with the mean-variance efficiency of the market portfolio—is a vanishingly small set as the number of assets grows without bound. In particular, in a CAPM equilibrium, covariances are also endogenously determined via supply and demand, despite the fact that most asset-pricing models focus exclusively on the properties of expected returns in equilibrium. Is it any wonder that the set of n means and $n(n+1)/2$ covariances that is consistent with capital-market equilibrium is apparently quite sparse?

To the disciples of general equilibrium theory, this may be heretical, but from a broader and more practical perspective, it should not be too surprising that the likelihood of simultaneous equality of supply and demand across a large number of markets is small, and increasingly less likely as the number of assets grows. With the techniques developed in this paper, we hope to be able to deduce other generic properties of financial market equilibria and their practical implications.

A Appendix

In this Appendix, we provide proofs for the main results of the paper.

A.1 Proof of Proposition 1

When there are only $n = 2$ assets, we may write the set of points on the frontier simply as

$$\mathcal{F} = \left\{ \boldsymbol{\omega} \colon \boldsymbol{\omega} = \begin{bmatrix} \frac{\mu_o - \mu_1}{\mu_2 - \mu_1} & \frac{\mu_2 - \mu_o}{\mu_2 - \mu_1} \end{bmatrix}^t, \text{for } \mu_o \geq \frac{B}{C} \right\} .$$

Thus, for all expected returns μ_o with $\mu_1 < \mu_o < \mu_2$, points on the frontier have positive weight in both components, but for all values of μ_o outside this range, every point on the frontier has exactly one negative component. If the minimum value of μ_o, namely $\mu_o = B/C$, is less than μ_2, then at least some point on the frontier has all positive weights, but if this value of μ_o is greater than μ_2, then all points on the frontier have at least one negative weight.

The condition that $B/C < \mu_2$ is the same as the condition that

$$\frac{\mu_1(\mathbf{e}_1^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}) + \mu_2(\mathbf{e}_2^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota})}{(\mathbf{e}_1^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}) + (\mathbf{e}_2^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota})} < \mu_2 .$$

The denominator on the left-hand side is non-negative, according to the Cauchy-Schwarz inequality, and so we may cross-multiply and collect terms to see that the inequality holds exactly when $\mathbf{e}_1^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota} > 0$. This, in turn, is the same as the inequality $\rho < \sigma_2/\sigma_1$, and so we see that a frontier will be impossible just when $\sigma_2/\sigma_1 < \rho \leq 1$ and $\mu_1 < \mu_2$, which is the assertion of Proposition 1.

A.2 Proof of Proposition 2

We calculate $\boldsymbol{\Sigma}^{-1}$ and reorganize terms to yield:

$$\boldsymbol{\omega}_g = \frac{\boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}}{\boldsymbol{\iota}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}} = \begin{bmatrix} (1 - a + b - c)(1 - b) \\ (1 - a - b + c)(1 - c) \\ (1 + a - b - c)(1 - a) \end{bmatrix} / (\boldsymbol{\iota}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}) . \tag{A.1}$$

Since the denominator is non-negative, by the Cauchy-Schwarz inequality, the proposition follows.

A.3 Proof of Corollary 1

The proof of the corollary is obtained by using the formula for $\boldsymbol{\omega}_g$ in (A.1) and simply plugging in the stated values of a, b, and c. This calculation shows that the shortsale amount in the second asset is

$$\frac{\mathbf{e}_2^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}}{\boldsymbol{\iota}^t \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}} = \frac{-2\varepsilon + 4\varepsilon^2 - \varepsilon^3}{4\varepsilon^2 - \varepsilon^3} = -\frac{1}{2\varepsilon} + \mathrm{O}(1) \ .$$

A.4 Proof of Lemma 1

The first inequality in (20) states that the i-th component of $\boldsymbol{\omega}_g$ is negative, and the second inequality states that the i-th component of $\boldsymbol{\omega}_P = B\boldsymbol{\omega}_{\boldsymbol{\mu}} - B\boldsymbol{\omega}_g$ is also negative. Together, these inequalities imply that the i-th component of each portfolio on the entire efficient frontier has a negative weight, since C and D are always positive, by the Cauchy-Schwartz inequality, and since frontier portfolios have the form described in (5). This demonstrates the sufficiency of the condition for impossibility in the i-th asset. The necessity also follows readily, since a negative i-th component of each portfolio is only possible if there is a negative i-th component in the minimum risk portfolio, $\boldsymbol{\omega}_g$, as well as in the high risk portfolios which tend toward a positive multiple of $\boldsymbol{\omega}_{\boldsymbol{\mu}} - \boldsymbol{\omega}_g$.

To deduce the equivalence between the conditions in (20) and (21), we note that

$$\boldsymbol{\Sigma}^{-1} = \left(\mathbf{A}^{-1}\right)^t \mathbf{M}^{-1} \mathbf{A}^{-1} \ .$$

As in Definition 2, we can express $\mathbf{M}$ in terms of coordinates as $(\mathbf{X}, \mathbf{W}, \mathbf{V})$, and we have

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ -\mathbf{X} & \mathbf{I}_{n-2} \end{pmatrix} \begin{pmatrix} \mathbf{W}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_2 & -\mathbf{X}^t \\ \mathbf{0} & \mathbf{I}_{n-2} \end{pmatrix} = \begin{pmatrix} \mathbf{W}^{-1} & -\mathbf{W}^{-1}\mathbf{X}^t \\ -\mathbf{X}\mathbf{W}^{-1} & \mathbf{V}^{-1} + \mathbf{X}\mathbf{W}^{-1}\mathbf{X}^t \end{pmatrix} .$$

From the definition of $\mathbf{A} = \mathbf{A}(c_1, \ldots, c_n)$ in (19), we see that

$$\mathbf{A}^{-1}\boldsymbol{\iota} = \mathbf{e}_1/c_1, \quad \mathbf{A}^{-1}\boldsymbol{\mu} = \mathbf{e}_2/c_2, \quad \text{and} \quad \mathbf{A}^{-1}\mathbf{e}_j = \mathbf{e}_j/c_j, \ \text{for } 3 \le j \le n \ .$$

We write $\mathbf{W} = [w_{ij}]$ so that

$$\mathbf{W}^{-1} = \frac{1}{\det(\mathbf{W})} \begin{pmatrix} w_{22} & -w_{12} \\ -w_{12} & w_{11} \end{pmatrix} .$$

After some algebraic rearrangements, we see that the conditions in (20) are equivalent to

$$x_{1,(i-2)}w_{22} - x_{2,(i-2)}w_{12} > 0 \quad \text{and} \quad x_{(i-2),2} > 0 \tag{A.2}$$

where we have used the facts that $\det(\mathbf{W}) > 0$ and $w_{22} > 0$, since $\mathbf{W}$ is positive definite.

With the notation from Definition 2, we write $w_{22} = v$, $w_{12} = yv$, $x_{1,i-2} = r_{i-2}\cos\theta_{i-2}$ and $x_{2,i-2} = r_{i-2}\sin\theta_{i-2}$. Equation (A.2) can be rewritten in terms of these new coordinates as

$$\cos\theta_{i-2} - y\sin\theta_{i-2} > 0 \quad \text{and} \quad \sin\theta_{i-2} > 0$$

since both $r_{i-2} > 0$ and $v > 0$, and this is the condition in (21).

A.5 Proof of Corollary 3

Substitution of the specified choice of $\varphi_{\boldsymbol{\gamma}}$ for into the result of Theorem 6 shows us that

$$p \geq \frac{1}{(2\pi s)^{(n-1)/2}} \int_{\mathbf{R}^{n-1}} F(\boldsymbol{\gamma}) \exp\left(-\frac{1}{2s}\boldsymbol{\gamma}^t\boldsymbol{\gamma}\right) d\boldsymbol{\gamma}\ .$$

This integral is simply an expression for the fraction of the unit sphere in $\mathbf{R}^{n-1}$ that does not have either all negative or all positive coordinates, and this fraction is $1 - 2^{2-n}$, as desired.

A.6 Proof of Theorem 1

From equation (21) of Lemma 1 we see that the probability, p_i, that a frontier is impossible with respect to the i-th coordinate, for $i > 2$, is just the probability that the conditions of (21) are fulfilled when θ_{i-2} is chosen from the uniform distribution on $S^1 = [0, 2\pi]$. The conditions are satisfied exactly when $\theta \in (0, \pi)$ and $y < \cot\theta_{i-2}$, and this corresponds to a probability of impossibility

$$p_i = \frac{1}{2\pi}\cot^{-1} y$$

where $\cot^{-1}$ denotes the branch of the inverse cotangent with values between 0 and π.

Equation (21) of Lemma 1 also shows that, for a fixed value of y, impossibility in the i-th coordinate is independent of impossibility in the j-th coordinate, for $i, j > 2$. Thus, the probability of impossibility in at least one of the coordinates $i > 2$ is bounded below as

$$p \;\geq\; 1 - \left(1 - \frac{1}{2\pi}\cot^{-1} y\right)^{n-2}$$

and this implies the first inequality of the theorem. The second inequality follows directly, since the inequality $\cot^{-1} y \geq \frac{1}{1+\max(0,y)}$ holds for all y.

A.7 Proof of Theorem 2

From Corollary 2, we see that the probability in the theorem is bounded below as

$$\begin{aligned} P_I \;&\geq\; 1 - \int_{\mathbf{R}} \left(1 - \frac{1}{2\pi\,(1+\max(0,y))}\right)^{n-2} \varphi_y(y) \\ &\geq\; 1 - \int_{-\infty}^{0} \left(1 - \frac{1}{2\pi}\right)^{n-2} \varphi_y(y) - c\int_0^\infty \left(1 - \frac{1}{2\pi(1+y)}\right)^{n-2} e^{-y^2}\, dy\ . \end{aligned}$$

The first integral in the last line is bounded above by $(1 - 1/(2\pi))^{n-2}$. The second integral is bounded above by the sum

$$c\int_0^2 \left(1 - \frac{1}{6\pi}\right)^{n-2} e^{-y^2}\, dy + c\int_2^\infty \left(1 - \frac{1}{3\pi y}\right)^{n-2} e^{-y^2}\, dy\ .$$

The first integral in this sum is bounded above by $c(1 - 1/(6\pi))^{n-2}$, and the second integral is bounded above by $2ce^{-\left(\frac{n-2}{3\pi}\right)^{2/3}}$. This last bound follows from the fact that $\left(1 - \frac{1}{3\pi y}\right)^{n-2}$ is bounded above by $e^{-\left(\frac{n-2}{3\pi}\right)^{2/3}}$ for $0 \leq y \leq \left(\frac{n-2}{3\pi}\right)^{1/3}$, as well as the fact that

$$\int_{\left(\frac{n-2}{3\pi}\right)^{1/3}}^{\infty} e^{-y^2}\, dy < e^{-\left(\frac{n-2}{3\pi}\right)^{2/3}}\ .$$

Combining these results, we see that the probability is bounded below by

$$P_I \;\geq\; 1 - \left(\frac{6}{7}\right)^{n-2} - c\left(\frac{19}{20}\right)^{n-2} - 2c\exp\left(-\left(\frac{n-2}{3\pi}\right)^{2/3}\right)$$

where we have made use of the fact that $6/7 > 1-1/(2\pi)$ and the fact that $19/20 > 1-1/(6\pi)$. Finally, numerical calculations show that

$$(19/20)^{n-2} \leq 2\exp(-((n-2)/(3\pi))^{2/3})$$

for small n, and this relationship continues to hold asymptotically. Thus, we can bound P_I below as

$$P_I \;\geq\; 1 - \left(\frac{6}{7}\right)^{n-2} - 4c\exp\left(-\left(\frac{n-2}{3\pi}\right)^{2/3}\right)$$

A.8 Proof of Theorem 3

The expected number of assets with respect to which an efficient frontier is impossible satisfies

$$E_n \;\geq\; \int_{\mathbf{R}} \left(\sum_{i=3}^{n} \frac{1}{2\pi\,(1+\max(0,y))}\right) \varphi_y(y)\;.$$

This follows from the proof of Theorem 1, which shows that the i-th summand in the integrand is a lower bound for the probability that a covariance matrix gives rise to an impossible frontier for a fixed value of y. We thus see that

$$E_n \;\geq\; (n-2)\int_{\mathbf{R}} \left(\frac{1}{2\pi\,(1+\max(0,y))}\right) \varphi_y(y)$$

and this last integral is the constant c' from the statement of the theorem. Also, in the case in which $\varphi_y(y)$ is a normal distribution with unit variance, we see from a numerical computation that

$$E_n \;\geq\; \frac{n-2}{8}$$

and this is the final claim of the theorem.

A.9 Proof of Theorem 4

If a frontier meets the necessary and sufficient conditions of Lemma 1 for the i-th coordinate, for $3 \leq i \leq n$, then it is an impossible frontier with respect to the i-th asset. In this case, the i-th components of both $\boldsymbol{\omega}_g$ and $\boldsymbol{\omega}_P$ are negative, and so the total amount of shortselling in the i-th asset throughout the frontier is bounded below by the amount of shortselling in the i-th asset for the minimum-variance portfolio. Thus we see that

$$S_i(\mathbf{\Sigma}) \geq -\mathbf{e}_i^t \boldsymbol{\omega}_g = -\frac{\mathbf{e}_i^t \mathbf{\Sigma}^{-1} \boldsymbol{\iota}}{\boldsymbol{\iota}^t \mathbf{\Sigma}^{-1} \boldsymbol{\iota}} = -\left(-r_{i-2} \cos \theta_{i-2} + y r_{i-2} \sin \theta_{i-2}\right)$$

where we have used the change of coordinates $\mathbf{\Sigma} = \mathbf{A} \mathbf{M} \mathbf{A}^t$ and the coordinates for $\mathbf{M}$ from Definition 2 to establish the final equality. We thus see that, the expected amount of shortselling with respect to the i-th asset satisfies

$$E[S_i] \geq \int_0^\infty \left(\int_0^\pi \int_{-\infty}^{\cot \theta_{i-2}} \left(r_{i-2} \cos \theta_{i-2} - y r_{i-2} \sin \theta_{i-2}\right) \varphi_y \frac{d\theta_{i-2}}{2\pi} \right) \varphi_{r_{i-2}} \tag{A.3}$$

where we have used the result from Lemma 1 that a frontier is impossible with respect to the i-th asset exactly when $\theta_{i-2} \in (0, \pi)$ and $y < \cot \theta_{i-2}$. We have also used the notation φ_y and $\varphi_{r_{i-2}}$ from Definition 3.

Since the integrand in (A.3) is positive throughout the region of integration, we can find a smaller lower bound by restricting the size of the region of integration. We calculate

$$\begin{aligned} E[S_i] &\geq \left(\int_0^\infty r_{i-2} \varphi_{r_{i-2}} \right) \left(\int_0^{\pi/2} \int_{-\infty}^0 \left(\cos \theta_{i-2} - y \sin \theta_{i-2}\right) \varphi_y \frac{d\theta_{i-2}}{2\pi} \right) \\ &= \frac{1}{2\pi} \left(\int_0^\infty r_{i-2} \varphi_{r_{i-2}} \right) \left(\int_{-\infty}^0 (1-y) \varphi_y \right) . \end{aligned}$$

This is the lower bound in the theorem for $E[S_i]$. The lower bound for $E[S]$ follows immeidately if the $\varphi_{r_{i-2}}$ are identical for $3 \leq i \leq n$.

A.10 Proof of Theorem 5

Let $\mathcal{F}_0$ be the frontier constrained to allow no shortselling that corresponds to $\mathcal{F}$ and $\mathcal{F}_b$. Let σ_0 be the risk of the minimum risk portfolio on $\mathcal{F}_0$, and let μ_0 be the expected return of the maximum expected return portfolio on $\mathcal{F}_0$. Each portfolio on $\mathcal{F}_b$ with a lower risk than σ_0 must involve shortselling, since σ_0 is the minimum possible risk without shortselling. Similarly, each portfolio on $\mathcal{F}_b$ with a higher expected return than μ_0 must involve shortselling, since μ_0 is the maximum possible expected return without shortselling. Thus, we need only show that each portfolio on $\mathcal{F}_b$ with a risk greater than or equal to σ_0 and an expected return less than or equal to μ_0 must involve shortselling.

Let $\boldsymbol{\omega}_b$ be a portfolio on $\mathcal{F}_b$ with risk and expected return, σ_b and μ_b, respectively, such that $\sigma_b \geq \sigma_0$ and $\mu_b \leq \mu_0$. There are weight vectors $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_U$ on $\mathcal{F}_0$ and $\mathcal{F}$, respectively, with the same expected return as $\boldsymbol{\omega}$. For $0 \leq \lambda \leq 1$, write

$$\boldsymbol{\omega}_\lambda = (1-\lambda)\boldsymbol{\omega}_0 + \lambda \boldsymbol{\omega}_U$$

so that each $\boldsymbol{\omega}_\lambda$ also has the same expected return μ_b. Let σ_λ denote the risk of $\boldsymbol{\omega}_\lambda$. Note that σ_λ is a decreasing function of λ, for $0 \le \lambda \le 1$, since σ_λ^2 is a quadratic function of λ, and since $\lambda = 1$ corresponds to the minimum risk portfolio for the level of expected return μ_o. We assume here that $\mathcal{F}$ is impossible so that $\boldsymbol{\omega}_U$ involves shortselling and is therefore distinct from $\boldsymbol{\omega}_0$. We thus see that each $\boldsymbol{\omega}_\lambda$ with $\lambda > 0$ has lower risk than $\boldsymbol{\omega}_0$ but the same level of return μ_b. Also, the amount of shortselling in $\boldsymbol{\omega}_\lambda$ is positive for all $\lambda > 0$ but goes to zero as $\lambda \to 0$. As a result, there is some $\lambda^* > 0$ such that the amount of shortselling on $\boldsymbol{\omega}_{\lambda^*}$ is no more than b. The existence of this $\boldsymbol{\omega}_{\lambda^*}$ implies that σ_b must be no greater than $\boldsymbol{\omega}_{\lambda^*}$, and hence strictly less than the risk of $\boldsymbol{\omega}_0$. Since the risk of $\boldsymbol{\omega}_b$ must be strictly less than the risk of $\boldsymbol{\omega}_0$, it follows that $\boldsymbol{\omega}_b$ must involve shortselling, as desired.

A.11 Proof of Theorem 6

From the condition for impossibility in Proposition 3 and from the expression for $\mathbf{x}$ in (39), we see that the probability is bounded below by

$$p \ge \int_{\mathbf{R}^{n-1}} F(\mathbf{x})\varphi_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}) = \int_{\mathbf{R}^{n-1}} F\left(\left(\frac{\Omega_{11}/\mu_1}{1+\Omega_{11}\left\|\widetilde{\boldsymbol{\Omega}}^{-1/2}\mathbf{z}\right\|^2}\right)\widetilde{\boldsymbol{\Omega}}^{-1}\mathbf{z}\right)\varphi_{\boldsymbol{\gamma}}(\boldsymbol{\gamma})$$

where $\mathbf{z} = \boldsymbol{\gamma} - \widetilde{\boldsymbol{\mu}}/\mu_1$. Because F depends only on the signs of the elements of its argument, we have

$$p \ge \int_{\mathbf{R}^{n-1}} F\left(\widetilde{\boldsymbol{\Omega}}^{-1}\mathbf{z}\right)\varphi_{\boldsymbol{\gamma}}(\boldsymbol{\gamma})$$

and after a change of varialbes, we see that

$$p \ge \det\left(\widetilde{\boldsymbol{\Omega}}\right)\int_{\mathbf{R}^{n-1}} F(\boldsymbol{\gamma})\varphi_{\boldsymbol{\gamma}}\left(\widetilde{\boldsymbol{\Omega}}\boldsymbol{\gamma} + \widetilde{\boldsymbol{\mu}}/\mu_1\right)$$

as desired.

References

Anderson, T., 1984, *An Introduction to Multivariate Statistical Analysis*, 2nd Edition. New York: John Wiley & Sons.

Bawa, V., Brown, S. and R. Klein, 1979, *Estimation Risk and Optimal Portfolio Choice.* Amsterdam: North-Holland.

Best, M. and R. Grauer, 1990, "The Efficient Set Mathematics When Mean-Variance Problems Are Subject to General Linear Constraints", *Journal of Economics and Business* 42, 105–120.

Black, F. and R. Litterman, 1992, "Global Portfolio Optimization", *Financial Analysts Journal* 47, 28–43.

Brito, N., 1978, "Portfolio Selection in an Economy with Marketability and Short Sales Restrictions", *Journal of Finance* 33, 589–601.

Brown, S., 1976, *Optimal Portfolio Choice Under Uncertainty: A Bayesian Approach.* Ph.D. dissertation, University of Chicago.

Box, G. and G. Tiao,, 1973, *Bayesian Inference in Statistical Analysis.* Reading, MA: Addison-Wesley.

DeMiguel, V., Garlappi, L. and R. Uppal, 2007, "Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?", to appear in *Review of Financial Studies.*

Detemple, J. and M. Shashidhar, 1997, "Equilibrium Asset Prices and No-Arbitrage with Portfolio Constraints", *Review of Financial Studies* 10, 1133–1174.

Diamond, D. and R. Verrecchia, 1987, "Constraints on Short-Selling and Asset Price Adjustment to Private Information", *Journal of Financial Economics* 18, 277–311.

Duffie, D., Garleanu, N., and L. Pedersen, 2002, "Securities Lending, Shorting, and Pricing", *Journal of Financial Economics* 66, 307–339.

Dybvig, P., 1984, "Short Sales Restrictions and Kinks on the Mean Variance Frontier", *Journal of Finance* 39, 239–244.

Fabozzi, F., Kolm, P., Pachamanova, D. and S. Focardi, 2007, *Robust Portfolio Optimization and Management.* New York: Wiley Finance.

Frost, P. and J. Savarino, 1986, "An Empirical Bayes Approach to Efficient Portfolio Selection", *Journal of Financial and Quantitative Analysis* 21, 293–305.

Garlappi, L., Uppal, R. and T. Wang, 2007, "Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach", *Review of Financial Studies* 20, 41–81.

Halmos, P., 1974, *Measure Theory.* Heidelberg: Springer-Verlag.

Heaton, J. and D. Lucas, 1996, "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing", *Journal of Political Economy* 104, 443–487.

Huang, C. and R. Litzenberger, 1988, *Foundations for Financial Economics.* New York: North-Holland.

Jarrow, R., 1980, "Heterogeneous Expectations, Restrictions on Short Sales, and Equilibrium Asset Prices", *Journal of Finance* 35, 1105–1113.

Jeffreys, H., 1961, *Theory of Probability*, 3rd ed. Oxford, UK: Oxford University Press.

Jorgenson, J. and S. Lang, 2005, $Pos_n(R)$ *and Eisenstein Series*. Berlin: Springer-Verlag.

Jorion, P., 1986, "Bayes-Stein Estimation For Portfolio Analysis", *Journal of Financial and Quantitative Analysis* 21, 279–292.

Kan, R. and G. Zhou, 2007, "Optimal Portfolio Choice with Parameter Uncertainty", *Journal of Financial and Quantitative Analysis* 42, 621–656.

Lintner, J., 1965, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets", *Review of Economics and Statistics* 47, 13–37.

Markowitz, H., 1952, "Portfolio Selection", *Journal of Finance* 7, 77–91.

Markowitz, H., 1959, *Portfolio Selection*. New Haven, CT: Cowles Foundation, Yale University.

Markowitz, H., 1987, *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. Oxford, UK: Basil Blackwell, Ltd.

Markowitz, H., 2005, "Market Efficiency: A Theoretical Distinction and So What?", *Financial Analysts Journal* 60, 17–30.

Merton, R., 1972, "An Analytic Derivation of the Efficient Portfolio Frontier", *Journal of Financial and Quantitative Analysis* 7, 1851–1872.

Michaud, R., 1998, *Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation*. Boston, MA: Harvard Business School Press.

Pogue, G., 1970, "An Extension of the Markowitz Portfolio Selection Model to Include Variable Transactions' Costs, Short Sales, Leverage Policies and Taxes", *Journal of Finance* 25, 1005–1027.

Ross, S., 1976, "The Arbitrage Theory of Capital Asset Pricing", *Journal of Economic Theory* 13, 341–360.

Sharpe, W., 1964, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk", *Journal of Finance* 19, 425–442.

Sun, Wa. and C. Wang, 2006, "The Mean-Variance Investment Problem in a Constrained Financial Market", *Journal of Mathematical Economics*42, 885–895.

Tobin, J., 1958, "Liquidity Preference As Behavior Towards Risk", *Review of Economic Studies* 67, 65–86.

Tu, J. and G. Zhou, 2004, "Data-generating Process Uncertainty: What Difference Does It Make in Portfolio Decisions?", *Journal of Financial Economics* 72, 385–421.

Tu, J. and G. Zhou, 2007, "Incorporating Economic Objectives into Bayesian Priors: Portfolio Choice Under Parameter Uncertainty", unpublished working paper, Olin School of Business, Washington University.

Tu, J. and G. Zhou, 2008, “Being Naive about Naive Diversification: Can Investment Theory Beat the 1/N Strategy?”, unpublished working paper, Olin School of Business, Washington University.

Wang, Z., 2005, “A Shrinkage Approach to Model Uncertainty and Asset Allocation”, *Review of Financial Studies* 18, 673–705.