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ABSTRACT

We propose a rational theory of momentum and reversal based on delegated portfolio management. An investor can hold assets through an index or an active fund. Investing in the active fund involves a time-varying cost, interpreted as managerial perk or ability. The investor responds to an increase in the cost by flowing out of the active and into the index fund. While prices of assets held by the active fund drop in anticipation of these outflows, the drop is expected to continue, leading to momentum. Because outflows push prices below fundamental values, expected returns eventually rise, leading to reversal. Besides momentum and reversal, fund flows generate comovement, lead-lag effects and amplification, with all effects being larger for assets with high idiosyncratic risk. The active-fund manager's concern with commercial risk makes prices more volatile.

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1 Introduction

Two of the most prominent financial-market anomalies are momentum and reversal. Momentum is the tendency of assets with good (bad) recent performance to continue overperforming (underperforming) in the near future. Reversal concerns predictability based on a longer performance history: assets that performed well (poorly) over a long period tend to subsequently underperform (overperform). Closely related to reversal is the value effect, whereby the ratio of an asset's price relative to book value is negatively related to subsequent performance. Momentum and reversal have been documented extensively and for a wide variety of assets.[1]

Momentum and reversal are viewed as anomalies because they are hard to explain within the standard asset-pricing paradigm with rational agents and frictionless markets. The prevalent explanations of these phenomena are behavioral, and assume that agents react incorrectly to information signals.[2] In this paper we show that momentum and reversal can arise in markets with rational agents. We depart from the standard paradigm by assuming that investors delegate the management of their portfolios to financial institutions, such as mutual funds and hedge funds.

Our explanation emphasizes the role of fund flows, and is loosely as follows. Suppose that a negative shock hits the fundamental value of some assets. Investment funds holding these assets realize low returns, triggering outflows by investors who update negatively about the ability of the managers running these funds. As a consequence of the outflows, funds sell assets they own, and this depresses further the prices of the assets hit by the original shock. If, in addition, outflows are gradual because of institutional constraints (e.g., lock-up periods, institutional decision lags), the selling pressure causes prices to decrease gradually, leading to momentum. At the same time, because outflows push prices below fundamental values, expected returns eventually rise, leading to reversal.

In addition to deriving momentum and reversal with rational agents, we contribute to the literature by building an equilibrium model with delegated portfolio management that is parsimonious and can speak to a broad range of phenomena. Delegation, to institutions such as mutual funds

[1] Jegadeesh and Titman (1993) document momentum for individual US stocks, predicting returns over horizons of 3-12 months by returns over the past 3-12 months. DeBondt and Thaler (1985) document reversal, predicting returns over horizons of up to 5 years by returns over the past 3-5 years. Fama and French (1992) document the value effect. This evidence has been extended to stocks in other countries (Fama and French 1998, Rouwenhorst 1998), industry-level portfolios (Grinblatt and Moskowitz 1999), country indices (Asness, Liew, and Stevens 1997, Bhojraj and Swaminathan 2006), bonds (Asness, Moskowitz and Pedersen 2008), currencies (Bhojraj and Swaminathan 2006) and commodities (Gorton, Hayashi and Rouwenhorst 2008). Asness, Moskowitz and Pedersen (2008) extend and unify much of this evidence and contain additional references.

[2] See, for example, Barberis, Shleifer and Vishny (1998), Daniel, Hirshleifer and Subrahmanyam (1998), Hong and Stein (1999), and Barberis and Shleifer (2003).

and hedge funds, is important in many markets. And while investors let fund managers invest on their behalf, they move across funds, generating flows that are large and linked to the funds' past performance.[3] Yet, incorporating delegation and fund flows into asset-pricing models is a daunting task: it entails modeling multiple assets and funds, portfolio choice by fund managers (over assets) and investors (over funds), and a motive for investors to be moving across funds, all in a dynamic equilibrium setting. Our model includes these elements, while allowing for a tractable analysis of fund flows and their price effects. The latter include not only momentum and reversal, but also comovement, lead-lag effects, amplification, and the effects of managers' concern with commercial risk.

Section 2 presents the model. We consider an infinite-horizon continuous-time economy with multiple risky assets, to which we refer as stocks, and one riskless asset. A competitive investor can invest in stocks through an index fund that holds the market portfolio, and through an active fund run by a competitive manager. The active fund can add value over the index fund because exogenous buy-and-hold investors hold stocks in different proportions than in the market portfolio: the active fund overweighs "large residual supply" stocks, which are in low demand by buy-and-hold investors and thus underpriced, and underweighs "small residual supply" stocks, which are in high demand and overpriced.[4] Flows between funds occur because the investor receives the return of the active fund net of an exogenous time-varying cost, which can be interpreted as a managerial perk or a reduced form for managerial ability. The manager determines the active fund's portfolio, and can invest his personal wealth in stocks through that fund. Both investor and manager are infinitely lived and maximize expected utility of intertemporal consumption.

Section 3 solves the model in the benchmark case of symmetric information, where the investor observes the manager's cost. When the cost increases, the investor flows out of the active and into to the index fund. This amounts to a net sale of stocks in large residual supply, which the active fund overweighs, and net purchase of stocks in small residual supply, which the active fund underweighs. The manager takes the other side of this transaction by raising his stake in the fund.[5] Because the manager is risk-averse, stocks in large residual supply become cheaper and offer

[3]According to the New York Stock Exchange Factbook, the fraction of stocks held directly by individuals in 2002 was less than 40%. The importance of fund flows and the link to past performance have been documented extensively. See, for example, Chevalier and Ellison (1997) and Sirri and Tufano (1998) for mutual funds, and Fund, Hsieh, Naik and Ramadorai (2008) and Ding, Getmansky, Liang and Wermers (2009) for hedge funds.

[4]The assumption of buy-and-hold investors ensures that the "true" market portfolio, which characterizes equilibrium asset returns, differs from the market index tracked by the index fund. Such a difference would arise even in the absence of buy-and-hold investors, provided that the market index is misconstructed, i.e., does not consist of one share of each stock. For example, the index might not be including some stocks, which are instead accessible to the active fund.

[5]The manager performs two roles in our model: select the active portfolio and take the other side of the investor's transactions. Separating the two roles would complicate the model without changing the main mechanisms. See Section 2 for further discussion.

higher expected return, while the opposite is true for stocks in small residual supply. Thus, the investor's flows generate return reversal, i.e., price changes forecast opposite changes in expected returns. Moreover, since flows impact stocks in large and stocks in small residual supply in opposite directions, they increase comovement within each group, while reducing comovement across groups.

The return reversal derived in Section 3 arises at any horizon. To generate momentum in the short run and reversal in the long run, we introduce the additional assumption that fund flows exhibit inertia. Section 4 models inertia through an exogenous convex cost that the investor incurs when changing her holdings of the active fund. In the presence of this adjustment cost, an increase in the manager's cost triggers gradual outflows from the active fund. Since these outflows are anticipated and amount to net sales of stocks in large residual supply, they cause the prices of these stocks to drop immediately. Yet, the drop is expected to continue, leading to momentum. This result is puzzling: why is the manager willing to hold—and even overweigh—stocks that are expected to underperform in the short run? The intuition is that these stocks offer the manager an attractive return over a long horizon because the anticipation of future outflows renders them underpriced. The manager could earn an even more attractive return, on average, by not holding the stocks until after the outflows occur. This, however, exposes him to the risk that the outflows might not occur, in which case the stocks would cease to be underpriced.[6] Thus, the short-run price drop is possible only because of the high long-run expected return; and more generally, momentum is possible only because of the subsequent reversal.

In addition to momentum, reversal and comovement, Sections 3 and 4 derive results on lead-lag effects, idiosyncratic risk and commercial risk. Because changes in the manager's cost impact the prices and subsequent expected returns of all stocks, past returns of one stock forecast subsequent returns of other stocks. For example, in Section 4, a price drop of a stock in high residual supply forecasts that other stocks in high residual supply will drop in the short run but have a high return in the long run. Momentum, reversal, comovement and lead-lag effects are larger for stocks with high idiosyncratic risk because these stocks are more sensitive to flows between the active and the index fund. Finally, when the manager receives a larger perk and is hence more concerned about commercial risk (i.e., future outflows), returns become more volatile.

Section 5 extends the analysis to the case of asymmetric information, where the investor does not observe the manager's cost and must infer it from fund performance. Asymmetric information

[6]The following three-period example illustrates the point. A stock is expected to pay off at 100 in Period 2. The stock price is 92 in Period 0, and 80 or 100 in Period 1 with equal probabilities. Buying the stock in Period 0 earns the manager a two-period expected capital gain of 8. Buying in Period 1 earns an expected capital gain of 20 if the price is 80 and 0 if the price is 100. A risk-averse manager might prefer earning 8 rather than 20 or 0 with equal probabilities, even though the expected capital gain between Periods 0 and 1 is negative.

generates a causal link from performance to flows: if, for example, the active fund underperforms relative to the index fund, the investor infers that the cost has increased and flows out of the active and into the index fund. Causality from performance to flows implies that the latter can be triggered by shocks to stocks' cashflows—in contrast to the case of symmetric information, where flows are driven only by changes in the cost.

The fund flows triggered by cashflow shocks amplify the effect of these shocks on stock returns, and generate momentum and reversal.[7] Under asymmetric information, momentum and reversal arise conditional not only on past returns, as under symmetric information, but also on past cashflow shocks. Moreover, asymmetric information generates new channels of comovement and lead-lag effects, as well as new effects of idiosyncratic and commercial risk. For example, a new channel of comovement is that a cashflow shock to one stock induces fund flows which affect the prices of other stocks. And a new effect of idiosyncratic risk is that cashflow shocks to stocks with high idiosyncratic risk generate a higher discrepancy between the active and the index return, and hence larger fund flows. Despite these new effects, the analysis of asymmetric information remains tractable and has many formal similarities to that of symmetric information. For example, the fund-flow-driven component of the covariance matrix of returns under asymmetric information is equal to its symmetric-information counterpart times a multiplicative scalar—which is larger than one because of the amplification effect of fund flows.

Momentum and reversal have mainly been derived in behavioral models.[8] In Barberis, Shleifer and Vishny (1998), momentum arises because investors view random-walk earnings as mean-reverting and under-react to news. In Hong and Stein (1999), prices under-react to news because information diffuses slowly across investors and those last to receive it do not infer it from prices. In Daniel, Hirshleifer and Subrahmanyam (1998), overconfident investors over-react to news because they underestimate the noise in their signals. Over-reaction builds up over time, leading to momentum, because the self-attribution bias makes investors gradually more overconfident.

Barberis and Shleifer (2003) is the behavioral model closest to our work. They assume that stocks belong in styles and are traded between switchers, who over-extrapolate performance trends, and fundamental investors. Following a stock's bad performance, switchers become pessimistic

[7] The mechanism for amplification is outlined in the third paragraph of the Introduction. The explanation in that paragraph assumes asymmetric information, while the mechanism for momentum and reversal is broader and present even under symmetric information (Section 4).

[8] Rational models of momentum include Berk, Green and Naik (1999), Johnson (2002) and Shin (2006), in which good news about a firm increase uncertainty and so raise the expected return required by investors. Albuquerque and Miao (2010) derive both momentum and reversal in a model where some investors receive a signal about dividends that is positively correlated with the return on a private investment technology. When the signal is high, the price goes up, but so does the investment in the technology. Since investors bear more risk overall, expected return increases.

about the future performance of the corresponding style, and switch to other styles. Because the extrapolation rule involves lags, switching is gradual and leads to momentum. Momentum requires additionally that fundamental investors are myopic and do not anticipate the switchers' flows.

The equilibrium implications of delegated portfolio management are the subject of a growing literature. In Shleifer and Vishny (1997), fund flows are an exogenous function of the funds' past performance, and amplify the effects of cashflow shocks. Amplification effects can also arise when the equity stake of fund managers must exceed a lower bound because of optimal contracting under moral hazard (He and Krishnamurthy 2009,2010), or when managers care about their reputation (Guerreri and Kondor 2010).[9] In Dasgupta, Prat and Verardo (2010), reputation concerns cause managers to herd, and this generates momentum and reversal under the additional assumption that the market makers trading with the managers are either monopolistic or myopic. In Basak and Pavlova (2010), flows by investors benchmarked against an index cause stocks in the index to comove.[10] Besides deriving momentum and reversal with competitive and rational agents, we contribute to that literature methodologically by bringing the analysis of delegation within a tractable normal-linear framework that can address a broad range of phenomena.

Finally, our emphasis on fund flows as generators of comovement and momentum is consistent with recent empirical findings. Coval and Stafford (2007) find that mutual funds experiencing large outflows engage in distressed selling of their stock portfolios. Anton and Polk (2010) and Greenwood and Thesmar (2010) find that comovement between stocks is larger when these are held by many mutual funds in common, controlling for style characteristics. Lou (2010) predicts flows into mutual funds by the funds' past performance, and imputes flows into individual stocks according to stocks' weight in funds' portfolios. He finds that flows into stocks can explain up to 50% of stock-level momentum, especially for large stocks and in recent data where mutual funds are more prevalent.

[9]Amplification effects can also arise when agents face margin constraints or have wealth-dependent risk aversion. See the survey by Gromb and Vayanos (2010).

[10]Other models exploring equilibrium implications of delegated portfolio management include Brennan (1993), Vayanos (2004), Dasgupta and Prat (2008), Petajisto (2009), Cuoco and Kaniel (2010), Kaniel and Kondor (2010) and Malliaris and Yan (2010). See also Berk and Green (2004), in which fund flows are driven by fund performance because investors learn about managers' ability, and feed back into performance because of exogenous decreasing returns to managing a large fund.

2 Model

Time t is continuous and goes from zero to infinity. There are N risky assets and a riskless asset. We refer to the risky assets as stocks, but they could also be interpreted as industry-level portfolios, asset classes, etc. The riskless asset has an exogenous, continuously compounded return r. The stocks pay dividends over time, and their prices are determined endogenously in equilibrium. We denote by D_{nt} the cumulative dividend per share of stock $n = 1,..,N$, and by S_{nt} the stock's price. We specify the stochastic process for dividends later in this section. By possibly redefining dividends, we normalize the supply of each stock to one share.

A competitive investor can invest in the riskless asset and in the stocks. The investor can access the stocks only through two investment funds. The first fund is passively managed and tracks mechanically the market index, i.e., holds stocks according to their supplies. Since all stocks are in supply of one share, the index fund holds an equal number of shares of each stock. The second fund is actively managed and selects an optimal portfolio in a way specified later in this section. We assume two investment funds, rather than only one, so that we can examine flows between funds. The assumption that one of the funds is indexed avoids the difficulty of having to solve for that fund's optimal portfolio.

Flows between the two funds can occur only if the funds hold different portfolios. To generate different portfolios, we assume that part of each stock's supply is held by an exogenous set of agents who do not trade. These agents could be the firm's managers or founding families, or unmodeled investors. We refer to them as buy-and-hold investors, and denote by $1-\theta_n$ the number of shares of stock n that they hold. The residual supply of stock n, left over from buy-and-hold investors, is θ_n shares. This is absorbed by the index fund, which holds an equal number of shares of each stock, and the active fund. If, therefore, residual supply differs across stocks, the active fund holds a different portfolio than the index fund in equilibrium: it overweighs stocks in large residual supply (high θ_n) and underweighs stocks in small residual supply (low θ_n). Moreover, the active portfolio dominates the index portfolio. Indeed, since prices adjust in equilibrium so that the active fund is induced to accommodate discrepancies in stocks' residual supplies, stocks in large residual supply (which are overweighed by the active fund) are cheap, while stocks in small residual supply (which are underweighed) are expensive.

The investor determines how to allocate her wealth between the riskless asset, the index fund, and the active fund. She maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-E\int_0^\infty \exp(-\alpha c_t - \beta t)dt, \tag{2.1}$$

where α is the coefficient of absolute risk aversion, c_t is consumption, and β is the discount rate. The investor's control variables are consumption c_t and the number of shares x_t and y_t of the index and active fund, respectively.

The active fund is run by a competitive manager, who can also invest his personal wealth in the fund. The manager determines the active portfolio and the allocation of his wealth between the riskless asset and the fund. He maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$-E\int_0^\infty \exp(-\bar{\alpha}\bar{c}_t - \bar{\beta} t)dt, \tag{2.2}$$

where $\bar{\alpha}$ is the coefficient of absolute risk aversion, $\bar{c}_t$ is consumption, and $\bar{\beta}$ is the discount rate. The manager's control variables are consumption $\bar{c}_t$, the number of shares $\bar{y}_t$ of the active fund, and the active portfolio $z_t \equiv (z_{1t}, .., z_{Nt})$, where z_{nt} denotes the number of shares of stock n included in one share of the active fund.

Under the assumptions introduced so far, and in the absence of other frictions, the equilibrium takes a simple form. As we show in Section 3, the investor holds stocks only through the active fund since its portfolio dominates the index portfolio. As a consequence, the active fund holds the entire residual supply of each stock, its portfolio is constant over time, and there are no flows between the two funds.

To generate fund flows, we introduce an additional element into our model. We assume that the investor's return from the active fund is equal to the gross return, made of the dividends and capital gains of the stocks held by the fund, net of a time-varying cost. We interpret this cost as a managerial perk, and discuss additional interpretations later in this section.[11] Empirical evidence on the existence of a time-varying cost impacting the returns to fund investors is provided in a number of papers.[12] For simplicity, we assume that the index fund entails no cost, so its gross and net returns coincide.

[11] An example of a managerial perk is late trading, whereby managers use their privileged access to the fund to buy or sell fund shares at stale prices. Late trading was common in many funds and led to the 2003 mutual-fund scandal. A related example is soft-dollar commissions, whereby funds inflate their brokerage commissions to pay for services that mainly benefit managers, e.g., promote the fund to new investors, or facilitate managers' late trading.

[12] Empirical papers measure the cost by the return gap, defined as the difference between a mutual fund's return over a given quarter and the return of a hypothetical portfolio invested in the stocks that the fund holds at the beginning of the quarter. Kacperczyk, Sialm and Zhang (2008) show that the return gap varies significantly across

We model the cost as a flow (i.e., the cost between t and $t+dt$ is of order dt), and assume that the flow cost is proportional to the number of shares y_t that the investor holds in the active fund. We denote the coefficient of proportionality by C_t and assume that it follows the process

$$dC_t = \kappa(\bar{C} - C_t)dt + s dB_t^C, \tag{2.3}$$

where κ is a mean-reversion parameter, $\bar{C}$ is a long-run mean, s is a positive scalar, and B_t^C is a Brownian motion. The mean-reversion of C_t is not essential for momentum and reversal, which occur even when $\kappa = 0$.

We allow the manager to derive a benefit from the investor's participation in the active fund. This benefit can be interpreted as a managerial perk or a fee. We model the benefit in the same way as the cost, i.e., a flow which is proportional to the number of shares y_t that the investor holds in the active fund. If the cost is a perk that the manager can extract efficiently, then the coefficient of proportionality for the benefit is C_t. We allow more generally the coefficient of proportionality to be $\lambda C_t + B$, where λ and B are scalars. The parameter λ can be interpreted as the efficiency of perk extraction, while the parameter B can derive from a constant fee.[13]

Varying the parameters λ and B generates a rich specification of the manager's objective. When $\lambda = B = 0$, the manager cares about fund performance only through his personal investment in the fund, and his objective is similar to the fund investor's. When instead λ and B are positive, the manager is also concerned with commercial risk, i.e., the risk that the investor might reduce her participation in the fund. The parameters λ and B are not essential for momentum and reversal, which occur even when $\lambda = B = 0$. As we show in later sections, λ affects the size of momentum relative to reversal, while B affects only the average mispricing.

The cost and benefit are assumed proportional to y_t for analytical convenience. At the same time, these variables are sensitive to how shares of the active fund are defined (e.g., they change with a stock split). We define one share of the fund by the requirement that its market value equals the equilibrium market value of the entire fund. Under this definition, the number of fund shares

funds and over time, and is persistent with a half-life of about three years. The high persistence indicates that the return gap is linked to underlying fund characteristics—and there is indeed a correlation with fund-specific measures of agency costs and trading costs. Because of its significant cross-sectional variation and persistence, the return gap a good forecaster of future returns: funds whose return gap is in the top decile outperform the market by an average 1.2% over the next year, while funds in the bottom decile underperform by 2.2%. Earlier studies that use the return gap and link it to fund characteristics include Grinblatt and Titman (1989) and Wermers (2000).

[13] If, for example, the cost $C_t y_t$ is the sum of a fee $F y_t$ and a perk $(C_t - F)y_t$, and the manager can extract a fraction λ of the perk, then the benefit is

$$[F + \lambda(C_t - F)]\, y_t = [\lambda C_t + (1-\lambda)F]\, y_t,$$

which has the assumed form with $B = (1-\lambda)F$.

held by the investor and the manager in equilibrium sum to one, i.e.,

$$y_t + \bar{y}_t = 1. \tag{2.4}$$

We define one share of the index fund to consist of one share of each stock, and refer to the corresponding vector $\mathbf{1} \equiv (1, .., 1)$ as the market portfolio. We refer to the vector $\theta \equiv (\theta_1, .., \theta_N)$ of the stocks' residual supplies as the residual-supply portfolio. We define the constant

$$\Delta \equiv \theta\Sigma\theta'\mathbf{1}\Sigma\mathbf{1}' - (\mathbf{1}\Sigma\theta')^2,$$

which is positive and becomes zero when the vectors $\mathbf{1}$ and θ are collinear.

The manager observes all the variables in the model. The investor observes the returns and share prices of the index and active funds, but not the same variables for the individual stocks. We study both the case of symmetric information, where the investor observes the cost C_t, and that of asymmetric information, where C_t is observable only by the manager. In the asymmetric-information case, the investor seeks to infer C_t from the returns and share prices of the index and active funds. The symmetric-information case is simpler analytically and delivers most of our main results, including momentum and reversal. The asymmetric-information case is more realistic and delivers some additional results.

We denote the vector of stocks' cumulative dividends by $D_t \equiv (D_{1t}, .., D_{Nt})'$ and the vector of stock prices by $S_t \equiv (S_{1t}, .., S_{Nt})'$, where v' denotes the transpose of the vector v. We assume that D_t follows the process

$$dD_t = F_t dt + \sigma dB_t^D, \tag{2.5}$$

where $F_t \equiv (F_{1t}, .., F_{Nt})'$ is a time-varying drift equal to the instantaneous expected dividend, σ is a constant matrix of diffusion coefficients, and B_t^D is a d-dimensional Brownian motion independent of B_t^C. The expected dividend F_t is observable only by the manager. Time-variation in F_t is not essential in the symmetric-information case, where momentum and reversal occur even when F_t is a constant parameter known to the investor. Time-variation in F_t becomes essential for the analysis of asymmetric information: with a constant F_t, the investor would infer C_t perfectly from the share price of the active fund, and information would be symmetric. We model time-variation in F_t through the process

$$dF_t = \kappa(\bar{F} - F_t)dt + \phi\sigma dB_t^F \tag{2.6}$$

where the mean-reversion parameter κ is the same as for C_t for simplicity, $\bar{F}$ is a long-run mean, ϕ is a positive scalar, and B_t^F is a d-dimensional Brownian motion independent of B_t^C and B_t^D. The diffusion matrices for D_t and F_t are proportional for simplicity.

We finally comment on the assumption that the manager can invest his personal wealth in the active fund. This assumption generates a simple objective that the manager maximizes when choosing the fund's portfolio.[14] It also ensures that the manager acts as trading counterparty to the investor: when C_t increases and the investor reduces her holdings of the active fund, effectively selling the stocks held by the fund, the manager takes the other side by raising his stake in the fund. Under the alternative assumption that the manager must invest his personal wealth in the riskless asset, we would need to introduce additional "smart-money" investors who could access stocks directly and act as counterparty to the fund investor. This would complicate the model without changing the basic mechanisms.

Note that in a model with smart-money investors, additional interpretations of the cost are possible. Two interpretations not emphasized so far are an operational cost (e.g., trading cost) and a reduced form for low managerial stock-picking ability. These additional interpretations are not consistent with the assumption that the manager can invest his personal wealth in the active fund. Indeed, the cost would then impact not only the investor's holdings in the fund y_t, as we are assuming, but also the manager's holdings $\bar{y}_t$. The additional interpretations, however, are consistent with a model with smart-money investors: since these investors do not invest in the active fund, their investments are not affected by the cost.

3 Symmetric Information

This section solves the model presented in the previous section in the case of symmetric information, where the cost C_t is observable by both the investor and the manager. We look for an equilibrium in which stock prices take the form

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r + \kappa} - (a_0 + a_1 C_t), \tag{3.1}$$

[14]Restricting the manager not to invest his personal wealth in the index fund is also in the spirit of generating a simple objective. Indeed, in the absence of this restriction, the active portfolio would be indeterminate: the manager could mix a given active portfolio with the index, and make that the new active portfolio, while achieving the same personal portfolio through an offsetting short position in the index. Note that restricting the manager not to invest in the index only weakly constrains his personal portfolio since he can always modify the portfolio of the active fund and his stake in that fund.

where (a_0, a_1) are constant vectors. The first two terms are the present value of expected dividends, discounted at the riskless rate r, and the last term is a risk premium linear in C_t. As we show later in this section, the risk premium moves in response to fund flows. The investor's holdings of the active fund in our conjectured equilibrium are

$$y_t = b_0 - b_1 C_t, \tag{3.2}$$

where (b_0, b_1) are constants. We expect b_1 to be positive, i.e., the investor reduces her holdings of the fund when C_t is high. We refer to an equilibrium satisfying (3.1) and (3.2) as linear.

3.1 Manager's Optimization

The manager chooses the active fund's portfolio z_t, the number $\bar{y}_t$ of fund shares that he owns, and consumption $\bar{c}_t$. The manager's budget constraint is

$$dW_t = rW_t dt + \bar{y}_t z_t (dD_t + dS_t - rS_t dt) + (\lambda C_t + B) y_t dt - \bar{c}_t dt. \tag{3.3}$$

The first term is the return from the riskless asset, the second term is the return from the active fund in excess of the riskless asset, the third term is the manager's benefit from the investor's participation in the fund, and the fourth term is consumption. To compute the return from the active fund, we note that since one share of the fund corresponds to z_t shares of the stocks, the manager's effective stock holdings are $\bar{y}_t z_t$ shares. These holdings are multiplied by the vector $dR_t \equiv dD_t + dS_t - rS_t dt$ of the stocks' excess returns per share (referred to as returns, for simplicity). Using (2.3), (2.5), (2.6) and (3.1), we can write the vector of returns as

$$dR_t = \left[ra_0 + (r+\kappa)a_1 C_t - \kappa a_1 \bar{C}\right] dt + \sigma \left(dB_t^D + \frac{\phi dB_t^F}{r+\kappa} \right) - s a_1 dB_t^C. \tag{3.4}$$

Returns depend only on the cost C_t, and not on the expected dividend F_t. The covariance matrix of returns is

$$Cov_t(dR_t, dR_t') = \left(f\Sigma + s^2 a_1 a_1'\right) dt, \tag{3.5}$$

where $f \equiv 1 + \phi^2/(r+\kappa)^2$ and $\Sigma \equiv \sigma\sigma'$. The matrix $f\Sigma$ represents the covariance driven purely by dividend (i.e., cashflow) news, and we refer to it as fundamental covariance. The matrix $s^2 a_1 a_1'$ represents the additional covariance introduced by fund flows, and we refer to it as non-fundamental covariance.

The manager's optimization problem is to choose controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (2.2) subject to the budget constraint (3.3) and the investor's holding policy (3.2). The active fund's portfolio z_t satisfies, in addition, the normalization

$$z_t S_t = (\theta - x_t \mathbf{1}) S_t. \tag{3.6}$$

This is because one share of the active fund is defined so that its market value equals the equilibrium market value of the entire fund. Moreover, the latter is $(\theta - x_t \mathbf{1}) S_t$ because in equilibrium the active fund holds the residual-supply portfolio θ minus the investor's holdings $x_t \mathbf{1}$ of the index fund. We conjecture that the manager's value function is

$$\bar{V}(W_t, C_t) \equiv -\exp\left[-\left(r\bar{\alpha}W_t + \bar{q}_0 + \bar{q}_1 C_t + \frac{1}{2}\bar{q}_{11} C_t^2\right)\right], \tag{3.7}$$

where $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$ are constants. The Bellman equation is

$$\max_{\bar{c}_t, \bar{y}_t, z_t} \left[-\exp(-\bar{\alpha}\bar{c}_t) + \mathcal{D}\bar{V} - \bar{\beta}\bar{V}\right] = 0, \tag{3.8}$$

where $\mathcal{D}\bar{V}$ is the drift of the process $\bar{V}$ under the controls $(\bar{c}_t, \bar{y}_t, z_t)$. Proposition 3.1 shows that the value function (3.7) satisfies the Bellman equation if $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$ satisfy a system of three scalar equations.

Proposition 3.1 *The value function (3.7) satisfies the Bellman equation (3.8) if $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$ satisfy a system of three scalar equations.*

In the proof of Proposition 3.1 we show that the optimization over $(\bar{c}_t, \bar{y}_t, z_t)$ can be reduced to optimization over the manager's consumption $\bar{c}_t$ and effective stock holdings $\hat{z}_t \equiv \bar{y}_t z_t$. Given $\hat{z}_t$, the decomposition between $\bar{y}_t$ and z_t is determined by the normalization (3.6). The first-order condition with respect to $\hat{z}_t$ is

$$E_t(dR_t) = r\bar{\alpha}Cov_t(dR_t, \hat{z}_t dR_t) + (\bar{q}_1 + \bar{q}_{11} C_t) Cov_t(dR_t, dC_t). \tag{3.9}$$

Eq. (3.9) links expected stock returns to the risk faced by the manager. The expected return that the manager requires from a stock depends on the stock's covariance with the manager's portfolio $\hat{z}_t$ (first term in the right-hand side), and on the covariance with changes to the cost C_t (second term). The latter effect reflects a hedging demand by the manager. We derive the implications of (3.9) for the cross section of expected returns later in this section.

3.2 Investor's Optimization

The investor chooses a number of shares x_t in the index fund and y_t in the active fund, and consumption c_t. The investor's budget constraint is

$$dW_t = rW_t dt + x_t \mathbf{1} dR_t + y_t \left(z_t dR_t - C_t dt \right) - c_t dt. \tag{3.10}$$

The first three terms are the returns from the riskless asset, the index fund, and the active fund (net of the cost C_t), and the fourth term is consumption. The investor's optimization problem is to choose controls (c_t, x_t, y_t) to maximize the expected utility (2.1) subject to the budget constraint (3.10). The investor takes the active fund's portfolio z_t as given and equal to its equilibrium value $\theta - x_t \mathbf{1}$. We conjecture that the investor's value function is

$$V(W_t, C_t) \equiv -\exp\left[-\left(r\alpha W_t + q_0 + q_1 C_t + \frac{1}{2} q_{11} C_t^2 \right) \right], \tag{3.11}$$

where (q_0, q_1, q_{11}) are constants. The Bellman equation is

$$\max_{c_t, x_t, y_t} \left[-\exp(-\alpha c_t) + \mathcal{D}V - \beta V \right] = 0, \tag{3.12}$$

where $\mathcal{D}V$ is the drift of the process V under the controls (c_t, x_t, y_t). Proposition 3.2 shows that the value function (3.11) satisfies the Bellman equation (3.12) if (q_0, q_1, q_{11}) satisfy a system of three scalar equations. The proposition shows additionally that the optimal control y_t is linear in C_t, as conjectured in (3.2).

Proposition 3.2 *The value function (3.11) satisfies the Bellman equation (3.12) if* (q_0, q_1, q_{11}) *satisfy a system of three scalar equations. The optimal control* y_t *is linear in* C_t.

In the proof of Proposition 3.2, we show that the first-order conditions with respect to x_t and y_t are

$$E_t(\mathbf{1} dR_t) = r\alpha Cov_t \left[\mathbf{1} dR_t, (x_t \mathbf{1} + y_t z_t) dR_t \right] + (q_1 + q_{11} C_t) Cov_t(\mathbf{1} dR_t, dC_t), \tag{3.13}$$

$$E_t(z_t dR_t) - C_t dt = r\alpha Cov_t \left[z_t dR_t, (x_t \mathbf{1} + y_t z_t) dR_t \right] + (q_1 + q_{11} C_t) Cov_t(z_t dR_t, dC_t), \tag{3.14}$$

respectively. Eqs. (3.13) and (3.14) are analogous to the manager's first-order condition (3.9) in that they equate expected returns to risk. The difference with (3.9) is that the investor is constrained to two portfolios rather than N individual stocks. Eq. (3.9) is a vector equation with N components, while (3.13) and (3.14) are scalar equations derived by pre-multiplying expected returns with the vectors $\mathbf{1}$ and z_t of index- and active-fund weights. Note that the investor's expected return from the active fund in (3.14) is net of the cost C_t.

3.3 Equilibrium

In equilibrium, the active fund's portfolio z_t is equal to $\theta - x_t\mathbf{1}$, and the shares held by the manager and the investor sum to one. Combining these equations with the first-order conditions (3.9), (3.13) and (3.14), and the value-function equations (Propositions 3.1 and 3.2), yields a system of equations characterizing a linear equilibrium. Proposition 3.3 shows that a linear equilibrium exists, and determines a sufficient condition for uniqueness.

Proposition 3.3 *There exists a linear equilibrium. The constant* b_1 *is positive and the vector* a_1 *is given by*

$$a_1 = \gamma_1 \Sigma p_f', \tag{3.15}$$

where γ_1 *is a positive constant and*

$$p_f \equiv \theta - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1} \tag{3.16}$$

is the "flow portfolio." There exists a unique linear equilibrium if $\lambda < \bar{\lambda}$ *for a constant* $\bar{\lambda} > 0$.[15]

Proposition 3.3 can be specialized to the benchmark case of costless delegation, where the investor's cost C_t of investing in the active fund is constant and equal to zero. This case can be derived by setting C_t, as well as its long-run mean $\bar{C}$ and diffusion coefficient s, to zero.

Corollary 3.1 (Costless Delegation) *When* $C_t = \bar{C} = s = 0$*, the investor holds* $y_t = \bar{\alpha}/(\alpha + \bar{\alpha})$ *shares of the active fund and* $x_t = 0$ *shares of the index fund. Stocks' expected returns are given by the one-factor model*

$$E_t(dR_t) = \frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}}\Sigma\theta' dt = \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}}Cov_t(dR_t, \theta dR_t), \tag{3.17}$$

with the factor being the residual-supply portfolio θ.

The investor holds only the active fund because it offers a superior portfolio than the index fund at no cost. The relative shares of the investor and the manager in the active fund are determined

[15] We conjecture that uniqueness holds even if $\lambda \geq \bar{\lambda}$. Moreover, most of the properties that we derive hold in any linear equilibrium: this applies, for example, to (3.15) and $\gamma_1 > 0$, as we show in the proof of Proposition 3.3, and to Corollaries 3.2-3.6.

by their risk-aversion coefficients, according to optimal risk-sharing. Stocks' expected returns are determined by the covariance with the residual-supply portfolio. The intuition for the latter result is that since the index fund receives zero investment, the residual-supply portfolio coincides with the active portfolio z_t, which is also the portfolio held by the manager. Since the manager determines the cross section of expected returns through the first-order condition (3.9), and there is no hedging demand because C_t is constant, the residual-supply portfolio is the only pricing factor. Note that when $C_t = \bar{C} = s = 0$, expected returns are constant over time. Thus, return predictability can arise only because of time-variation in C_t. We next allow C_t to vary over time, and determine the effects on fund flows, prices and expected returns.

Corollary 3.2 (Fund Flows) *The change in the investor's effective stock holdings, caused by a change in C_t, is proportional to the flow portfolio p_f:*

$$\frac{\partial(x_t \boldsymbol{1} + y_t z_t)}{\partial C_t} = -b_1 p_f. \tag{3.18}$$

Following an increase in the cost C_t of investing in the active fund, the investor flows out of that fund and into the index fund. The net change in the investor's effective stock holdings is proportional to the flow portfolio p_f, defined in (3.16). This portfolio consists of the residual-supply portfolio θ, plus a position in the market portfolio $\mathbf{1}$ that renders the covariance with the market equal to zero.[16] The intuition why the flow portfolio characterizes fund flows is as follows. Following an increase in C_t, the investor reduces her investment in the active fund, thus selling a slice of the residual-supply portfolio. She also increases her investment in the index fund, thus buying a slice of the market portfolio. Because investing in the index fund is costless, the investor maintains a constant overall exposure to the market. Therefore, the net change in her portfolio is uncorrelated with the market, which means that she is selling a slice of the flow portfolio.

In selling a slice of the flow portfolio, the investor is effectively selling some stocks and buying others. The stocks being sold are in large residual supply and correspond to long positions in the flow portfolio, while the stocks being bought are in small residual supply and correspond to short positions. Thus, when the investor flows out of the active fund and into the index fund, she sells stocks that the active fund overweighs relative to the index fund, and buys stocks that the active fund underweighs.

[16] The zero covariance between the market and the flow portfolio follows from the more general result of Corollary 3.3: premultiply the last equality in (3.19) by $\mathbf{1}$ and note that $\mathbf{1}\epsilon_t = 0$.

Corollary 3.3 (Prices) *The change in stock prices, caused by a change in C_t, is proportional to stocks' covariance with the flow portfolio p_f:*

$$\frac{\partial S_t}{\partial C_t} = -\gamma_1 \Sigma p_f' = -\frac{\gamma_1}{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} Cov(dR_t, p_f dR_t) = -\frac{\gamma_1}{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} Cov_t(d\epsilon_t, p_f d\epsilon_t), \tag{3.19}$$

where $d\epsilon_t \equiv (d\epsilon_{1t}, .., d\epsilon_{Nt})'$ denotes the residual from a regression of stock returns dR_t on the market return $\mathbf{1}dR_t$.

An increase in C_t lowers the prices of stocks that covary positively with the flow portfolio and raises the prices of stocks covarying negatively. This price impact arises because of two distinct mechanisms: an intuitive mechanism involving fund flows, and a more subtle mechanism involving the manager's hedging demand that we discuss at the end of this section. The fund-flows mechanism is as follows. When C_t increases, the investor sells a slice of the flow portfolio, which is acquired by the manager. As a result, the manager requires higher expected returns from stocks that covary positively with the flow portfolio, and the price of these stocks decreases. Conversely, the expected returns of stocks that covary negatively with the flow portfolio decrease, and their price increases.

A stock's covariance with the flow portfolio can be characterized in terms of idiosyncratic risk. The last equality in Corollary 3.3 implies that the covariance is positive if the stock's idiosyncratic movement $d\epsilon_{nt}$ (i.e., the part of its return that is orthogonal to the index) covaries positively with the idiosyncratic movement of the flow portfolio. This is likely to occur when the stock is in large residual supply, because it then corresponds to a long position in the flow portfolio. Thus, stocks in large residual supply, which the active fund overweighs, are likely to drop when the investor flows out of the active fund and into the index fund. Conversely, stocks in small residual supply, which the active fund underweighs, are likely to rise.

While residual supply influences the sign of a stock's covariance with the flow portfolio, idiosyncratic risk influences the magnitude: stocks with high idiosyncratic risk have higher covariance with the flow portfolio in absolute value, and are therefore more affected by changes in C_t. The intuition can be seen from the extreme case of a stock with no idiosyncratic risk. Since changes in C_t do not change the investor's overall exposure to the market, they also do not change her willingness to carry market risk. Therefore, they do not affect the price of the market portfolio, or of a stock carrying only market risk.

Since changes in C_t, and the fund flows they trigger, affect prices, they contribute to comovement between stocks. Recall from (3.5) that the covariance matrix of stock returns is the sum of a fundamental covariance, driven purely by cashflows, and a non-fundamental covariance, introduced by fund flows. Using Proposition 3.3, we can compute the non-fundamental covariance.

Corollary 3.4 (Comovement) *The covariance matrix of stock returns is*

$$Cov_t(dR_t, dR_t') = \left(f\Sigma + s^2\gamma_1^2\Sigma p_f' p_f \Sigma\right) dt. \tag{3.20}$$

The non-fundamental covariance is positive for stock pairs whose covariance with the flow portfolio has the same sign, and is negative otherwise.

The non-fundamental covariance between a pair of stocks is proportional to the product of the covariances between each stock in the pair and the flow portfolio. It is thus large in absolute value when the stocks have high idiosyncratic risk, because they are more affected by changes in C_t. Moreover, it can be positive or negative: positive for stock pairs whose covariance with the flow portfolio has the same sign, and negative otherwise. Intuitively, two stocks move in the same direction in response to fund flows if they are both overweighed or both underweighed by the active fund, but move in opposite directions if one is overweighed and the other underweighed.

The effect of C_t on expected returns goes in the opposite direction than the effect on prices. We next determine more generally the cross section of expected returns.

Corollary 3.5 (Expected Returns) *Stocks' expected returns are given by the two-factor model*

$$E_t(dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}} \frac{\boldsymbol{1}\Sigma\theta'}{\boldsymbol{1}\Sigma\boldsymbol{1}'} Cov_t(dR_t, \boldsymbol{1}dR_t) + \Lambda_t Cov_t(dR_t, p_f dR_t), \tag{3.21}$$

with the factors being the market and the flow portfolio. The factor risk premium Λ_t associated to the flow portfolio is

$$\Lambda_t = \frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}} + \frac{\gamma_1}{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} \left[(r+\kappa)C_t - \frac{s^2(\alpha\bar{q}_1 + \bar{\alpha}q_1)}{\alpha+\bar{\alpha}}\right]. \tag{3.22}$$

The presence of the flow portfolio as a priced factor can be viewed as a mispricing relative to a CAPM in which the only factor is the market portfolio. The factor risk premium Λ_t associated to the flow portfolio measures the severity of the mispricing. Note that the mispricing exists even when delegation is costless (and it is because of this mispricing that the active fund is attractive to the investor). Indeed, Corollary 3.1 shows that with costless delegation, expected returns are described by a one-factor model, with the factor being the residual-supply portfolio rather than the market. This one-factor model is also implied from Corollary 3.5. Indeed, setting $C_t = \bar{C} = s = 0$ in (3.22), we find that the risk premia associated to the two factors are constant over time, and therefore the two factors can be reduced to one. Moreover, the factor risk premium Λ_t, which measures the mispricing, is positive. When C_t varies over time, so does Λ_t, and the two factors cannot be reduced to one. An increase in C_t raises Λ_t and renders the mispricing more severe: stocks overweighed by the active fund become more underpriced and their expected returns increase, while stocks underweighed by the active fund become more overpriced and their expected returns decrease. Note that changes in C_t are the only driver of time-variation in expected returns.

The time-variation in expected returns gives rise to predictability. We examine predictability based on past returns. As in the rest of our analysis, we evaluate returns over an infinitesimal time period; returns thus concern a single point in time. We compute the covariance between the vector of returns at time t and the same vector at time $t' > t$. Corollary 3.6 shows that this autocovariance matrix is equal to the non-fundamental (contemporaneous) covariance matrix times a negative scalar.

Corollary 3.6 (Return Predictability) *The covariance between stock returns at time t and those at time $t' > t$ is*

$$Cov_t(dR_t, dR'_{t'}) = -s^2(r+\kappa)\gamma_1^2 e^{-\kappa(t'-t)}\Sigma p'_f p_f \Sigma (dt)^2. \tag{3.23}$$

A stock's return predicts negatively the stock's subsequent return (return reversal). It predicts negatively the subsequent return of another stock when the covariance between each stock in the pair and the flow portfolio has the same sign (negative lead-lag effect), and positively otherwise (positive lead-lag effect).

Since the diagonal elements of the autocovariance matrix are negative, stocks exhibit negative autocovariance, i.e., return reversal. This is because expected returns vary over time only in response to changes in C_t, and these changes move prices in the opposite direction. Thus, a lower-than-expected price predicts a higher-than-expected subsequent return, and vice-versa.

The non-diagonal elements of the autocovariance matrix characterize lead-lag effects, i.e., whether the past return of one stock predicts the future return of another. Lead-lag effects are negative for stock pairs whose covariance with the flow portfolio has the same sign, and are positive otherwise. For example, when the sign is the same, changes in C_t move the prices of both stocks in the same direction and their expected returns in the opposite direction. Therefore, a lower-than-expected price of one stock predicts a higher-than-expected subsequent return of the other, and vice-versa.

We next examine how prices and expected returns depend on the manager's concern with commercial risk, i.e., the risk that the investor might reduce her participation in the fund. Recall that the manager derives the benefit $(\lambda C_t + B)y_t$ from the investor's participation, where y_t is the number of shares owned by the investor, λ is the efficiency of perk extraction, and B is a fee.

Corollary 3.7 (Commercial Risk) *An increase in λ raises γ_1, and thus increases the non-fundamental volatility of stock returns and the extent of return reversal. An increase in B has no effect on γ_1, but raises Λ_t, and thus increases the average mispricing.*

Since B raises Λ_t, it exacerbates the mispricing that the active fund seeks to exploit: stocks that the active fund overweighs become more underpriced, while stocks that it underweighs become more overpriced. Thus, a manager concerned with losing his fee is less willing to trade against mispricings. A common intuition for this result is that the manager fears that mispricings might worsen, in which case the fund will perform poorly and outflows will occur.[17] In the symmetric-information case, where the investor observes C_t, the causality is not from performance to flows, as the previous intuition requires, but from flows to performance: an increase in C_t triggers outflows from the active fund, and the negative price pressure these exert on the stocks that the fund overweighs impairs fund performance. The intuition for the effect of B is different as well: a manager concerned with losing his fee seeks to hedge against increases in C_t since these trigger outflows. Hedging requires the manager to hold stocks that perform well when C_t increases. These are the stocks that the active fund underweighs, and the manager's hedging demand renders these stocks more overpriced.

The parameter B has an effect only on the average mispricing, but not on how the mispricing varies with C_t. By contrast, λ renders the mispricing more sensitive to C_t, i.e., raises γ_1. Indeed, $\lambda > 0$ implies that when C_t increases, the manager can extract a larger perk from each share of the fund held by the investor, and is therefore more willing to hedge against future changes in C_t. Thus, an increase in C_t not only generates outflows, but also makes the manager more

[17] This is, for example, the mechanism in Shleifer and Vishny (1997), who assume that fund flows are an exogenous function of fund performance. Causality from performance to flows is endogenous in our model, and arises in the asymmetric-information case, where the investor does not observe C_t and seeks to infer it from fund performance. In the asymmetric-information case, B raises Λ_t because of a mechanism similar to that in Shleifer and Vishny.

concerned with future outflows.[18] As a consequence, the manager's hedging demand increases, and this adds to the mispricing caused by current outflows. Note that since λ raises γ_1, it also increases non-fundamental volatility and comovement (Corollary 3.4), as well as return reversal (Corollary 3.6). Thus, the manager's demand to hedge against outflows can have the perverse effect to render returns more volatile.

4 Gradual Adjustment

Section 3 shows that returns exhibit reversal at any horizon. To generate short-run momentum and long-run reversal, we need the additional assumption that fund flows exhibit inertia, i.e., the investor can adjust her fund holdings to new information only gradually. Gradual adjustment can result from contractual restrictions or institutional decision lags.[19] We model these frictions as a flow cost $\psi(dy_t/dt)^2/2$ that the investor must incur when changing the number y_t of active-fund shares that she owns. The advantage of the quadratic cost over other formulations (such as an upper bound on $|dy_t/dt|$) is that it preserves the linearity of the model.

We maintain the assumption that information about C_t is symmetric, and look for an equilibrium in which stock prices take the form

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r+\kappa} - (a_0 + a_1 C_t + a_2 y_t), \tag{4.1}$$

where (a_0, a_1, a_2) are constant vectors. The number y_t of active-fund shares that the investor owns becomes a state variable and affects prices since it cannot be set instantaneously to its optimal level. The investor's speed of adjustment $v_t \equiv dy_t/dt$ in our conjectured equilibrium is

$$v_t = b_0 - b_1 C_t - b_2 y_t, \tag{4.2}$$

where (b_0, b_1, b_2) are constants. We expect (b_1, b_2) to be positive, i.e., the investor reduces her investment in the active fund faster when C_t or y_t are large. We refer to an equilibrium satisfying (4.1) and (4.2) as linear.

[18] The same effect would arise under the non-perk interpretations of the cost, discussed at the end of Section 2, if the manager's benefit is assumed concave in the number of shares y_t owned by the investor.

[19] An example of contractual restrictions is lock-up periods, often imposed by hedge funds, which require investors not to withdraw capital for a pre-specified time period. Institutional decision lags can arise for investors such as pension funds, foundations or endowments, where decisions are made by boards of trustees that meet infrequently. The inertia in capital flows and its relevance for asset prices are emphasized in Duffie's (2010) presidential address to the American Finance Association.

4.1 Optimization

The manager chooses controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (2.2) subject to the budget constraint (3.3), the normalization (3.6), and the investor's holding policy (4.2). Since stock prices depend on (C_t, y_t), the same is true for the manager's value function. We conjecture that the value function is

$$\bar{V}(W_t, \bar{X}_t) \equiv -\exp\left[-\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right)\right], \tag{4.3}$$

where $\bar{X}_t \equiv (C_t, y_t)'$, $(\bar{q}_0, \bar{q}_1, \bar{q}_2)$ are constants, and $\bar{Q}$ is a constant symmetric 2×2 matrix.

Proposition 4.1 *The value function (4.3) satisfies the Bellman equation (3.8) if* $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{Q})$ *satisfy a system of six scalar equations.*

The investor chooses controls (c_t, x_t, v_t) to maximize the expected utility (2.1) subject to the budget constraint

$$dW_t = rW_t dt + x_t \mathbf{1} dR_t + y_t \left(z_t dR_t - C_t dt\right) - \frac{1}{2}\psi v_t^2 dt - c_t dt \tag{4.4}$$

and the manager's portfolio policy $z_t = \theta - x_t\mathbf{1}$. We study this optimization problem in two steps. In a first step, we optimize over (c_t, x_t), assuming that v_t is given by (4.2). We solve this problem using dynamic programming, and conjecture the value function

$$V(W_t, X_t) \equiv -\exp\left[-\left(r\alpha W_t + q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t'QX_t\right)\right], \tag{4.5}$$

where $X_t \equiv (C_t, y_t)'$, (q_0, q_1, q_2) are constants, and Q is a constant symmetric 2×2 matrix. The Bellman equation is

$$\max_{c_t, x_t} \left[-\exp(-\alpha c_t) + \mathcal{D}V - \beta V\right] = 0, \tag{4.6}$$

where $\mathcal{D}V$ is the drift of the process V under the controls (c_t, x_t). In a second step, we derive conditions under which the control v_t given by (4.2) is optimal.

Proposition 4.2 *The value function (4.5) satisfies the Bellman equation (4.6) if* (q_0, q_1, q_2, Q) *satisfy a system of six scalar equations. The control* v_t *given by (4.2) is optimal if* (b_0, b_1, b_2) *satisfy a system of three scalar equations.*

4.2 Equilibrium

The system of equations characterizing a linear equilibrium is higher-dimensional than under instantaneous adjustment, and so more complicated. Proposition 4.3 shows that a unique linear equilibrium exists when the diffusion coefficient s of C_t is small. This is done by computing explicitly the linear equilibrium for $s = 0$ and applying the implicit function theorem. Our numerical solutions for general values of s seem to generate a unique linear equilibrium. Moreover, the properties that we derive for small s in the rest of this section seem to hold for general values of s.[20]

Proposition 4.3 *For small s, there exists a unique linear equilibrium. The constants (b_1, b_2) are positive, and the vectors (a_1, a_2) are given by*

$$a_i = \gamma_i \Sigma p_f', \tag{4.7}$$

where γ_1 is a positive and γ_2 a negative constant. Eq. (4.7) holds in any linear equilibrium for general values of s.

Since $\gamma_1 > 0$, an increase in C_t lowers the prices of stocks that covary positively with the flow portfolio and raises the prices of stocks covarying negatively. This effect is the same as under instantaneous adjustment (Corollary 3.3) but the mechanism is slightly different. Under instantaneous adjustment, an increase in C_t triggers an immediate outflow from the active fund by the investor. In flowing out of the fund, the investor sells the stocks that the fund overweighs, and the prices of these stocks drop so that the manager is induced to buy them. Under gradual adjustment, the outflow is expected to occur in the future, and so are the sales of the stocks that the fund overweighs. The prices of these stocks drop immediately in anticipation of the future sales.

We next examine how C_t impacts stocks' expected returns. As in the case of instantaneous adjustment, expected returns are given by a two-factor model, with the factors being the market and the flow portfolio. The key difference with instantaneous adjustment lies in the properties of the factor risk premium associated with the flow portfolio.

Corollary 4.1 (Expected Returns) *Stocks' expected returns are given by the two-factor model (3.21), with the factors being the market and the flow portfolio. The factor risk premium Λ_t*

[20] This applies to $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 < 0$, and to Corollaries 4.1 and 4.2 (with a different threshold λ^R).

associated to the flow portfolio is

$$\Lambda_t = r\bar{\alpha} + \frac{1}{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}\left(\gamma_1^R C_t + \gamma_2^R y_t - \gamma_1 s^2 \bar{q}_1\right), \tag{4.8}$$

where (γ_1^R, γ_2^R) *are constants. For small* s*, the constant* γ_1^R *is negative if*

$$\lambda < \lambda^R \equiv \frac{\bar{\alpha}}{2(\alpha+\bar{\alpha}) + \frac{\psi\mathbf{1}\Sigma\mathbf{1}'}{2f\Delta}\left[r + (r+2\kappa)\sqrt{1+\frac{4(\alpha+\bar{\alpha})f\Delta}{r\psi\mathbf{1}\Sigma\mathbf{1}'}}\right]}, \tag{4.9}$$

and is positive otherwise, and the constant γ_2^R *is negative.*

When $\gamma_1^R < 0$, the effect of C_t on expected returns goes in the same direction as the effect on prices. For example, an increase in C_t not only lowers the prices of stocks that covary positively with the flow portfolio, but also lowers their subsequent expected returns. This seems paradoxical: given that C_t does not affect cash flows, shouldn't the drop in price be accompanied by an increase in expected return? The explanation is that while expected return decreases in the short run, it increases in the long run, in response to the gradual outflows triggered by the increase in C_t.

Figure 1 illustrates the dynamic behavior of fund flows and expected returns following a shock to C_t at time t. We assume that the shock is positive, and trace its effects for $t' > t$. We set the realized values of all shocks occurring subsequent to time t to zero: given the linearity of our model, this amounts to taking expectations over the future shocks. To better illustrate the main effects, we assume no mean-reversion in C_t, i.e., $\kappa = 0$. Thus, the shock to C_t generates an equal increase in $C_{t'}$ for all $t' > t$. We assume parameter values for which the constant γ_1^R of Corollary 4.1 is negative. The constant γ_2^R is also negative for these parameter values, a result which our numerical solutions suggest is general.

The solid line in Figure 1 plots the investor's holdings of the active fund, $y_{t'}$. Holdings decrease to a lower constant level, and the decrease happens gradually because of the adjustment cost. The dashed line in Figure 1 plots the instantaneous expected return $E(dR_{t'})/dt$ of a stock that covaries positively with the flow portfolio. Immediately following the increase in C_t, expected return decreases because $\gamma_1^R < 0$. Over time, however, as outflows occur, expected return increases. This is because the manager must be induced to absorb the outflows and buy the stock—an effect which can also be seen from Corollary 4.1 by noting that $y_{t'}$ decreases over time and $\gamma_2^R < 0$. The increase in expected return eventually overtakes the initial decrease, and the overall effect becomes

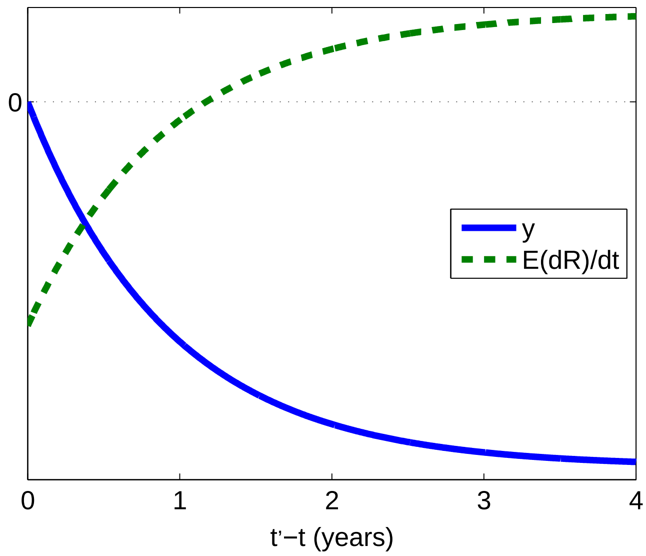

Figure 1: Effect of a positive shock to C_t on the investor's holdings of the active fund $y_{t'}$ (solid line) and on the instantaneous expected return $E(dR_{t'})/dt$ of a stock that covaries positively with the flow portfolio (dashed line) for $t' > t$. Time is measured in years. The figure is drawn for $(r, \kappa, \bar{\alpha}/\alpha, \psi/\alpha, \phi^2, \Delta/(\mathbf{1}\Sigma\mathbf{1}'), s^2, \lambda) = (0.04, 0, 4, 4, 0.1, 0.1, 1, 0)$. The equations describing the dynamics of $y_{t'}$ and $E(dR_{t'})/dt$ are derived in the proof of Corollary 4.2.

an increase. It is the long-run increase in expected return that causes the initial price drop at time t.

While Figure 1 reconciles the initial price drop with the behavior of expected return, it does not explain why expected return decreases in the short run. The latter effect is, in fact, puzzling: why is the manager willing to hold at time t—and even overweigh—a stock that is expected to underperform in the short run? The intuition is that the manager prefers to guarantee a "bird in the hand." Indeed, the anticipation of future outflows causes the stock to become underpriced at time t and offer an attractive return over a long horizon. The manager could earn an even more attractive return, on average, by buying the stock after the outflows occur. This, however, exposes him to the risk that the outflows might not occur, in which case the stock would cease to be underpriced. Thus, the manager might prefer to guarantee an attractive long-horizon return (bird in the hand), and pass up on the opportunity to exploit an uncertain short-run price drop. Note that in seeking to guarantee the long-horizon return, the manager is, in effect, causing the short-run drop. Indeed, the manager's buying pressure prevents the price at time t from dropping to a level that fully reflects the future outflows, i.e., from which a short-run drop is not expected.

The bird-in-the-hand effect can be seen formally in the manager's first-order condition (3.9), which in the case of gradual adjustment becomes

$$E_t(dR_t) = r\bar{\alpha}Cov_t(dR_t, \hat{z}_t dR_t) + (\bar{q}_1 + \bar{q}_{11}C_t + \bar{q}_{12}y_t)Cov_t(dR_t, dC_t). \tag{4.10}$$

Following an increase in C_t, the expected return of a stock that covaries positively with the flow portfolio decreases, lowering the left-hand side of (4.10). Therefore, the manager remains willing to hold the stock only if its risk, described by the right-hand side of (4.10), also decreases. The decrease in risk is not caused by a lower covariance between the stock and the manager's portfolio $\hat{z}_t$ (first term in the right-hand side). Indeed, since outflows are gradual, $\hat{z}_t$ remains constant immediately following the increase in C_t. The decrease in risk is instead driven by the manager's hedging demand (second term in the right-hand side), which means that a stock covarying positively with the flow portfolio becomes a better hedge for the manager when C_t increases. The intuition is that when C_t increases, mispricing becomes severe, and the manager has attractive investment opportunities. Hedging against a reduction in these opportunities requires holding stocks that perform well when C_t decreases, and these are the stocks covarying positively with the flow portfolio. Holding such stocks guarantees the manager an attractive long-horizon return—the bird-in-the-hand effect.

The manager's hedging demand is influenced not only by the bird-in-the-hand effect, but also by the concern with commercial risk (Corollary 3.7). The two effects work in opposite directions when $\lambda > 0$. Indeed, a stock covarying positively with the flow portfolio is a bad hedge for the manager because it performs poorly when C_t increases, which is also when outflows occur. Moreover, $\lambda > 0$ implies that the hedge tends to worsen when C_t increases because the manager becomes more concerned with future outflows. When λ is small, the bird-in-the-hand effect dominates the commercial-risk effect in influencing how the manager's hedging demand depends on C_t. Thus, when λ is small, changes in C_t impact prices and short-run expected returns in the same direction ($\gamma_1^R < 0$), as Corollary 4.1 confirms in the case of small s.[21]

The time-variation in expected returns implied by Corollary 4.1 gives rise to predictability. As in the case of instantaneous adjustment, the autocovariance matrix of returns is equal to the non-fundamental covariance matrix times a scalar. But while the scalar is negative for all lags under instantaneous adjustment, it can be positive for short lags under gradual adjustment.

Corollary 4.2 (Return Predictability) *The covariance between stock returns at time t and*

[21]Note that in a model with smart-money investors, sketched at the end of Section 2, λ would naturally be small: since these investors invest their own wealth, their hedging demand would be influenced only by the bird-in-the-hand effect.

those at time $t' > t$ is

$$Cov_t(dR_t, dR'_{t'}) = \left[\chi_1 e^{-\kappa(t'-t)} + \chi_2 e^{-b_2(t'-t)}\right] \Sigma p'_f p_f \Sigma (dt)^2, \tag{4.11}$$

where (χ_1, χ_2) are constants. For small s, the term in the square bracket of (4.11) is positive if $t' - t < \hat{u}$ and negative if $t' - t > \hat{u}$, for a threshold $\hat{u}$ which is positive if $\lambda < \lambda^R$ and zero if $\lambda > \lambda^R$. A stock's return predicts positively the stock's subsequent return for $t' - t < \hat{u}$ (short-run momentum) and negatively for $t' - t > \hat{u}$ (long-run reversal). It predicts in the same manner the subsequent return of another stock when the covariance between each stock in the pair and the flow portfolio has the same sign, and in the opposite manner otherwise.

When λ is small, stocks exhibit positive autocovariance for short lags and negative for long lags, i.e., short-run momentum and long-run reversal. This is because expected returns vary over time only in response to changes in C_t and the changes in y_t that these trigger. Moreover, changes in C_t move prices and short-run expected returns in the same direction, but long-run expected returns in the opposite direction. When instead λ is large, autocovariance is negative for all lags because changes in C_t move even short-run expected returns in the opposite direction to prices.[22] Lead-lag effects have the same sign as autocovariance for stock pairs whose covariance with the flow portfolio has the same sign. This is because changes in C_t influence both stocks in the same manner.

5 Asymmetric Information

This section treats the case of asymmetric information, where the investor does not observe the cost C_t and seeks to infer it from the returns and share prices of the index and active funds. Asymmetric information involves the additional complexity of having to solve for the investor's dynamic inference problem. Yet, this complexity does not come at the expense of tractability: the equilibrium has a similar formal structure and many properties in common with symmetric information. For example, the autocovariance and non-fundamental covariance matrices are identical to their symmetric-information counterparts up to multiplicative scalars.

We maintain the adjustment cost assumed in Section 4, and look for an equilibrium with the following characteristics. The investor's conditional distribution of C_t is normal with mean $\hat{C}_t$. The

[22] The result that stocks exhibit short-run momentum and long-run reversal when λ is small, but reversal for all lags when λ is large is consistent with the implication of Corollary 3.7 that an increase in λ increases the extent of reversal.

variance of the conditional distribution is, in general, a deterministic function of time, but we focus on a steady state where it is constant.[23] Stock prices take the form

$$S_t = \frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r+\kappa} - (a_0 + a_1\hat{C}_t + a_2 C_t + a_3 y_t), \tag{5.1}$$

where (a_0, a_1, a_2, a_3) are constant vectors. The conditional mean $\hat{C}_t$ becomes a state variable and affects prices because it determines the investor's target holdings of the active fund. The true value C_t, which is observed by the manager, also affects prices because it forecasts the investor's target holdings in the future. We conjecture that the effects of $(\hat{C}_t, C_t, y_t)$ on prices depend on the covariance with the flow portfolio, as is the case for (C_t, y_t) under symmetric information. That is, there exist constants $(\gamma_1, \gamma_2, \gamma_3)$ such that for $i = 1, 2, 3$,

$$a_i = \gamma_i \Sigma p_f'. \tag{5.2}$$

The investor's speed of adjustment $v_t \equiv dy_t/dt$ in our conjectured equilibrium is

$$v_t = b_0 - b_1\hat{C}_t - b_2 y_t, \tag{5.3}$$

where (b_0, b_1, b_2) are constants. Eq. (5.3) is identical to its symmetric-information counterpart (4.2), except that C_t is replaced by its mean $\hat{C}_t$. We refer to an equilibrium satisfying (5.1)-(5.3) as linear.

5.1 Investor's Inference

The investor seeks to infer the cost C_t from fund returns and share prices. The share prices of the index and active fund are $z_t S_t$ and $\mathbf{1}S_t$, respectively, and are informative about C_t because C_t affects the vector of stock prices S_t. Prices do not reveal C_t perfectly, however, because they also depend on the time-varying expected dividend F_t that the investor does not observe.

In addition to prices, the investor observes the net-of-cost return of the active fund, $z_t dR_t - C_t dt$, and the return of the index fund, $\mathbf{1}dR_t$. Because the investor observes prices, she also observes capital gains, and therefore can deduce net dividends (i.e., dividends minus C_t). Net dividends are the incremental information that returns provide to the investor.

In equilibrium, the active fund's portfolio z_t is equal to $\theta - x_t\mathbf{1}$. Since the investor knows x_t, observing the price and net dividends of the index and active funds is informationally equivalent

[23] The steady state is reached in the limit when time t becomes large.

to observing the price and net dividends of the index fund and of a hypothetical fund holding the residual-supply portfolio θ. Therefore, we can take the investor's information to be the net dividends of the residual-supply portfolio $\theta dD_t - C_t dt$, the dividends of the index fund $\mathbf{1}dD_t$, the price of the residual-supply portfolio θS_t, and the price of the index fund $\mathbf{1}S_t$.[24] We solve the investor's inference problem using recursive (Kalman) filtering.

Proposition 5.1 *The mean $\hat{C}_t$ of the investor's conditional distribution of C_t evolves according to the process*

$$
\begin{aligned}
d\hat{C}_t =& \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left\{ p_f \left[dD_t - E_t(dD_t) \right] - (C_t - \hat{C}_t)dt \right\} \\
& - \beta_2 p_f \left[dS_t + a_1 d\hat{C}_t + a_3 dy_t - E_t(dS_t + a_1 d\hat{C}_t + a_3 dy_t) \right], \qquad (5.4)
\end{aligned}
$$

where

$$\beta_1 \equiv T \left[1 - (r + k) \frac{\gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right] \frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta}, \qquad (5.5)$$

$$\beta_2 \equiv \frac{s^2 \gamma_2}{\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'}}, \qquad (5.6)$$

and T denotes the distribution's steady-state variance. The variance T is the unique positive solution of the quadratic equation

$$T^2 \left[1 - (r + \kappa) \frac{\gamma_2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'} \right]^2 \frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} + 2\kappa T - \frac{\frac{s^2 \phi^2}{(r+\kappa)^2}}{\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2 \gamma_2^2 \Delta}{\mathbf{1}\Sigma\mathbf{1}'}} = 0. \qquad (5.7)$$

The term in β_1 in (5.4) represents the investor's learning from net dividends. Recalling the definition (3.16) of the flow portfolio, we can write this term as

$$-\beta_1 \left\{ \theta dD_t - C_t dt - E_t \left(\theta dD_t - C_t dt \right) - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \left[\mathbf{1}dD_t - E_t(\mathbf{1}dD_t) \right] \right\}. \qquad (5.8)$$

The investor lowers her estimate of the cost C_t if the net dividends of the residual-supply portfolio $\theta dD_t - C_t dt$ are above expectations. Of course, net dividends can be high not only because C_t is

[24] We are assuming that the investor's information is the same in and out of equilibrium, i.e., the manager cannot manipulate the investor's beliefs by deviating from his equilibrium strategy and choosing a portfolio $z_t \neq \theta - x_t \mathbf{1}$. This is consistent with the assumption of a competitive manager. Indeed, one interpretation of this assumption is that there exists a continuum of managers, each with the same C_t. A deviation by one manager would then not affect the investors' beliefs about C_t because these would depend on averages across managers.

low, but also because gross dividends are high. The investor adjusts for this by comparing with the dividends $\mathbf{1}D_t$ of the index fund. The adjustment is made by computing the regression residual of $\theta dD_t - C_t dt$ on $\mathbf{1}D_t$, which is the term in curly brackets in (5.8).

The term in β_2 in (5.4) represents the investor's learning from prices. The investor lowers her estimate of C_t if the price of the residual-supply portfolio is above expectations. Indeed, the price can be high because the manager knows privately that C_t is low, and anticipates that the investor will increase her participation in the fund, causing the price to rise, as she learns about C_t. As with dividends, the investor needs to account for the fact that the price of the residual-supply portfolio can be high not only because C_t is low, but also because the manager expects future dividends to be high (F_t small). She adjusts for this by comparing with the price of the index fund. Note that if the expected dividend F_t is constant ($\phi = 0$), learning from prices is perfect: (5.7) implies that the conditional variance T is zero.

Because the investor compares the performance of the residual-supply portfolio, and hence of the active fund, to that of the index fund, she is effectively using the index as a benchmark. Note that benchmarking is not part of an explicit contract tying the manager's compensation to the index. Compensation is tied to the index only implicitly: if the active fund outperforms the index, the investor infers that C_t is low and increases her participation in the fund.

5.2 Optimization

The manager chooses controls $(\bar{c}_t, \bar{y}_t, z_t)$ to maximize the expected utility (2.2) subject to the budget constraint (3.3), the normalization (3.6), and the investor's holding policy (5.3). Since stock prices depend on $(\hat{C}_t, C_t, y_t)$, the same is true for the manager's value function. We conjecture that the value function is

$$\bar{V}(W_t, \bar{X}_t) \equiv -\exp\left[-\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2, \bar{q}_3)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right)\right], \tag{5.9}$$

where $\bar{X}_t \equiv (\hat{C}_t, C_t, y_t)'$, $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)$ are constants, and $\bar{Q}$ is a constant symmetric 3×3 matrix.

Proposition 5.2 *The value function (5.9) satisfies the Bellman equation (3.8) if* $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q})$ *satisfy a system of ten scalar equations.*

The investor chooses controls (c_t, x_t, v_t) to maximize the expected utility (2.1) subject to the budget constraint (4.4) and the manager's portfolio policy $z_t = \theta - x_t \mathbf{1}$. As in the case of symmetric information, we study this optimization problem in two steps: first optimize over (c_t, x_t), assuming that v_t is given by (5.3), and then derive conditions under which (5.3) is optimal. We solve the first problem using dynamic programming, and conjecture the value function (4.5), where $X_t \equiv (\hat{C}_t, y_t)'$, (q_0, q_1, q_2) are constants, and Q is a constant symmetric 2×2 matrix.

Proposition 5.3 *The value function (4.5) satisfies the Bellman equation (4.6) if* (q_0, q_1, q_2, Q) *satisfy a system of six scalar equations. The control* v_t *given by (5.3) is optimal if* (b_0, b_1, b_2) *satisfy a system of three scalar equations.*

5.3 Equilibrium

Proposition 5.4 shows that a unique linear equilibrium exists when the diffusion coefficient s of C_t is small. Our numerical solutions for general values of s seem to generate a unique linear equilibrium, with properties similar to those derived in the rest of this section for small s.[25]

Proposition 5.4 *For small s, there exists a unique linear equilibrium. The constants* (b_1, b_2, γ_1) *are positive, and the constant γ_3 is negative. The constant γ_2 is positive if* $\lambda \geq 0$.

When information is asymmetric, cashflow news affect the investor's estimate of the cost C_t, and so trigger fund flows. These flows, in turn, impact stock returns. We refer to the effect that cashflow news have on returns through fund flows as an indirect effect, to distinguish from the direct effect computed by holding flows constant. To illustrate the two effects, consider the dividend shock dD_t at time t. The shock's direct effect is to add dD_t to returns $dR_t = dD_t + dS_t - rS_t dt$. The shock's indirect effect is to trigger fund flows which impact returns dR_t through the price change dS_t. Eqs. (5.1), (5.2) and (5.4) imply that the indirect effect is $\beta_1 \gamma_1 \Sigma p_f' p_f dD_t$.

The indirect effect amplifies the direct effect. Suppose, for example, that a stock experiences a negative cashflow shock. If the stock is in large residual supply, and so overweighed by the active fund, then the shock lowers the return of the active fund more than of the index fund. As a consequence, the investor infers that C_t has increased, and flows out of the active and into the index fund. Since the active fund overweighs the stock, the investor's flows cause the stock to be

[25]This applies to $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 < 0$, and to Corollaries 5.1, 5.2 and 5.3.

sold and push its price down. Conversely, if the stock is in small residual supply, then the investor infers that C_t has decreased, and flows out of the index and into the active fund. Since the active fund underweighs the stock, the investor's flows cause again the stock to be sold and push its price down. Thus, in both cases, fund flows amplify the direct effect that the cashflow shock has on returns.

Amplification is related to comovement. Recall that under symmetric information fund flows generate comovement between a pair of stocks because they affect the expected return of each stock in the pair. This channel of comovement, to which we refer as ER/ER (where ER stands for expected return) is also present under asymmetric information. Asymmetric information introduces an additional channel involving fund flows, to which we refer as CF/ER (where CF stands for cashflow). This is that cashflow news of one stock in a pair trigger fund flows which affect the expected return of the other stock. The CF/ER channel is the one related to amplification.

While the ER/ER and CF/ER channels are conceptually distinct, their effects are formally similar: the covariance matrix generated by CF/ER is equal to that generated by ER/ER times a positive scalar (Corollary 5.1). Thus, if ER/ER generates a positive covariance between a pair of stocks, so does CF/ER, and if the former covariance is large, so is the latter. Consider, for example, two stocks that are in large residual supply. Since outflows from the active fund (triggered by, e.g., a cashflow shock to a third stock) push down the prices of both stocks, ER/ER generates a positive covariance. Moreover, since a negative cashflow shock to one stock triggers outflows from the active fund and this pushes down the price of the other stock, CF/ER also generates a positive covariance. The former covariance is large if the two stocks have high idiosyncratic risk since this makes them more sensitive to fund flows. But high idiosyncratic risk also renders the latter covariance large: cashflow shocks to stocks having low correlation with the index generate a large discrepancy between the active and the index return, hence triggering large fund flows.

Corollary 5.1 computes the covariance matrix of stock returns. The fundamental covariance is identical to that under symmetric information, while the non-fundamental covariance is proportional. The intuition for proportionality is that the covariance matrices generated by ER/ER and CF/ER are proportional, the non-fundamental covariance under symmetric information is generated by ER/ER, and that under asymmetric information is generated by ER/ER and CF/ER. Corollary 5.1 shows, in addition, that for small s the non-fundamental covariance matrix is larger under asymmetric information, i.e., the proportionality coefficient with the symmetric-information matrix is larger than one. This result, which our numerical solutions suggest is general, implies that the non-fundamental volatility of each stock is larger under asymmetric information, and so is the absolute value of the non-fundamental covariance between any pair of stocks. Intuitively, these quantities are larger under asymmetric information because the amplification channel CF/ER is present only in that case.

Corollary 5.1 (Comovement and Amplification) *The covariance matrix of stock returns is*

$$Cov_t(dR_t, dR_t') = \left(f\Sigma + k\Sigma p_f' p_f \Sigma\right) dt, \tag{5.10}$$

where k is a positive constant. The fundamental covariance is identical to that under symmetric information, while the non-fundamental covariance is proportional. Moreover, for small s, the proportionality coefficient is larger than one.

The cross section of expected returns is explained by the same two factors as under symmetric information.

Corollary 5.2 (Expected Returns) *Stocks' expected returns are given by the two-factor model (3.21), with the factors being the market and the flow portfolio. The factor risk premium Λ_t associated to the flow portfolio is*

$$\Lambda_t = r\bar{\alpha} + \frac{1}{f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} \left(\gamma_1^R \hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - k_1\bar{q}_1 - k_2\bar{q}_2\right), \tag{5.11}$$

where $(\gamma_1^R, \gamma_2^R, \gamma_3^R, k_1, k_2)$ are constants. For small s, the constants (γ_1^R, γ_3^R) are negative and the constant γ_2^R has the same sign as λ.

Using Corollary 5.2, we can examine how expected returns respond to shocks. Consider a cashflow shock, which we assume is negative and hits a stock in large residual supply. The shock raises $\hat{C}_t$, the investor's estimate of C_t. The increase in $\hat{C}_t$ lowers the prices of stocks covarying positively with the flow portfolio (including the stock hit by the cashflow shock) since $\gamma_1 > 0$, and lowers the subsequent expected returns of these stocks since $\gamma_1^R < 0$. The simultaneous decrease in prices and expected returns is consistent because expected returns increase in the long run. Expected returns decrease in the short run because of the bird-in-the-hand effect.

The time-variation in expected returns following cashflow shocks can be characterized in terms of the covariance between cashflow shocks and subsequent returns. Corollary 5.3 computes the covariance between the vectors (dD_t, dF_t) of cashflow shocks at time t and the vector of returns at time $t' > t$. Both covariance matrices are equal to the non-fundamental covariance matrix times a scalar which is positive for short lags and negative for long lags. Thus, cashflow shocks generate short-run momentum and long-run reversal in returns, consistent with the discussion in the previous paragraph. Note that predictability based on cashflows arises only under asymmetric information because only then cashflow shocks trigger fund flows.

Corollary 5.3 (Return Predictability Based on Cashflows) *The covariance between cashflow shocks (dD_t, dF_t) at time t and returns at time $t' > t$ is given by*

$$Cov_t(dD_t, dR'_{t'}) = \frac{\beta_1(r+\kappa)Cov_t(dF_t, dR'_{t'})}{\beta_2\phi^2} = \left[\chi_1^D e^{-(\kappa+\rho)(t'-t)} + \chi_2^D e^{-b_2(t'-t)}\right] \Sigma p'_f p_f \Sigma (dt)^2, \tag{5.12}$$

where (χ_1^D, χ_2^D) are constants. For small s, the term in the square bracket of (4.11) is positive if $t'-t < \hat{u}^D$ and negative if $t'-t > \hat{u}^D$, for a threshold $\hat{u}^D > 0$. A stock's cashflow shocks predict positively the stock's subsequent return for $t'-t < \hat{u}^D$ (short-run momentum) and negatively for $t'-t > \hat{u}^D$ (long-run reversal). They predict in the same manner the subsequent return of another stock when the covariance between each stock in the pair and the flow portfolio has the same sign, and in the opposite manner otherwise.

We finally examine predictability based on past returns rather than cashflows. This predictability is driven both by cashflow shocks and by shocks to C_t. Predictability based on past returns has the same form as under symmetric information (Corollary 4.2), except that short-run momentum arises even for large λ.[26]

Corollary 5.4 (Return Predictability) *The covariance between stock returns at time t and those at time $t' > t$ is*

$$Cov_t(dR_t, dR'_{t'}) = \left[\chi_1 e^{-(\kappa+\rho)(t'-t)} + \chi_2 e^{-\kappa(t'-t)} + \chi_3 e^{-b_2(t'-t)}\right] \Sigma p'_f p_f \Sigma (dt)^2, \tag{5.13}$$

[26] The latter result relies on the assumption that s is small. Recall that when information is symmetric, short-run momentum does not arise for large λ because of commercial risk. Indeed, an increase in C_t lowers the prices of stocks covarying positively with the flow portfolio because of the anticipation of future outflows from the active fund. Moreover, the subsequent expected returns of these shocks increase, even in the short run, because the manager becomes more concerned with commercial risk (and this effect dominates the bird-in-the-hand effect for large λ). Both effects are also present when information is asymmetric. Under asymmetric information, however, predictability is driven not only by shocks to C_t but also by cashflow shocks. Moreover, the latter have a dominating effect when shocks to C_t have small variance (small s). Indeed, for small s, shocks to C_t are not only small but also trigger a small price reaction holding size constant. This is because the price reaction is driven by the anticipation of future flows as the investor learns about C_t, and learning is limited for small s.

where $(\chi_1, \chi_2, \chi_3, \rho)$ *are constants. For* $\lambda \geq 0$ *and small* s*, the term in the square bracket of (5.13) is positive if* $t' - t < \hat{u}$ *and negative if* $t' - t > \hat{u}$*, for a threshold* $\hat{u} > 0$*. Given* $\hat{u}$*, predictability is as in Corollary 4.2.*

6 Conclusion

This paper proposes a rational theory of momentum and reversal based on delegated portfolio management. Momentum arises because prices do not fully adjust to reflect future fund flows, and reversal arises because these flows push prices away from fundamental values. Besides momentum and reversal, fund flows generate comovement, lead-lag effects and amplification, with all effects being larger for assets with high idiosyncratic risk. Moreover, managers' concern with commercial risk makes prices more volatile. Our model provides a parsimonious and tractable framework to study the price effects of fund flows.

In focusing on flows between investment funds as a driver of momentum, we do not intend to suggest that they are the only driver. Indeed, momentum could be also generated by gradual and anticipated changes in leverage or irrational sentiment. At the same time, flows between investment funds seem to be a relevant driver of momentum as recent empirical findings indicate, and can be modeled in a manner that might be more tractable than alternatives. Moreover, the basic intuitions identified in this paper, e.g., momentum is driven by the bird-in-the-hand effect and is possible only because of the subsequent reversal, seem general and could carry over to other settings.

Our emphasis in this paper is to develop a framework that allows for a general analysis of the price effects of fund flows. An important next step, left for future work, is to examine more systematically the empirical implications of our analysis, both to confront existing empirical facts and to suggest new tests. For example, is momentum larger for individual assets or asset classes? Are momentum winners correlated and is there a momentum factor? If so, how do momentum and value factors correlate?

Appendix

A Symmetric Information

Proof of Proposition 3.1: Eqs. (2.3), (3.2), (3.3) and (3.4) imply that

$$d\left(r\bar{\alpha}W_t+\bar{q}_0+\bar{q}_1C_t+\frac{1}{2}\bar{q}_{11}C_t^2\right)=\bar{G}dt+r\bar{\alpha}\hat{z}_t\sigma\left(dB_t^D+\frac{\phi dB_t^F}{r+\kappa}\right)-s\left[r\bar{\alpha}\hat{z}_ta_1-\bar{f}_1(C_t)\right]dB_t^C, \quad (A.1)$$

where

$$\begin{aligned}\bar{G}\equiv & r\bar{\alpha}\left\{rW_t+\hat{z}_t\left[ra_0+(r+\kappa)a_1C_t-\kappa a_1\bar{C}\right]+(\lambda C_t+B)(b_0-b_1C_t)-\bar{c}_t\right\}\\ & +\bar{f}_1(C_t)\kappa(\bar{C}-C_t)+\frac{1}{2}s^2\bar{q}_{11},\end{aligned}$$

$$\bar{f}_1(C_t)\equiv\bar{q}_1+\bar{q}_{11}C_t.$$

Eqs. (3.7) and (A.1) imply that

$$\mathcal{D}\bar{V}=-\bar{V}\left\{\bar{G}-\frac{1}{2}(r\bar{\alpha})^2f\hat{z}_t\Sigma\hat{z}_t'-\frac{1}{2}s^2\left[r\bar{\alpha}\hat{z}_ta_1-\bar{f}_1(C_t)\right]^2\right\}. \quad (A.2)$$

Substituting (A.2) into (3.8), we can write the first-order conditions with respect to $\bar{c}_t$ and $\hat{z}_t$ as

$$\bar{\alpha}\exp(-\bar{\alpha}\bar{c}_t)+r\bar{\alpha}\bar{V}=0, \quad (A.3)$$

$$\bar{h}(C_t)=r\bar{\alpha}(f\Sigma+s^2a_1a_1')\hat{z}_t', \quad (A.4)$$

respectively, where

$$\bar{h}(C_t)\equiv ra_0+(r+\kappa)a_1C_t-\kappa a_1\bar{C}+s^2a_1\bar{f}_1(C_t). \quad (A.5)$$

Eq. (A.4) is equivalent to (3.9) because of (2.3), (3.4) and (3.5). Using (A.2) and (A.3), we can simplify (3.8) to

$$\bar{G}-\frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma+s^2a_1a_1')\hat{z}_t'+r\bar{\alpha}s^2\hat{z}_ta_1\bar{f}_1(C_t)-\frac{1}{2}s^2\bar{f}_1(C_t)^2+\bar{\beta}-r=0. \quad (A.6)$$

Eqs. (3.7) and (A.3) imply that

$$\bar{c}_t=rW_t+\frac{1}{\bar{\alpha}}\left[\bar{q}_0+\bar{q}_1C_t+\frac{1}{2}\bar{q}_{11}C_t^2-\log(r)\right]. \quad (A.7)$$

Substituting (A.7) into (A.6) the terms in W_t cancel, and we are left with

$$r\bar{\alpha}\hat{z}_t\left[ra_0 + (r+\kappa)a_1C_t - \kappa a_1\bar{C}\right] + r\bar{\alpha}(\lambda C_t + B)(b_0 - b_1C_t) - r\left(\bar{q}_0 + \bar{q}_1C_t + \frac{1}{2}\bar{q}_{11}C_t^2\right)$$
$$+ \bar{f}_1(C_t)\kappa(\bar{C} - C_t) + \frac{1}{2}s^2\bar{q}_{11} + \bar{\beta} - r + r\log(r)$$
$$- \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma + s^2a_1a_1')\hat{z}_t' + r\bar{\alpha}s^2\hat{z}_ta_1\bar{f}_1(C_t) - \frac{1}{2}s^2\bar{f}_1(C_t)^2 = 0. \tag{A.8}$$

The terms in (A.8) that involve $\hat{z}_t$ can be written as

$$r\bar{\alpha}\hat{z}_t\left[ra_0 + (r+\kappa)a_1C_t - \kappa a_1\bar{C}\right] - \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma + s^2a_1a_1')\hat{z}_t' + r\bar{\alpha}s^2\hat{z}_ta_1\bar{f}_1(C_t)$$
$$= r\bar{\alpha}\hat{z}_t\bar{h}(C_t) - \frac{1}{2}(r\bar{\alpha})^2\hat{z}_t(f\Sigma + s^2a_1a_1')\hat{z}_t'$$
$$= \frac{1}{2}r\bar{\alpha}\hat{z}_t\bar{h}(C_t)$$
$$= \frac{1}{2}\bar{h}(C_t)'(f\Sigma + s^2a_1a_1')^{-1}\bar{h}(C_t), \tag{A.9}$$

where the first step follows from (A.5) and the last two from (A.4). Substituting (A.9) into (A.8), we find

$$\frac{1}{2}\bar{h}(C_t)'(f\Sigma + s^2a_1a_1')^{-1}\bar{h}(C_t) + r\bar{\alpha}(\lambda C_t + B)(b_0 - b_1C_t) - r\left(\bar{q}_0 + \bar{q}_1C_t + \frac{1}{2}\bar{q}_{11}C_t^2\right)$$
$$+ \bar{f}_1(C_t)\kappa(\bar{C} - C_t) + \frac{1}{2}s^2\left[\bar{q}_{11} - \bar{f}_1(C_t)^2\right] + \bar{\beta} - r + r\log(r) = 0. \tag{A.10}$$

Eq. (A.10) is quadratic in C_t. Identifying terms in C_t^2, C_t, and constants, yields three scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_{11})$. We defer the derivation of these equations until the proof of Proposition 3.3 (see (A.40) and (A.41)). ∎

Proof of Proposition 3.2: Eqs. (2.3), (3.4) and (3.10) imply that

$$d\left(r\alpha W_t + q_0 + q_1C_t + \frac{1}{2}q_{11}C_t^2\right) = Gdt + r\alpha(x_t\mathbf{1} + y_tz_t)\sigma\left(dB_t^D + \frac{\phi dB_t^F}{r+\kappa}\right)$$
$$- s\left[r\alpha(x_t\mathbf{1} + y_tz_t)a_1 - f_1(C_t)\right]dB_t^C, \tag{A.11}$$

where

$$G \equiv r\alpha\left\{rW_t + (x_t\mathbf{1} + y_tz_t)\left[ra_0 + (r+\kappa)a_1C_t - \kappa a_1\bar{C}\right] - y_tC_t - c_t\right\} + f_1(C_t)\kappa(\bar{C} - C_t) + \frac{1}{2}s^2q_{11},$$

$$f_1(C_t) \equiv q_1 + q_{11}C_t.$$

Eqs. (3.11) and (A.11) imply that

$$\mathcal{D}V = -V\left\{G - \frac{1}{2}(r\alpha)^2 f(x_t\mathbf{1} + y_t z_t)\Sigma(x_t\mathbf{1} + y_t z_t)' - \frac{1}{2}s^2\left[r\alpha(x_t\mathbf{1} + y_t z_t)a_1 - f_1(C_t)\right]^2\right\}. \quad \text{(A.12)}$$

Substituting (A.12) into (3.12), we can write the first-order conditions with respect to c_t, x_t and y_t as

$$\alpha\exp(-\alpha c_t) + r\alpha V = 0, \quad \text{(A.13)}$$

$$\mathbf{1}h(C_t) = r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)', \quad \text{(A.14)}$$

$$z_t h(C_t) - C_t = r\alpha z_t(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)', \quad \text{(A.15)}$$

respectively, where

$$h(C_t) \equiv ra_0 + (r+\kappa)a_1 C_t - \kappa a_1\bar{C} + s^2 a_1 f_1(C_t). \quad \text{(A.16)}$$

Eqs. (A.14) and (A.15) are equivalent to (3.13) and (3.14) because of (2.3), (3.4) and (3.5). Solving for c_t, and proceeding as in the proof of Proposition 3.1, we can simplify (3.12) to

$$\begin{aligned}
&r\alpha(x_t\mathbf{1} + y_t z_t)\left[ra_0 + (r+\kappa)a_1 C_t - \kappa a_1\bar{C}\right] - r\alpha y_t C_t - r\left(q_0 + q_1 C_t + \frac{1}{2}q_{11}C_t^2\right) \\
&+ f_1(C_t)\kappa(\bar{C} - C_t) + \frac{1}{2}s^2 q_{11} + \beta - r + r\log(r) \\
&- \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)' + r\alpha s^2(x_t\mathbf{1} + y_t z_t)a_1 f_1(C_t) - \frac{1}{2}s^2 f_1(C_t)^2 = 0.
\end{aligned} \quad \text{(A.17)}$$

Eq. (A.17) is the counterpart of (A.8) for the investor. The terms in (A.17) that involve (x_t, y_t) can be written as

$$\begin{aligned}
&r\alpha(x_t\mathbf{1} + y_t z_t)\left[ra_0 + (r+\kappa)a_1 C_t - \kappa a_1\bar{C}\right] - r\alpha y_t C_t \\
&- \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)' + r\alpha s^2(x_t\mathbf{1} + y_t z_t)a_1 f_1(C_t) \\
&= r\alpha(x_t\mathbf{1} + y_t z_t)h(C_t) - r\alpha y_t C_t - \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)' \\
&= \frac{1}{2}r\alpha(x_t\mathbf{1} + y_t z_t)h(C_t) - \frac{1}{2}r\alpha y_t C_t,
\end{aligned} \quad \text{(A.18)}$$

where the first step follows from (A.16) and the second from

$$(x_t\mathbf{1} + y_t z_t)h(C_t) - y_t C_t = r\alpha(x_t\mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)', \quad \text{(A.19)}$$

which in turn follows by multiplying (A.14) by x_t, (A.15) by y_t, and adding up. To eliminate x_t and y_t in (A.18), we use (A.14) and (A.15). Noting that in equilibrium $z_t = \theta - x_t\mathbf{1}$, we can write (A.14) as

$$\mathbf{1}h(C_t) = r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')\left[x_t(1-y_t)\mathbf{1} + y_t\theta\right]'. \tag{A.20}$$

Multiplying (A.14) by x_t and adding to (A.15), we similarly find

$$\theta h(C_t) - C_t = r\alpha\theta(f\Sigma + s^2 a_1 a_1')\left[x_t(1-y_t)\mathbf{1} + y_t\theta\right]'. \tag{A.21}$$

Eqs. (A.20) and (A.21) form a linear system in $x_t(1-y_t)$ and y_t. Solving the system, we find

$$x_t(1-y_t) = \frac{1}{r\alpha D}\left\{\mathbf{1}h(C_t)\theta(f\Sigma + s^2 a_1 a_1')\theta' - \left[\theta h(C_t) - C_t\right]\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'\right\}, \tag{A.22}$$

$$y_t = \frac{1}{r\alpha D}\left\{\left[\theta h(C_t) - C_t\right]\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1'} - \mathbf{1}h(C_t)\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'\right\}, \tag{A.23}$$

where

$$D \equiv \theta(f\Sigma + s^2 a_1 a_1')\theta'\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1'} - \left[\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'\right]^2.$$

Eq. (A.23) implies that the optimal control y_t is linear in C_t. Using (A.22) and (A.23), we can write (A.18) as

$$\begin{aligned}
&\frac{1}{2}r\alpha(x_t\mathbf{1} + y_t z_t)h(C_t) - \frac{1}{2}r\alpha y_t C_t \\
&= \frac{1}{2}r\alpha\left[x_t\mathbf{1} + y_t(\theta - x_t\mathbf{1})\right]h(C_t) - \frac{1}{2}r\alpha y_t C_t \\
&= \frac{1}{2D}\Big\{\left[\mathbf{1}h(C_t)\right]^2\theta(f\Sigma + s^2 a_1 a_1')\theta' - 2\left[\theta h(C_t) - C_t\right]\mathbf{1}h(C_t)\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta' \\
&+ \left[\theta h(C_t) - C_t\right]^2\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1'}\Big\}.
\end{aligned} \tag{A.24}$$

Substituting (A.24) into (A.17), we find

$$\begin{aligned}
&\frac{1}{2D}\Big\{\left[\mathbf{1}h(C_t)\right]^2\theta(f\Sigma + s^2 a_1 a_1')\theta' - 2\left[\theta h(C_t) - C_t\right]\mathbf{1}h(C_t)\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta' \\
&+ \left[\theta h(C_t) - C_t\right]^2\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1'}\Big\} - r\left(q_0 + q_1 C_t + \frac{1}{2}q_{11}C_t^2\right) \\
&+ f_1(C_t)\kappa(\bar{C} - C_t) + \frac{1}{2}s^2\left[q_{11} - f_1(C_t)^2\right] + \beta - r + r\log(r) = 0.
\end{aligned} \tag{A.25}$$

Eq. (A.25) is quadratic in C_t. Identifying terms in C_t^2, C_t, and constants, yields three scalar equations in (q_0, q_1, q_{11}). We defer the derivation of these equations until the proof of Proposition 3.3 (see (A.44) and (A.45)). ∎

Proof of Proposition 3.3: We first impose market clearing and derive the constants (a_0, a_1, b_0, b_1) as functions of $(\bar{q}_1, \bar{q}_{11}, q_1, q_{11})$. For these derivations, as well as for later proofs, we use the following properties of the flow portfolio:

$$\mathbf{1}\Sigma p_f' = 0,$$

$$\theta\Sigma p_f' = p_f\Sigma p_f' = \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}.$$

Setting $z_t = \theta - x_t\mathbf{1}$ and $\bar{y}_t = 1 - y_t$, we can write (A.4) as

$$\bar{h}(C_t) = r\bar{\alpha}(f\Sigma + s^2 a_1 a_1')(1 - y_t)(\theta - x_t\mathbf{1})'. \tag{A.26}$$

Premultiplying (A.26) by $\mathbf{1}$, dividing by $r\bar{\alpha}$, and adding to (A.20) divided by $r\alpha$, we find

$$\mathbf{1}\left[\frac{h(C_t)}{r\alpha} + \frac{\bar{h}(C_t)}{r\bar{\alpha}}\right] = \mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'. \tag{A.27}$$

Eq. (A.27) is linear in C_t. Identifying terms in C_t, we find

$$\left(\frac{r + \kappa + s^2 q_{11}}{r\alpha} + \frac{r + \kappa + s^2\bar{q}_{11}}{r\bar{\alpha}}\right)\mathbf{1}a_1 = 0 \Rightarrow \mathbf{1}a_1 = 0. \tag{A.28}$$

Identifying constant terms, and using (A.28), we find

$$\mathbf{1}a_0 = \frac{\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}}\mathbf{1}\Sigma\theta'. \tag{A.29}$$

Substituting (A.28) and (A.29) into (A.20), we find

$$\frac{r\alpha\bar{\alpha}f}{\alpha + \bar{\alpha}}\mathbf{1}\Sigma\theta' = r\alpha f\mathbf{1}\Sigma\left[x_t(1 - y_t)\mathbf{1} + y_t\theta\right]' \Rightarrow x_t = \frac{\frac{\bar{\alpha}}{\alpha+\bar{\alpha}} - y_t}{1 - y_t}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}. \tag{A.30}$$

Substituting (A.30) into (A.26), we find

$$\begin{aligned}\bar{h}(C_t) &= r\bar{\alpha}(f\Sigma + s^2 a_1 a_1')\left[\frac{\alpha}{\alpha + \bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1} + (1 - y_t)p_f\right]' \\ &= r\bar{\alpha}(f\Sigma + s^2 a_1 a_1')\left[\frac{\alpha}{\alpha + \bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1} + (1 - b_0 + b_1 C_t)p_f\right]',\end{aligned} \tag{A.31}$$

where the second step follows from (3.2). Eq. (A.31) is linear in C_t. Identifying terms in C_t, we find

$$(r+\kappa+s^2\bar{q}_{11})a_1 = r\bar{\alpha}b_1\left(f\Sigma p_f' + s^2 a_1' p_f' a_1\right). \tag{A.32}$$

Therefore, a_1 is collinear to the vector $\Sigma p_f'$, as in (3.15). Substituting (3.15) into (A.32), we find

$$(r+\kappa+s^2\bar{q}_{11})\gamma_1 = r\bar{\alpha}b_1\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right). \tag{A.33}$$

Identifying constant terms in (A.31), and using (3.15), we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\Sigma\mathbf{1}' + \left[\frac{\gamma_1(\kappa\bar{C}-s^2\bar{q}_1)}{r} + \bar{\alpha}(1-b_0)\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\right]\Sigma p_f'. \tag{A.34}$$

Using (3.2) and (A.30), we can write (A.21) as

$$\begin{aligned}\theta h(C_t) - C_t &= r\alpha\theta(f\Sigma + s^2 a_1 a_1')\left[\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1} + (b_0 - b_1 C_t)p_f\right]' \\ &= \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{(\mathbf{1}\Sigma\theta')^2}{\mathbf{1}\Sigma\mathbf{1}'} + r\alpha(b_0-b_1C_t)\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'},\end{aligned} \tag{A.35}$$

where the second step follows from (3.15). Eq. (A.35) is linear in C_t. Identifying terms in C_t, and using (3.15), we find

$$(r+\kappa+s^2q_{11})\frac{\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - 1 = -r\alpha b_1\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}. \tag{A.36}$$

Identifying constant terms, and using (3.15) and (A.34), we find

$$b_0 = \frac{\bar{\alpha}}{\alpha+\bar{\alpha}} + \frac{s^2\gamma_1(q_1-\bar{q}_1)}{r(\alpha+\bar{\alpha})\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)}. \tag{A.37}$$

Substituting b_0 from (A.37) into (A.34), we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\Sigma\mathbf{1}' + \left[\frac{\gamma_1\kappa\bar{C}}{r} - \frac{s^2\gamma_1(\alpha\bar{q}_1+\bar{\alpha}q_1)}{r(\alpha+\bar{\alpha})} + \frac{\alpha\bar{\alpha}\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)}{\alpha+\bar{\alpha}}\right]\Sigma p_f'. \tag{A.38}$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, a_1, b_0, b_1, \gamma_1, \bar{q}_0, \bar{q}_1, \bar{q}_{11}, q_0, q_1, q_{11})$. The equations linking them are (3.15), (A.33), (A.36), (A.37), (A.38), the three equations derived from (A.10) by identifying terms in C_t^2, C_t, and constants, and the three equations derived from (A.25) through the same procedure. To simplify the system, we note that the variables $(\bar{q}_0, q_0)$ enter only in the equations derived from (A.10) and (A.25) by identifying constants. Therefore they can be determined separately, and we need to consider only the equations derived from (A.10) and (A.25) by identifying linear and quadratic terms. We next simplify these equations, using implications of market clearing.

Using (A.31), we find

$$
\begin{aligned}
&\frac{1}{2}\bar{h}(C_t)'(f\Sigma + s^2 a_1 a_1')^{-1}\bar{h}(C_t) \\
&= \frac{r^2\alpha^2\bar{\alpha}^2 f(\mathbf{1}\Sigma\theta')^2}{2(\alpha+\bar{\alpha})^2\mathbf{1}\Sigma\mathbf{1}'} + \frac{1}{2}r^2\bar{\alpha}^2(1-b_0+b_1C_t)^2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\
&= \frac{r^2\alpha^2\bar{\alpha}^2 f(\mathbf{1}\Sigma\theta')^2}{2(\alpha+\bar{\alpha})^2\mathbf{1}\Sigma\mathbf{1}'} + \frac{1}{2}r^2\bar{\alpha}^2\left[\frac{\alpha}{\alpha+\bar{\alpha}} + \frac{s^2\gamma_1(\bar{q}_1-q_1)}{r(\alpha+\bar{\alpha})\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)} + b_1C_t\right]^2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'},
\end{aligned}
\tag{A.39}
$$

where the second step follows from (A.37). Substituting (A.39) into (A.10), and identifying terms in C_t^2 and C_t, we find

$$
(r+2\kappa)\bar{q}_{11} + s^2\bar{q}_{11}^2 - r^2\bar{\alpha}^2 b_1^2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} + r\bar{\alpha}\lambda b_1 = 0, \tag{A.40}
$$

$$
(r+\kappa)\bar{q}_1 + s^2\bar{q}_1\bar{q}_{11} - r\bar{\alpha}b_1\left[\frac{r\alpha\bar{\alpha}\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)}{\alpha+\bar{\alpha}} + \frac{\bar{\alpha}s^2\gamma_1(\bar{q}_1-q_1)}{\alpha+\bar{\alpha}}\right]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - \kappa\bar{C}\bar{q}_{11} + r\bar{\alpha}(Bb_1-\lambda b_0) = 0, \tag{A.41}
$$

respectively. Using (3.15) and (A.30), we can write (A.20) as

$$
\mathbf{1}h(C_t) = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\mathbf{1}\Sigma\theta'. \tag{A.42}
$$

Eq. (3.15) implies that the denominator D in (A.25) is

$$
D = f\Delta\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right). \tag{A.43}
$$

Using (3.15), (A.35), (A.37), (A.42) and (A.43), we find that the equations derived from (A.25) by identifying terms in C_t^2 and C_t are

$$(r+2\kappa)q_{11} + s^2 q_{11}^2 - r^2\alpha^2 b_1^2 \left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}\right)\frac{\Delta}{\mathbf{1\Sigma 1'}} = 0, \tag{A.44}$$

$$(r+\kappa)q_1 + s^2 q_1 q_{11} + r\alpha b_1 \left[\frac{r\alpha\bar{\alpha}\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}\right)}{\alpha+\bar{\alpha}} + \frac{\alpha s^2\gamma_1(q_1-\bar{q}_1)}{\alpha+\bar{\alpha}}\right]\frac{\Delta}{\mathbf{1\Sigma 1'}} - \kappa\bar{C}q_{11} = 0, \tag{A.45}$$

respectively.

Solving for equilibrium amounts to solving the system of (3.15), (A.33), (A.36), (A.37), (A.38), (A.40), (A.41), (A.44) and (A.45) in the unknowns $(a_0, a_1, b_0, b_1, \gamma_1, \bar{q}_1, \bar{q}_{11}, q_1, q_{11})$. This reduces to solving the system of (A.33), (A.36), (A.40) and (A.44) in the unknowns $(b_1, \gamma_1, \bar{q}_{11}, q_{11})$: given $(b_1, \gamma_1, \bar{q}_{11}, q_{11})$, a_1 can be determined from (3.15), $(\bar{q}_1, q_1)$ from the linear system of (A.41) and (A.45), and (a_0, b_0) from (A.38) and (A.37). Replacing the unknown b_1 by

$$\hat{b}_1 \equiv r\bar{\alpha}b_1\sqrt{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}},$$

we can write the system of (A.33), (A.36), (A.40) and (A.44) as

$$(r+\kappa+s^2\bar{q}_{11})\gamma_1 = \hat{b}_1\sqrt{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}}, \tag{A.46}$$

$$\frac{r+\kappa+s^2 q_{11}}{r\alpha}\frac{\gamma_1\Delta}{\mathbf{1\Sigma 1'}} + \frac{\hat{b}_1\sqrt{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}}}{r\bar{\alpha}}\frac{\Delta}{\mathbf{1\Sigma 1'}} = \frac{1}{r\alpha}, \tag{A.47}$$

$$(r+2\kappa)\bar{q}_{11} + s^2\bar{q}_{11}^2 - \frac{\hat{b}_1^2\Delta}{\mathbf{1\Sigma 1'}} + \frac{\lambda\hat{b}_1}{\sqrt{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}}} = 0, \tag{A.48}$$

$$(r+2\kappa)q_{11} + s^2 q_{11}^2 - \frac{\alpha^2\hat{b}_1^2\Delta}{\bar{\alpha}^2\mathbf{1\Sigma 1'}} = 0. \tag{A.49}$$

To show that the system of (A.46)-(A.49) has a solution, we reduce it to a single equation in $\hat{b}_1$. Eq. (A.49) is quadratic in q_{11} and has a unique positive solution $q_{11}(\hat{b}_1)$, which is increasing in $\hat{b}_1 \in (0,\infty)$, and is equal to zero for $\hat{b}_1 = 0$ and to ∞ for $\hat{b}_1 = \infty$.[27] Substituting $q_{11}(\hat{b}_1)$ into (A.47), we find

$$\frac{r+\kappa+s^2 q_{11}(\hat{b}_1)}{r\alpha}\frac{\gamma_1\Delta}{\mathbf{1\Sigma 1'}} + \frac{\hat{b}_1\sqrt{f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}}}{r\bar{\alpha}}\frac{\Delta}{\mathbf{1\Sigma 1'}} = \frac{1}{r\alpha}. \tag{A.50}$$

[27]The positive solution of (A.49) is the relevant one. Indeed, under the negative solution, the investor's certainty equivalent would converge to $-\infty$ when $|C_t|$ goes to ∞. The investor can, however, achieve a certainty equivalent converging to ∞ by holding a large short position in the active fund when C_t goes to ∞, or a large long position when C_t goes to $-\infty$.

The left-hand side of (A.50) is increasing in $\gamma_1 \in (0,\infty)$, and is equal to $\hat{b}_1\sqrt{f}\Delta/(r\bar{\alpha}\mathbf{1}\Sigma\mathbf{1}')$ for $\gamma_1 = 0$ and to ∞ for $\gamma_1 = \infty$. Therefore, (A.50) has a unique positive solution $\gamma_1(\hat{b}_1)$ if $\hat{b}_1 \in (0, \hat{b}_1^*)$, where $\hat{b}_1^* \equiv \bar{\alpha}\mathbf{1}\Sigma\mathbf{1}'/(\alpha\sqrt{f}\Delta)$, and no solution if $\hat{b}_1 \in (\hat{b}_1^*, \infty)$. The solution is decreasing in $\hat{b}_1$ since the left-hand side of (A.50) is increasing in $\hat{b}_1$, and is equal to $\mathbf{1}\Sigma\mathbf{1}'/[(r+\kappa)\Delta]$ for $\hat{b}_1 = 0$ and to zero for $\hat{b}_1 = \hat{b}_1^*$. Substituting $\gamma_1(\hat{b}_1)$, and $\bar{q}_{11}$ from (A.46), into (A.48), we find

$$-\frac{(r+\kappa)\kappa}{s^2} - \frac{r\hat{b}_1\sqrt{f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}}{\gamma_1(\hat{b}_1)s^2} + \frac{\hat{b}_1^2 f}{\gamma_1^2(\hat{b}_1)s^2} + \frac{\lambda\hat{b}_1}{\sqrt{f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}} = 0. \tag{A.51}$$

Eq. (A.51) is the single equation in $\hat{b}_1$ to which the system of (A.46)-(A.49) reduces. Since the left-hand side of (A.51) is equal to $-(r+\kappa)\kappa/s^2$ for $\hat{b}_1 = 0$ and to ∞ for $\hat{b}_1 = \hat{b}_1^*$, (A.51) has a solution $\hat{b}_1 \in (0, \hat{b}_1^*)$. Therefore, a linear equilibrium exists. The equilibrium is unique if the solution $\hat{b}_1$ of (A.51) is unique, which is the case if the derivative of the left-hand side with respect to $\hat{b}_1$ and evaluated at the solution is positive. The derivative is

$$\frac{1}{\hat{b}_1}\left[-\frac{r\hat{b}_1\sqrt{f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}}{\gamma_1(\hat{b}_1)s^2} + \frac{2\hat{b}_1^2 f}{\gamma_1^2(\hat{b}_1)s^2} + \frac{\lambda\hat{b}_1}{\sqrt{f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}}\right]$$
$$+\frac{d\gamma_1(\hat{b}_1)}{d\hat{b}_1}\frac{1}{\gamma_1(\hat{b}_1)}\left[\frac{r\hat{b}_1\sqrt{f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}}{\gamma_1(\hat{b}_1)s^2} - \frac{r\hat{b}_1\frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}{\gamma_1(\hat{b}_1)s^2\sqrt{f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}} - \frac{2\hat{b}_1^2 f}{\gamma_1^2(\hat{b}_1)s^2} - \frac{\lambda\hat{b}_1\frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}{\left(f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)^{\frac{3}{2}}}\right]. \tag{A.52}$$

If $\hat{b}_1$ solves (A.51), we can write (A.52) as

$$\frac{1}{\hat{b}_1}\left[\frac{(r+\kappa)\kappa}{s^2} + \frac{\hat{b}_1^2 f}{\gamma_1^2(\hat{b}_1)s^2}\right]$$
$$+\frac{d\gamma_1(\hat{b}_1)}{d\hat{b}_1}\frac{1}{\gamma_1(\hat{b}_1)}\left[-\frac{(r+\kappa)\kappa}{s^2} - \frac{r\hat{b}_1\frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}{\gamma_1(\hat{b}_1)s^2\sqrt{f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}} - \frac{\hat{b}_1^2 f}{\gamma_1^2(\hat{b}_1)s^2} + \frac{\lambda\hat{b}_1 f}{\left(f + \frac{s^2\gamma_1^2(\hat{b}_1)\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)^{\frac{3}{2}}}\right]. \tag{A.53}$$

The term inside the first squared bracket is positive. The term inside the second squared bracket is negative for $\lambda = 0$ and by continuity for $\lambda < \bar{\lambda}$ for a $\bar{\lambda} > 0$. Since $\gamma_1(\hat{b}_1)$ is decreasing in $\hat{b}_1$, (A.53) is positive for $\lambda < \bar{\lambda}$. ∎

Proof of Corollary 3.1: Eq. $y_t = \bar{\alpha}(\alpha+\bar{\alpha})$ follows from (3.2) and (A.37). Eq. $x_t = 0$ follows from (A.30) and $y_t = \bar{\alpha}(\alpha+\bar{\alpha})$. The first equality in (3.17) follows from (3.4) and (A.38), and the second equality follows from (3.5). ∎

Proof of Corollary 3.2: The investor's effective stock holdings are

$$\begin{aligned} x_t\mathbf{1} + y_t z_t &= x_1\mathbf{1} + y_t(\theta - x_t\mathbf{1}) \\ &= y_t p_f + \frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}, \end{aligned} \tag{A.54}$$

where the second step follows from (A.30). Eq. (3.18) follows from (3.2) and (A.54). ∎

Proof of Corollary 3.3: The first equality in (3.19) follows from (3.1) and (3.15). The second equality follows from (3.5) and (3.15). To derive the third equality, we note from (3.5) and (3.15) that

$$Cov_t(\mathbf{1}dR_t, p_f dR_t) = 0.$$

Therefore, if β denotes the regression coefficient of dR_t on $\mathbf{1}dR_t$, then

$$\begin{aligned} Cov_t(dR_t, p_f dR_t) &= Cov_t\left(dR_t - \beta\mathbf{1}dR_t, p_f dR_t\right) \\ &= Cov_t\left(d\epsilon_t, p_f dR_t\right) \\ &= Cov_t\left[d\epsilon_t, p_f(dR_t - \beta\mathbf{1}dR_t)\right] \\ &= Cov_t\left(d\epsilon_t, p_f d\epsilon_t\right), \end{aligned}$$

where the second and fourth steps follow from the definition of $d\epsilon_t$, and the third step follows because $d\epsilon_t$ is independent of $\mathbf{1}dR_t$. ∎

Proof of Corollary 3.4: The corollary follows by substituting (3.15) into (3.5). ∎

Proof of Corollary 3.5: Stocks' expected returns are

$$\begin{aligned} E_t(dR_t) &= \left[ra_0 + (r+\kappa)a_1C_t - \kappa a_1\bar{C}\right]dt \\ &= \left\{\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\Sigma\mathbf{1}' + \left[(r+\kappa)\gamma_1C_t - \frac{s^2\gamma_1(\alpha\bar{q}_1+\bar{\alpha}q_1)}{\alpha+\bar{\alpha}} + \frac{r\alpha\bar{\alpha}\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)}{\alpha+\bar{\alpha}}\right]\Sigma p_f'\right\}dt \\ &= \left[\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}(f\Sigma + s^2a_1a_1')\mathbf{1}' + \Lambda_t(f\Sigma + s^2a_1a_1')p_f'\right]dt, \end{aligned} \tag{A.55}$$

where the first step follows from (3.4), the second from (3.15) and (A.38), and the third from (3.15) and (3.22). Eq. (A.55) is equivalent to (3.21) because of (3.5). ∎

Proof of Corollary 3.6: The autocovariance matrix is

$$
\begin{aligned}
&Cov_t(dR_t, dR'_{t'})\\
&= Cov_t\left\{\sigma\left(dB_t^D + \frac{\phi dB_t^F}{r+\kappa}\right) - sa_1 dB_t^C, \left[(r+\kappa)a_1 C_{t'}dt + \sigma\left(dB_{t'}^D + \frac{\phi dB_{t'}^F}{r+\kappa}\right) - sa_1 dB_{t'}^C\right]'\right\}\\
&= Cov_t\left[\sigma\left(dB_t^D + \frac{\phi dB_t^F}{r+\kappa}\right) - sa_1 dB_t^C, (r+\kappa)a_1' C_{t'}dt\right]\\
&= Cov_t\left[-sa_1 dB_t^C, (r+\kappa)a_1' C_{t'}dt\right]\\
&= -s(r+\kappa)\gamma_1^2 Cov_t\left(dB_t^C, C_{t'}\right)\Sigma p_f p_f' \Sigma dt, \qquad \text{(A.56)}
\end{aligned}
$$

where the first step follows by using (3.4) and omitting quantities known at time t, the second step follows because the increments $(dB_{t'}^D, dB_{t'}^F, dB_{t'}^C)$ are independent of information up to time t', the third step follows because B_t^C is independent of (B_t^D, B_t^F), and the fourth step follows from (3.15). Eq. (2.3) implies that

$$
C_{t'} = e^{-\kappa(t'-t)}C_t + \left[1 - e^{-\kappa(t'-t)}\right]\bar{C} + s\int_t^{t'} e^{-\kappa(t'-u)}dB_u^C. \qquad \text{(A.57)}
$$

Substituting (A.57) into (A.56), and noting that the only non-zero covariance is between dB_t^C and dB_t^C, we find (3.23). ∎

Proof of Corollary 3.7: The left-hand side of (A.48) is increasing in λ. Since, in addition, the derivative (A.53) is positive, the solution $\hat{b}_1$ of (A.48) is decreasing in λ. Since $\gamma_1(\hat{b}_1)$ is decreasing in $\hat{b}_1$, it is increasing in λ.

Since B does not enter into the system of (A.46)-(A.49), it does not affect $(b_1, \gamma_1, \bar{q}_{11}, q_{11})$. Therefore, its effect on Λ_t is only through $(\bar{q}_1, q_1)$. Differentiating (A.41) and (A.45) with respect to B, we find

$$
(r+\kappa+s^2\bar{q}_{11})\frac{\partial \bar{q}_1}{\partial B} - r\bar{\alpha}b_1 \frac{\bar{\alpha}s^2\gamma_1\left(\frac{\partial \bar{q}_1}{\partial B} - \frac{\partial q_1}{\partial B}\right)}{\alpha+\bar{\alpha}}\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} + r\bar{\alpha}b_1 = 0, \qquad \text{(A.58)}
$$

$$
(r+\kappa+s^2 q_{11})\frac{\partial q_1}{\partial B} + r\alpha b_1 \frac{\alpha s^2\gamma_1\left(\frac{\partial q_1}{\partial B} - \frac{\partial \bar{q}_1}{\partial B}\right)}{\alpha+\bar{\alpha}}\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = 0. \qquad \text{(A.59)}
$$

The system of (A.58) and (A.59) is linear in $(\partial\bar{q}_1/\partial B, \partial q_1/\partial B)$. Its solution satisfies

$$
\alpha\frac{\partial \bar{q}_1}{\partial B} + \bar{\alpha}\frac{\partial q_1}{\partial B} = -\frac{Y}{Z}, \qquad \text{(A.60)}
$$

where

$$Y \equiv r\alpha\bar{\alpha}b_1\left(r+\kappa+s^2q_{11}+\frac{r\alpha s^2b_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right),$$

$$Z \equiv (r+\kappa+s^2\bar{q}_{11})(r+\kappa+s^2q_{11})+\frac{r\alpha^2s^2b_1\gamma_1\Delta(r+\kappa+s^2\bar{q}_{11})}{(\alpha+\bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'}-\frac{r\bar{\alpha}^2s^2b_1\gamma_1\Delta(r+\kappa+s^2q_{11})}{(\alpha+\bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'}$$

$$= r\bar{\alpha}b_1\left[\frac{f}{\gamma_1}+\frac{\alpha s^2\gamma_1\Delta}{(\alpha+\bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'}\right](r+\kappa+s^2q_{11})+\frac{r^2\alpha^2\bar{\alpha}s^2b_1^2\left(f+\frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)}{(\alpha+\bar{\alpha})\mathbf{1}\Sigma\mathbf{1}'},$$

and where the second equation for Z follows from (A.33). Since (b_1,γ_1,q_{11}) are positive, so are (Y,Z). Therefore, $\alpha\bar{q}_1+\bar{\alpha}q_1$ is decreasing in B, and (3.22) implies that Λ_t is increasing in B. ∎

B Gradual Adjustment

Proof of Proposition 4.1: Eqs. (2.3), (2.5), (2.6), (4.1) and (4.2) imply that the vector of returns is

$$dR_t = \left(ra_0+a_1^RC_t+a_2^Ry_t-\kappa a_1\bar{C}-b_0a_2\right)dt+\sigma\left(dB_t^D+\frac{\phi dB_t^F}{r+\kappa}\right)-sa_1dB_t^C, \tag{B.1}$$

where

$$a_1^R \equiv (r+\kappa)a_1+b_1a_2,$$

$$a_2^R \equiv (r+b_2)a_2.$$

Eqs. (2.3), (3.3), (4.2), (4.3) and (B.1) imply the following counterpart of (A.2):

$$\mathcal{D}\bar{V} = -\bar{V}\left\{\bar{G}-\frac{1}{2}(r\bar{\alpha})^2f\hat{z}_t\Sigma\hat{z}_t'-\frac{1}{2}s^2\left[r\bar{\alpha}\hat{z}_ta_1-\bar{f}_1(\bar{X}_t)\right]^2\right\}, \tag{B.2}$$

where

$$\bar{G} \equiv r\bar{\alpha}\left[rW_t+\hat{z}_t\left(ra_0+a_1^RC_t+a_2^Ry_t-\kappa a_1\bar{C}-b_0a_2\right)+(\lambda C_t+B)y_t-\bar{c}_t\right]$$
$$+\bar{f}_1(\bar{X}_t)\kappa(\bar{C}-C_t)+\bar{f}_2(\bar{X}_t)v_t+\frac{1}{2}s^2\bar{q}_{11},$$

$$\bar{f}_1(\bar{X}_t) \equiv \bar{q}_1+\bar{q}_{11}C_t+\bar{q}_{12}y_t,$$

$$\bar{f}_2(\bar{X}_t) \equiv \bar{q}_2+\bar{q}_{12}C_t+\bar{q}_{22}y_t,$$

and $\bar{q}_{ij}$ denotes the (i,j)'th element of $\bar{Q}$. Substituting (B.2) into (3.8), we can write the first-order conditions with respect to $\bar{c}_t$ and $\hat{z}_t$ as (A.3) and

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + s^2 a_1 a_1')\hat{z}_t', \tag{B.3}$$

respectively, where

$$\bar{h}(\bar{X}_t) \equiv ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2 + s^2 a_1 \bar{f}(\bar{X}_t). \tag{B.4}$$

Proceeding as in the proof of Proposition 3.1, we find the following counterpart of (A.10):

$$\begin{aligned}&\frac{1}{2}\bar{h}(\bar{X}_t)'(f\Sigma + s^2 a_1 a_1')^{-1}\bar{h}(\bar{X}_t) + r\bar{\alpha}(\lambda C_t + B)y_t - r\left[\bar{q}_0 + (\bar{q}_1, \bar{q}_2)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right]\\ &+ \bar{f}_1(\bar{X}_t)\kappa(\bar{C} - C_t) + \bar{f}_2(\bar{X}_t)v_t + \frac{1}{2}s^2\left[\bar{q}_{11} - \bar{f}_1(\bar{X}_t)^2\right] + \bar{\beta} - r + r\log(r) = 0.\end{aligned} \tag{B.5}$$

Eq. (B.5) is quadratic in $\bar{X}_t$. Identifying quadratic, linear and constant terms yields six scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{Q})$. We defer the derivation of these equations until the proof of Proposition 4.3 (see (B.38)-(B.40)). ∎

Proof of Proposition 4.2: Suppose that the investor optimizes over (c_t, x_t) but follows the control v_t given by (4.2). Eqs. (2.3), (4.2), (4.4), (4.5) and (B.1) imply the following counterpart of (A.12):

$$\mathcal{D}V = -V\left\{G - \frac{1}{2}(r\alpha)^2 f(x_t\mathbf{1} + y_t z_t)\Sigma(x_t\mathbf{1} + y_t z_t)' - \frac{1}{2}s^2\left[r\alpha(x_t\mathbf{1} + y_t z_t)a_1 - f_1(X_t)\right]^2\right\}, \tag{B.6}$$

where

$$\begin{aligned}G \equiv & r\alpha\left[rW_t + (x_t\mathbf{1} + y_t z_t)\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1\bar{C} - b_0 a_2\right) - y_t C_t - \frac{1}{2}\psi v_t^2 - c_t\right]\\ &+ f_1(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2 q_{11},\end{aligned}$$

$$f_1(X_t) \equiv q_1 + q_{11}C_t + q_{12}y_t,$$

$$f_2(X_t) \equiv q_2 + q_{12}C_t + q_{22}y_t,$$

and q_{ij} denotes the (i,j)'th element of Q. Substituting (B.6) into (4.6), we can write the first-order conditions with respect to c_t and x_t as (A.13) and

$$\mathbf{1}h(X_t) = r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)', \tag{B.7}$$

respectively, where

$$h(X_t) \equiv ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2 + s^2 a_1 f_1(X_t). \tag{B.8}$$

The counterpart of (A.17) is

$$\begin{aligned} & r\alpha(x_t\mathbf{1} + y_t z_t)\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) - r\alpha y_t C_t - \frac{1}{2} r\alpha\psi v_t^2 \\ & - r\left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t\right] + f_1(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2 q_{11} + \beta - r + r\log(r) \\ & - \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)' + r\alpha s^2(x_t\mathbf{1} + y_t z_t)a_1 f_1(X_t) - \frac{1}{2}s^2 f_1(X_t)^2 = 0. \end{aligned} \tag{B.9}$$

The terms in (B.9) that involve $x_t\mathbf{1} + y_t z_t$ can be written as

$$\begin{aligned} & r\alpha(x_t\mathbf{1} + y_t z_t)\left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) \\ & - \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)' + r\alpha s^2(x_t\mathbf{1} + y_t z_t)a_1 f_1(X_t) \\ & = r\alpha(x_t\mathbf{1} + y_t z_t)h(X_t) - \frac{1}{2}(r\alpha)^2(x_t\mathbf{1} + y_t z_t)(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)' \\ & = r\alpha y_t \theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2 \theta(f\Sigma + s^2 a_1 a_1')\theta' \\ & + r\alpha x_t(1 - y_t)\left\{\mathbf{1}h(X_t) - \frac{1}{2}r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')\left[x_t(1 - y_t)\mathbf{1} + 2y_t\theta\right]'\right\}', \end{aligned} \tag{B.10}$$

where the first step follows from (B.8) and the second from the equilibrium condition $z_t = \theta - x_t\mathbf{1}$. Using $z_t = \theta - x_t\mathbf{1}$, we can write (B.7) as

$$\mathbf{1}h(X_t) = r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')\left[x_t(1 - y_t)\mathbf{1} + y_t\theta\right]' \tag{B.11}$$

$$\Rightarrow x_t(1 - y_t) = \frac{\mathbf{1}h(X_t) - r\alpha y_t\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'}{r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}'}. \tag{B.12}$$

Eqs. (B.11) and (B.12) imply that

$$\begin{aligned} & r\alpha x_t(1 - y_t)\left\{\mathbf{1}h(X_t) - \frac{1}{2}r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')\left[x_t(1 - y_t)\mathbf{1} + 2y_t\theta\right]'\right\}' \\ & = \frac{1}{2}\left[r\alpha x_t(1 - y_t)\right]^2\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}' \\ & = \frac{1}{2}\frac{\left[\mathbf{1}h(X_t) - r\alpha y_t\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'\right]^2}{\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}'}. \end{aligned} \tag{B.13}$$

Substituting (B.10) and (B.13) into (B.9), we find

$$r\alpha y_t \theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2 \theta(f\Sigma + s^2 a_1 a_1')\theta' + \frac{1}{2}\frac{\left[\mathbf{1}h(X_t) - r\alpha y_t \mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'\right]^2}{\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}'} - r\alpha y_t C_t - \frac{1}{2} r\alpha\psi v_t^2$$

$$- r\left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t\right] + f_1(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2\left[q_{11} - f_1(X_t)^2\right]$$

$$+ \beta - r + r\log(r) = 0. \tag{B.14}$$

Since v_t in (4.2) is linear in X_t, (B.14) is quadratic in X_t. Identifying quadratic, linear and constant terms yields six scalar equations in (q_0, q_1, q_2, Q). We defer the derivation of these equations until the proof of Proposition 4.3 (see (B.42)-(B.44)).

We next study optimization over v_t, and derive a first-order condition under which the control (4.2) is optimal. We use a perturbation argument, which consists in assuming that the investor follows the control (4.2) except for an infinitesimal deviation over an infinitesimal internal.[28] Suppose that the investor adds $\omega d\epsilon$ to the control (4.2) over the interval $[t, t+d\epsilon]$ and subtracts $\omega d\epsilon$ over the interval $[t+dt-d\epsilon, t+dt]$, where the infinitesimal $d\epsilon > 0$ is $o(dt)$. The increase in adjustment cost over the first interval is $\psi v_t \omega (d\epsilon)^2$ and over the second interval is $-\psi v_{t+dt}\omega(d\epsilon)^2$. These changes reduce the investor's wealth at time $t + dt$ by

$$\begin{aligned}
&\psi v_t \omega (d\epsilon)^2 (1 + rdt) - \psi v_{t+dt}\omega(d\epsilon)^2 \\
&= \psi\omega(d\epsilon)^2 (rv_t dt - dv_t) \\
&= \psi\omega(d\epsilon)^2 (rv_t dt + b_1 dC_t + b_2 dy_t) \\
&= \psi\omega(d\epsilon)^2 \left\{(r + b_2)v_t dt + b_1\left[\kappa(\bar{C} - C_t)dt + s dB_t^C\right]\right\},
\end{aligned} \tag{B.15}$$

where the second step follows from (4.2) and the third from (2.3). The change in the investor's wealth between t and $t + dt$ is derived from (4.4) and (B.1), by subtracting (B.15) and replacing y_t by $y_t + \omega(d\epsilon)^2$:

$$\begin{aligned}
dW_t =& G_\omega dt - \psi\omega(d\epsilon)^2 b_1\left[\kappa(\bar{C} - C_t)dt + s dB_t^C\right] \\
&+ \left\{x_t \mathbf{1} + \left[y_t + \omega(d\epsilon)^2\right] z_t\right\}\left[\sigma\left(dB_t^D + \frac{\phi dB_t^F}{r + \kappa}\right) - s a_1 dB_t^C\right],
\end{aligned} \tag{B.16}$$

[28] The perturbation argument is simpler than the dynamic programming approach, which assumes that the investor can follow any control v_t over the entire history. Indeed, under the dynamic programming approach, the state variable y_t which describes the investor's holdings in the active fund must be replaced by two state variables: the holdings out of equilibrium, and the holdings in equilibrium. This is because the latter affect the equilibrium price, which the investor takes as given.

where

$$G_\omega \equiv rW_t + \left\{x_t\mathbf{1} + \left[y_t + \omega(d\epsilon)^2\right] z_t\right\} \left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) - \left[y_t + \omega(d\epsilon)^2\right] C_t$$
$$- \frac{\psi v_t^2}{2} - c_t - \psi\omega(d\epsilon)^2(r + b_2)v_t.$$

The investor's position in the active fund at $t + dt$ is the same under the deviation as under no deviation. Therefore, the investor's expected utility at $t + dt$ is given by the value function (4.5) with the wealth W_{t+dt} determined by (B.16). The drift $\mathcal{D}V$ corresponding to the change in the value function between t and $t + dt$ is given by the following counterpart of (B.6):

$$\mathcal{D}V = -V\left\{G - \frac{1}{2}(r\alpha)^2 f\left\{x_t\mathbf{1} + \left[y_t + \omega(d\epsilon)^2\right] z_t\right\} \Sigma \left\{x_t\mathbf{1} + \left[y_t + \omega(d\epsilon)^2\right] z_t\right\}'\right.$$
$$\left. -\frac{1}{2}s^2\left[r\alpha\left\{x_t\mathbf{1} + \left[y_t + \omega(d\epsilon)^2\right] z_t\right\} a_1 - f_{1\omega}(X_t)\right]^2\right\}, \quad \text{(B.17)}$$

where

$$G \equiv r\alpha G_\omega + f_{1\omega}(X_t)\kappa(\bar{C} - C_t) + f_2(X_t)v_t + \frac{1}{2}s^2 q_{11},$$

$$f_{1\omega}(X_t) \equiv f_1(X_t) - r\alpha\psi\omega(d\epsilon)^2 b_1.$$

The drift is maximum for $\omega = 0$, and this yields the first-order condition

$$z_t h(X_t) - r\alpha\psi b_1 s^2(x_t\mathbf{1} + y_t z_t)a_1 - C_t = r\alpha z_t(f\Sigma + s^2 a_1 a_1')(x_t\mathbf{1} + y_t z_t)' + \psi h_\psi(X_t), \quad \text{(B.18)}$$

where

$$h_\psi(X_t) \equiv (r + b_2)v_t + b_1\kappa(\bar{C} - C_t) - b_1 s^2 f_1(X_t).$$

Using (B.7) and the equilibrium condition $z_t = \theta - x_t\mathbf{1}$, we can write (B.18) as

$$\theta h(X_t) - r\alpha\psi b_1 s^2\left[x_t(1 - y_t)\mathbf{1} + y_t\theta\right] a_1 - C_t = r\alpha\theta(f\Sigma + s^2 a_1 a_1')\left[x_t(1 - y_t)\mathbf{1} + y_t\theta\right]' + \psi h_\psi(X_t). \quad \text{(B.19)}$$

Using (B.12), we can write (B.19) as

$$\theta\left[h(X_t) - r\alpha\psi b_1 s^2 y_t a_1\right] - r\alpha\psi b_1 s^2 \frac{\mathbf{1}h(X_t) - r\alpha y_t\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'}{r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}'}\mathbf{1}a_1 - C_t$$
$$= r\alpha\theta(f\Sigma + s^2 a_1 a_1')\left[y_t\theta + \frac{\mathbf{1}h(X_t) - r\alpha y_t\mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'}{r\alpha\mathbf{1}(f\Sigma + s^2 a_1 a_1')\mathbf{1}'}\mathbf{1}\right]' + \psi h_\psi(X_t). \quad \text{(B.20)}$$

Eq. (B.20) is linear in X_t. Identifying linear and constant terms, yields three scalar equations in (b_0, b_1, b_2). We defer the derivation of these equations until the proof of Proposition 4.3 (see (B.29)-(B.35)). ∎

Proof of Proposition 4.3: We first impose market clearing and follow similar steps as in the proof of Proposition 3.3 to derive the constants $(a_0, a_1, a_2, b_0, b_1, b_2)$ as functions of $(\bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_2, Q)$. Setting $z_t = \theta - x_t\mathbf{1}$ and $\bar{y}_t = 1 - y_t$, we can write (B.3) as

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + s^2 a_1 a_1')(1 - y_t)(\theta - x_t\mathbf{1})'. \tag{B.21}$$

Premultiplying (B.21) by $\mathbf{1}$, dividing by $r\bar{\alpha}$, and adding to (B.11) divided by $r\alpha$, we find

$$\mathbf{1}\left[\frac{h(X_t)}{r\alpha} + \frac{\bar{h}(\bar{X}_t)}{r\bar{\alpha}}\right] = \mathbf{1}(f\Sigma + s^2 a_1 a_1')\theta'. \tag{B.22}$$

Eq. (B.22) is linear in (C_t, y_t). Identifying terms in C_t and y_t, we find

$$\left(\frac{r + \kappa + s^2 q_{11}}{r\alpha} + \frac{r + \kappa + s^2 \bar{q}_{11}}{r\bar{\alpha}}\right)\mathbf{1}a_1 + \frac{b_1(\alpha + \bar{\alpha})}{r\alpha\bar{\alpha}}\mathbf{1}a_2 = 0, \tag{B.23}$$

$$\left(\frac{s^2 q_{12}}{r\alpha} + \frac{s^2 \bar{q}_{12}}{r\bar{\alpha}}\right)\mathbf{1}a_1 + \frac{(r + b_2)(\alpha + \bar{\alpha})}{r\alpha\bar{\alpha}}\mathbf{1}a_2 = 0, \tag{B.24}$$

respectively. Eqs. (B.23) and (B.24) imply

$$\mathbf{1}a_1 = \mathbf{1}a_2 = 0. \tag{B.25}$$

Identifying constant terms in (B.22), and using (B.25), we find (A.29). Substituting (A.29) and (B.25) into (B.11), we find (A.30).

Substituting (A.30) into (B.21), we find

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + s^2 a_1 a_1')\left[\frac{\alpha}{\alpha + \bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1} + (1 - y_t)p_f\right]'. \tag{B.26}$$

Eq. (A.31) is linear in $\bar{X}_t$. Identifying terms in C_t and y_t, we find

$$(r + \kappa + s^2 \bar{q}_{11})a_1 + b_1 a_2 = 0, \tag{B.27}$$

$$s^2 \bar{q}_{12} a_1 + (r + b_2)a_2 = -r\bar{\alpha}\left(f\Sigma p_f' + s^2 a_1' p_f' a_1\right), \tag{B.28}$$

respectively. Therefore, (a_1, a_2) are collinear to the vector $\Sigma p_f'$, as in (4.7). Substituting (4.7) into (B.27) and (B.28), we find

$$(r+\kappa+s^2\bar{q}_{11})\gamma_1 + b_1\gamma_2 = 0, \tag{B.29}$$

$$s^2\bar{q}_{12}\gamma_1 + (r+b_2)\gamma_2 = -r\bar{\alpha}\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right), \tag{B.30}$$

respectively. Identifying constant terms in (B.26), and using (4.7), we find

$$a_0 = \frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\Sigma\mathbf{1}' + \left[\frac{\gamma_1(\kappa\bar{C} - s^2\bar{q}_1) + b_0\gamma_2}{r} + \bar{\alpha}\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\right]\Sigma p_f'. \tag{B.31}$$

Using (A.30), we can write (B.19) as

$$\begin{aligned}
&\theta h(X_t) - r\alpha\psi b_1 s^2\left(\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1} + y_t p_f\right)a_1 - C_t \\
&= r\alpha\theta(f\Sigma + s^2 a_1 a_1')\left(\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1} + y_t p_f\right)' + \psi h_\psi(X_t) \\
&\Rightarrow \theta h(X_t) - r\alpha\psi b_1 s^2\gamma_1\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}y_t - C_t = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{(\mathbf{1}\Sigma\theta')^2}{\mathbf{1}\Sigma\mathbf{1}'} + r\alpha\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}y_t + \psi h_\psi(X_t),
\end{aligned} \tag{B.32}$$

where the second step follows from (4.7). Eq. (B.32) is linear in (C_t, y_t). Identifying terms in C_t and y_t, and using (4.2) and (4.7), we find

$$\left[(r+\kappa+s^2 q_{11})\gamma_1 + b_1\gamma_2\right]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - 1 = -\psi b_1(r+\kappa+b_2+s^2 q_{11}), \tag{B.33}$$

$$\left[(r+b_2)\gamma_2 + (q_{12} - r\alpha\psi b_1)s^2\gamma_1\right]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = r\alpha\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - \psi\left[(r+b_2)b_2 + b_1 s^2 q_{12}\right], \tag{B.34}$$

respectively. Identifying constant terms, and using (4.2), (4.7) and (B.31), we find

$$\left[s^2\gamma_1(q_1 - \bar{q}_1) + r\bar{\alpha}\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\right]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = \psi\left[(r+b_2)b_0 + b_1(\kappa\bar{C} - s^2 q_1)\right]. \tag{B.35}$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, a_1, a_2, b_0, b_1, b_2, \gamma_1, \gamma_2, \bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_2, Q)$. (As in Proposition 3.3, we can drop $(\bar{q}_0, q_0)$.) The equations linking them are (4.7), (B.29)-(B.31), (B.33)-(B.35), the five equations derived from (B.5) by identifying linear and quadratic terms, and the five equations derived from (B.14) through the same procedure. We next simplify the latter two sets of equations, using implications of market clearing.

Using (B.26), we find

$$\frac{1}{2}\bar{h}(\bar{X}_t)'(f\Sigma + s^2 a_1 a_1')^{-1}\bar{h}(\bar{X}_t) = \frac{r^2\alpha^2\bar{\alpha}^2 f(\mathbf{1}\Sigma\theta')^2}{2(\alpha+\bar{\alpha})^2\mathbf{1}\Sigma\mathbf{1}'} + \frac{1}{2}r^2\bar{\alpha}^2(1-y_t)^2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}. \quad \text{(B.36)}$$

We next substitute (B.36) into (B.5), and identify terms. Identifying terms in C_t^2, $C_t y_t$ and y_t^2, we find

$$\frac{1}{2}\bar{X}_t'\left(\bar{Q}\bar{\mathcal{R}}_2\bar{Q} + \bar{Q}\bar{\mathcal{R}}_1 + \bar{\mathcal{R}}_1'\bar{Q} - \bar{\mathcal{R}}_0\right)\bar{X}_t = 0, \quad \text{(B.37)}$$

where

$$\bar{\mathcal{R}}_2 \equiv \begin{pmatrix} s^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\bar{\mathcal{R}}_1 \equiv \begin{pmatrix} \frac{r}{2}+\kappa & 0 \\ b_1 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\bar{\mathcal{R}}_0 \equiv \begin{pmatrix} 0 & r\bar{\alpha}\lambda \\ r\bar{\alpha}\lambda & r^2\bar{\alpha}^2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \end{pmatrix}.$$

Eq. (B.37) must hold for all $\bar{X}_t$. Since the square matrix in (B.37) is symmetric, it must equal zero, and this yields the algebraic Riccati equation

$$\bar{Q}\bar{\mathcal{R}}_2\bar{Q} + \bar{Q}\bar{\mathcal{R}}_1 + \bar{\mathcal{R}}_1'\bar{Q} - \bar{\mathcal{R}}_0 = 0. \quad \text{(B.38)}$$

We next identify terms in C_t and y_t, which yield

$$(r+\kappa+s^2\bar{q}_{11})\bar{q}_1 + b_1\bar{q}_2 - \kappa\bar{C}\bar{q}_{11} - b_0\bar{q}_{12} = 0, \quad \text{(B.39)}$$

$$(r+b_2)\bar{q}_2 + s^2\bar{q}_1\bar{q}_{12} + r^2\bar{\alpha}^2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - r\bar{\alpha}B - \kappa\bar{C}\bar{q}_{12} - b_0\bar{q}_{22} = 0, \quad \text{(B.40)}$$

respectively. Using (3.15) and (A.30), we can write (B.11) as

$$\mathbf{1}h(X_t) = \frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\mathbf{1}\Sigma\theta'. \quad \text{(B.41)}$$

Using (4.2), (4.7), (B.32) and (B.41), we find that the equation derived from (B.14) by identifying terms in C_t^2, $C_t y_t$ and y_t^2 is

$$Q\mathcal{R}_2 Q + Q\mathcal{R}_1 + \mathcal{R}_1' Q - \mathcal{R}_0 = 0, \tag{B.42}$$

where

$$\mathcal{R}_2 \equiv \begin{pmatrix} s^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{R}_1 \equiv \begin{pmatrix} \frac{r}{2} + \kappa & r\alpha\psi b_1 s^2 \\ b_1 & \frac{r}{2} + b_2 \end{pmatrix},$$

$$\mathcal{R}_0 \equiv \begin{pmatrix} -r\alpha\psi b_1^2 & -r\alpha\psi b_1(r+\kappa+2b_2) \\ -r\alpha\psi b_1(r+\kappa+2b_2) & r^2\alpha^2\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}\right)\frac{\Delta}{\mathbf{1\Sigma 1'}} + 2r^2\alpha^2\psi b_1 s^2\gamma_1\frac{\Delta}{\mathbf{1\Sigma 1'}} - r\alpha\psi b_2(2r+3b_2) \end{pmatrix},$$

and the equations derived by identifying terms in C_t and y_t are

$$(r+\kappa+s^2 q_{11})q_1 + b_1 q_2 - r\alpha\psi b_0 b_1 - \kappa\bar{C} q_{11} - b_0 q_{12} = 0, \tag{B.43}$$

$$(r+b_2)q_2 + s^2(q_{12} + r\alpha\psi b_1)q_1 - r\alpha\psi\left[(r+2b_2)b_0 + b_1\kappa\bar{C}\right] - \kappa\bar{C} q_{12} - b_0 q_{22} = 0, \tag{B.44}$$

respectively.

Solving for equilibrium amounts to solving the system of (4.7), (B.29)-(B.31), (B.33)-(B.35), (B.38)-(B.40) and (B.42)-(B.44) in the unknowns $(a_0, a_1, a_2, b_0, b_1, b_2, \gamma_1, \gamma_2, \bar{q}_1, \bar{q}_2, \bar{Q}, q_1, q_2, Q)$. This reduces to solving the system of (B.29), (B.30), (B.33), (B.34), (B.38) and (B.42) in the unknowns $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q)$: given $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q)$, (a_1, a_2) can be determined from (4.7), $(b_0, \bar{q}_1, \bar{q}_2, q_1, q_2)$ from the linear system of (B.35), (B.39), (B.40), (B.43) and (B.44), and a_0 from (B.31). We replace the system of (B.29), (B.30), (B.33), (B.34), (B.38) and (B.42) by the equivalent system of (B.29), (B.30), (B.38), (B.42),

$$\psi b_1(r+\kappa+b_2+s^2 q_{11}) = 1 + s^2\gamma_1(\bar{q}_{11} - q_{11})\frac{\Delta}{\mathbf{1\Sigma 1'}}, \tag{B.45}$$

$$\psi\left[(r+b_2)b_2 + b_1 s^2 q_{12}\right] - r\alpha\psi b_1 s^2\gamma_1\frac{\Delta}{\mathbf{1\Sigma 1'}} = r(\alpha+\bar{\alpha})\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1\Sigma 1'}}\right)\frac{\Delta}{\mathbf{1\Sigma 1'}} + s^2\gamma_1(\bar{q}_{12} - q_{12})\frac{\Delta}{\mathbf{1\Sigma 1'}}. \tag{B.46}$$

For $s = 0$, (B.29), (B.30), (B.38), (B.42), (B.45) and (B.46) become

$$(r+\kappa)\gamma_1 + b_1\gamma_2 = 0, \tag{B.47}$$

$$(r+b_2)\gamma_2 = -r\bar{\alpha}f, \tag{B.48}$$

$$\bar{Q}\bar{\mathcal{R}}_1^0 + \bar{\mathcal{R}}_1^{0'}\bar{Q} - \bar{\mathcal{R}}_0^0 = 0, \tag{B.49}$$

$$Q\mathcal{R}_1^0 + \mathcal{R}_1^{0'}Q - \mathcal{R}_0^0 = 0, \tag{B.50}$$

$$\psi b_1(r+\kappa+b_2) = 1, \tag{B.51}$$

$$\psi(r+b_2)b_2 = r(\alpha+\bar{\alpha})f\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}, \tag{B.52}$$

respectively, where

$$\bar{\mathcal{R}}_1^0 = \mathcal{R}_1^0 \equiv \begin{pmatrix} \frac{r}{2}+\kappa & 0 \\ b_1 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\bar{\mathcal{R}}_0^0 \equiv \begin{pmatrix} 0 & r\bar{\alpha}\lambda \\ r\bar{\alpha}\lambda & r^2\bar{\alpha}^2 f\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \end{pmatrix},$$

$$\mathcal{R}_0^0 \equiv \begin{pmatrix} -r\alpha\psi b_1^2 & -r\alpha\psi b_1(r+\kappa+2b_2) \\ -r\alpha\psi b_1(r+\kappa+2b_2) & r^2\alpha^2 f\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} - r\alpha\psi b_2(2r+3b_2) \end{pmatrix}.$$

Eq. (B.52) is quadratic and has a unique positive solution b_2.[29] Given b_2, b_1 is determined uniquely from (B.51), γ_2 from (B.48), γ_1 from (B.47), $\bar{Q}$ from (B.49) (which is linear in $\bar{Q}$), and Q from (B.50) (which is linear in Q). We denote this solution by $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \bar{Q}^0, Q^0)$.

To show that the system of (B.29), (B.30), (B.38), (B.42), (B.45) and (B.46) has a solution for small s, we apply the implicit function theorem. We move all terms in each equation to the left-hand side, and stack all left-hand sides into a vector $\mathcal{F}$, in the order (B.46), (B.45), (B.30), (B.29), (B.38), (B.42). Treated as a function of $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q, s)$, $\mathcal{F}$ is continuously differentiable around the point $\mathcal{A} \equiv (b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \bar{Q}^0, Q^0, 0)$ and is equal to zero at $\mathcal{A}$. To show that the Jacobian matrix of $\mathcal{F}$ with respect to $(b_1, b_2, \gamma_1, \gamma_2, \bar{Q}, Q)$ has non-zero determinant at $\mathcal{A}$, we note that $\mathcal{F}$ has a triangular structure for $s = 0$: $\mathcal{F}_1$ depends only on b_2, $\mathcal{F}_2$ only on (b_1, b_2), $\mathcal{F}_3$ only on (b_2, γ_2), $\mathcal{F}_4$ only on $(b_1, \gamma_1, \gamma_2)$, $\mathcal{F}_5$ only on $(b_1, b_2, \bar{Q})$, and $\mathcal{F}_6$ only on (b_1, b_2, Q). Therefore, the Jacobian matrix of $\mathcal{F}$ has non-zero determinant at $\mathcal{A}$ if the derivatives of $\mathcal{F}_1$ with respect to b_2, $\mathcal{F}_2$ with

[29] The positive solution is the relevant one. Indeed, since the negative solution satisfies $r + 2b_2 < 0$, (B.49) implies that $\bar{q}_{22} < 0$. Therefore, the manager's certainty equivalent would converge to $-\infty$ at the rate y_t^2 when $|y_t|$ goes to ∞ and C_t is held constant. The manager can, however, achieve higher certainty equivalent by not investing in the active fund.

respect to b_1, $\mathcal{F}_3$ with respect to γ_2, and $\mathcal{F}_4$ with respect to γ_1 are non-zero, and the Jacobian matrices of $\mathcal{F}_5$ with respect to $\bar{Q}$ and $\mathcal{F}_6$ with respect to Q have non-zero determinants. These results follow from (B.47)-(B.52) and the positivity of (b_1^0, b_2^0). Therefore, the implicit function theorem applies, and the system of (B.29), (B.30), (B.38), (B.42), (B.45) and (B.46) has a solution for small s. This solution is unique in a neighborhood of $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \bar{Q}^0, Q^0)$, which corresponds to the unique equilibrium for $s = 0$. Since $b_1^0 > 0$, $b_2^0 > 0$, $\gamma_1^0 > 0$, $\gamma_2^0 < 0$, continuity implies that $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 < 0$ for small s. ∎

Proof of Corollary 4.1: Stocks' expected returns are

$$\begin{aligned} E_t(dR_t) &= \left(ra_0 + a_1^R C_t + a_2^R y_t - \kappa a_1 \bar{C} - b_0 a_2\right) dt \\ &= \left\{ \frac{r\alpha\bar{\alpha} f}{\alpha+\bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\Sigma\mathbf{1}' + \left[\gamma_1^R C_t + \gamma_2^R y_t + r\bar{\alpha}\left(f + \frac{s^2\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) - \gamma_1 s^2 \bar{q}_1\right]\Sigma p_f' \right\} dt \\ &= \left[\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}} \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}(f\Sigma + s^2 a_1 a_1')\mathbf{1}' + \Lambda_t (f\Sigma + s^2 a_1 a_1') p_f'\right] dt, \end{aligned} \tag{B.53}$$

where

$$\gamma_1^R \equiv (r+\kappa)\gamma_1 + b_1\gamma_2,$$

$$\gamma_2^R \equiv (r+b_2)\gamma_2.$$

The first step in (B.53) follows from (B.1), the second from (4.7) and (B.31), and the third from (4.7) and (4.8). Eq. (B.53) is equivalent to (3.21) because of (3.5).

Eq. (B.29) implies that γ_1^R has the opposite sign of $\gamma_1 \bar{q}_{11}$. For small s, $\gamma_1 > 0$ and $\bar{q}_{11}$ has the same sign as its value $\bar{q}_{11}^0$ for $s = 0$. Eq. (B.49) implies that

$$\begin{aligned} \bar{q}_{11}^0 &= -\frac{2b_1^0 \bar{q}_{12}^0}{r+2\kappa} \\ &= -\frac{2b_1^0}{(r+2\kappa)(r+\kappa+b_2^0)}\left(r\bar{\alpha}\lambda - b_1^0 \bar{q}_{22}^0\right), \\ &= -\frac{2r\bar{\alpha}b_1^0}{(r+2\kappa)(r+\kappa+b_2^0)}\left[\lambda - \frac{r\bar{\alpha}b_1^0 f\Delta}{(r+2b_2^0)\mathbf{1}\Sigma\mathbf{1}'}\right], \\ &= -\frac{2r\bar{\alpha}b_1^0}{(r+2\kappa)(r+\kappa+b_2^0)}\left[\lambda - \frac{r\bar{\alpha} f\Delta}{\psi(r+\kappa+b_2^0)(r+2b_2^0)\mathbf{1}\Sigma\mathbf{1}'}\right], \end{aligned} \tag{B.54}$$

where the last step follows from (B.51). Using (B.52), we find

$$\psi(r+\kappa+b_2^0)(r+2b_2^0) = 2r(\alpha+\bar{\alpha})f\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} + \psi\left[(r+2\kappa)b_2^0 + r(r+\kappa)\right]$$

$$= 2r(\alpha+\bar{\alpha})f\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{\psi r}{2}\left[r + (r+2\kappa)\sqrt{1+\frac{4(\alpha+\bar{\alpha})f\Delta}{r\psi\mathbf{1}\Sigma\mathbf{1}'}}\right]. \qquad \text{(B.55)}$$

Eqs. (B.54) and (B.55) imply that $\bar{q}_{11}^0$ is positive if (4.9) holds, and is negative otherwise. Therefore, for small s, γ_1^R is negative if (4.9) holds, and is positive otherwise. Moreover, $\gamma_2^R < 0$ since $b_2 > 0$ and $\gamma_2 < 0$. ∎

Proof of Corollary 4.2: Using (B.1) and proceeding as in the derivation of (A.56), we find

$$Cov_t(dR_t, dR_{t'}') = Cov_t\left[-sa_1 dB_t^C, \left(a_1^R C_{t'} + a_2^R y_{t'}\right)' dt\right]$$

$$= -s\gamma_1 Cov_t\left(dB_t^C, \gamma_1^R C_{t'} + \gamma_2^R y_{t'}\right)\Sigma p_f' p_f \Sigma dt, \qquad \text{(B.56)}$$

where the last step follows from (4.7). Using the dynamics (2.3) and (4.2), we can express $(C_{t'}, y_{t'})$ as a function of their time t values and the Brownian shocks dB_u^C for $u \in [t, t']$. The covariance (B.56) depends only on how the Brownian shock dB_t^C impacts $(C_{t'}, y_{t'})$. (See the proof of Corollary 3.6.) To compute this impact, we solve the "impulse-response" dynamics

$$dC_t = -\kappa C_t dt,$$

$$dy_t = -(b_1 C_t + b_2 y_t)dt,$$

with the initial conditions

$$C_t = s dB_t^C,$$

$$y_t = 0.$$

The solution to these dynamics is

$$C_{t'} = e^{-\kappa(t'-t)} s dB_t^C, \qquad \text{(B.57)}$$

$$y_{t'} = -\frac{b_1\left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right]}{b_2-\kappa} s dB_t^C, \qquad \text{(B.58)}$$

and the implied dynamics of expected return are

$$\frac{E(dR_{t'})}{dt} = \left\{\gamma_1^R e^{-\kappa(t'-t)} - \gamma_2^R \frac{b_1\left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right]}{b_2-\kappa}\right\} s\Sigma p_f' dB_t^C. \qquad \text{(B.59)}$$

Eqs. (B.58) and (B.59) are used to plot the solid and dashed lines, respectively, in Figure 1. Substituting (B.57) and (B.58) into (B.56), we find (4.11) with

$$\chi_1 \equiv s^2\gamma_1 \left(\frac{b_1\gamma_2^R}{b_2-\kappa} - \gamma_1^R \right) = s^2(r+\kappa)\gamma_1 \left(\frac{b_1\gamma_2}{b_2-\kappa} - \gamma_1 \right), \tag{B.60}$$

$$\chi_2 \equiv -\frac{s^2 b_1\gamma_1\gamma_2^R}{b_2-\kappa} = -\frac{s^2(r+b_2)b_1\gamma_1\gamma_2}{b_2-\kappa}. \tag{B.61}$$

The function $\chi(u) \equiv \chi_1 e^{-\kappa u} + \chi_2 e^{-b_2 u}$ can change sign only once, is equal to $-s^2\gamma_1\gamma_1^R$ when $u=0$, and has the sign of χ_1 if $b_2 > \kappa$ and of χ_2 if $b_2 < \kappa$ when u goes to ∞. For small s, γ_1^R is negative if (4.9) holds, and is positive otherwise. The opposite is true for $\chi(0)$ since $\gamma_1 > 0$. Since, in addition, $b_1 > 0$, $b_2 > 0$ and $\gamma_2 < 0$, (B.60) and (B.61) imply that $\chi_1 < 0$ if $b_2 > \kappa$ and $\chi_2 < 0$ if $b_2 < \kappa$. Therefore, there exists a threshold $\hat{u} \geq 0$, which is positive if (4.9) holds and is zero otherwise, such that $\chi(u) > 0$ for $0 < u < \hat{u}$ and $\chi(u) < 0$ for $u > \hat{u}$. ∎

C Asymmetric Information

Proof of Proposition 5.1: We use Theorem 10.3 of Liptser and Shiryaev (LS 2000). The investor learns about C_t, which follows the process (2.3). She observes the following information:

- The net dividends of the residual-supply portfolio $\theta D_t - C_t dt$. This corresponds to the process $\xi_{1t} \equiv \theta D_t - \int_0^t C_s ds$.
- The dividends of the index fund $\mathbf{1}dD_t$. This corresponds to the process $\xi_{2t} \equiv \mathbf{1}D_t$.
- The price of the residual-supply portfolio θS_t. Given the conjecture (5.1) for stock prices, this is equivalent to observing the process $\xi_{3t} \equiv \theta(S_t + a_1\hat{C}_t + a_3 y_t)$.
- The price of the index portfolio $\mathbf{1}S_t$. This is equivalent to observing the process $\xi_{4t} \equiv \mathbf{1}(S_t + a_1\hat{C}_t + a_3 y_t)$.

The dynamics of ξ_{1t} are

$$
\begin{aligned}
d\xi_{1t} &= \theta(F_t dt + \sigma dB_t^D) - C_t dt \\
&= \left[(r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r+\kappa)\xi_{3t} + (r+\kappa)\theta a_2 C_t - C_t\right] dt + \theta\sigma dB_t^D \\
&= \left[(r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r+\kappa)\xi_{3t} - \left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) C_t\right] dt + \theta\sigma dB_t^D,
\end{aligned} \tag{C.1}
$$

where the first step follows from (2.5), the second from (5.1), and the third from (5.2). Likewise, the dynamics of ξ_{2t} are

$$
d\xi_{2t} = \left[(r+\kappa)\mathbf{1}a_0 - \frac{\kappa\mathbf{1}\bar{F}}{r} + (r+\kappa)\xi_{4t}\right] dt + \mathbf{1}\sigma dB_t^D. \tag{C.2}
$$

The dynamics of ξ_{3t} are

$$
\begin{aligned}
d\xi_{3t} &= d\left\{\theta\left[\frac{\bar{F}}{r} + \frac{F_t - \bar{F}}{r+\kappa} - (a_0 + a_2 C_t)\right]\right\} \\
&= \theta\left[\frac{\kappa(\bar{F} - F_t)dt + \phi\sigma dB_t^F}{r+\kappa} - a_2\left[\kappa(\bar{C} - C_t)dt + s dB_t^C\right]\right] \\
&= \kappa\left[\theta\left(\frac{\bar{F}}{r} - a_0 - a_2\bar{C}\right) - \xi_{3t}\right] dt + \frac{\phi\theta\sigma dB_t^F}{r+\kappa} - s\theta a_2 dB_t^C \\
&= \kappa\left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\mathbf{1}\Sigma\mathbf{1}'} - \xi_{3t}\right) dt + \frac{\phi\theta\sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2\Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'},
\end{aligned} \tag{C.3}
$$

where the first step follows from (5.1), the second from (2.6) and (2.3), and the fourth from (5.2). Likewise, the dynamics of ξ_{4t} are

$$
d\xi_{4t} = \kappa\left(\frac{\mathbf{1}\bar{F}}{r} - \mathbf{1}a_0 - \xi_{4t}\right) dt + \frac{\phi\mathbf{1}\sigma dB_t^F}{r+\kappa}. \tag{C.4}
$$

The dynamics (2.3) and (C.1)-(C.4) map into the dynamics (10.62) and (10.63) of LS by setting $\theta_t \equiv C_t$, $\xi_t \equiv (\xi_{1t}, \xi_{2t}, \xi_{3t}, \xi_{4t})'$, $W_{1t} \equiv \begin{pmatrix} B_t^D \\ B_t^F \end{pmatrix}$, $W_{2t} \equiv B_t^C$, $a_0(t) \equiv \kappa\bar{C}$, $a_1(t) \equiv -\kappa$, $a_2(t) \equiv 0$, $b_1(t) \equiv 0$, $b_2(t) \equiv s$, $\gamma_t \equiv R$,

$$
A_0(t) \equiv \begin{pmatrix} (r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} \\ (r+\kappa)\mathbf{1}a_0 - \frac{\kappa\mathbf{1}\bar{F}}{r} \\ \kappa\left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\mathbf{1}\Sigma\mathbf{1}'}\right) \\ \kappa\left(\frac{\mathbf{1}\bar{F}}{r} - \mathbf{1}a_0\right) \end{pmatrix},
$$

$$A_1(t) \equiv -\begin{pmatrix} 1-\frac{(r+\kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A_2(t) \equiv \begin{pmatrix} 0 & 0 & r+\kappa & 0 \\ 0 & 0 & 0 & r+\kappa \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & -\kappa \end{pmatrix},$$

$$B_1(t) \equiv \begin{pmatrix} \theta\sigma & 0 \\ \mathbf{1}\sigma & 0 \\ 0 & \frac{\phi\theta\sigma}{r+\kappa} \\ 0 & \frac{\phi\mathbf{1}\sigma}{r+\kappa} \end{pmatrix},$$

$$B_2(t) \equiv -\begin{pmatrix} 0 \\ 0 \\ \frac{s\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \\ 0 \end{pmatrix}.$$

The quantities $(b \circ b)(t)$, $(b \circ B)(t)$, and $(B \circ B)(t)$, defined in LS (10.80) are

$$(b \circ b)(t) = s^2,$$

$$(b \circ B)(t) = -\begin{pmatrix} 0 & 0 & \frac{s^2\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & 0 \end{pmatrix},$$

$$(B \circ B)(t) = \begin{pmatrix} \theta\Sigma\theta' & \mathbf{1}\Sigma\theta' & 0 & 0 \\ \mathbf{1}\Sigma\theta' & \mathbf{1}\Sigma\mathbf{1}' & 0 & 0 \\ 0 & 0 & \frac{\phi^2\theta\Sigma\theta'}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta^2}{(\mathbf{1}\Sigma\mathbf{1}')^2} & \frac{\phi^2\mathbf{1}\Sigma\theta'}{(r+\kappa)^2} \\ 0 & 0 & \frac{\phi^2\mathbf{1}\Sigma\theta'}{(r+\kappa)^2} & \frac{\phi^2\mathbf{1}\Sigma\mathbf{1}'}{(r+\kappa)^2} \end{pmatrix}.$$

Theorem 10.3 of LS (first subequation of (10.81)) implies that

$$\begin{aligned} d\hat{C}_t =& \kappa(\bar{C}-\hat{C}_t)dt - \beta_1 \Bigg\{ d\xi_{1t} - \left[(r+\kappa)\theta a_0 - \frac{\kappa\theta\bar{F}}{r} + (r+\kappa)\xi_{3t} - \left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\hat{C}_t \right] dt \\ & - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \left[d\xi_{2t} - \left[(r+\kappa)\mathbf{1}a_0 - \frac{\kappa\mathbf{1}\bar{F}}{r} + (r+\kappa)\xi_{4t} \right] dt \right] \Bigg\} \\ & - \beta_2 \Bigg\{ d\xi_{3t} - \kappa\left(\frac{\theta\bar{F}}{r} - \theta a_0 - \frac{\gamma_2\Delta\bar{C}}{\mathbf{1}\Sigma\mathbf{1}'} - \xi_{3t} \right) dt \\ & - \frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'} \left[d\xi_{4t} - \kappa\left(\frac{\mathbf{1}\bar{F}}{r} - \mathbf{1}a_0 - \xi_{4t} \right) dt \right] \Bigg\}. \end{aligned} \tag{C.5}$$

Eq. (5.4) follows from (C.5) by noting that the term in dt after each $d\xi_{it}$, $i = 1, 2, 3, 4$, is $E_t(d\xi_{it})$. In subsequent proofs we use a different form of (5.4), where we replace each $d\xi_{it}$, $i = 1, 2, 3, 4$, by its value in (C.1)-(C.4):

$$d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 \left[p_f \sigma dB_t^D - \left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(C_t - \hat{C}_t)dt \right] - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2 \Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'} \right). \tag{C.6}$$

Eq. (5.7) follows from Theorem 10.3 of LS (second subequation of (10.81)). ∎

Proof of Proposition 5.2: Eqs. (2.3), (2.5), (2.6), (5.1)-(5.3) and (C.6) imply that the vector of returns is

$$dR_t = \left\{ ra_0 + \left[\gamma_1^R \hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0\gamma_3 \right] \Sigma p_f' \right\} dt + \left(\sigma + \beta_1\gamma_1 \Sigma p_f' p_f \sigma\right) dB_t^D$$
$$+ \frac{\phi}{r+\kappa}\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma\right) dB_t^F - s\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\Sigma p_f' dB_t^C, \tag{C.7}$$

where

$$\gamma_1^R \equiv (r + \kappa + \rho)\gamma_1 + b_1\gamma_3,$$
$$\gamma_2^R \equiv (r+\kappa)\gamma_2 - \rho\gamma_1,$$
$$\gamma_3^R \equiv (r + b_2)\gamma_3,$$

and

$$\rho \equiv \beta_1 \left(1 - \frac{(r+\kappa)\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right). \tag{C.8}$$

Eqs. (2.3), (3.3), (5.3), (C.6) and (C.7) imply that

$$d\left(r\bar{\alpha}W_t + \bar{q}_0 + (\bar{q}_1, \bar{q}_2, \bar{q}_3)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right)$$
$$= \bar{G}dt + \left[r\bar{\alpha}\hat{z}_t\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f\sigma\right) - \beta_1 \bar{f}_1(\bar{X}_t)p_f\sigma\right] dB_t^D$$
$$+ \frac{\phi}{r+\kappa}\left[r\bar{\alpha}\hat{z}_t\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f\sigma\right) - \beta_2 \bar{f}_1(\bar{X}_t)p_f\sigma\right] dB_t^F$$
$$- s\left[r\bar{\alpha}\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} - \bar{f}_2(\bar{X}_t)\right] dB_t^C, \tag{C.9}$$

where

$$\bar{G} \equiv r\bar{\alpha}\left(rW_t + \hat{z}_t\left\{ra_0 + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right]\Sigma p_f'\right\} + (\lambda C_t + B)y_t - \bar{c}_t\right)$$

$$+ \bar{f}_1(\bar{X}_t)\left[\kappa(\bar{C} - \hat{C}_t) + \rho(C_t - \hat{C}_t)\right] + \bar{f}_2(\bar{X}_t)\kappa(\bar{C} - C_t) + \bar{f}_3(\bar{X}_t)v_t$$

$$+ \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta\bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{s^2\beta_2\gamma_2\Delta\bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{1}{2}s^2\bar{q}_{22},$$

$$\bar{f}_1(\bar{X}_t) \equiv \bar{q}_1 + \bar{q}_{11}\hat{C}_t + \bar{q}_{12}C_t + \bar{q}_{13}y_t,$$

$$\bar{f}_2(\bar{X}_t) \equiv \bar{q}_2 + \bar{q}_{12}\hat{C}_t + \bar{q}_{22}C_t + \bar{q}_{23}y_t,$$

$$\bar{f}_3(\bar{X}_t) \equiv \bar{q}_3 + \bar{q}_{13}\hat{C}_t + \bar{q}_{23}C_t + \bar{q}_{33}y_t.$$

Eqs. (5.9) and (C.9) imply that

$$\mathcal{D}\bar{V} = -\bar{V}\left\{\bar{G} - \frac{1}{2}(r\bar{\alpha})^2 f\hat{z}_t\Sigma\hat{z}_t'\right.$$

$$-\frac{1}{2}\beta_1\left[r\bar{\alpha}\gamma_1\hat{z}_t\Sigma p_f' - \bar{f}_1(\bar{X}_t)\right]\left[r\bar{\alpha}\left(2 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_1\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right]$$

$$-\frac{1}{2}\frac{\phi^2\beta_2}{(r+\kappa)^2}\left[r\bar{\alpha}\gamma_1\hat{z}_t\Sigma p_f' - \bar{f}_1(\bar{X}_t)\right]\left[r\bar{\alpha}\left(2 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_2\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right]$$

$$\left.-\frac{1}{2}s^2\left[r\bar{\alpha}\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\hat{z}_t\Sigma p_f' - \frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} - \bar{f}_2(\bar{X}_t)\right]^2\right\}. \tag{C.10}$$

Substituting (C.10) into (3.8), we can write the first-order conditions with respect to $\bar{c}_t$ and $\hat{z}_t$ as (A.3) and

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p_f' p_f\Sigma)\hat{z}_t', \tag{C.11}$$

respectively, where

$$\bar{h}(\bar{X}_t) \equiv ra_0 + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3 + k_1\bar{f}_1(\bar{X}_t) + k_2\bar{f}_2(\bar{X}_t)\right]\Sigma p_f', \tag{C.12}$$

$$k \equiv \beta_1\gamma_1\left(2 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + \frac{\phi^2\beta_2\gamma_1}{(r+\kappa)^2}\left(2 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + s^2\gamma_2^2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)^2, \tag{C.13}$$

$$k_1 \equiv \beta_1\left(1 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + \frac{\phi^2\beta_2}{(r+\kappa)^2}\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + \frac{s^2\beta_2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right), \tag{C.14}$$

$$k_2 \equiv s^2\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right). \tag{C.15}$$

Proceeding as in the proof of Proposition 3.1, we find the following counterpart of (A.10):

$$
\begin{aligned}
&\frac{1}{2}\bar{h}(\bar{X}_t)'(f\Sigma + k\Sigma p_f' p_f \Sigma)^{-1}\bar{h}(\bar{X}_t) + r\bar{\alpha}(\lambda C_t + B)y_t - r\left[\bar{q}_0 + (\bar{q}_1, \bar{q}_2, \bar{q}_3)\bar{X}_t + \frac{1}{2}\bar{X}_t'\bar{Q}\bar{X}_t\right] \\
&+ \bar{f}_1(\bar{X}_t)\left[\kappa(\bar{C} - \hat{C}_t) + \rho(C_t - \hat{C}_t)\right] + \bar{f}_2(\bar{X}_t)\kappa(\bar{C} - C_t) + \bar{f}_3(\bar{X}_t)v_t \\
&+ \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta\bar{q}_{11}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{s^2\beta_2\gamma_2\Delta\bar{q}_{12}}{\mathbf{1}\Sigma\mathbf{1}'} + \frac{1}{2}s^2\bar{q}_{22} + \bar{\beta} - r + r\log(r) \\
&- \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{f}_1(\bar{X}_t)^2}{\mathbf{1}\Sigma\mathbf{1}'} - \frac{1}{2}s^2\left[\frac{\beta_2\gamma_2\Delta\bar{f}_1(\bar{X}_t)}{\mathbf{1}\Sigma\mathbf{1}'} + \bar{f}_2(\bar{X}_t)\right]^2 = 0. \qquad (C.16)
\end{aligned}
$$

Eq. (C.16) is quadratic in $\bar{X}_t$. Identifying quadratic, linear and constant terms yields ten scalar equations in $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q})$. We defer the derivation of these equations until the proof of Proposition 5.4 (see (C.40)-(C.43)). ∎

Proof of Proposition 5.3: Dynamics under the investor's filtration can be deduced from those under the manager's by replacing C_t by the investor's expectation $\hat{C}_t$. Eq. (C.6) implies that the dynamics of $\hat{C}_t$ are

$$
d\hat{C}_t = \kappa(\bar{C} - \hat{C}_t)dt - \beta_1 p_f \sigma d\hat{B}_t^D - \beta_2\left(\frac{\phi p_f \sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2\Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'}\right), \qquad (C.17)
$$

where $\hat{B}_t^D$ is a Brownian motion under the investor's filtration. Eq. (C.7) implies that the net-of-cost return of the active fund is

$$
\begin{aligned}
z_t dR_t - C_t dt = {}& z_t\left\{ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right]\Sigma p_f'\right\}dt - \hat{C}_t dt \\
&+ z_t\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f\sigma\right)d\hat{B}_t^D + z_t\frac{\phi}{r+\kappa}\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma\right)dB_t^F - s\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)z_t\Sigma p_f' dB_t^C, \qquad (C.18)
\end{aligned}
$$

and the return of the index fund is

$$
\begin{aligned}
\mathbf{1}dR_t = {}& \mathbf{1}\left\{ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right]\Sigma p_f'\right\}dt \\
&+ \mathbf{1}\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f\sigma\right)d\hat{B}_t^D + \mathbf{1}\frac{\phi}{r+\kappa}\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f\sigma\right)dB_t^F - s\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\mathbf{1}\Sigma p_f' dB_t^C. \qquad (C.19)
\end{aligned}
$$

Suppose that the investor optimizes over (c_t, x_t) but follows the control v_t given by (5.3). Eqs. (4.4), (5.3), (C.17), (C.18) and (C.19) imply that

$$\begin{aligned}
& d\left(r\alpha W_t + q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t\right) \\
&= G dt + \left[r\alpha(x_t\mathbf{1} + y_t z_t)\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f \sigma\right) - \beta_1 f_1(X_t) p_f \sigma\right] d\hat{B}_t^D \\
&+ \frac{\phi}{r+\kappa}\left[r\alpha(x_t\mathbf{1} + y_t z_t)\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma\right) - \beta_2 f_1(X_t) p_f \sigma\right] dB_t^F \\
&- s\left[r\bar{\alpha}\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p_f' - \frac{\beta_2\gamma_2\Delta f_1(X_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right] dB_t^C, \qquad \text{(C.20)}
\end{aligned}$$

where

$$\begin{aligned}
G \equiv & r\alpha\left[rW_t + (x_t\mathbf{1} + y_t z_t)\left\{ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right]\Sigma p_f'\right\} - y_t\hat{C}_t \right. \\
& \left. -\frac{\psi v_t^2}{2} - c_t\right] + f_1(X_t)\kappa(\bar{C} - \hat{C}_t) + f_2(X_t) v_t + \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta q_{11}}{\mathbf{1}\Sigma\mathbf{1}'},
\end{aligned}$$

$$f_1(X_t) \equiv q_1 + q_{11}\hat{C}_t + q_{12} y_t,$$

$$f_2(X_t) \equiv q_2 + q_{12}\hat{C}_t + q_{22} y_t.$$

Eqs. (4.5) and (C.20) imply that

$$\begin{aligned}
\mathcal{D}V = & -V\left\{G - \frac{1}{2}(r\alpha)^2 f(x_t\mathbf{1} + y_t z_t)\Sigma(x_t\mathbf{1} + y_t z_t)' \right. \\
& - \frac{1}{2}\beta_1\left[r\alpha\gamma_1(x_t\mathbf{1} + y_t z_t)\Sigma p_f' - f_1(X_t)\right]\left[r\alpha\left(2 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p_f' - \frac{\beta_1\Delta f_1(X_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right] \\
& - \frac{1}{2}\frac{\phi^2\beta_2}{(r+\kappa)^2}\left[r\alpha\gamma_1(x_t\mathbf{1} + y_t z_t)\Sigma p_f' - f_1(X_t)\right]\left[r\alpha\left(2 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p_f' - \frac{\beta_2\Delta f_1(X_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right] \\
& \left. -\frac{1}{2}s^2\left[r\alpha\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)(x_t\mathbf{1} + y_t z_t)\Sigma p_f' - \frac{\beta_2\gamma_2\Delta f_1(X_t)}{\mathbf{1}\Sigma\mathbf{1}'}\right]^2\right\}. \qquad \text{(C.21)}
\end{aligned}$$

Substituting (C.21) into (4.6), we can write the first-order conditions with respect to c_t and x_t as (A.13) and

$$\mathbf{1}h(X_t) = r\alpha\mathbf{1}(f\Sigma + k\Sigma p_f' p_f \Sigma)(x_t\mathbf{1} + y_t z_t)', \qquad \text{(C.22)}$$

respectively, where

$$h(X_t) \equiv ra_0 + \left[(g_1^R + g_2^R)\hat{C}_t + g_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3 + k_1 f_1(X_t)\right]\Sigma p_f'. \qquad \text{(C.23)}$$

Proceeding as in the proof of Proposition 4.2, we find the following counterpart of (B.14):

$$r\alpha y_t\theta h(X_t) - \frac{1}{2}(r\alpha)^2 y_t^2\theta(f\Sigma + k\Sigma p_f' p_f\Sigma)\theta' + \frac{[\mathbf{1}h(X_t) - r\alpha f y_t \mathbf{1}\Sigma\theta']^2}{2f\mathbf{1}\Sigma\mathbf{1}'} - r\alpha y_t\hat{C}_t - \frac{1}{2}r\alpha\psi v_t^2$$
$$- r\left[q_0 + (q_1, q_2)X_t + \frac{1}{2}X_t' Q X_t\right] + f_1(X_t)\kappa(\bar{C} - \hat{C}_t) + f_2(X_t)v_t$$
$$+ \frac{1}{2}\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta[q_{11} - f_1(X_t)^2]}{\mathbf{1}\Sigma\mathbf{1}'} + \beta - r + r\log(r) = 0. \tag{C.24}$$

Since v_t in (4.2) is linear in X_t, (C.24) is quadratic in X_t. Identifying quadratic, linear and constant terms yields six scalar equations in (q_0, q_1, q_2, Q). We defer the derivation of these equations until the proof of Proposition 5.4 (see (C.44)-(C.46)).

We next study optimization over v_t, using the same perturbation argument as in the proof of Proposition 4.2. The counterparts of (B.19) and (B.20) are

$$\theta\left[h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p_f'\right] - \hat{C}_t = r\alpha\theta(f\Sigma + k\Sigma p_f' p_f\Sigma)\left[x_t(1-y_t)\mathbf{1} + y_t\theta\right]' + \psi h_\psi(X_t), \tag{C.25}$$

$$\theta\left[h(X_t) - r\alpha\psi b_1 k_1 y_t \Sigma p_f'\right] - \hat{C}_t = r\alpha\theta(f\Sigma + k\Sigma p_f' p_f\Sigma)\left[y_t\theta + \frac{\mathbf{1}h(X_t) - r\alpha y_t f\mathbf{1}\Sigma\theta'}{r\alpha f\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1}\right]' + \psi h_\psi(X_t), \tag{C.26}$$

respectively, where

$$h_\psi(X_t) \equiv (r + b_2)v_t + b_1\kappa(\bar{C} - \hat{C}_t) - b_1\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta f_1(X_t)}{\mathbf{1}\Sigma\mathbf{1}'}.$$

Eq. (C.26) is linear in X_t. Identifying linear and constant terms, yields three scalar equations in (b_0, b_1, b_2). We defer the derivation of these equations until the proof of Proposition 5.4 (see (C.36)-(C.38)). ∎

Proof of Proposition 5.4: We first impose market clearing and derive the constants $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3)$ as functions of $(\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. Setting $z_t = \theta - x_t\mathbf{1}$ and $\bar{y}_t = 1 - y_t$, we can write (C.11) and (C.22) as

$$\bar{h}(\bar{X}_t) = r\bar{\alpha}(f\Sigma + k\Sigma p_f' p_f\Sigma)(1 - y_t)(\theta - x_t\mathbf{1})', \tag{C.27}$$

$$\mathbf{1}h(X_t) = r\alpha\mathbf{1}(f\Sigma + k\Sigma p_f' p_f\Sigma)\left[x_t(1-y_t)\mathbf{1} + y_t\theta\right]', \tag{C.28}$$

respectively. Premultiplying (C.27) by $\mathbf{1}$, dividing by $r\bar{\alpha}$, and adding to (C.28) divided by $r\alpha$, we find

$$\mathbf{1}\left[\frac{h(X_t)}{r\alpha}+\frac{\bar{h}(\bar{X}_t)}{r\bar{\alpha}}\right]=\mathbf{1}(f\Sigma+k\Sigma p_f'p_f\Sigma)\theta'. \tag{C.29}$$

Eq. (C.29) is linear in $(\hat{C}_t, C_t, y_t)$. The terms in $\hat{C}_t$, C_t and y_t are zero because $\mathbf{1}\Sigma p_f'=0$. Identifying constant terms, we find (A.29). Substituting (A.29) into (C.28), we find (A.30).

Substituting (A.30) into (C.27), we find

$$\bar{h}(\bar{X}_t)=r\bar{\alpha}(f\Sigma+k\Sigma p_f'p_f\Sigma)\left[\frac{\alpha}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1}+(1-y_t)p_f\right]'. \tag{C.30}$$

Eq. (C.30) is linear in $\bar{X}_t$. Identifying terms in $\hat{C}_t$, C_t and y_t, we find

$$(r+\kappa+\rho)\gamma_1+b_1\gamma_3+k_1\bar{q}_{11}+k_2\bar{q}_{12}=0, \tag{C.31}$$

$$(r+\kappa)\gamma_2-\rho\gamma_1+k_1\bar{q}_{12}+k_2\bar{q}_{22}=0, \tag{C.32}$$

$$(r+b_2)\gamma_3+k_1\bar{q}_{13}+k_2\bar{q}_{23}=-r\bar{\alpha}\left(f+\frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right), \tag{C.33}$$

respectively. Identifying constant terms, we find

$$a_0=\frac{\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\Sigma\mathbf{1}'+\left[\frac{\kappa(\gamma_1+\gamma_2)\bar{C}+b_0\gamma_3-k_1\bar{q}_1-k_2\bar{q}_2}{r}+\bar{\alpha}\left(f+\frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\right]\Sigma p_f'. \tag{C.34}$$

Using (A.30), we can write (C.26) as

$$\theta h(X_t)-r\alpha\psi b_1k_1\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}y_t-\hat{C}_t=r\alpha\theta(f\Sigma+k\Sigma p_f'p_f\Sigma)\left(\frac{\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\mathbf{1}+y_tp_f\right)'+\psi h_\psi(X_t)$$

$$\Rightarrow\theta h(X_t)-r\alpha\psi b_1k_1\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}y_t-\hat{C}_t=\frac{r\alpha\bar{\alpha}f}{\alpha+\bar{\alpha}}\frac{(\mathbf{1}\Sigma\theta')^2}{\mathbf{1}\Sigma\mathbf{1}'}+r\alpha\left(f+\frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}y_t+\psi h_\psi(X_t). \tag{C.35}$$

Eq. (C.35) is linear in $(\hat{C}_t, y_t)$. Identifying terms in $\hat{C}_t$ and y_t, and using (5.3), we find

$$[(r+\kappa)(\gamma_1+\gamma_2)+b_1\gamma_3+k_1q_{11}]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}-1$$

$$=-\psi b_1\left\{r+\kappa+b_2+\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta q_{11}}{\mathbf{1}\Sigma\mathbf{1}'}\right\}, \tag{C.36}$$

$$[(r+b_2)\gamma_3+(q_{12}-r\alpha\psi b_1)k_1]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}$$

$$=r\alpha\left(f+\frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}-\psi\left\{(r+b_2)b_2+b_1\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta q_{12}}{\mathbf{1}\Sigma\mathbf{1}'}\right\}, \tag{C.37}$$

respectively. Identifying constant terms, and using (5.3) and (C.34), we find

$$\left[k_1(q_1-\bar{q}_1)-k_2\bar{q}_2+r\bar{\alpha}\left(f+\frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\right]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}$$

$$=\psi\left\{(r+b_2)b_0+b_1\kappa\bar{C}-b_1\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta q_1}{\mathbf{1}\Sigma\mathbf{1}'}\right\}. \tag{C.38}$$

The system of equations characterizing equilibrium is as follows. The endogenous variables are $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. (As in Propositions 3.3 and 4.3, we can drop $(\bar{q}_0, q_0)$.) The equations linking them are (5.5)-(5.7), (C.31)-(C.34), (C.36)-(C.38), the nine equations derived from (C.16) by identifying linear and quadratic terms, and the five equations derived from (C.24) by identifying linear and quadratic terms. We next simplify the latter two sets of equations, using implications of market clearing.

Using (C.30), we find

$$\frac{1}{2}\bar{h}(\bar{X}_t)'(f\Sigma+k\Sigma p_fp_f'\Sigma)^{-1}\bar{h}(\bar{X}_t)=\frac{r^2\alpha^2\bar{\alpha}^2f(\mathbf{1}\Sigma\theta')^2}{2(\alpha+\bar{\alpha})^2\mathbf{1}\Sigma\mathbf{1}'}+\frac{1}{2}r^2\bar{\alpha}^2(1-y_t)^2\left(f+\frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'}. \tag{C.39}$$

We next substitute (C.39) into (C.16), and identify terms. Quadratic terms yield the algebraic Riccati equation

$$\bar{Q}\bar{\mathcal{R}}_2\bar{Q}+\bar{Q}\bar{\mathcal{R}}_1+\bar{\mathcal{R}}_1'\bar{Q}-\bar{\mathcal{R}}_0=0, \tag{C.40}$$

where

$$\bar{\mathcal{R}}_2\equiv\begin{pmatrix}\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & \frac{s^2\beta_2\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & 0\\ \frac{s^2\beta_2\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'} & s^2 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\bar{\mathcal{R}}_1 \equiv \begin{pmatrix} \frac{r}{2}+\kappa+\rho & -\rho & 0 \\ 0 & \frac{r}{2}+\kappa & 0 \\ b_1 & 0 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\bar{\mathcal{R}}_0 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r\bar{\alpha}\lambda \\ 0 & r\bar{\alpha}\lambda & r^2\bar{\alpha}^2\left(f+\frac{k\Delta}{\mathbf{1\Sigma1'}}\right)\frac{\Delta}{\mathbf{1\Sigma1'}} \end{pmatrix}.$$

Terms in $\hat{C}_t$, C_t and y_t yield

$$(r+\kappa+\rho)\,\bar{q}_1 + b_1\bar{q}_3 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{q}_1\bar{q}_{11}}{\mathbf{1\Sigma1'}} + s^2\left(\frac{\beta_2\gamma_2\Delta\bar{q}_1}{\mathbf{1\Sigma1'}}+\bar{q}_2\right)\left(\frac{\beta_2\gamma_2\Delta\bar{q}_{11}}{\mathbf{1\Sigma1'}}+\bar{q}_{12}\right)$$
$$-\,\kappa\bar{C}(\bar{q}_{11}+\bar{q}_{12}) - b_0\bar{q}_{13} = 0, \tag{C.41}$$

$$(r+\kappa)\bar{q}_2 - \rho\bar{q}_1 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{q}_1\bar{q}_{12}}{\mathbf{1\Sigma1'}} + s^2\left(\frac{\beta_2\gamma_2\Delta\bar{q}_1}{\mathbf{1\Sigma1'}}+\bar{q}_2\right)\left(\frac{\beta_2\gamma_2\Delta\bar{q}_{12}}{\mathbf{1\Sigma1'}}+\bar{q}_{22}\right)$$
$$-\,\kappa\bar{C}(\bar{q}_{12}+\bar{q}_{22}) - b_0\bar{q}_{23} = 0, \tag{C.42}$$

$$(r+b_2)\bar{q}_3 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2}\right]\frac{\Delta\bar{q}_1\bar{q}_{13}}{\mathbf{1\Sigma1'}} + s^2\left(\frac{\beta_2\gamma_2\Delta\bar{q}_1}{\mathbf{1\Sigma1'}}+\bar{q}_2\right)\left(\frac{\beta_2\gamma_2\Delta\bar{q}_{13}}{\mathbf{1\Sigma1'}}+\bar{q}_{23}\right)$$
$$+\,r^2\bar{\alpha}^2\left(f+\frac{k\Delta}{\mathbf{1\Sigma1'}}\right)\frac{\Delta}{\mathbf{1\Sigma1'}} - r\bar{\alpha}B - \kappa\bar{C}(\bar{q}_{13}+\bar{q}_{23}) - b_0\bar{q}_{33} = 0, \tag{C.43}$$

respectively. Using (A.30), we can write (C.28) as (B.41). Using (5.3), (B.41) and (C.35), we find that the equation derived from (C.24) by identifying quadratic terms is

$$Q\mathcal{R}_2Q + Q\mathcal{R}_1 + \mathcal{R}_1'Q - \mathcal{R}_0 = 0, \tag{C.44}$$

where

$$\mathcal{R}_2 \equiv \begin{pmatrix} \left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1\Sigma1'}}\right]\frac{\Delta}{\mathbf{1\Sigma1'}} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{R}_1 \equiv \begin{pmatrix} \frac{r}{2}+\kappa & r\alpha\psi b_1\left[\beta_1^2+\frac{\phi^2\beta_2^2}{(r+\kappa)^2}+\frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1\Sigma1'}}\right]\frac{\Delta}{\mathbf{1\Sigma1'}} \\ b_1 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\mathcal{R}_0 \equiv \begin{pmatrix} -r\alpha\psi b_1^2 & -r\alpha\psi b_1(r+\kappa+2b_2) \\ -r\alpha\psi b_1(r+\kappa+2b_2) & r^2\alpha^2\left(f+\frac{k\Delta}{\mathbf{1\Sigma1'}}\right)\frac{\Delta}{\mathbf{1\Sigma1'}} + 2r^2\alpha^2\psi b_1k_1\frac{\Delta}{\mathbf{1\Sigma1'}} - r\alpha\psi b_2(2r+3b_2) \end{pmatrix},$$

and the equations derived by identifying terms $\hat{C}_t$ and y_t are

$$(r+\kappa)q_1 + b_1 q_2 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1\Sigma 1'}}\right]\frac{\Delta q_1 q_{11}}{\mathbf{1\Sigma 1'}} - \kappa\bar{C}q_{11} - b_0 q_{12} - r\alpha\psi b_0 b_2 = 0, \tag{C.45}$$

$$(r+b_2)q_2 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1\Sigma 1'}}\right]\frac{\Delta(q_{12}+r\alpha\psi b_1)q_1}{\mathbf{1\Sigma 1'}} - \kappa\bar{C}q_{12} - b_0 q_{22}$$

$$- r\alpha\psi\left[b_0(r+2b_2) + b_1\kappa\bar{C}\right] = 0, \tag{C.46}$$

respectively.

Solving for equilibrium amounts to solving the system of (5.5)-(5.7), (C.31)-(C.34), (C.36)-(C.38), (C.40)-(C.43), (C.44)-(C.46) in the unknowns $(a_0, b_0, b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{Q}, q_1, q_2, Q)$. This reduces to solving the system of (5.5)-(5.7), (C.31)-(C.33), (C.36), (C.37), (C.40), (C.44) in the unknowns $(b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{Q}, Q)$: given $(b_1, b_2, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, T, \bar{Q}, Q)$, $(b_0, \bar{q}_1, \bar{q}_2, \bar{q}_3, q_1, q_2)$ can be determined from the linear system of (C.38), (C.41)-(C.43), (C.45), (C.46), and a_0 from (C.34). We replace the system of (5.5)-(5.7), (C.31)-(C.33), (C.36), (C.37), (C.40), (C.44) by the equivalent system of (5.5)-(5.7), (C.31)-(C.33), (C.40), (C.44),

$$\psi b_1\left\{r+\kappa+b_2+\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1\Sigma 1'}}\right]\frac{\Delta q_{11}}{\mathbf{1\Sigma 1'}}\right\}$$

$$= 1 + \left[k_1(\bar{q}_{11}+\bar{q}_{12}) + k_2(\bar{q}_{12}+\bar{q}_{22}) - k_1 q_{11}\right]\frac{\Delta}{\mathbf{1\Sigma 1'}}, \tag{C.47}$$

$$\psi\left\{(r+b_2)b_2 + b_1\left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1\Sigma 1'}}\right]\frac{\Delta q_{12}}{\mathbf{1\Sigma 1'}}\right\} - r\alpha\psi b_1 k_1\frac{\Delta}{\mathbf{1\Sigma 1'}}$$

$$= r(\alpha+\bar{\alpha})\left(f + \frac{k\Delta}{\mathbf{1\Sigma 1'}}\right)\frac{\Delta}{\mathbf{1\Sigma 1'}} + (k_1\bar{q}_{13} + k_2\bar{q}_{23} - k_1 q_{12})\frac{\Delta}{\mathbf{1\Sigma 1'}}. \tag{C.48}$$

For $s = 0$, the unique non-negative solution of (5.7) is $T = 0$. Eqs. (5.5), (5.6), (C.8) and (C.13)-(C.15) imply that $\beta_1 = \beta_2 = \rho = k = k_1 = k_2 = 0$. Eqs. (C.31)-(C.33), (C.40), (C.44), (C.47) and (C.48) become

$$(r+\kappa)\gamma_1 + b_1\gamma_3 = 0, \tag{C.49}$$

$$(r+\kappa)\gamma_2 = 0, \tag{C.50}$$

$$(r+b_2)\gamma_3 = -r\bar{\alpha}f, \tag{C.51}$$

(B.49), (B.50), (B.51) and (B.52), respectively, where

$$\bar{\mathcal{R}}_1^0 \equiv \begin{pmatrix} \frac{r}{2}+\kappa & 0 & 0 \\ 0 & \frac{r}{2}+\kappa & 0 \\ b_1 & 0 & \frac{r}{2}+b_2 \end{pmatrix},$$

$$\bar{\mathcal{R}}_0 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r\bar{\alpha}\lambda \\ 0 & r\bar{\alpha}\lambda & r^2\bar{\alpha}^2 f \frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} \end{pmatrix},$$

and $(\mathcal{R}_1^0, \mathcal{R}_0^0)$ are as under symmetric information (Proposition 4.3). Given the unique positive solution b_2 of (B.52), $(b_1, \gamma_3, \gamma_1, \bar{Q}, Q)$ are determined uniquely from (B.51), (C.51), (C.49), (B.49) and (B.50), respectively, and (C.50) implies that $\gamma_2 = 0$. We denote the solution for $s = 0$ by $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \gamma_3^0, \beta_1^0, \beta_2^0, T^0, \bar{Q}^0, Q^0)$. The variables $(b_1^0, b_2^0, \gamma_1^0, \gamma_3^0, Q^0)$ coincide with $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, Q^0)$ under symmetric information. Proceeding as in the proof of Proposition 4.3, we can apply the implicit function theorem and show that the system of (5.5)-(5.7), (C.31)-(C.33), (C.40), (C.44), (C.47), (C.48) has a solution for small s. Moreover, this solution is unique in a neighborhood of $(b_1^0, b_2^0, \gamma_1^0, \gamma_2^0, \gamma_3^0, \beta_1^0, \beta_2^0, T^0, \bar{Q}^0, Q^0)$, which corresponds to the unique equilibrium for $s = 0$. Since $b_1^0 > 0$, $b_2^0 > 0$, $\gamma_1^0 > 0$, $\gamma_3^0 < 0$, continuity implies that $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_3 < 0$ for small s. Since $\gamma_2^0 = 0$, continuity does not establish the sign of γ_2 for small s, so we need to study the asymptotic behavior of the solution. Eqs. (5.7), (5.5) and (5.6) imply that

$$T = \frac{s^2}{2\kappa} + o(s^2), \tag{C.52}$$

$$\beta_1 = \frac{\mathbf{1}\Delta\mathbf{1}'}{2\kappa\Delta} s^2 + o(s^2) \equiv \hat{\beta}_1^0 s^2 + o(s^2), \tag{C.53}$$

$$\beta_2 = o(s^2), \tag{C.54}$$

respectively, where $\frac{o(s^2)}{s^2}$ converges to zero when s goes to zero. Eqs. (C.8) and (C.13)-(C.15) imply that

$$\rho = \hat{\beta}_1^0 s^2 + o(s^2), \tag{C.55}$$

$$k = 2\hat{\beta}_1^0\gamma_1^0 s^2 + o(s^2), \tag{C.56}$$

$$k_1 = \hat{\beta}_1^0 s^2 + o(s^2), \tag{C.57}$$

$$k_2 = o(s^2), \tag{C.58}$$

respectively, and (C.40) implies that

$$\bar{Q}^0 = \begin{pmatrix} \frac{2r^2\bar{\alpha}^2(b_1^0)^2 f\Delta}{(r+2\kappa)(r+\kappa+b_2^0)(r+2b_2^0)\mathbf{1}\Sigma\mathbf{1}'} & -\frac{r\bar{\alpha}b_1^0\lambda}{(r+2\kappa)(r+\kappa+b_2^0)} & -\frac{r^2\bar{\alpha}^2 b_1^0 f\Delta}{(r+\kappa+b_2^0)(r+2b_2^0)\mathbf{1}\Sigma\mathbf{1}'} \\ -\frac{r\bar{\alpha}b_1^0\lambda}{(r+2\kappa)(r+\kappa+b_2^0)} & 0 & \frac{r\bar{\alpha}\lambda}{r+\kappa+b_2^0} \\ -\frac{r^2\bar{\alpha}^2 b_1^0 f\Delta}{(r+\kappa+b_2^0)(r+2b_2^0)\mathbf{1}\Sigma\mathbf{1}'} & \frac{r\bar{\alpha}\lambda}{r+\kappa+b_2^0} & \frac{r^2\bar{\alpha}^2 f\Delta}{(r+2b_2^0)\mathbf{1}\Sigma\mathbf{1}'} \end{pmatrix}. \tag{C.59}$$

Eqs. (C.32), (C.55), (C.57), (C.58) and (C.59) imply that

$$\gamma_2 = \hat{\beta}_1^0 \left[\gamma_1^0 + \frac{r\bar{\alpha}b_1^0\lambda}{(r+2\kappa)(r+\kappa+b_2^0)}\right] s^2 + o(s^2).$$

Therefore, $\gamma_2 > 0$ if $\lambda \geq 0$. ∎

Proof of Corollary 5.1: Eq. (C.7) implies that the covariance matrix of stock returns is

$$\begin{aligned} Cov_t(dR_t, dR_t') &= \left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f \sigma\right)\left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f \sigma\right)' \\ &\quad + \frac{\phi^2}{(r+\kappa)^2}\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma\right)\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma\right)' \\ &\quad + s^2\gamma_2^2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)^2 \Sigma p_f' p_f \Sigma, \end{aligned}$$

which is equal to (5.10) because of (C.13). Eqs. (3.20) (which is also valid under gradual adjustment) and (5.10) imply that the proportionality coefficient between the non-fundamental covariance matrices under asymmetric and symmetric information is larger than one if $k > s^2\gamma_{1sym}^2$, where γ_{1sym} denotes the value of γ_1 under symmetric information. Rearranging (C.13), we find

$$k = 2\left\{\beta_1 + \beta_2\left[\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\right\}\gamma_1 + s^2\gamma_2^2 + \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\gamma_1^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}. \tag{C.60}$$

Rearranging (5.6), we find

$$\beta_2\left[\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right] = s^2\gamma_2, \tag{C.61}$$

and rearranging (5.7), we find

$$T^2\left[1 - (r+k)\frac{\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]^2 \frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} + \frac{\frac{s^4\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}}{\frac{\phi^2}{(r+\kappa)^2} + \frac{s^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}} = s^2 - 2\kappa T$$

$$\Rightarrow \left[\beta_1^2 + \frac{\phi^2\beta_2^2}{(r+\kappa)^2} + \frac{s^2\beta_2^2\gamma_2^2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right]\frac{\Delta}{\mathbf{1}\Sigma\mathbf{1}'} = s^2 - 2\kappa T, \tag{C.62}$$

where the second step follows from (5.5) and (5.6). Substituting (C.61) and (C.62) into (C.60), we find

$$\begin{aligned} k &= 2\beta_1\gamma_1 + s^2(\gamma_1+\gamma_2)^2 - 2\kappa T\gamma_1^2 \\ &= s^2(\gamma_1+\gamma_2)^2 + 2T\gamma_1\left[\frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} - \kappa\gamma_1 - (r+\kappa)\gamma_2\right], \end{aligned} \tag{C.63}$$

where the second step follows from (5.5).

Eqs. (C.52), (C.63) and $\gamma_2^0 = 0$ imply that for small s,

$$k = s^2(\gamma_1^0)^2 + \frac{s^2\gamma_1^0}{\kappa}\left(\frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} - \kappa\gamma_1^0\right) + o(s^2). \tag{C.64}$$

The variables (b_1^0, b_2^0) are identical under symmetric and asymmetric information. Moreover, (B.47), (B.48), (C.49) and (C.51) imply that the same is true for γ_1^0. Therefore, $k > s^2\gamma_{1sym}^2$ for small s if

$$\begin{aligned} &\frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} - \kappa\gamma_1^0 > 0 \\ \Leftrightarrow\ &\frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta} - \frac{\kappa r\bar{\alpha} f b_1^0}{(r+\kappa)(r+b_2^0)} > 0 \\ \Leftrightarrow\ &\frac{\mathbf{1}\Sigma\mathbf{1}'}{\Delta}\left[1 - \frac{\kappa\bar{\alpha} b_2^0}{(r+\kappa)(\alpha+\bar{\alpha})(r+\kappa+b_2^0)}\right] > 0, \end{aligned} \tag{C.65}$$

where the second step follows from (C.49) and (C.51), and the third from (B.51) and (B.52). Since $b_2^0 > 0$, (C.65) holds. ∎

Proof of Corollary 5.2: Stocks' expected returns are

$$\begin{aligned} E_t(dR_t) &= \left\{ra_0 + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t - \kappa(\gamma_1+\gamma_2)\bar{C} - b_0\gamma_3\right]\Sigma p_f'\right\}dt \\ &= \left\{\frac{r\alpha\bar{\alpha} f}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\Sigma\mathbf{1}' + \left[\gamma_1^R\hat{C}_t + \gamma_2^R C_t + \gamma_3^R y_t + r\bar{\alpha}\left(f + \frac{k\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) - k_1\bar{q}_1 - k_2\bar{q}_2\right]\Sigma p_f'\right\}dt \\ &= \left[\frac{r\alpha\bar{\alpha}}{\alpha+\bar{\alpha}}\frac{\mathbf{1}\Sigma\theta'}{\mathbf{1}\Sigma\mathbf{1}'}\left(f\Sigma + k\Sigma p_f' p_f\Sigma\right)\mathbf{1}' + \Lambda_t\left(f\Sigma + k\Sigma p_f' p_f\Sigma\right)p_f'\right]dt, \end{aligned} \tag{C.66}$$

where the first step follows from (C.7), the second from (C.34), and the third from (5.11). Eq. (C.66) is equivalent to (3.21) because of (5.10).

Eqs. (C.31) and (C.32) imply that γ_1^R and γ_2^R have the opposite sign of $k_1\bar{q}_{11} + k_2\bar{q}_{12}$ and $k_1\bar{q}_{12} + k_2\bar{q}_{22}$, respectively. Eqs. (C.57) and (C.58) imply that for small s, the latter variables have the same sign as $\bar{q}_{11}^0$ and $\bar{q}_{12}^0$, respectively. Since $b_1^0 > 0$ and $b_2^0 > 0$, (C.59) implies that $\bar{q}_{11}^0 > 0$ and $\bar{q}_{12}^0$ has the same sign as $-\lambda$. Therefore, for small s, $\gamma_1^R < 0$ and γ_2^R has the same sign as λ. Moreover, $\gamma_3^R < 0$ since $b_2 > 0$ and $\gamma_3 < 0$. ∎

Proof of Corollary 5.3: Using (C.7) and proceeding as in the derivation of (A.56), we find

$$Cov_t(dD_t, dR'_{t'}) = \sigma Cov_t\left(dB_t^D, \gamma_1^R \hat{C}_{t'} + \gamma_2^R C_{t'} + \gamma_3^R y_{t'}\right) p_f \Sigma dt, \tag{C.67}$$

$$Cov_t(dF_t, dR'_{t'}) = \phi\sigma Cov_t\left(dB_t^F, \gamma_1^R \hat{C}_{t'} + \gamma_2^R C_{t'} + \gamma_3^R y_{t'}\right) p_f \Sigma dt. \tag{C.68}$$

The covariances (C.67) and (C.68) depend only on how the Brownian shocks dB_t^D and dB_t^F, respectively, impact $(\hat{C}_{t'}, C_{t'}, y_{t'})$. To compute the impact of these shocks, as well as of dB_t^C for the next corollary, we solve the impulse-response dynamics

$$dC_t = -\kappa C_t dt,$$

$$d\hat{C}_t = \left[-\kappa \hat{C}_t + \rho(C_t - \hat{C}_t)\right] dt,$$

$$dy_t = -\left(b_1 \hat{C}_t + b_2 y_t\right) dt,$$

with the initial conditions

$$C_t = s dB_t^C,$$

$$\hat{C}_t = -\beta_1 p_f \sigma dB_t^D - \beta_2 \left(\frac{\phi p_f \sigma dB_t^F}{r+\kappa} - \frac{s\gamma_2 \Delta dB_t^C}{\mathbf{1}\Sigma\mathbf{1}'}\right),$$

$$y_t = 0.$$

The solution to these dynamics is (B.57),

$$\hat{C}_{t'} = e^{-\kappa(t'-t)} s dB_t^C - e^{-(\kappa+\rho)(t'-t)} \left[\beta_1 p_f \sigma dB_t^D + \frac{\phi\beta_2 p_f \sigma dB_t^F}{r+\kappa} + s\left(1 - \frac{\beta_2\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) dB_t^C\right] \tag{C.69}$$

$$\begin{aligned} y_{t'} = &-\frac{b_1}{b_2-\kappa}\left[e^{-\kappa(t'-t)} - e^{-b_2(t'-t)}\right] s dB_t^C \\ &+ \frac{b_1}{b_2-\kappa-\rho}\left[e^{-(\kappa+\rho)(t'-t)} - e^{-b_2(t'-t)}\right]\left[\beta_1 p_f \sigma dB_t^D + \frac{\phi\beta_2 p_f \sigma dB_t^F}{r+\kappa} + s\left(1 - \frac{\beta_2\gamma_2\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) dB_t^C\right]. \end{aligned} \tag{C.70}$$

Substituting (B.57), (C.69) and (C.70) into (C.67) and (C.68), and using the mutual independence of (dB_t^D, dB_t^F, dB_t^C), we find (5.12) with

$$\chi_1^D \equiv \beta_1\left(\frac{b_1\gamma_3^R}{b_2-\kappa-\rho} - \gamma_1^R\right) = (r+\kappa+\rho)\beta_1\left(\frac{b_1\gamma_3}{b_2-\kappa-\rho} - \gamma_1\right), \tag{C.71}$$

$$\chi_2^D \equiv -\frac{b_1\beta_1\gamma_3^R}{b_2-\kappa-\rho} = -\frac{(r+b_2)b_1\beta_1\gamma_3}{b_2-\kappa-\rho}. \tag{C.72}$$

The function $\chi^D(u) \equiv \chi_1^D e^{-(\kappa+\rho)u} + \chi_2^D e^{-b_2 u}$ can change sign only once, is equal to $-\beta_1\gamma_1^R$ when $u=0$, and has the sign of χ_1 if $b_2 > \kappa+\rho$ and of χ_2 if $b_2 < \kappa+\rho$ when u goes to ∞. For small s, $\chi(0) > 0$ since $\gamma_1^R < 0$. Since, in addition, $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_3 < 0$ and $\rho > 0$, (C.71) and (C.72) imply that $\chi_1 < 0$ if $b_2 > \kappa+\rho$ and $\chi_2 < 0$ if $b_2 < \kappa+\rho$. Therefore, there exists a threshold $\hat{u}^D > 0$ such that $\chi(u) > 0$ for $0 < u < \hat{u}^D$ and $\chi(u) < 0$ for $u > \hat{u}^D$. ∎

Proof of Corollary 5.4: Using (C.7) and proceeding as in the derivation of (A.56), we find

$$\begin{aligned}
Cov_t(dR_t, dR_{t'}') &= \left(\sigma + \beta_1\gamma_1\Sigma p_f' p_f \sigma\right) Cov_t\left(dB_t^D, \gamma_1^R\hat{C}_{t'} + \gamma_2^R C_{t'} + \gamma_3^R y_{t'}\right) p_f \Sigma dt \\
&\quad + \frac{\phi}{r+\kappa}\left(\sigma + \beta_2\gamma_1\Sigma p_f' p_f \sigma\right) Cov_t\left(dB_t^F, \gamma_1^R\hat{C}_{t'} + \gamma_2^R C_{t'} + \gamma_3^R y_{t'}\right) p_f \Sigma dt \\
&\quad - s\gamma_2\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) Cov_t\left(dB_t^C, \gamma_1^R\hat{C}_{t'} + \gamma_2^R C_{t'} + \gamma_3^R y_{t'}\right) \Sigma p_f' p_f \Sigma dt.
\end{aligned} \tag{C.73}$$

Substituting (B.57), (C.69) and (C.70) into (C.73), and using (5.6) and the mutual independence of (dB_t^D, dB_t^F, dB_t^C), we find (5.13) with

$$\chi_1 \equiv \chi_1^D\left(1 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right), \tag{C.74}$$

$$\begin{aligned}
\chi_2 &\equiv s^2\gamma_2\left(\frac{b_1\gamma_3^R}{b_2-\kappa} - \gamma_1^R - \gamma_2^R\right)\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) \\
&= s^2(r+\kappa)\gamma_2\left(\frac{b_1\gamma_3}{b_2-\kappa} - \gamma_1 - \gamma_2\right)\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right),
\end{aligned} \tag{C.75}$$

$$\begin{aligned}
\chi_3 &\equiv -b_1\gamma_3^R\left[\frac{\beta_1}{b_2-\kappa-\rho}\left(1 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + \frac{s^2\gamma_2}{b_2-\kappa}\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\right] \\
&= -(r+b_2)b_1\gamma_3\left[\frac{\beta_1}{b_2-\kappa-\rho}\left(1 + \frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + \frac{s^2\gamma_2}{b_2-\kappa}\left(1 + \frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)\right].
\end{aligned} \tag{C.76}$$

The function $\chi(u) \equiv \chi_1 e^{-(\kappa+\rho)u} + \chi_2 e^{-\kappa u} + \chi_3 e^{-b_2 u}$ has the same sign as $\hat{\chi}(u) \equiv \chi_1 e^{-\rho u} + \chi_2 + \chi_3 e^{-(b_2-\kappa)u}$. The latter function is equal to

$$-\beta_1\gamma_1^R\left(1+\frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) - s^2\gamma_2(\gamma_1^R+\gamma_2^R)\left(1+\frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right)$$

when $u = 0$, and has the sign of χ_2 if $b_2 > \kappa$ and $\rho > 0$ and of χ_3 if $b_2 < \kappa$ and $\rho > 0$ when u goes to ∞. Moreover, its derivative $\hat{\chi}'(u) = -\chi_1\rho e^{-\rho u} - \chi_3(b_2-\kappa)e^{-(b_2-\kappa)u}$ is equal to

$$-\chi_1\rho - \chi_3(b_2-\kappa) = \beta_1(\rho\gamma_1^R + b_1\gamma_3^R)\left(1+\frac{\beta_1\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) + s^2 b_1\gamma_2\gamma_3^R\left(1+\frac{\beta_2\gamma_1\Delta}{\mathbf{1}\Sigma\mathbf{1}'}\right) \tag{C.77}$$

when $u = 0$. For small s, $\chi(0) > 0$ since $\gamma_1^R < 0$ and $s^2\gamma_2/\beta_1 = o(1)$. Since, in addition, $b_1 > 0$, $b_2 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 < 0$, $\gamma_3^R < 0$ and $\rho > 0$, (C.75) and (C.76) imply that $\chi_2 < 0$ if $b_2 > \kappa$ and $\chi_3 < 0$ if $b_2 < \kappa$, and (C.77) implies that $\hat{\chi}'(0) < 0$. Since $\hat{\chi}'(u)$ can change sign only once, it is either negative or negative and then positive. Therefore, $\hat{\chi}(u)$ is positive and then negative. The same is true for $\chi(u)$, which means that there exists a threshold $\hat{u} > 0$ such that $\chi(u) > 0$ for $0 < u < \hat{u}$ and $\chi(u) < 0$ for $u > \hat{u}$. ∎

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