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INTERNATIONAL POLICY COORDINATION IN DYNAMIC MACROECONOMIC MODELS

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ABSTRACT

Recent analyses of the gains to policy coordination have focussed on the strategic aspects of macroeconomic policymaking in a static setting. A major theme is that noncooperative policy making is likely to be Pareto inefficient because of the presence of beggar-thy-neighbor policies. This paper extends the analysis to a dynamic setting, thereby introducing three important points of realism to the static game. First, the payoffs to beggar-thy-neighbor policies look very different in one-period and multiperiod games, and thus so do the gains to coordination. Second, we show that policy coordination may reduce economic welfare if governments are myopic in their policy making, as is sometimes claimed. Third, governments act under a fundamental constraint that they cannot bind the actions of later governments, and we investigate how this constraint alters the gains to policy coordination.

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I. Introduction

In an earlier essay (Oudiz and Sachs, 1984) we investigated the quantitative gains to international policy coordination in a static environment. In this paper, we begin to extend the analysis to a dynamic setting. However, because of several new methodological issues, this first step is more theoretical than empirical. The extension to dynamics introduces three important points of realism to the static game. First, the payoffs to beggar-thy-neighbor policies may look very different in one-period and multiperiod games, so that the need for policy coordination may be different in the two games. Second, it is often claimed that governments are shortsighted in macroeconomic planning, and support for this view has come from the literature on political business cycles.[1] We should therefore investigate whether international policy coordination is likely to exacerbate or meliorate this shortsighted behavior. Third, governments act under a fundamental constraint that they cannot bind the actions of later governments (or even of themselves at a future date). In principle, therefore, optimizing governments must take into account how future governments will behave in view of the economic environment that they inherit. We study the implications for policy coordination of this inability to bind future governments.

Let us consider these three points in turn. In the static game, uncoordinated macroeconomic policy-making is typically inefficient because of a prisoner's dilemma in policy choices. Consider, for example, two countries that are attempting to move optimally along a short-run Phillips curve. It may be that each country will choose contractionary policies no matter what the other country selects, though the policy pair (expand, expand) is better for both countries than the non-cooperative equilibrium (contract, contract). As we showed in our earlier study, this situation arises naturally under flexible exchange rates, since by contracting while the other country is expanding, a country can appreciate its currency and export some of its inflation abroad. It is this beggar-thy-neighbor action that gives rise to the prisoner's dilemma. Cooperation, say in the form of a binding international commitment to expand, may be useful in moving the countries to the efficient equilibrium.

The question arises whether the payoff structure in a multiperiod, or infinite-horizon game will look the same. The reason for doubt is simple. In almost all macroeconomic models, policies which lead to a short-run real appreciation also lead to long-run real depreciation, or at least a return to the initial real exchange rate. In this circumstance, farsighted players would understand that a short-run beggar-thy-neighbor appreciation is less attractive than it looks, since it will be reversed in the long run, at which point the country reimports the inflation that it earlier sent abroad. To this extent, the beggar-thy-neighbor policy loses its appeal, and the need for coordination is reduced.

The second theme introduced in a multiperiod setting is the myopic behavior of governments. In considering public welfare in a multiperiod game, it is natural to consider a payoff of the form:

$$(1) \qquad U_0^i = \Sigma_{t=0}^{T} \beta^t u(T_t^i)$$

Here, U_0^i is the intertemporal utility of country i as of time zero. $u(T_t^i)$ is the instantaneous utility of the country at time t, as a function of a vector of macroeconomic targets T_t^i. β is a pure rate of time preference, with $\beta < 1$, so that the future is discounted relative to the present.

In view of the evidence on political business cycles, in which governments attempt to manipulate T_t^i in conjunction with upcoming elections, it seems natural to suggest that if (1) is the "true" social welfare function, the government's social welfare function takes the form:

$$U_0^{gi} = \Sigma_{t=0}^{T^g} \beta^{gt} u(T_t^i) \tag{2}$$

where $T^g < T$ and $\beta^g < \beta$. That is, its planning horizon is shorter than the economy's, or its discounting of the future is higher.

In this view, the public is partly a hostage of a self-serving government. The policy choices reflect the incumbent government's goals, and not the public's. If this is so, we can ask whether international policy coordination is likely to improve or worsen this sub-optimal situation. At an abstract level, the arguments seem to fall on both sides. Some critics, for example, have characterized policy coordination as a cartel of the incumbents, in which each policymaker helps the others to manipulate the political business cycle. As an example of this, policymakers may have a short-run expansionary bias if expansion shows up as output today and as inflation only many years in the future. To some extent, the fear of currency depreciation following a unilateral expansion keeps this bias in check. That is, the flexible exchange rate provides discipline on the shortsighted government. With policy coordination, the fear of currency depreciation can be removed by a commitment of all countries to expand. In this way, policy coordination may give incumbent governments a free hand to undertake overly inflationary policies.

On the other hand, we can think of circumstances in which policy coordination ties the hand of incumbents, and thus prevents such self-serving policies. An international gold standard, for example, might impose discipline on governments that would not exist in each country alone. To analyze this possibility fully we would have to examine each government's incentive to stick with a particular rule, and the extent to which internationally certified rules are more or less durable than rules undertaken unilaterally. For example, each country on its own could adopt a gold standard. What, if anything, is added by a multicountry commitment?

The third theme introduced in a multiperiod setting is that of "time-consistency" of optimal plans. Even in circumstances in which the current government (or current administration) has the public's interest at heart, its ability to maximize social welfare may be limited by its inability to pre-commit the actions of (well-meaning) future governments. In these circumstances, the current government must choose its optimal policy <u>taking as given</u> the policy rules that will be pursued in the future. That is, it must optimize today, assuming that future governments will optimize under the assumption that yet future governments will optimize, and so on. In general this constrained optimization yields a lower level of social welfare than does the case in which the government can choose not only its own policies but those of future governments as well.

Many authors, including Barro and Gordon (1983) and Rogoff (1983), have given examples in which the inability to bind future policies imparts an inflationary bias to the economy. In these examples, wage setters set wages before macroeconomic policy is set. Once the wages are set, policymakers have an incentive to expand the economy to reduce real wages, and raise output. Wage setters anticipate these policies, and choose inflationary wage settlements in anticipation. If the government can pre-commit to avoid inflationary policies, the economy can get the same ex post output levels at a lower rate of inflation. Unfortunately, such a pre-commitment is not credible since the government has an incentive to renege on it after the wages are set.

As Rogoff stresses, this time consistency problem may have important consequences for international policy coordination. If the inability to bind future policies leads to an inflationary bias, international policy coordination may further exacerbate this bias by eliminating each country's concern about currency depreciation. Thus, even when a sequence of governments within each country is trying to maximize that country's true social welfare function, policy coordination may make the situation worse rather than better.

We consider later on several factors that tend to weaken this pessimistic conclusion. First, in infinite-horizon games, governments may be able to invest in a "reputation" in order to overcome the time-inconsistency problem (as illustrated in Barro and Gordon (1983)). In other words, a government's credibility may be judged by its willingness to honor a program laid down by an earlier government, so much that it continues the policy rather than reoptimizing during its incumbency. We will provide an example of this solution to the time inconsistency problem. Second, to the extent that the time inconsistency problem revolves around the exchange rate, policy coordination may actually eliminate the problem. In examples later in the paper, optimal coordinated policies in our a two-country model turn out to be time-consistent.

The plan of the paper is as follows. In the next section we set out a simple dynamic macroeconomic model characterized by flexible exchange rates and perfect foresight on the part of the private and public sectors. In Section III, we describe various equilibria in a one-country version of the model, to highlight the implications of time inconsistency. Next, in Section IV, we describe the various equilibria in the two-country version of the game, including the welfare gains or losses from policy coordination. Extensions and conclusions are discussed in a final section.

II. A Simple Dynamic Macroeconomic Model

We consider a simple model of the sort explored by Dornbusch (1976). The home country produces output Q, at price P, and trades with a foreign country, which produces Q* at price P*. The domestic exchange rate E measures units of home currency per unit of foreign currency, so that the relative price of the home good is P/EP*. Demand for the home good is a decreasing function of P/EP* and of the real interest rate, and an increasing function of Q*. Letting lower case variables p, q, and e represent the logarithms of their upper-case counterparts, we write demand for home goods as:

$$q_t = -\delta(p_t - e_t - p_t^*) - \sigma[i_t - (p_{t+1}^e - p_t)] + \gamma q_t^* \tag{3}$$

Here, i is the nominal interest rate, and $i_t - (p_{t+1}^e - p_t)$ the home real interest rate at time t (p_{t+1}^e is the expectation of p_{t+1} at time t). Under the perfect foresight assumption, which we hereafter maintain, $p_{t+1}^e = p_{t+1}$ for all $t > 0$.

The money demand equations take the standard transactions form:

(4) $$m_t - p_t = \alpha q_t - \varepsilon i_t$$

For convenience, we will invert this equation and write

(5) $$i_t = \mu q_t - \rho(m_t - p_t)$$

with $\mu = \alpha/\varepsilon$ and $\rho = 1/\varepsilon$. Following Dornbusch, we assume perfect capital mobility, so that uncovered interest arbitrage holds:

(6) $$e^e_{t+1} - e_t = i_t - i^*_t$$

Again, assuming perfect foresight, we solve for equilibria with $e^e_{t+1} = e_{t+1}$ for all t.

It remains to specify wage and price dynamics. First, the (log) consumer price index (p^c) is written as a weighted average of home (p) and foreign (p^*+e) prices:

(7) $$p^c_t = \lambda p_t + (1-\lambda)(p^*_t + e_t)$$

Home prices are written as a fixed markup over wages:

(8) $$p_t = w_t$$

Finally, nominal wage change, $w_{t+1} - w_t$, is made a function of lagged nominal price change, $p^c_t - p^c_{t-1}$, output, and output change:

(9) $$(w_{t+1} - w_t) = (p^c_t - p^c_{t-1}) + \psi q_t + \theta(q_t - q_{t-1})$$

Note that since $w_{t+1} - w_t$ is a function of <u>lagged</u> rather than contemporaneous price change, the system will display typical Keynesian features, particularly the non-neutrality of q_t with respect to contemporaneous and future anticipated changes in m_t. This is the standard presumption in the Dornbusch model that the labor market clears more slowly than the asset markets.

In the next section, we will introduce corresponding equations for the second country, in order to construct a two-country model. Here, we focus on the one-country case by making the small-country assumption for the home economy that p^*, i^*, and q^* are given for all $t \geq 0$. By doing so, we can write the one-country model as a four-dimensional difference equation system as in (10):[2]

$$(10) \qquad \begin{bmatrix} p_{t+1} \\ p^c_t \\ q_t \\ e_{t+1} \end{bmatrix} = A \begin{bmatrix} p_t \\ p^c_{t-1} \\ q_{t-1} \\ e_t \end{bmatrix} + B\, m_t + C \begin{bmatrix} p^*_t \\ i^*_t \\ q^*_t \end{bmatrix}$$

In any given period, p_t, p^c_{t-1}, and q_{t-1} are given by the past history of the economy. These are the "pre-determined" variables of the economy. m_t, and indeed the entire sequence of m, is chosen as a policy variable. p^*_t, i^*_t, and q^*_t are exogenous forcing variables of the system from the point of view of the home economy.

As is typical of perfect foresight models, an asset price such as e_t is determined not by past history but by forward-looking behavior of asset holders. In particular, for given values of p_t, p^c_{t-1}, q_{t-1}, and given sequences of p^*, i^*, and q^* from t to infinity, there is typically a unique value of e_t such that the exchange rate does not grow or collapse explosively (technically, this unique value of e_t puts the economy on its stable manifold). Such a unique value of e_t exists as long as the eigenvalue associated with e_t in the A matrix is is outside of the unit circle, and the remaining eigenvalues are on or within the unit circle. In the simulations reported below, this condition is always satisfied.

The goal of economic policy in our model will be to maximize a social welfare function as in (1) or (2), subject to the constraint in (10). The assumption that e_t is always such as to keep the economy on the saddlepoint path (or stable manifold) requires that economic agents have complete knowledge as to the path of future policies. In this sense, the government is like a Stackelberg leader with respect to the private sector, choosing monetary policy with a view to affecting e_t and thereby more basic economic targets, while e_t is chosen taking as given the future sequence of m. This is not to say, however, that governments can necessarily choose any sequence of m that they desire. A large part of the discussion that follows describes the "admissible" sequences of policies.

As a concrete example of this model, we will suppose that instantaneous utility $u(T_t^i)$ is a quadratic function of inflation, $\pi_t = p_t^c - p_{t-1}^c$, and the deviation of output from full employment q_t. That is, $u_t = -(1/2)(q_t^2 + \phi\pi_t^2)$. Thus, intertemporal utility is

$$(11) \qquad U_0 = -(1/2)\Sigma_{t=0}^{\infty}\beta^t(q_t^2+\phi\pi_t^2)$$

Note that ϕ is a parameter reflecting the weight attached to π_t relative to q_t. β is the discount factor. We have written the utility function with an infinite horizon, and we will point out shortly some special features of the problem that arise with such a formulation.

We now turn to the optimal policy for m. It may seem straightforward to maximize (11) subject to (10), but as Phelps and Pollak (1968) first explained, and Kydland and Prescott (1977) further elucidated, the maximization is quite problematic. Here we sketch the problem, and treat it in greater detail below.

Suppose that we apply optimal control techniques to the problem of maximizing U_0 subject to (10), taking as given p_0, p^c_{-1}, q_{-1}. For simplicity, we set $p^*_t = i^*_t = q^*_t = 0$ for all $t \geq 0$. The result of this straightforward control problem will be an infinite sequence $m_0, m_1, \ldots$, denoted hereafter $\{m\}^\infty_0$, that maximizes U_0. Let us write this optimal choice of monetary policy as $\{\hat{m}\}^\infty_0$. We have already noted that e_0 will in general be a function of p_0, p^c_{-1}, q_{-1} and the <u>entire</u> sequence $\{\hat{m}\}^\infty_0$. The first step of this sequence is $\hat{m}_0$.

Given $\hat{m}_0$, e_0, p_0, p^c_{-1}, and q_{-1}, we can use (10) to find p^c_1, p_0, q_0. Suppose now that at time 1 the policymakers reoptimize, in order to maximize U_1 subject (10). Once again, a simple control problem will yield a sequence $m_1, m_2, \ldots$, now denoted as $\{\tilde{m}\}^\infty_1$. In general, $\hat{m}_t$ will not equal $\tilde{m}_t$ for $t \geq 1$, so that the government at time 1 will not want to carry on with the optimal plan as of time zero. If the government at time 1 is not bound (e.g. by a constitution) to carry out $\{\hat{m}\}^\infty_1$, the earlier plan will be scrapped.

As Kydland and Prescott stressed, we cannot simply assume away this problem by letting the initial government choose $\hat{m}_0$, the next choose $\tilde{m}_1$, etc.; i.e. by letting each succeeding government optimize anew, using the optimal control solution (this is close to what Buiter (1983) proposes, incorrectly we believe, as discussed below). The problem is much deeper, for the following reason. The choice $\hat{m}_0$ is optimal only under the assumption that it is followed by $\hat{m}_1, \hat{m}_2, \ldots$ It has no particular attractiveness given that it will be followed by $\tilde{m}_1$ and other $m_t \neq \hat{m}_t$ for $t \geq 2$. Moreover, the exchange rate e_0

will be a function not of $\{\hat{m}_0\}$, as the original government's solution assumed, but rather of the actual m_t that will be selected.

Phelps and Pollak, and Kydland and Prescott, provided the answer to this difficulty. Unless the original government can act to bind all future governments, it must optimize with the full knowledge that all future governments will be free to optimize. A time consistent equilibrium is one in which each government optimizes its policy choice taking as given the policy rules (or specific policy actions) that future governments will use. With a finite time horizon, such an optimization is easy to carry out. Let x_T represent the inherited state of the economy in the final period T. In our example x_T would be the vector $<p_T, p^c_{T-1}, q_{T-1}>$. Given x_T, it is easy to find the best policy $m_T = f_T(x_T)$ that maximizes $\Sigma^T_{t=T}\beta^T U_t$. At time T-1, the penultimate government knows that its successor will follow $m_T = f_T(x_T)$. It is then an easy task to maximize $\Sigma^T_{t=T-1}\beta^T U_t$ subject to (10) and the constraint $m_T = f_T(x_T)$. This second optimization will yield the rule $m_{T-1} = f_{T-1}(x_{T-1})$. By backward recursion, every government could thereby find a policy rule $f_i(x_i)$ that is optimal given the rules that succeeding administrations will follow. Such rules will be credible to the private sector (e.g. the asset holders in the foreign exchange market) because each government is doing the best that it can given the freedom of action of future governments.

In an infinite-horizon setting, the solution of the time-consistency issue is a bit more complex, as we shall soon see. The problem is that there is likely to be a multiplicity, perhaps an infinity, of policy rules that have the property that they are optimal given that future governments will also choose the rule. There is an embarrassing abundance of time-consistent policies. Not only is it hard to find all of these solutions, but it is not necessarily straightforward to choose among them.

In summary, there are typically two types of equilibria in multiperiod planning problems. The first type assumes that the initial government can pre-commit to an entire sequence of moves, or to a policy rule. For this type of problem, optimal control suffices. The second type of problem more realistically assumes that each government can make its "move," but cannot bind the hand of future governments. It must therefore optimize, taking as given the freedom of choice of future governments. Before proceeding to the multicountry setting, it is useful to study some more technical aspects of these two approaches.

Pre-commitment Equilibria

There are two types of pre-commitment equilibria. In the first, the government selects an entire sequence $\{\hat{m}\}_0^\infty$ that by assumption will be carried out at all future dates. In the second, the initial government selects a rule $m_t = f(x_t, x_{t-1}, \ldots)$ that is also assumed to bind all future governments. The first equilibrium is termed an open-loop solution, and the second, a closed-loop solution. Both solutions will tend to be time-inconsistent, except in special cases, in the sense that future governments will want to deviate from the original sequence (in the open-loop case), or the original rule (in the closed-loop case), even if they believe that other governments would abide by the original plans.

We now calculate the optimal open-loop equilibrium in order to pinpoint the source of the time inconsistency. Starting with (10), we write the elements of the A matrix as a_{ij}, the B matrix as b_{ij}, and the C matrix as c_{ij} (the specific values of a_{ij}, b_{ij}, and c_{ij} are given in the footnote preceding equation (10)). In fact C can be ignored under our simplifying assumption that $p_t^* = q_t^* = i_t^* = 0$ for $t > 0$. Thus $p_{t+1} = a_{11}p_t + a_{12}p_{t-1}^c + a_{13}q_{t-1} + a_{14}e_t + b_{11}m_t$, while similar expressions hold for p_t^c, q_t, and e_{t+1}. The goal is to choose the sequence $\{m\}_0^\infty$ that maximizes U_0 in (11) subject to (10). To solve this problem, we write down the Lagrangian $\pounds$ as follows:

$$\max_{\{m\}_0^\infty} \pounds = -(1/2)\Sigma_{t=0}^{\infty}\beta^t\{[q_t^2 + \phi\pi_t^2] \tag{12}$$

$$+ \mu_{1,t+1}[a_{11}p_t+a_{12}p_{t-1}^c+a_{13}q_{t-1}+a_{14}e_t+b_{11}m_t-p_{t+1}]$$

$$+ \mu_{2,t+1}[a_{21}p_t+a_{22}p_{t-1}^c+a_{23}q_{t-1}+a_{24}e_t+b_{21}m_t-p_t^c]$$

$$+ \mu_{3,t+1}[a_{31}p_t+a_{32}p_{t-1}^c+a_{33}q_{t-1}+a_{34}e_t+b_{31}m_t-q_t]$$

$$+ \mu_{4,t+1}[a_{41}p_t+a_{42}p_{t-1}^c+a_{43}q_{t-1}+a_{44}e_t+b_{41}m_t-e_{t+1}]\}$$

As is well known, $\mu_{1,0}$, $\mu_{2,0}$, and $\mu_{3,0}$ are shadow values which describe how U_0 is affected by different inherited values of p_0, p_{-1}^c, and q_{-1}. In particular, $\mu_{1,0} = \partial U_0/\partial p_0$; $\mu_{2,0} = \partial U_0/\partial p_{-1}^c$; and $\mu_{3,0} = \partial U_0/\partial q_{-1}$.

By analogy, $\mu_{4,0}$ equals $\partial U_0/\partial e_0$; that is $\mu_{4,0}$ measures the change in intertemporal utility for a small change in e_0. Unlike p_0, p_{-1}^c, and q_{-1}, however, the policymaker does not inherit e_0, but rather determines e_0 as a function of the policies that are selected. Because e_0 is a policy choice, a necessary condition of the optimization must therefore be that $\partial U_0/\partial e_0 = \mu_{4,0} = 0$. At the optimum, μ_{4t} will equal zero at $t = 0$.

The time consistency problem arises because along the optimum sequence $\{m\}_0^\infty$, μ_{4t} will (in general) not always equal zero. (μ_{4t} will follow a difference equation of the form described in the Appendix). Since μ_{4t} will tend to move away from zero, reoptimization at any t when $\mu_{4t} \neq 0$ would lead to a new sequence of m such that μ_{4t} would again start at zero (a necessary condition of the optimization). From a technical point of view, the open-loop sequence is time consistent if and only if the equation for μ_{4t} can be satisfied with $\mu_{4,t} \equiv 0$ for all $t > 0$. If this condition is met, then future governments will choose $\{\hat{m}\}_0^\infty$ at all dates t even if they are not bound by the original government. If the condition is not satisfied, the open-loop solution makes sense only if future governments are not allowed to reoptimize.

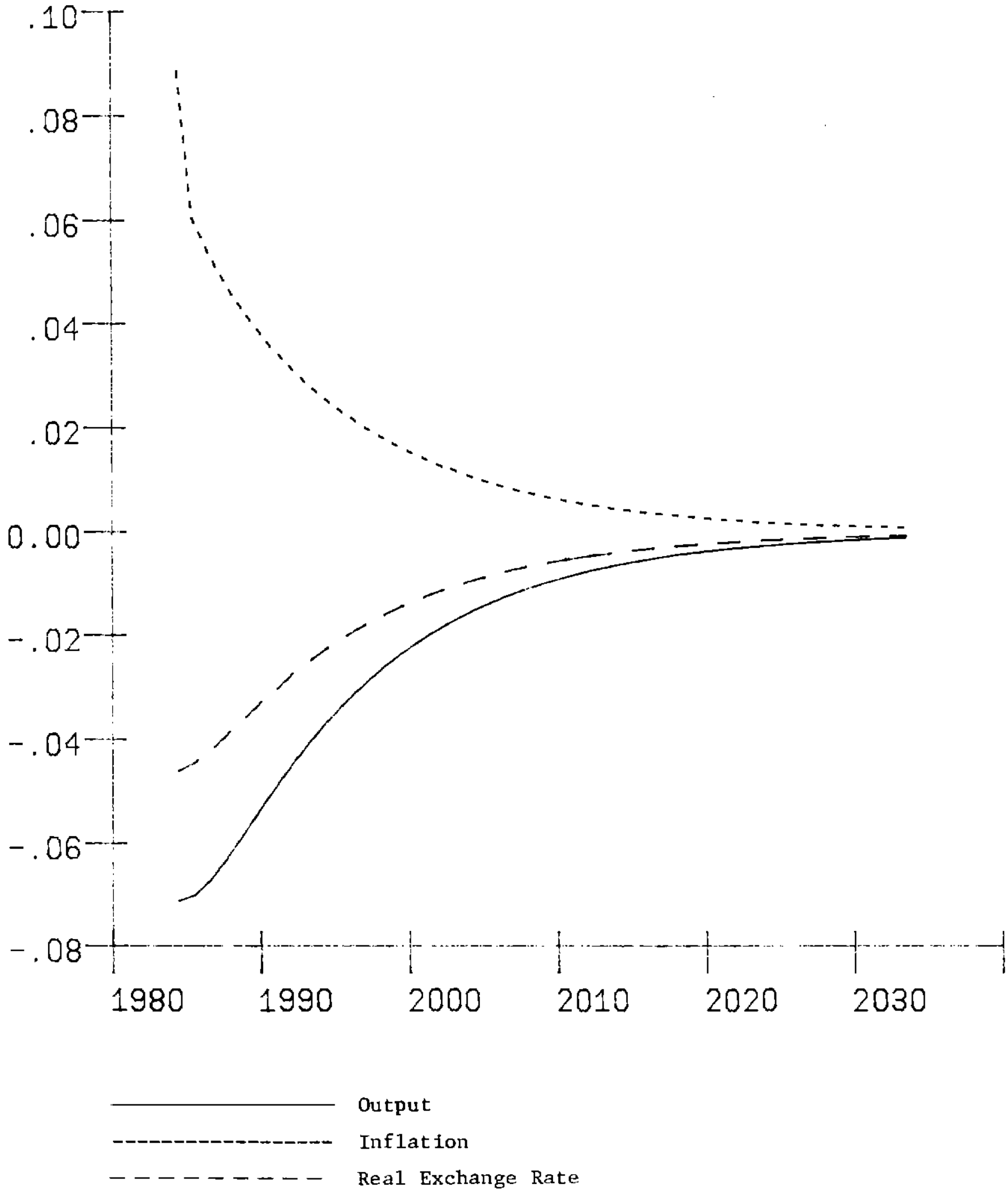
Consider a simple illustration using our model. We select simulation values for the key parameters of the model, as shown in Table 1. The economy inherits a ten-percent domestic inflation rate, and lagged full employment (i.e. $p_0 = 0.10$; $p_{-1} = 0.0$; $p^c_{-1} = 0.0$; $q_{-1} = 0.0$). With a constant exchange rate ($e_0 = 0$), CPI inflation will equal ten percent in period zero (i.e. $\pi_0 = 0.10$), while a currency appreciation can reduce the initial CPI inflation rate. Given our parameter values, the optimal sequence $\{\hat{m}\}_0^\infty$ is sharply contractionary at $t = 0$, so that output is pushed below zero, with the goal of reducing inflation. The real exchange rate $p_0 - e_0 - p^*_0$ appreciates at $t = 0$, 4.7 percent above its long run value, with the currency appreciation helping to export inflation abroad. Figure 1 shows the optimal paths of inflation, output, and the real exchange rate. (1984 is taken as $t = 0$).

Consider the behavior of $\mu_{4,t}$, as shown in Figure 2. After $t = 0$, μ_4 turns

Table 1. Parameter Values

α	= 1.00
β	= 0.75
γ	= 0.00
δ	= 1.50
ϵ	= 0.50
θ	= 0.30
λ	= 0.75
σ	= 1.50
ψ	= 0.10
ϕ	= 2.00

Figure 1. Open-Loop Control in the One-Country Model



positive, meaning that an increase in e would raise welfare. From the point of view of the government at time t = 3 (1987), for example, the original plan is too contractionary, since a currency depreciation would raise welfare. A new optimization at t = 3 would lead to a new sequence $\{\bar{m}\}_0^\infty$, with $\bar{m}_3 > \hat{m}_3$. This is shown in Figure 3, where we superimpose $\{\hat{m}\}_0^\infty$ and $\{\bar{m}\}_3^\infty$. Loosely speaking, the initial government, at t = 0, has an incentive to announce a stern set of future monetary policies in order to induce a currency appreciation at t = 0, and thereby to reduce π_0 (which is otherwise very high). Of course, e_0 can be reduced by extremely low m_0 and higher m_t for $t \geq 1$, or by more moderate m_0 and somewhat lower m_t for $t \geq 1$. The optimal policy is to opt for moderate m_0 and low future m, rather than extremely restrictive m_0, since the approach with restrictive future m achieves the same currency appreciation with a somewhat lower loss of initial output, q_0.

Thus, from the perspective at t = 0, it is worthwhile to commit future m to low values for the sake of e_0. However, from the perspective of future governments, e_0 is a bygone, and m should reflect tradeoffs in the present and future, not the past. Thus, by the time a future government assumes office, part of the original incentive to keep m low has disappeared, and the new optimization in period t consequently yields a higher value of m_t.

It is interesting to note that there is a single special case in which the open-loop policy is also time consistent, and that is when $\sigma = 0$ in the original model (i.e. output is not affected by the real interest rate). In that case, $\mu_{4t} \equiv 0$ satisfies the equation for μ_{4t} derived in the Appendix.[3] From an economic point of view, when $\sigma = 0$, only the exchange rate e_0, but not the

Figure 2. Shadow Price on the Exchange Rate (μ_{4t})
In Open-Loop Control
(one-country model)

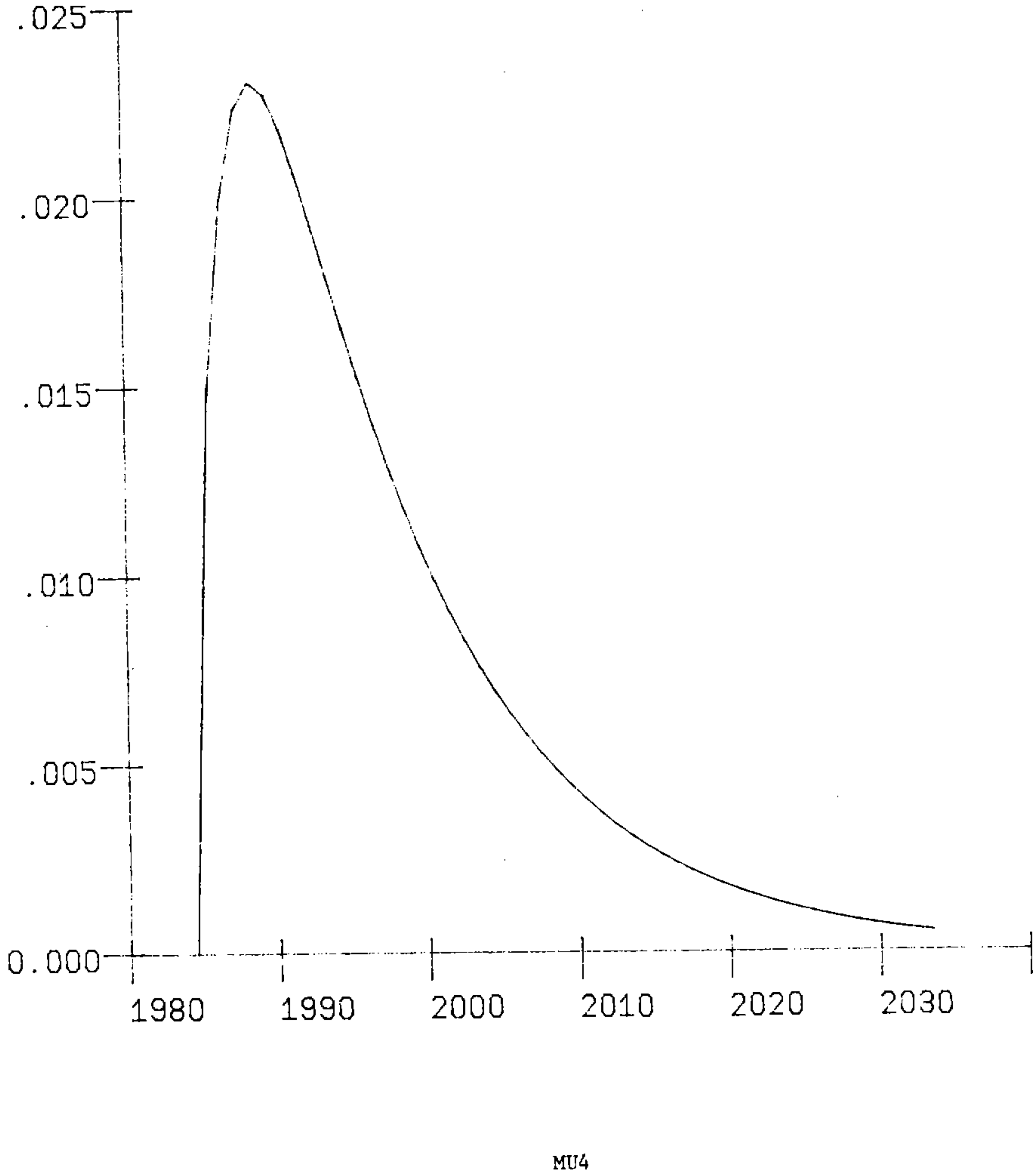
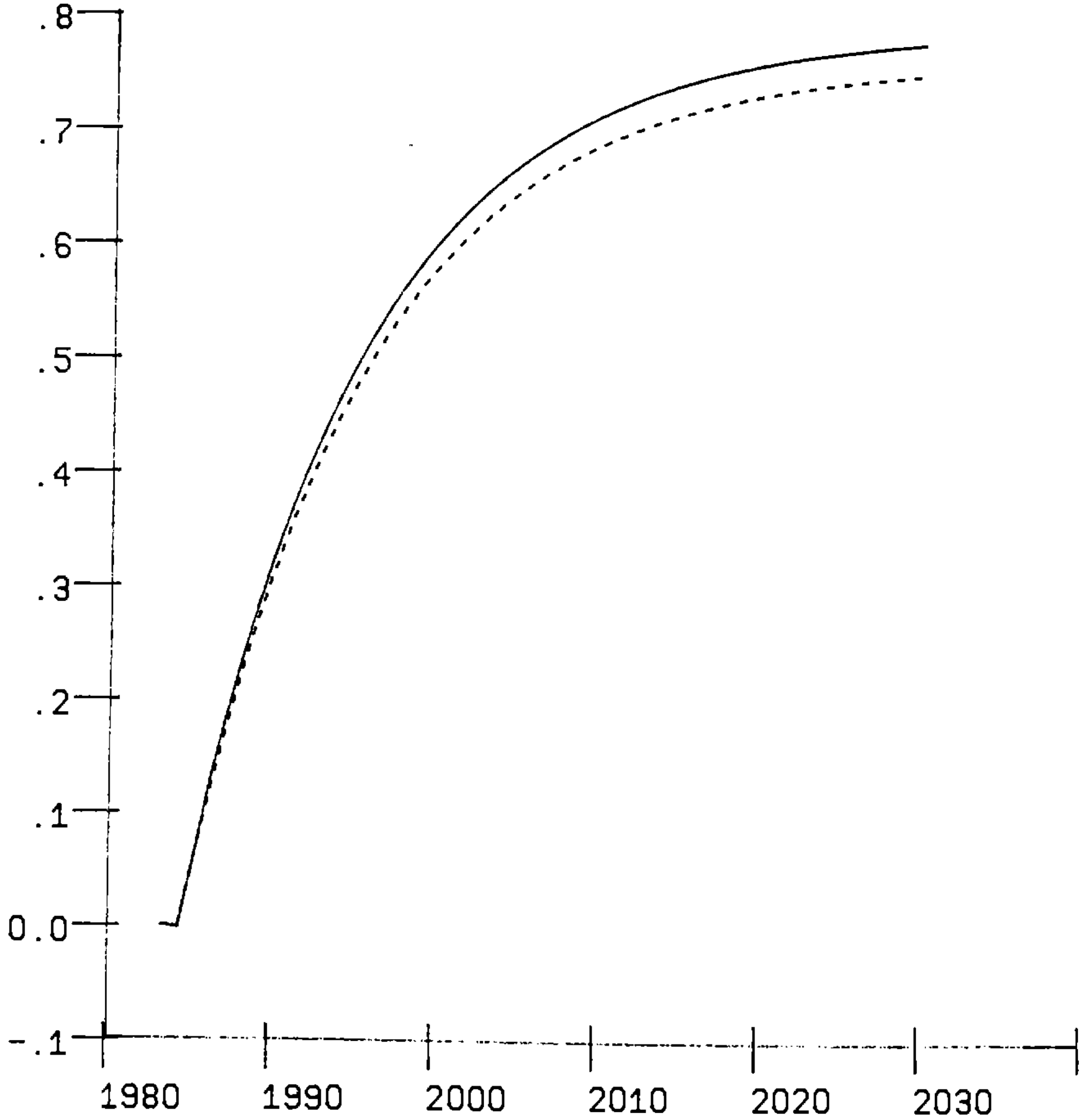


Figure 3. Reoptimization of Open-Loop Control in 1987
(Comparison with original solution; one-country model)



————————— $\tilde{M}$: Open-loop policy with reoptimization in 1987

- - - - - - - - - - - $\hat{M}$: Open-loop policy sequence of future m, affects q_0 amd π_0, so that there is no reason to prefer one path of m over another as long as they both lead to the same e_0. The same is true about all future e_t. This property allows the original government to specify a path $\{\hat{m}\}_0^\infty$ that all future governments will be content to honor.

The open-loop equilibrium is the best pre-commitment equilibrium available. It is sometimes argued, however, that while governments cannot credibly pre-commit future governments to a sequence of policy moves, they may be able to pre-commit governments to a specific policy rule for m_t. Such a closed-loop rule might not be as good as the open-loop result, but it might be better than no rule at all. There is some merit to this argument, as we shall soon see. The rule can of course be of varying complexity. We illustrate this case by choosing a simple rule, which links m_t to the current state of the economy, as described by the vector x_t $\langle p_t, p_{t-1}^c, q_{t-1}\rangle$. Such a rule is termed memoryless, in that the past history of the economy, in arriving at $\langle p_t, p_{t-1}^c, q_{t-1}\rangle$, is not permitted to affect m_t. We simplify further by specifying m_t as a linear function of p_t, p_{t-1}^c, and q_{t-1}:

(13) $$m_t = \beta_0 + \beta_1 p_t + \beta_2 p_{t-1}^c + \beta_3 q_{t-1}$$

Our method of solution is straightforward. A solution of the form (13) is guessed. Using (10) and the assumption that e_0 places the economy on the stable manifold, we find U_0 as a function of the rule. Implicitly then $U_0 = U_0(\beta_0, \beta_1, \beta_2, \beta_3)$. Using a standard numerical optimization technique, we then proceed to maximize U_0 with respect to $\beta_0, \beta_1, \beta_2, \beta_3$, to arrive at the optimal rule $m_t = \hat{\beta}_0 + \hat{\beta}_1 p_t + \hat{\beta}_2 p_{t-1}^c + \hat{\beta}_3 q_{t-1}$. Given our assumed parameter values for the structural model, we find:

(14) $m_t = -.038\ p_t + 1.027\ p^c_{t-1} + 0.322\ q_{t-1}$

Note that this is the optimal linear rule for a given $x_0 = \langle p_0, p^c_{-1}, q_{-1}\rangle = \langle 0.1, 0.0, 0.0\rangle$. For a different starting point, we would find a different rule.

Time-Consistent Equilibria

The previous equilibria depend on the unsatisfactory assumption that future governments can be bound by rules made at an earlier date. Some writers have suggested that macroeconomic policies must therefore be formulated as constitutional rules, in order to bind successfully at a later date. For many reasons, including conflicting views about the correct rules, unwillingness to tamper with a constitution, and the realization that even constitutions can be amended at a later date, there is little likelihood the macroeconomic policy will soon be etched in constitutional stone. In practice, therefore, governments must operate with the knowledge that future governments have freedom to change course and will have incentives to do so, relative to the open-loop or closed-loop optimum, even when the future governments share the goals of the earlier governments.

In this circumstance, we can reformulate the policy problem as a game among an infinite number of players (i.e., governments), who are identified by the time period in which they act. The initial move is made by the government at t = 0 (hereafter G_0), then by G_1, and so on. The payoff functions for G_t is $\Sigma^{\infty}_{i=t}\beta^t U_t(T^i_t)$, and the move is m_t.

Now, we can think of various types of Nash equilibria among these governments. In analogy to the pre-commitment case, we can think of Nash equilibria in which each government takes as given the moves of other governments, or Nash equilibria in which each government takes as given policy rules of other governments. A Nash equilibrium in moves will be called "open-loop," and a Nash equilibrium in strategies or policy rules will be called "closed-loop."

Consider first the case of open-loop Nash equilibrium. Let $\{m\}_{-t}$ denote the sequence of moves before and after, but not including, period t: $m_0, m_1, \ldots, m_{t-1}, m_{t+1}, m_{t+2}, \ldots$. An open-loop Nash equilibrium is a sequence $\{m^N\}_0^\infty$, with the property that for all governments, m^N is optimal taking as given $\{m^N\}_{-t}$:

(15) $\{m_0^N\}^\infty$ is an open-loop Nash equilibrium if and only if for all t, m_t^N maximizes $\Sigma_{i=t}^\infty \beta^i U_i$ subject to (10) and given $\{m^N\}_{-t}$.

In performing the optimization at period t, the government assumes that e_t adjusts to keep the economy on the stable manifold, given the past history of m, the current policy choice m_t, and the assumed future path $m_{t+1}^N, m_{t+2}^N, \ldots$

With this definition, the problem with the precommitment equilibrium is that the resulting path is not a Nash equilibrium among the infinite sequence of governments (this was verified in Figure 3). Taking as given that other governments will play $\hat{m}_t$ (the open-loop sequence), only the initial government will want its part of the sequence (i.e. $\hat{m}_0$). For all other governments (in general), there will exist a superior choice of policy.

Now, consider the "closed-loop" version of Nash equilibrium, in which we assume that G_t plays a rule (or strategy) f_t, which maps $(x_t, x_{t-1}, \ldots)$ to

m_t, rather than just a move m_t. As before, define the sequence $\{f\}_{-t}$ as f_0, $f_1, \ldots f_{t-1}, f_{t+1}, \ldots)$. Now, we define a Nash equilibrium in this strategy space as follows:

(16) $\{f^N\}_0^\infty$ is a closed-loop Nash equilibrium if and only if for all t, $m_t = f_t^N(x_t, x_{t-1}, \ldots)$ maximizes $\Sigma_{i=t}^{\infty}\beta^i U_i$ subject to (10), and given $\{f^N\}_{-t}$.

In general, there will be many such Nash equilibria, some of which (as we shall see) are not very desirable.

As is typical in such circumstances, we further refine the nature of the equilibrium to include only Nash perfect equilibria. A strategy sequence $\{f\}_0^\infty$ is said to be a perfect equilibrium if for any history of the economy from time 0 to t (even histories not resulting from a Nash equilibrium during periods 0 to t), strategies $\{f\}_0^\infty$ constitute a Nash equilibrium in the sub-game from t to ∞. We now define time consistency:

(17) $\{f\}_0^\infty$ time consistent if and only if $\{f\}_0^\infty$ is a Nash perfect equilibrium.

In general, open-loop Nash equilibria, as in (15), will not be perfect equilibria. Suppose, for example, that the sequence $\tilde{m}_1, \tilde{m}_2, \ldots$ has the Nash property. In most models, including those in our paper, the sequence $\tilde{m}_2, \tilde{m}_3, \ldots$ will not be subgame Nash (starting at period 2), if m_1 is set differently from $\tilde{m}_1$. Thus, from this point on, we restrict our search for time-consistent equilibria to closed-loop Nash equilibria, in which governments take as given the policy rules of other governments.

Unfortunately, even the perfectness concept does not eliminate the problem of a multiplicity of equilibria. There will in general be many truly time-consistent equilibria. To narrow the search, we begin with the simplest case, in which m_t is a function of the current state x_t ($= \langle p^c_{t-1}, p_t, q_{t-1} \rangle$) alone (see Maskin and Tirole (1983) for some justification for restricting our search to such "memoryless" strategies). Thus, we are searching for a function $m_t = f(x_t)$ such that:

(18) $m_t = f(x_t)$ maximizes $\Sigma^{\infty}_{i=t} \beta^i u_i$
subject to (10) and to the restriction that
$m_i = f(x_i)$ for all $i \neq t$.

(Note that in this case the government at time t does not actually care about the rules up to time t, since the past is fully summarized in x_t). Implicit throughout is the assumption that e_t is always such as to keep the economy on the stable manifold. In practice, this means that along with f there is another function h linking e_t and x_t: $e_t = h(x_t)$.

Our strategy is to search for f among the class of linear functions. Although we cannot prove that the resulting function is the unique memoryless, time-consistent equilibrium, we suspect that it is in fact unique, in view of the linear-quadratic structure of the underlying problem. Consider the necessary conditions for a time-consistent optimum. Let $m_t = \gamma_0 + \gamma_1 p_t + \gamma_2 p^c_{t-1} + \gamma_3 q_{t-1}$ be a candidate solution (call it the γ-rule). Plugging this rule into (10), we can also determine a unique linear rule $e_t = h_0 + h_1 p_t + h_2 p^c_{t-1} + h_3 q_{t-1}$ that keeps the economy on the stable manifold. Now, suppose that these rules hold for all $t > 1$. It is possible to calculate $\Sigma^{\infty}_{t=1} \beta^t U_t$ as a function of the rule and the state of the economy at t = 1, i.e. x_1. Let us call the value of the utility function $V_1^\gamma(x_1)$, where V^γ denotes the dependence of utility on the rule γ.

At time zero, the 0^{th} government wants to maximize $\Sigma_{t=0}^{\infty}\beta^t U_t$, which equals $U_0 + \beta V_1^\gamma(x_1)$ under the assumption that future governments will use the γ-rule. Note that $x_1 = \langle p_1, p_0^c, q_0 \rangle$. Specifically, the initial government solves the following:

$$\max_{m_0} U_0 + \beta V_1^\gamma(p_1, p_0^c, q_0) \qquad (19)$$

Subject to:

(a) $e_1 = h_0 + h_1 p_1 + h_2 p_0^c + h_3 q_0$

(b) $p_1 = a_{11} p_0 + a_{12} p_{-1}^c + a_{13} q_{-1} + a_{14} e_0 + b_{11} m_0$

(c) $p_0^c = a_{21} p_0 + a_{22} p_{-1}^c + a_{23} q_{-1} + a_{24} e_0 + b_{21} m_0$

(d) $q_0 = a_{31} p_0 + a_{32} p_{-1}^c + a_{33} q_{-1} + a_{34} e_0 + b_{31} m_0$

(e) $e_1 = a_{41} p_0 + a_{42} p_{-1}^c + a_{43} q_{-1} + a_{44} e_0 + b_{41} m_0$

(f) $U_0 = -(q_0^2 + \phi \pi_0^2)$

(g) p_0, p_{-1}^c, q_{-1} and V_1^γ given

In this optimization problem, (a) is determined by the candidate γ-rule. (b)-(e) are the structural dynamic equations summarized in (10). (f) is the instantaneous utility function (note that $\pi_0 = p_0^c - p_{-1}^c$). Finally, (g) defines the state of the economy for the initial government.

The optimization is straightforward. Using (a) and (e) we can write $e_0 = (1/a_{44})[h_0 + h_1 p_1 + h_2 p_0^c + h_3 q_0 - a_{41} p_0 - a_{42} p_{-1}^c - a_{43} q_{-1} - b_{41} m_0]$. Now using (b), (c) and (d) together with the new equation for e_0, we have four equations that make e_0, p_0^c, q_0, and p_1 linear functions of m_0 and the predetermined variables p_0, p_{-1}^c, q_{-1}. Let us write this system as:

$$(20)\qquad \begin{aligned} e_0 &= d_{11}p_0 + d_{12}p_{-1}^c + d_{13}q_{-1} + d_{14}m_0 \\ p_0^c &= d_{21}p_0 + d_{22}p_{-1}^c + d_{23}q_{-1} + d_{24}m_0 \\ q_0 &= d_{31}p_0 + d_{32}p_{-1}^c + d_{33}q_{-1} + d_{34}m_0 \\ p_1 &= d_{41}p_0 + d_{42}p_{-1}^c + d_{43}q_{-1} + d_{44}m_0 \end{aligned}$$

Now simply impose the first-order condition that $d[-(q_0^2+\phi\pi_0^2) + \beta V_1^\gamma(p_1,p_0^c,q_0)]/dm_0$ equals zero. By direct substitution we have:

$$(21)\qquad \begin{aligned} 0 = &-2d_{34}(d_{31}p_0+d_{32}p_{-1}^c+d_{33}q_{-1}+d_{34}m_0) \\ &-2\phi d_{24}(d_{21}p_0+d_{22}p_{-1}^c+d_{23}q_{-1}+d_{24}m_0-p_{-1}^c) \\ &+\beta(\partial V_1^\gamma/\partial p_1)d_{44} \\ &+\beta(\partial V_1^\gamma/\partial p_0^c)d_{24} \\ &+\beta(\partial V_1^\gamma/\partial q_0)d_{34} \end{aligned}$$

This gives us a linear rule for m_0 as a function of p_0, p_{-1}^c, q_{-1} and implicitly (through V_1^γ) the γ rule:

$$(22)\qquad \begin{aligned} m_0 = &[1/(d_{34}+\phi d_{24}^2)][(d_{34}d_{31}+\phi d_{24}d_{21})p_0 \\ &+ (d_{34}d_{32}+\phi d_{24}d_{22})p_{-1}^c + (d_{33}^2+\phi d_{23}^2)q_{-1} \\ &+ 1/2\beta(\partial V_1^\gamma/\partial p_1)d_{44} + 1/2\beta(\partial V_1^\gamma/\partial p_0^c)d_{24} \\ &+ 1/2\beta(\partial V_1^\gamma/\partial q_0)d_{34}] \end{aligned}$$

Under our assumptions, the partial derivatives of V_1^γ are linear functions of p_0, p_{-1}^c, and q_{-1} (though not easy to write down analytically!). Thus, m_0 is a linear rule in p_0, p_{-1}^c, and q_{-1}:

$$(23) \qquad m_0 = \delta_0 + \delta_1 p_0 + \delta_2 p_{-1}^c + \delta_3 q_{-1}$$

As long as (23) is the same as the γ rule, we have found a stationary, time-consistent rule. That is, for $\delta_0 = \gamma_0$, $\delta_1 = \gamma_1$, $\delta_2 = \gamma_2$, $\delta_3 = \gamma_3$, the γ rule is validated as a time-consistent policy. Starting at <u>any</u> period t and any state t, the t^{th} government will choose the γ rule given that all future governments will make that choice.

In general, the time-consistent rule must be found numerically (see Cohen and Michel (1984) for an elegant treatment of the one-dimensional case for the state vector x, for which an analytical solution is found). To do so, we start with a finite-period problem, in which $\Sigma_{t=0}^{T}\beta^t u_t$. It is then easy to find the optimal final period rule $m_T = f_T(x_T)$. Given f_T, f_{T-1} is readily found by the type of backward recursion just described. For each T, we can readily compute $f_0(x_0)$. Denote this rule as $f_0^T(x_0)$ to denote the dependence of the rule on the periods remaining. Then it is a simple matter to find the limiting value of $f_0^T(x_0)$ as $T \to \infty$. The rule $f(x_0) = \lim_{T\to\infty} f_0^T(x_0)$ can then be verified directly to have the time-consistency, Nash equilibrium property for the infinite-horizon game. We provide details of this method in the Appendix.

Using the parameter values described earlier, the time-consistent rule is calculated to be:

$$(24) \qquad m_t = -.032\ p_t + 1.032\ p_{t-1}^c + .275\ q_{t-1}$$

As is shown in the Appendix, the open-loop optimal policy can be written as a linear function of the state variables <u>and</u> μ_{4t}:

(25) $$m_t = -.019\ p_t + 1.019\ p^c_{t-1} + .272\ q_{t-1} + .389\ \mu_{4t}$$

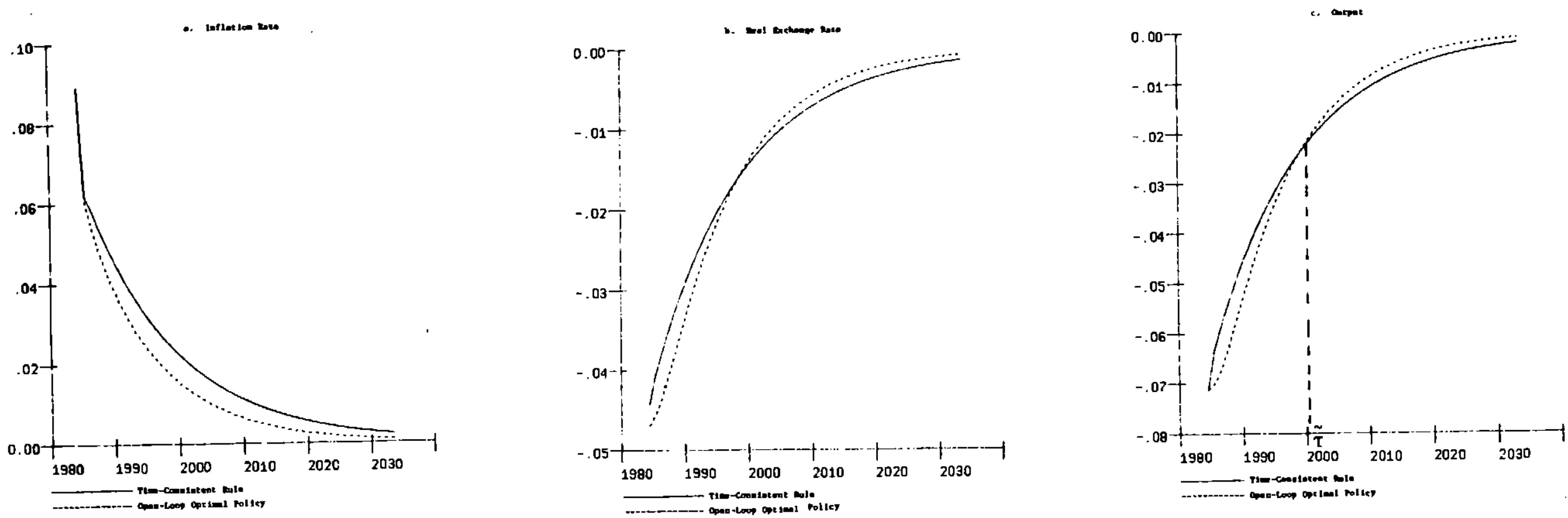
Starting, as before, with 10 percent inflation, we can compute the path of output and inflation for the time-consistent policy, for comparison with the open-loop pre-commitment equilibrium. In Figure 4a, we compare the inflation performance in the two cases; in Figure 4b, we compare the exchange rates; and in Figure 4c, we compare the output paths. We have already seen that the open-loop control holds future governments to an <u>over-contractionary</u> policy relative to the one that they would select upon reoptimization. Since the time-consistent policy explicitly allows for (expansionary) reoptimization in the future, it is not surprising that the real exchange rate is less appreciated in the time-consistent (TC) case than in the open-loop (OL) case. Simply, agents recognize that future governments will select more expansionary m, and e_t is an increasing function of the entire sequence of m. Thus, $\pi_0^{OL} < \pi_0^{T}$, via the exchange rate effect. In general, $q_t^{OL} < q_t^{TC}$ in the early periods, as governments in the OL case pursue a steady, contractionary policy. After a certain period (shown as $\tilde{t}$ in Figure 4c), the inequality is reversed. Both policies reduce the inherited inflation to zero in the long run.

Before turning to a welfare ranking of the various policies, we must note a key feature of the disinflation process (pointed out earlier in Buiter and Miller (1982) and elsewhere). The price equation is:

$$(p_{t+1}-p_t) = (p^c_t-p^c_{t-1}) + \psi q_t + \theta(q_t-q_{t-1}).$$

Also $p^c_t = p_t + (1-\lambda)(p^*_t+e_t-p_t) = p_t + (1-\lambda)r_t$, where $r_t(= p^*_t+e_t-p_t)$ is the real exchange rate. Thus,

Figure 4. A Comparison of Open-Loop and Time-Consistent Policies
(One-country model)



(26) $$(p_{t+1}-p_t) = (p_t-p_{t-1}) + (1-\lambda)(r_t-r_{t-1}) + \psi q_t + \theta(q_t-q_{t-1})$$

Suppose an economy inherits an inflation rate of $\Delta_0 = p_0 - p_{-1}$, with $r_{-1} = q_{-1} = 0$. By simple forward integration of (26) from t = 0, we have

(27) $$(p_{t+1}-p_t) = \Delta_0 + (1-\lambda)r_t + \psi\Sigma_{i=0}^{t}q_i + \theta q_t$$

Now, for all of the equilibria so far considered, $p_{t+1} - p_t$ equals zero in the long run (i.e. inflation is eliminated), r_t returns to zero (i.e. no long-run change in competitiveness), and q_t returns to zero (i.e. long-run full employment). Thus, taking limits of (27), we find $0 = \Delta_0 + \psi\Sigma_{i=0}^{\infty}q_i$, or

(28) $$\Sigma_{i=0}^{\infty}q_i = \Delta_0/\psi$$

All policies have the same cumulative output loss, no matter what is the time path of exchange rates, money, etc.! Thus, the welfare issue is always one of timing, rather than the overall magnitude of lost output.

On purely logical grounds, we can rank the welfare achieved by the three policies so far studied: open-loop control, closed-loop control (with pre-commitment), and time-consistent control. The open-loop control is clearly first best, since both of the other solutions reflect the same optimization, but under additional constraints. The closed-loop, linear feedback rule also must produce higher utility than the time-consistent rule. Both the linear rule and time-consistent solution choose m_t as a linear function of x_t; the linear rule is chosen as the best among this class of functions, so in particular it is better than the time-consistent rule. Thus we know that $U_0^{OL} > U_0^{CL} > U_0^{TC}$. In general, the inequalities will be strict, though we have already noted special cases (e.g. $\sigma = 0$) in which all of the policies are identical.

Buiter (1983) has recently proposed an alternative strategy for finding a time-consistent linear rule (we describe his approach at length in the appendix). His reasoning is as follows. Consider the open-loop control solution, with shadow prices μ_1, μ_2, and μ_3 on the state variables, and μ_4 on the exchange rate. At t = 0, the initial government chooses policies so that $\mu_{4,0} = 0$. For t > 0, we know that $\mu_{4,t}$ will tend to deviate from zero. Each government in period t would like to reset $\mu_{4,t} = 0$. Buiter proposes, therefore, that a time-consistent solution is found by assuming that $\mu_{4,t} \equiv 0$ for all t, and dropping the open-loop dynamic equation for $\mu_{4,t}$. When this procedure is followed, we obtain the following linear rule:

(29) $$m_t = .237\ p_t + .763\ p^c_{t-1} + .229\ q_{t-1}$$

There are two counts against this proposed solution. Most important, it is simply not time consistent. If all governments for t ≥ 1 adopt the Buiter rule, the government at t = 0 would not choose this rule. By following the procedures described earlier (for calculating the best rule at t = 0 for a given rule at t ≥ 1) we find that the initial government would choose:

(30) $$m_0 = -.147\ p_0 + 1.147\ p^c_{-1} + .309\ q_{-1}$$

The logic underlying the Buiter solution seems problematic as well. The merit for a government to choose $\mu_{4,t} = 0$ comes if the sequence of m corresponding to $\mu_{4,t} = 0$ will in fact be carried out by future governments. But, by construction, each succeeding government alters the chosen sequence of m. There is simply no attraction to choosing $\mu_{4,t} = 0$ if the government knows that its plans will not be carried forward. The private sector understands this point perfectly, by setting e_t to correspond to the actual sequence of m rather than to the sequence planned by each government. In a nutshell, Buiter's government is naive in assuming that future governments will carry out its open-loop optimum, at the same time that the private sector is completely on top of the policy-making process, and knows that future governments will reoptimize.

Reputation and Time-Consistency

In the previous section we simplified our search for a time-consistent policy to "memoryless" rules. Such rules make m_t a function of the contemporaneous state vector x_t, but not of the past history of x and m. Many policies in the real world depend on the history of a game as much as the current state. In competitive environments, for example, aggressive behavior by one player at time t-1 might bring forward retaliation by others at period t, as in "tit-for-tat" strategies. Game theorists have long understood that such history-dependent strategies can help competing players to achieve more efficient outcomes than those obtainable from memoryless strategies alone.

It turns out that similar complex strategies can help a sequence of governments to achieve a better equilibrium than the one obtained by the memoryless rule $m_t = f(x_t)$. Consider a compound rule of the sort:

(31) (a) Government t chooses its policy according to $m_t = g(x_t)$, as long as all governments $j < t$ have also selected policy this way;

(b) If any government $j < t$ selects $m_j \neq g(x_j)$, then government t selects $m_t = f(x_t)$, where f is the memoryless, time-consistent rule.

Suppose now that the rule $g(x_t)$ is better than $f(x_t)$ in the sense that if all governments $t > 0$ choose $g(x_t)$ they achieve utility $U_t^g > U_t^f$. Also, suppose that $g(x_t)$ itself is not time consistent in the sense of (19): If all governments $t \geq 1$ are known to choose $g(x_t)$, it is not optimal for the government at $t = 0$ to select $g(x_0)$.

The surprising result is that while $g(x_t)$ is not time consistent, a compound strategy like (31)(a)-(b) can be time consistent with the result that all governments end up playing $g(x_t)$, leading to higher social welfare. In the memoryless time-consistency problem, each government takes as given the choice of policy rule followed by future governments. If future governments are going to choose $m_t = g(x_t)$, the current government may have no particular incentive to choose g. With a compound rule as in (31), the government at time t knows that it affects the policy rule selected by future governments. It takes as given the two-part decision mechanism (a)-(b), but it recognizes that if it is the first government to deviate from $g(x_t)$, it will cause all future governments to choose $f(x_t)$ instead of $g(x_t)$. Since $U^g > U^f$ by assumption, this deviation from $g(x_t)$ imposes a cost, which deters the government from deviating from $g(x_t)$.

Thus, each government operates under a "threat" that future governments will revert to $f(x_t)$ if the current government fails to play $m_t = g(x_t)$. Game theorists have long recognized that such a threat mechanism is viable only if the reversion to $f(x_t)$ is credible. For example, suppose that the rule is "let money growth obey the open-loop strategy or else each future government lets money grow by one million percent." If every government takes it as given that future governments hold this rule, then money growth will indeed obey the open-loop strategy (governments would seek to avoid the hyperinflation that they fear would otherwise ensue).A true intertemporal Nash equilibrium is obtained, in which the open-loop sequence is carried out by every government. The problem here, of course, is that the threat of hyperinflation is not rational. Surely, if any government does violate the open-loop rule, the next government will not exercise the threat. Knowing this, no government really has an incentive to persist in the open-loop path.

Game theorists therefore restrict the threats to actions that would indeed be carried out if deviations from $g(x_t)$ occur (even if, as in the example, the threats need never actually be carried out). It is here that the assumption of perfection of equilibrium becomes important. In the hyperinflation example just cited, not all subgames are Nash, and thus the proposed equilibrium is not perfect. To see this, suppose that G_0 deviates. Even if G_1 assumes that all future governments will play the hyperinflation threat, it is not optimal for government 1 to play the threat. Thus the subgame in which government 0 deviates, and all G_t ($t > 1$) let m grow by 1 million percent per period, is not a Nash equilibrium. G_1 can do better unilaterally, taking as given the actions of other G_t.

As long as the reversion is to $f(x_t)$, i.e. the threat is to return to the time-consistent rule, the threat is credible. After all, if a government believes that all future governments will play $f(x_t)$, it is optimal for the government itself to play $f(x_t)$. Every subgame consisting of the infinite sequence of governments playing $f(x_t)$ is therefore a Nash equilibrium.

Now we argue that by this mechanism the sequence of governments can sustain any linear rule $m_t = \ell(x_t)$, as long as the utility from this rule is higher than the utility from the memoryless time-consistent rule for any x_t. We want to show, therefore, that the following strategy for each government constitutes a perfect Nash equilibrium, in which $m_t = \ell(x_t)$ is always played.

(32) (a) Each government chooses $m_t = \ell(x_t)$ as long as all governments $j < t$ have also selected this rule;

(b) If any government $j < t$ selects a different m_t, then all governments t select $m_t = f(x_t)$.

Now let us examine the incentive of any government to deviate from $m_t = \ell(x_t)$. It knows that all future governments will then play $f(x_t)$. But knowing that all future governments will play $f(x_t)$, it is optimal for the government in question to choose $m_t = f(x_t)$ as well, by the definition of f. In other words, if a government is going to deviate, the best deviation is simply to revert to $f(x_t)$ immediately. Thus, the cost of defecting from the $m_t = \ell(x_t)$ rule is to revert immediately and permanently to the $m_t = f(x_t)$ rule. Since utility is higher under ℓ than f, there is never an incentive to deviate from ℓ. The equilibrium is perfect, since in any subgame in which a defection from $m_t = \ell(x_t)$ has occurred, it will be a Nash equilibrium for all governments to revert to $f(x_t)$.

For the case $\theta = 0.0$, we have found a rule $m_t = \ell(x_t)$ that has the property that $U_t^{\ell}(x_t) > U_t^{f}(x_t)$, and thus have verified that such reputational equilibria exist in our model. With $\theta = 0$, and all other parameter values as in Table 1, the time consistent rule is:

$$m_t = f(x_t) = -.165\ p_t + 1.165\ p^c_{t-1}$$

The following rule has higher utility for all x_t:

$$m_t = \ell(x_t) = -.185\ p_t + 1.185\ p^c_{t-1}$$

The loss functions corresponding to these rules are:

$$U^f(x_t) = -(\tfrac{1}{2})x_t' \begin{bmatrix} 1.726 & -1.726 \\ -1.726 & 1.726 \end{bmatrix} x_t = -x_t' S^f x_t$$

$$U^\ell(x_t) = -(\tfrac{1}{2})x_t' \begin{bmatrix} 1.725 & -1.725 \\ -1.725 & 1.725 \end{bmatrix} x_t = -x_t' S^\ell x_t$$

Since $S^f - S^\ell$ is positive definite, we have for all x_t that

$$U^\ell - U^f = x_t'(S^f - S^\ell)x_t > 0.$$

We have not found such an example for $\theta > 0.0$.

In an important sense, then, the time inconsistency problem is exaggerated, in that many "pre-commitment" equilibria can probably be sustained even in situations where actions of future governments cannot be bound. The memoryless time-consistent equilibrium is the lower limit of what can be obtained by a sequence of governments, not the only outcome. We should stress, however, that time consistency does impose costs, since the first-best, open-loop strategy almost surely cannot be sustained as a perfect equilibrium. The reason is as follows. Suppose that the sequence of governments pursues the open-loop solution under the threat of reversion to $m_t = f(x_t)$ if it ever violates the open loop rule. We know that it will follow the sequence $\{\hat{m}\}_0^\infty$, to which corresponds a sequence of states, denoted $\{\hat{x}\}_0^\infty$. At each t, we may calculate the utility of continuing with the open-loop sequence, $U_t^{OL}(\hat{x}_t)$, with the utility of reverting to the time-consistent equilibrium, $U_t^{TC}(\hat{x}_t)$. The threat of reverting to f will continue to work only when $U_t^{OL}(\hat{x}_t) < U_t^{TC}(\hat{x}_t)$. However, at some point this equality is reversed, and the government at that date actually prefers to revert to the time-consistent equilibrium. Knowing that such a date will be reached, earlier governments will also know that the open-loop path cannot be sustained. This phenomenon is shown in Figure 5, where at each t, we graph $U_t^{OL}(\hat{x}_t) - U^{TC}(\hat{x}_t)$, with the $\hat{x}_t$ calculated along the open-loop path. As long as $U_t^{OL}(\hat{x}_t) - U_t^{TC}(\hat{x}_t)$ is positive, the government at t does not have an incentive to deviate. At time $\tilde{t}$ (here 1987), the government prefers to revert to the time-consistent solution.

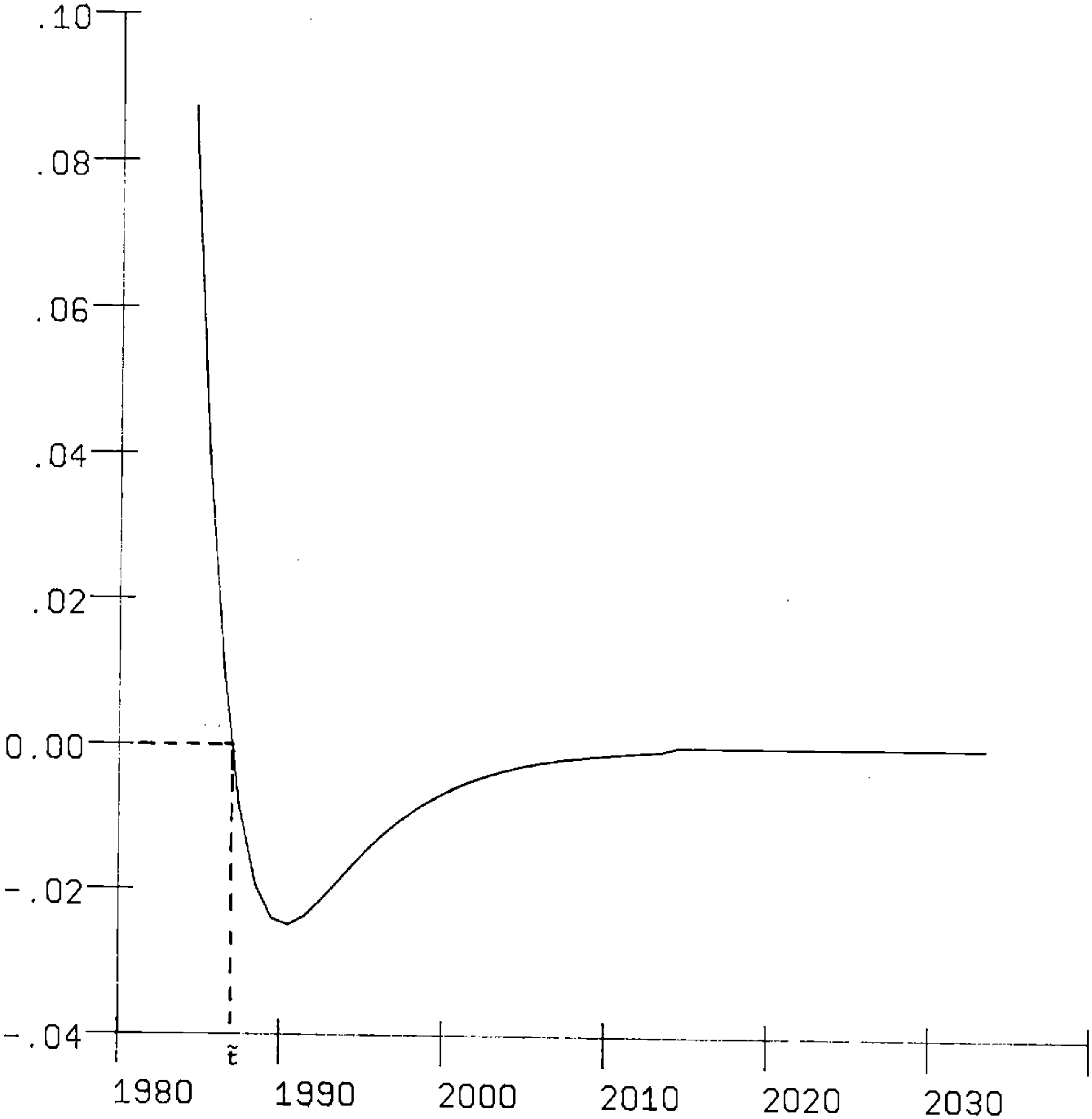
III. Policy Coordination in the Two-Country Model

The first part of the paper has dealt with economic policy in a single economy. We now extend the same set of techniques to a two-country setting. The goal is to compare "non-cooperative" equilibria (NC), in which each country optimizes while taking as given the policies abroad, with "cooperative" equilibria (C), in which binding commitments can be made between the two countries. Formally, we treat the cooperative case as one in which a single controller chooses the policies of the two countries. As in the early section, we must treat two separate types of equilibria: (1) the pre-commitment case, in which the two countries (in NC) or the single controller (in C), can credibly pre-commit to a rule or to an infinite sequence of actions; and (2) the time-consistent case, in which no pre-commitment in future periods is possible. We turn first to the pre-commitment case.

Figure 5. The Cost of Reversion to Time Consistent Control[a]

$[U_t^{OL}(\hat{x}_t) - U_t^{TC}(\hat{x}_t)]$

(One-country model)



[a]Note that the y-axis has been adjusted by a multiplicative factor for graphical convenience.

Open-Loop Control and Policy Coordination

The open-loop case is most easily dealt with. We first append a symmetric foreign-country model to the home-country model just discussed. The model is shown in Table 2. In the NC solution, each government at $t = 0$ solves for an optimal sequence of monetary policies taking as given the sequence selected from abroad. In the C solution, a single controller chooses $\{m\}_0^{\infty}$ and $\{m^*\}_0^{\infty}$ to maximize a weighted average of intertemporal utilities at home and abroad. In view of the symmetry assumed between the countries, $\{m\}_0^{\infty}$ will equal $\{m^*\}_0^{\infty}$ as a feature of both solutions, with the adjustment paths at home and abroad identical. The key result is that non-cooperative control leads to over-contractionary anti-inflation policies relative to the social optimum. Both countries are made better off by a coordinated policy of less rapid disinflation.

In general, the dimensionality of the control problem is too high to analyze the NC case analytically. An important special case, however, allows us to establish analytically the key features of the NC versus C solutions. Since the findings are insightful, we begin with that special case. In particular, we first assume that aggregate demand and money demand are not interest sensitive ($\sigma = \varepsilon = 0$ in the original model). This simplification allows us to determine e_t as a function of the current state vector together with m_t and m_t^*, rather than as a forward-looking variable dependent on the entire future sequence of policies. Also, to reduce further the dimensionality, we set $\theta = 0$, so that wage change depends on the level of output but not its lagged rate of change.

Denoting the real exchange rate as $r_t = p_t^* + e_t - p_t$, we can write $p_t^c = p_t + (1-\lambda)r_t$, and $\pi_t = p_t^c - p_{t-1}^c = (p_t - p_{t-1}) + (1-\lambda)(r_t - r_{t-1})$. Therefore, from the

Table 2. Two-Country Model

Aggregate Demand

$$q_t = -\delta(p_t - e_t - p_t^*) + \gamma q_t^* - \sigma[i_t - (p_{t+1} - p_t)]$$

$$q_t^* = -\delta(p_t^* + e_t - p_t) + \gamma q_t - \sigma[i_t^* - (p_{t+1}^* - p_t^*)]$$

Money Demand

$$m_t - p_t = \alpha q_t - \varepsilon i$$

$$m_t^* - p_t^* = \alpha q_t^* - \varepsilon i^*$$

Consumer Price Index

$$p_t^c = \lambda p_t + (1-\lambda)(p_t^* + e_t)$$

$$p_t^{c*} = \lambda p_t^* + (1-\lambda)(p_t - e_t)$$

Domestic Price Level

$$p_t = w_t$$

$$p_t^* = w_t^*$$

Nominal Wage Change

$$(w_{t+1} - w_t) = \pi_t + \psi q_t + \theta(q_t - q_{t-1})$$

$$(w_{t+1}^* - w_t^*) = \pi_t^* + \psi q_t^* + \theta(q_t^* - q_{t-1}^*)$$

Inflation

$$\pi_t = p_t^c - p_{t-1}^c$$

$$\pi_t^* = p_t^{c*} - p_{t-1}^{c*}$$

Exchange Rate

$$e_{t+1} = e_t + i_t - i_t^*$$

wage equation, and the fact that $p_t = w_t$, we have $\pi_{t+1} = \pi_t + (1-\lambda)(r_{t+1}-r_t) + \psi q_t$. Note from this expression that inflation accelerated when $r_{t+1} > r_t$ or $q_t > 0$. In other words, a real depreciation between periods t and t+1 causes inflation to accelerate, basically because real import prices rise. Carrying out the same manipulation for the foreign country yields $\pi^*_{t+1} = \pi^*_t - (1-\lambda)(r_{t+1}-r_t) + \psi q^*_t$. Note that a real depreciation at home causes inflation to <u>fall</u> abroad, while an appreciation at home causes foreign inflation to rise. Here is the nub of the coordination problem: each country may have an incentive to contract the economy in order to appreciate the currency and thereby export inflation abroad at the expense of the other country. Since the exchange rate effects are bound to cancel out if each country chooses contractionary policies to appreciate its currency, a coordinated policy can avoid the contractionary policies, to the mutual benefit of both countries.

It only remains to determine r_t before solving for the two equilibria. Subtracting the foreign aggregate demand schedule from the home schedule we find:

(33) $$r_t = \alpha(q_t-q^*_t) \qquad \alpha = (1+\gamma)/2\delta > 0$$

From (33), we see that the key to a real appreciation is to be more contractionary than one's neighbor. The effort towards contraction leads to the inefficiency of the non-cooperative outcome.

In any period, p_t and p^*_t are predetermined variables, so that the choice of m_t and m^*_t fix q_t and q^*_t respectively, in view of the money demand schedules. Thus, we may think of the policy authorities as controlling q_t and q^*_t directly, and then use the sequences $\{q_t\}_0^\infty$ and $\{q^*\}_0^\infty$ to find the paths of prices and the policies m_t and m_t^* as $p_t + \alpha q_t$ and $p_t^* + \alpha q_t^*$.

We now write the home country's optimization problem in canonical form. At any moment, there are two state variables, p_t and p_{t-1}^c, and we write the dynamic system in terms of these states:

$$(34)\qquad \begin{bmatrix} p_{t+1} \\ p_t^c \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_t \\ p_{t-1}^c \end{bmatrix} + \begin{bmatrix} \alpha(1-\gamma) + \psi \\ \alpha(1-\gamma) \end{bmatrix} q_t - \begin{bmatrix} \alpha(1-\gamma) \\ \alpha(1-\gamma) \end{bmatrix} q_t^*$$

Note that q_t is the control variable, and q_t^* is an exogenous forcing variable from the point of view of the home country. The objective function is again a discounted sum of quadratic loss functions in q_t and π_t:

$$(35)\qquad U_0 = -(1/2)\Sigma_{t=0}^{\infty}\beta^t(q_t^2+\phi\pi_t^2)$$

Note that $\pi_t = p_t^c - p_{t-1}^c = (p_t-p_{t-1}^c) + \alpha(1-\gamma)(q_t-q_t^*)$.

We set up a Lagrangian $\mathcal{L}$ and take first-order conditions in the standard way (note that μ_{1t} is the co-state variable for p_t, and μ_{2t} for p_{t-1}^c).

$$(36)\qquad \mathcal{L} = -1/2\Sigma_{t=0}^{\infty}\beta^t\{q_t^2 + \phi[(p_t-p_{t-1}^c) + \alpha(1-\gamma)(q_t-q_t^*)]^2$$

$$+ \mu_{1t}[2p_t - p_{t-1}^c + \psi q_t + \alpha(1-\gamma)(q_t-q_t^*) - p_{t+1}]$$

$$+ \mu_{2t}[p_t + \alpha(1-\gamma)(q_t-q_t^*) - p_t^c]$$

First order conditions are:

$$\partial\mathcal{L}/\partial q_t = 0 \Rightarrow q_t + \phi\alpha(1-\gamma)[(p_t-p_{t-1}^c) + \alpha(1-\gamma)(q_t-q_t^*)]$$

$$+ \mu_{1t}\psi + \mu_{1t}\alpha(1-\gamma) + \mu_{2t}\alpha(1-\gamma) = 0$$

$$\partial\mathcal{L}/\partial p_t = 0 \Rightarrow \phi[(p_t-p_{t-1}^c) + \alpha(1-\gamma)(q_t-q_t^*)] + 2\mu_{1t} - \mu_{1t-1}/\beta + \mu_{2t} = 0$$

$$\partial\pounds/\partial\mu_{1t} = 0 \Rightarrow p_{t+1} = 2p_t - p^c_{t-1} + \psi q_t + \alpha(1-\gamma)(q_t - q^*_t)$$

$$\partial\pounds/\partial\mu_{2t} = 0 \Rightarrow p^c_t = p_t + \alpha(1-\gamma)(q_t - q^*_t)$$

$$\partial\pounds/\partial p^c_{t-1} = 0 \Rightarrow -\phi[(p_t - p^c_{t-1}) + \alpha(1-\gamma)(q_t - q^*_t)] - \mu_{1t} - \mu_{2t-1}/\beta = 0$$

We now invoke a sleight of hand. The foreign country is carrying out an identical optimization, which by symmetry must yield $q_t = q^*_t$. Without specifying the foreign country's problem, we simply invoke this symmetry condition as a property of the equilibrium, in order to simplify the first-order conditions. Note that when $q_t = q^*_t$, p^c_t equals p_t, so that $\pi_t = p^c_t - p^c_{t-1} = p_t - p^c_{t-1}$. Using these facts, we rewrite the first-order conditions as:

(37)
$$\mu_{1t}[\psi + \alpha(1-\gamma)] + \mu_{2t}\alpha(1-\gamma) + \phi\alpha(1-\gamma)\pi_t + q_t = 0$$
$$2\mu_{1t} - \mu_{1t-1}/\beta + \mu_{2t} + \phi\pi_t = 0$$
$$\mu_{1t} + \mu_{2t-1}/\beta + \phi\pi_t = 0$$
$$\pi_{t+1} - \pi_t - \psi q_t = 0$$

By direct inspection of (37)(b) and (c), we can see that the system will satisfy $\mu_{2t} = -\mu_{1t}$.[4] We now make that substitution and also substitute for q_t, to write a 2x2 system in μ_{1t} and π_t:

(38)
$$\begin{bmatrix} \mu_{1t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1/\beta + \phi\psi^2 & \phi^2\psi\alpha(1-\gamma) - \phi \\ -\psi^2 & 1 - \psi\alpha(1-\gamma)\phi \end{bmatrix} \begin{bmatrix} \mu_{1t} \\ \pi_t \end{bmatrix}$$

As long as $\beta < [1-\psi\alpha(1-\gamma)\phi]$, this system has a single root within the unit circle and a single root outside the unit circle (the condition is sufficient, though not necessary).[5] Denote the stable root as λ^N_1 (the superscript N denotes non-cooperative case). Thus, the dynamics of inflation are:

(39) $$\pi_{t+1} = \lambda_1^N \pi_t$$

Starting from an inherited inflation rate π_0, the two economies converge to zero inflation, with a mean lag of $\lambda_1^N/(1-\lambda_1^N)$ years.

Now let us consider the cooperative case. Here, a single controller chooses q_t and q_t^* to maximize an average of utilities in the two countries. Since the countries are identical, we may assume simply that the controller maximizes domestic utility subject to the constraint that $q_t = q_t^*$ for all t. With this constraint, the inflation equation is $\pi_{t+1} = \pi_t + \psi q_t$. The Lagrangian for the single controller problem is therefore:

(40) $$\max_{\{q\}_{t=0}^{\infty}} \pounds = -1/2\Sigma_{t=0}^{\infty}\beta^t\{q_t^2 + \phi\pi_t^2 + \mu_{1t}[\pi_t + \psi q_t - \pi_{t+1}]\}$$

The dynamic equations for the first-order conditions of (40) are:

(41) $$\begin{bmatrix} \mu_{1t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1/\beta + \phi\psi^2 & -\phi \\ -\psi^2 & 1 \end{bmatrix} \begin{bmatrix} \mu_{1t} \\ \pi_t \end{bmatrix}$$

Note the relationship between (38) and (41). The cooperative dynamics are found by setting $\alpha = 0$ in (38). α is the parameter which measures how large a real appreciation is achieved for a given contraction of q relative to q*. It thus indicates the importance of the "beggar-thy-neighbor" phenomenon, in which each country (vainly) attempts to keep output lower at home than abroad in order to export inflation. Since the single controller recognizes the futility of each country, in a closed system, trying to export inflation, the controller simply sets $\alpha = 0$. That is the root of the gain to cooperation.

The matrix in (41) again has a single stable root, this time denoted λ_1^C.[6] The dynamics of inflation are now

$$\pi_{t+1} = \lambda_1^C \pi_t \tag{42}$$

It is a simple matter to prove that $\lambda_1^C > \lambda_1^N$ for $\alpha > 0$, so that cooperative control results in <u>slower disinflation</u> than non-cooperative control.[7] Figure 6 illustrates the inflation and output paths of the home economy under cooperation and non-cooperation. The faster disinflation under NC is clearly brought about by increased unemployment (i.e. reduced output) in the early years of the disinflation process. Remember from our earlier discussion that the <u>cumulative</u> output loss is the same for all paths that asymptotically reduce inflation to zero.

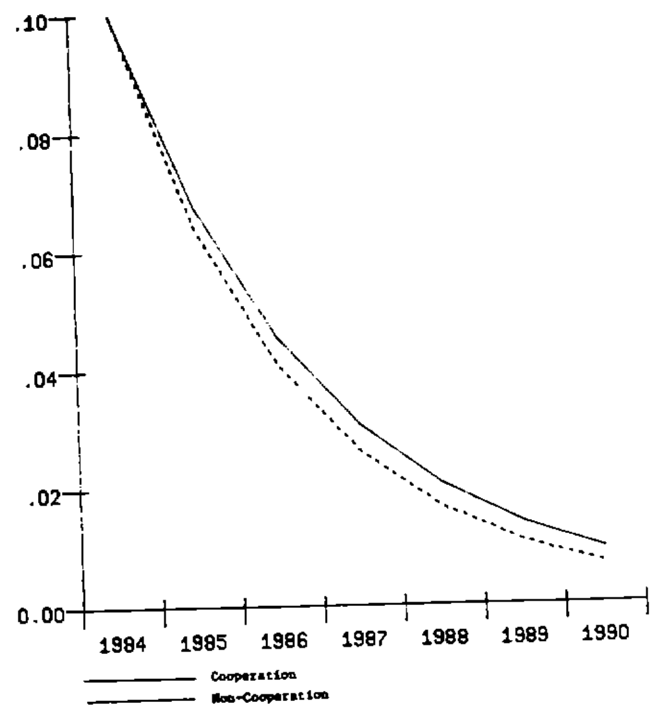
<u>Welfare Aspects of Cooperation</u>

Assuming that governments are pursuing appropriate objectives (e.g. that they use the "right" discount rate), it is easy to show that the cooperative path, with less extreme disinflation, dominates the non-cooperative path. A simple argument is as follows (direct computation would also make the same point). Define the set of pareto efficient (E) pairs of sequences $[\{q\}_0^\infty, \{q^*\}_0^\infty]^E$ that have the property that U_0 is maximized given U_0^*, and U_0^* is maximized given U_0. It is well known that the set of pareto efficient pairs may be found by maximizing $wU_0 + (1-w)U_0^*$ with respect to $\{q\}_0^\infty$ and $\{q^*\}_0^\infty$ for all weights $w \in [0,1]$. Every pareto efficient sequence pair maximizes some weighted average of U_0 and U_0^*, and every sequence pair that maximizes $wU_0 + (1-w)U_0^*$ is pareto efficient.

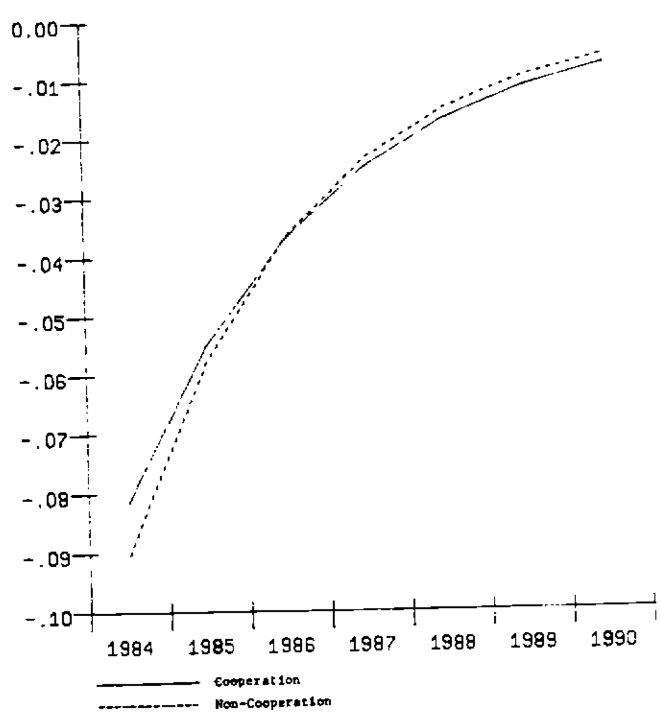
Figure 6.

A Comparison of Non-Cooperative and Cooperative Control
(Simplified two-country model)

a. Inflation



b. Output

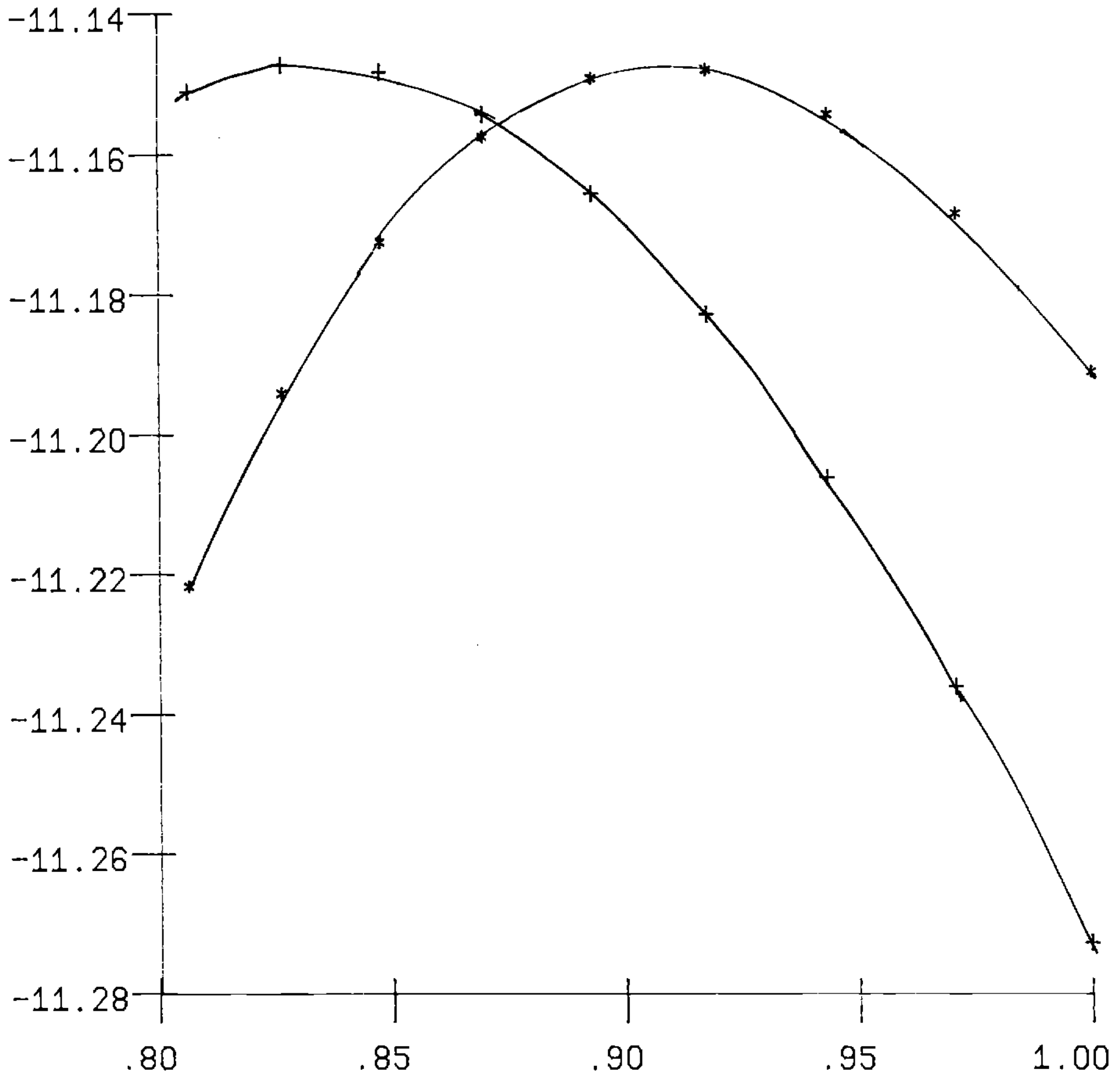


The cooperative solution, by construction, gives the sequence pair corresponding to w = 0.5 (i.e. equal weighting of the countries). It is the unique solution to the problem. Since the non-cooperative solution also yields a symmetric equilibrium, with $U_0 = U_0^*$, it must be that $U_0^{NC} < U_0^C$, since otherwise the non-cooperative solution would pareto dominate a known pareto efficient solution.

We mentioned in the introduction that some critics of cooperation are dubious of the assumption that governments maximize the proper social welfare function. In particular, plausible arguments have been made that the government's discount rate β^G is less than the "true" β. If so, cooperation might exacerbate rather than meliorate social welfare. The point is that cooperation allows governments to pursue a more "leisurely" disinflation. However, short-sighted governments might already be postponing the necessary disinflation, in return for short-run gains to output. In an already distorted policy environment, cooperation might further retard the necessary adjustment.

To examine this view, we computed the open-loop cooperative and non-cooperative intertemporal utilities for a range of β^G, holding fixed the "true" β at $(1.1)^{-1}$ (we use the simplified version of the two-country model for these calculations). For each β^G, we calculate the two equilibria and then evaluate the social welfare of the resulting paths using $\beta = (1.1)^{-1}$. As seen from Figure 7 non-cooperation dominates cooperation when β^G is sufficiently smaller than β, and cooperation dominates non-cooperation as long as β^G is "close enough" or somewhat greater than β. Of course, for any $\beta^G = \beta$, open-loop cooperation will necessarily be superior to open-loop non-cooperation. It is

Figure 7. The Gains from Cooperation with Myopic Governments[a]
$[\beta^G < \beta = (1.1)^{-1}]$



* Cooperation
\+ Non-cooperation

[a]Note that the welfare scale on the y-axis has been adjusted by a multiplicative factor for graphical convenience.

not the level of β^G but the difference of β^G and β which might cause cooperation to be welfare reducing.

Policy Coordination and Time Consistency

We now leave the case of open-loop control and return to the more realistic assumption that governments cannot bind their successors. In the non-cooperative setting we are looking for an equilibrium characterized by rules $m_t = f(x_t)$ and $m_t^* = f^*(x_g)$ that have the following property: for the home country, f is optimal at time t given that all future governments at home play f and that abroad the contemporaneous and all future governments play f*; while for the foreign country, f* is optimal under the analogous conditions. Note that x_t is the state vector including predetermined variables of both the home and foreign economy. In particular, $x_t = \langle p_t, p_t^*, p_{t-1}^c, p_{t-1}^{c*}, q_{t-1}, q_{t-1}^* \rangle$.

There are two key differences with the open-loop model previously described. First, of course, is the inability of G_0 and G_0^* to bind the entire sequence of future moves. Second is the assumption that each government takes as given the foreign rule rather than the foreign actions, so that optimal moves today take into account the effects of today's actions on tomorrow's state vector, and thus on the foreign governments' moves. It would be possible instead to calculate a time-consistent multicountry equilibrium in which each government takes as given the sequence of future moves (i.e. open-loop time consistency), but we have not pursued that choice here.

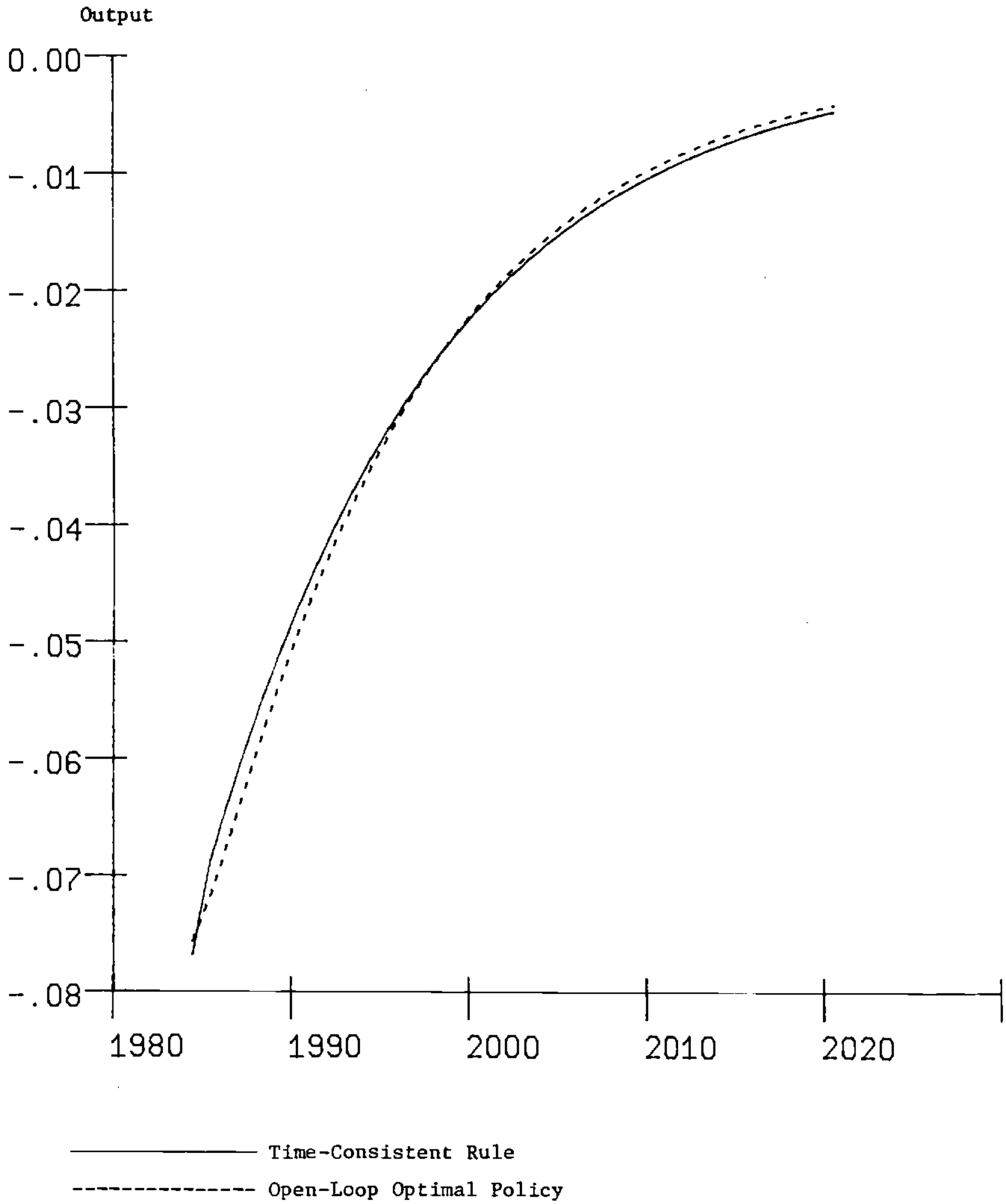
As in the one-country case, the time-consistent equilibrium is solved as the limit of a backward recursion. (For the calculations that follow, we revert to the complete two-country model, with non-zero values of σ, ε, and θ). Using the parameter values of the one-country model, we arrive at the following rules:

$$(43) \qquad m_t = -.286p_t + .953\, p^c_{t-1} - .132\, p^*_t + .246\, p^*_{t-1} + .23\, q_t + .072q^*_t$$

Figure 8 compares the paths of the home economy output for the non-cooperative open-loop and non-cooperative time-consistent equilibria. As in the one-country model, output losses are smaller in the early periods for TC than OL. The inability to bind one's successors causes a bias towards more expansionary policies and thus more rapid inflation, relative to the open-loop solution.

Significantly, it is no longer possible to rank social welfare under open-loop versus time-consistent policies (for non-cooperative equilibria), as it was in the one-country model. Remember the argument in the one-country context. Open-loop control, by definition, picks the optimal sequence; time-consistent policy, on the other hand, reflects an optimization under additional constraints and therefore is inferior to the open-loop control. In the two-country setting, the same logic does not apply. The open-loop sequence is no longer the optimal sequence. Indeed we have seen that open-loop, non-cooperative control is typically pareto inefficient. There is no presumption that adding constraints to the optimization will now lower welfare, particularly since constraints are being added abroad as well as at home. It is true that the home country can no longer pre-commit to a sequence of moves, but now neither can the foreign country. It is true that the home country prefers an open-loop to time consistent policy assuming that the other country is <u>fixed</u> at one or the other. With the other country's policy fixed, an open-loop policy

Figure 8. A Comparison of Non-Cooperative Control: Open-Loop versus Time-Consistent Solutions (Two-country model)



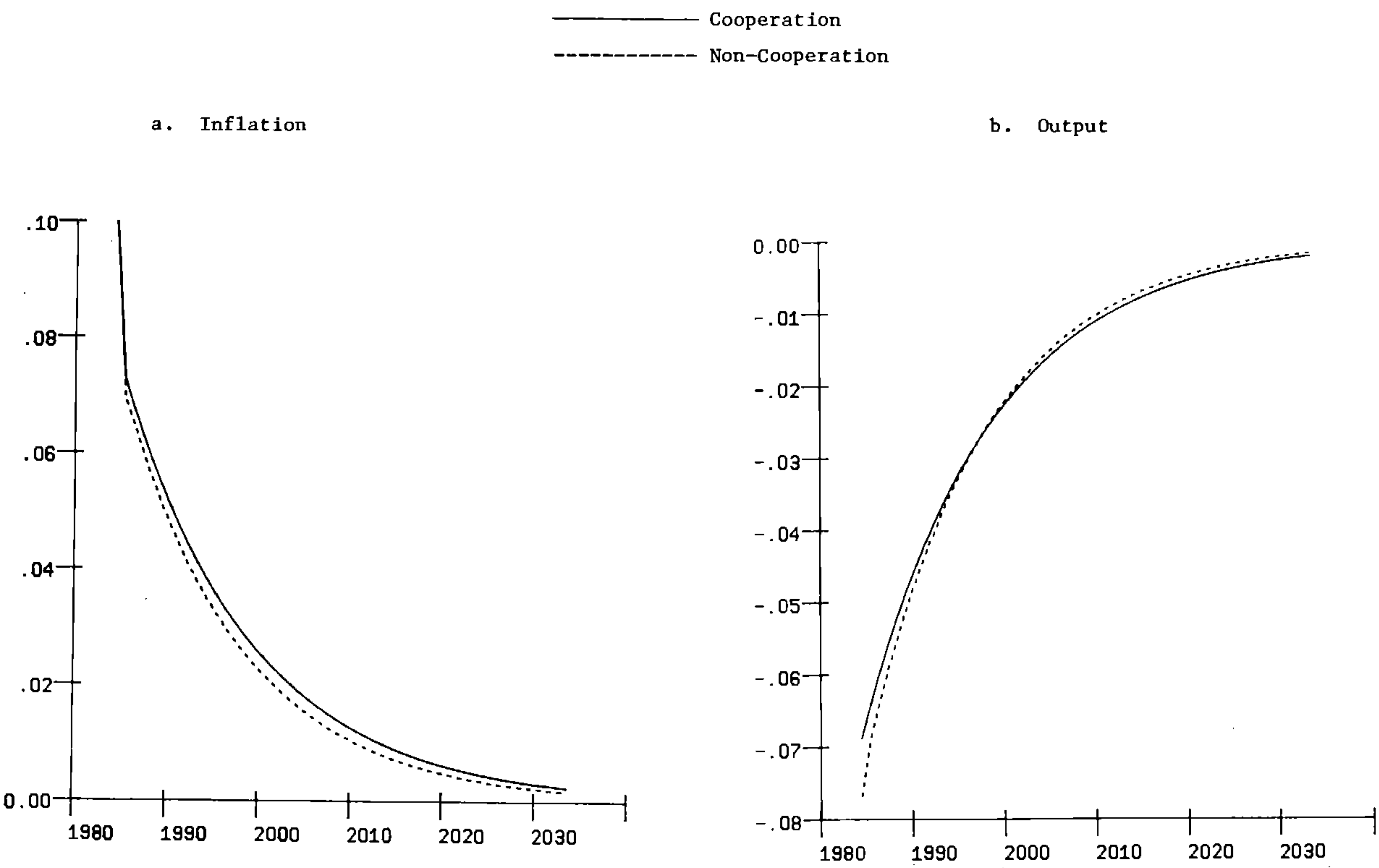
at home can exactly replicate the time-consistent sequence, and presumably it can do it better.

There are good economic reasons to believe that the time-consistent policy may actually dominate the open-loop solution in the non-cooperative game. The open-loop policy, we know, is over-contractionary relative to the efficient equilibrium. Moving from open-loop control to time consistency causes policy to become less contractionary and therefore pushes the economy towards the efficient equilibrium.

Now, let us consider the time-consistent cooperative equilibrium. Here we imagine that a single controller each period sets m and m*, but now subject to the time-consistency constraint. The single cooperative controller must optimize while taking as giving the actions of single cooperative controllers in later periods. We should like to determine whether time-consistent cooperation is superior to time-consistent non-cooperation. As we have noted in several places Rogoff (1983) has devised an ingenious example where cooperation reduces welfare. Simply, time-consistency leads governments to be over-inflationary relative to the open-loop pre-commitment equilibrium. Cooperation further exacerbates this over-inflationary bias by removing each government's fear of currency depreciation.

Interestingly, our results run counter to Rogoff's: cooperation is superior in welfare terms to non-cooperation. While the cooperative solution is more inflationary (see Figure 9), as we might expect, it is not overly inflationary in a welfare sense. The less rapid disinflation merely corrects the contractionary bias of the non-cooperative case. The key point here is as

Figure 9. A Comparison of Non-Cooperative versus Cooperative Control: The Case of Time Consistency (Two-country model)



follows. In the symmetric country model, the single controller always adopts symmetric rules so that $e_t = 0$ for all t. Since the exchange rate is the sole potential source of time inconsistency in this model, and since it is always equal to zero, the cooperative time-consistent solution is also the open-loop cooperative solution. For a cooperative controller, there is no time-consistency problem in our model (since the countries are symmetric). The single controller can reach the first-best optimum solution for open-loop cooperative control.

In sum, we have shown examples where cooperative control is more inflationary than open-loop non-cooperative control and time-consistent non-cooperative control. In both cases, the cooperative solution is welfare improving relative to the non-cooperative equilibrium. In view of Rogoff's example, it will be difficult indeed to set out general principles on the gains from cooperation under the constraint of time consistency. Comparing our example with his, the key difference seems to rest on the source of the time-consistency problem. In Rogoff's case, the problem arises from forward-looking wage setters and cooperation exacerbates the problem. In our model, the problem arises from forward-looking exchange market participants, and cooperation eliminates the problem.

Conclusions

This study represents work in progress on the gains to coordination in dynamic macroeconomic models. Our focus has been purely methodological, and preparatory to attempts at a quantitative assessment of international policy coordination. The methodological issues arise from the wide variety of possible equilibrium concepts in multicountry dynamic games. The games can be solved under the assumption of pre-commitment versus time-consistency; open-loop versus closed-loop behavior; and non-cooperative versus cooperative decision-making. These three dimensions are all independent, so any choice along each dimension is possible.

Moreover, in some cases there may be multiple equilibria. For example, there are probably many time-consistent, non-cooperative equilibria that depend on the "threat-reputation" mechanism outlined in the paper. As yet, we have made no systematic attempt to search for such equilibria.

This work should now be used to gain empirical insight into the cooperation issue. For all of the discussion surrounding time consistency, for example, there is not a single empirical investigation of its importance in the macroeconomics literature. Similarly, there are no reliable measures of the gains to cooperation in the simpler, pre-commitment equilibria. Such quantitative work deserves a high priority.

Appendix

We shall present in this appendix the derivation of the four policy rules discussed in this paper. All of these rules are obtained as the stationary limit of backward recursions using a methodology similar to Basar and Olsder (1982) or Kydland (1975). The only significant difference with these authors is the fact the followers' actions are represented here by a forwardlooking variable, the exchange rate.

Let us consider a two-country world. The world economy is characterized by an n-dimensional vector of state variables, x_t and the domestic currency price of the foreign currency is e_t. In each country the authorities seek to maximize a welfare function W_i, i = 1,2, and can use a set of policy instruments denoted U_{it}, where U_{it} is an m_i-dimensional vector. The dynamics of the world economy can be represented by a system of difference equations.

(A1)
$$x_{t+1} = Ax_t + Be_t + CU_t$$
$$e_{t+1} = Dx_t + Fe_t + GU_t$$

where U_t denotes the stacked vector of instruments for the world economy and A, B, C, D, F and G are matrixes of parameters. Note that matrixes A, B, C are defined differently than matrixes A, B, C in the rest of the paper.

Let us denote by τ_{it} the vectors of targets for each country. τ_{1t} and τ_{2t} are linear functions of the state variables, the exchange rate and the values of the policy instruments:

(A2)
$$\tau_{it} = M_i x_t + L_i e_t + N_i U_t \qquad i = 1,2$$

Pages 57 and 58

are missing.

where

(A14) $$J_t = (F-H_{t+1}B)^{-1}(H_{t+1}A-D)$$

$$K_t = (F-H_{t+1}B)^{-1}(H_{t+1}C-G)$$

The value function of country 1 for period t is defined by:

(A15) $$V_{1t}(x_t) = \underset{U_{1t}}{\text{Min}} -(1/2)\tau_{1t}'\Omega_1\tau_{1t} + \beta_1 V_{1t+1}(x_{t+1}), \text{ given } x_t$$

Substituting (A13) into (A1) and (A2) leads to the following first order conditions:

(A16) $$[(N_{11}+L_1K_{1t})'\Omega_1(N_1+L_1K_t) + \beta_1(C_1+BK_{1t})'S_{1t+1}(C+BK_t)]U_t$$
$$= -[(N_{11}+L_1K_{1t})'\Omega_1(M_1+L_1J_t) + \beta_1(C_1+BK_{1t})'S_{1t+1}(A+BJ_t)]x_t$$

where K_{1t} and Z_1 is the submatrixes of K_t and $\bar{C}$ corresponding to U_{1t}.

A similar set of conditions holds for country 2. We thus obtain:

(A17) $$MM_tU_t = -NN_tx_t$$

where MM_t is an $(m_1+ m_2)x(m_1+ m_2)$ dimensional matrix and NN_t is an $(m_1+ m_2)x\ n$ dimensional matrix.

Let us divide MM_t and NN_t in submatrixes corresponding to U_{1t} and U_{2t}:

(A18) $$MM_t = \begin{bmatrix} MM_{11t} & MM_{12t} \\ MM_{21t} & MM_{22t} \end{bmatrix}; \quad NN_t = \begin{bmatrix} NN_{1t} \\ NN_{2t} \end{bmatrix}$$

Then we have:

(A19) $$MM_{ijt} = (N_{ii}+L_iK_{it})'\Omega_i(N_{ij}+L_iK_{jt}) + \beta_i(C_{it}+BK_{it})'S_{it+1}(C_j+BK_{jt})$$

(A20) $$NN_{it} = (N_{ii}+L_iK_{it})'\Omega_i(M_i+L_iJ_t) + \beta_i(C_i+BK_{it})'S_{it+1}(A+BJ_t)$$

These formula hold for period T with J_T and K_T defined as above and $S_{iT+1} = 0$.

Finally we can derive Γ_t, H_t and S_{it}:

(A21) $\Gamma_t = -MM_t^{-1}NN_t$

(A22) $H_t = J_t + K_t\Gamma_t$

(A23) $S_{it} = (M_i+L_iH_t+N_i\Gamma_t)'\Omega_i(M_i+L_iH_t+N_i\Gamma_t) + \beta_i(A+BH_t+C\Gamma_t)'S_{it+1}(A+BH_t+C\Gamma_t);$ $i = 1,2$

We have thus obtained both recursion rules and starting values for the set of matrixes Γ_t, H_t, S_{1t} and S_{2t}. We define as the time consistent solution the stationary solution to which this system converges for t = 0 as T goes to infinity. We do not know of any general result concerning the convergence of this process. However in our empirical applications we have not run into major problems. Cohen and Michel (1984) show that in a one dimensional case this kind of a recursion does have a fix-point.

The Open-Loop Solution

The open-loop solution corresponds to a one-shot game where the authorities announce at time zero the whole path of their policies. It thus does not by definition require the use of a backward recursion procedure. The set of dynamic equations formed by the state variable difference equations and the first-order conditions corresponding to the optimal control problem of the authorities could for example be solved explicitly by using the method proposed in Blanchard and Kahn (1980) or numerically with a multiple shooting algorithm (see Lipton, Poterba, Sachs and Summers (1982)). However, we shall present here a backward recursion procedure which leads to a simple algorithm.

The optimal control problem faced by the authorities of country i leads to the definition of the Hamiltonian H_{it}:

(A24) $$H_{it} = (1/2)\tau'_{it}\beta_i^t\Omega_i\tau_{it} + \beta_i^{t+1}p'_{it+1}(Ax_t+Be_t+CU_t-x_{t+1}) + \beta_i^{t+1}\mu_{it+1}(Dx_t+Fe_t+GU_t-e_{t+1})$$

where p_{it+1} is the vector of co-state variables or shadow costs which the authorities of country i associate with each of the state variables and, similarly μ_{it+1} is the co-state variable corresponding to the exchange rate.[8]

The set of first-order conditions is then:

(A25) $$\partial H_{it}/\partial U_{it} = N'_{ii}\Omega_i\tau_{it} + \beta_i C'_i p_{it+1} + \beta_i G'_i\mu_{it+1} = 0$$

(A26) $$\partial H_{it}/\partial x_t = M'_i\Omega_i\tau_{it} + \beta_i A' p_{it+1} + \beta_i D'\mu_{it+1} = p_{it}$$

(A27) $$\partial H_{it}/\partial e_t = L'_i\Omega_i\tau_{it} + \beta_i B' p_{it+1} + \beta_i F'\mu_{it+1} = q_{it}$$

Let us first of all derive the recursion equations at period t. One major difference with the time consistent case is the existence of μ_t, the co-state variable corresponding to the exchange rate at time t. Since e_0 is not pre-determined, it can be set freely by the authorities in the initial period by announcing a proper path of future policies. Its shadow cost in the first period, μ_1, is zero. μ_t is thus a predetermined variable equal to zero in the first period and has to be added to the vector of state variables, x_t, when the recursion relations are defined.

More precisely we shall assume that the problem is solved for t+1 and that the following relations hold:

(A28) $$e_{t+1} = H_{t+1}x_{t+1} + h_{t+1}\mu_{t+1}$$

(A29) $$p_{t+1} = \Delta_{t+1}x_{t+1} + \delta_{t+1}\mu_{t+1}$$

(A30) $$U_{t+1} = \Gamma_{t+1}x_{t+1} + \gamma_{t+1}\mu_{t+1}$$

Let us now define the following matrixes:

$$A^i_{11} = N'_{ii}\Omega_i;\ A^i_{21} = M'_i\Omega_i;\ A^i_{31} = L'_i\Omega_i$$

(A31) $$A^i_{12} = \beta_i C'_i;\ A^i_{22} = \beta_i A';\ A^i_{32} = \beta_i B';\ i = 1,2$$

$$A^i_{13} = \beta_i G'_i;\ A^i_{23} = \beta_i D';\ A^i_{33} = \beta_i F$$

(A32) $$A_{k\ell} = \begin{bmatrix} A^1_{k\ell} & 0 \\ 0 & A^2_{k\ell} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Equations (A25) to (A27) can be rewritten in matrix form:

(A34) $$A_1 \begin{bmatrix} \tau_t \\ p_{t+1} \\ \mu_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{2n} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_t \\ \mu_{t+1} \\ p_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} x_t \\ \mu_t \end{bmatrix}$$

where I_{2n} and I_2 denote identity matrixes of dimensions 2n and 2 respectively. Then using equations (A28) to (A30) we get:

(A35) $$e_t = J_t x_t + k_t U_t + R_t \mu_{t+1}$$

(A36) $$\tau_t = B_{1t} x_t + B_{2t} U_t + B_{3t} \mu_{t+1}$$

(A37) $$\begin{bmatrix} \tau_t \\ p_{t+1} \\ \mu_{t+1} \end{bmatrix} = A_{2t} \begin{bmatrix} U_t \\ \mu_{t+1} \\ p_t \end{bmatrix} + A_{3t} \begin{bmatrix} x_t \\ \mu_t \end{bmatrix}$$

where τ_t and p_{t+1} are the stacked vectors of targets and co-state variables and

$$A_{2t} = \begin{bmatrix} B_{2t} & 0 & 0 \\ \Delta_{t+1}(C+BK_t) & \Delta_{t+1}BR_t+\delta_{t+1} & 0 \\ 0 & I_2 & 0 \end{bmatrix}$$

$$A_{3t} = \begin{bmatrix} B_{1t} & B_{3t} \\ \Delta_{t+1}(A+BJ_t) & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_{1t} = \begin{bmatrix} M_1 + L_1J_t \\ M_2 + L_2J_t \end{bmatrix}; \quad B_{2t} = \begin{bmatrix} N_1 + L_1K_1 \\ N_2 + L_2K_t \end{bmatrix}; \quad B_{3t} = \begin{bmatrix} L_1R_t \\ L_2R_t \end{bmatrix}$$

$$J_t = (F-H_{t+1}\bar{B})^{-1}(H_{t+1}\bar{A}-D)$$

$$K_t = (F-H_{t+1}\bar{B})^{-1}(H_{t+1}\bar{C}-G)$$

$$R_t = (F-H_{t+1}\bar{B})^{-1}h_{t+1}$$

(A38) $$\begin{bmatrix} U_t \\ \mu_{t+1} \\ p_t \end{bmatrix} = -MM_t^{-1}NN_t \begin{bmatrix} x_t \\ \mu_t \end{bmatrix}$$

where:

$$MM_t = A_1A_{2t} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{2n} \\ 0 & 0 & 0 \end{bmatrix}$$

$$NN_t = A_1A_{3t} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I_2 \end{bmatrix}$$

From (A38) we can derive Γ_t, γ_t, Δ_t, δ_t, Λ_t and λ_t where the two last variables are defined by:

$$\mu_{t+1} = \Lambda_t x_t + \lambda_t \mu_t$$

Lastly we get:

$$H_t = J_t + K_t \Gamma_t + R_t \Lambda_t$$

$$h_t = K_t \gamma_t + R_t \lambda_t$$

We now need to obtain starting values for the recursions thus defined. If we assume as above that the exchange rate stabilizes at time T and that $p_{T+1} = 0$, we get:

$$J_T = (1-F)^{-1}D;\ K_t = (1-F)^{-1}G;\ R_T = 0;\ \Delta_{t+1} = 0;\ \delta_{T+1} = 0$$

The open-loop solution is the stationary limit to which this recursion converges. It should be noted that here the policy rule is not only a function of the state variables, x_t, but also of the costate variables μ_t.

Let us give a simple example in the case where each country has a single policy instrument. The policy rule is $U_t = \Gamma x_t + \gamma \mu_t$, where γ is a (2x2) matrix. We also have:

$$\mu_t = \Lambda x_{t-1} + \lambda \mu_{t-1}$$

which, given the policy rule, yields:

$$\mu_t = \Lambda x_{t-1} + \lambda(\gamma^{-1}U_{t-1} - \gamma^{-1}\Gamma x_{t-1})$$

Thus we finally obtain

$$U_t = \gamma\lambda\gamma^{-1}U_{t-1} + \gamma(\Lambda-\lambda\gamma^{-1})x_{t-1} + \Gamma x_t$$

The policy rule appears to be of a more complicated form than the time consistent rule. It is a function not only of the current state variables but of the lagged values of these state variables and of the lagged moves.

The Buiter Solution

Buiter (1983) proposes a solution to the time inconsistency problem which we discuss in the paper. Formally his strategy amounts to setting μ_t equal to zero and suppressing equations (A37).

Using the same notation the set of first order conditions becomes:

$$A_1 \begin{bmatrix} \tau_t \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ p_t \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Equations (A28) to (A30) become:

(A28´) $e_{t+1} = H_{t+1}x_{t+1}$

(A29´) $p_{t+1} = \Delta_{t+1}x_{t+1}$

(A30´) $U_{t+1} = \Gamma_{t+1}x_{t+1}$

Then we get:

(A35´) $e_t = J_t x_t + K_t U_t = H_t x_t$

(A36´) $\tau_t = B_{1t}x_t + B_{2t}U_t$

$$(A37') \quad \begin{bmatrix} \tau_t \\ p_{t+1} \end{bmatrix} = A_{2t} \begin{bmatrix} U_t \\ p_t \end{bmatrix} + A_{3t} x_t$$

where

$$A_{2t} \begin{bmatrix} B_{2t} & 0 \\ \Delta_{t+1}(C+Bk_t) & 0 \end{bmatrix} ; \; A_{3t} = \begin{bmatrix} B_{1t} \\ \Delta_{t+1}(A+BJ_t) \end{bmatrix}$$

and finally:

$$(A38) \quad \begin{bmatrix} U_t \\ P_t \end{bmatrix} = -MM_t^{-1} NN_t x_t$$

where

$$MM_t = A_1 A_{2t} - \begin{bmatrix} 0 & 0 \\ 0 & I_{2n} \end{bmatrix}$$

$$NN_t = A_1 A_{3t}$$

From (A38) we derive Γ_t and Δ_t which give H_t:

$$H_t = J_t + K_t \Gamma_t$$

The system of recursive equations thus obtained is solved backward from T with the same starting values as above:

$$J_T = (1-F)^{-1} D; \; K_T (1-F)^{-1} G; \; \Delta_{T+1} = 0$$

The Optimal Linear Rule

The problem here is to derive the optimal linear rule, i.e. the constant feedback rule which yields the higher welfare for the authorities of each country. It can be divided into two steps. The first step consists in obtaining for a given rule

$$\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \text{ such that}$$

$U_t = \Gamma x_t$, the value of the welfare for each country, $W_1(\Gamma)$ and $W_2(\Gamma)$. Then, in a second step, the optimal values of Γ_1 and Γ_2 are calculated using a numerical gradient method. We shall not discuss here the second step for which we refer the reader to Roth (1979). The first step is again solved by backward recursion which proved more tractable for the repeated calculations imposed by the gradient method.

Substituting $U_t = \Gamma x_t$ into (A1) yields:

$$\text{(A39)} \qquad \begin{bmatrix} x_{t+1} \\ e_{t+1} \end{bmatrix} = \begin{bmatrix} A+C\Gamma & B \\ D+G\Gamma & F \end{bmatrix} \begin{bmatrix} x_t \\ e_t \end{bmatrix}$$

For period T assuming $e_{T+1} = e_T$ yields

$$\text{(A40)} \qquad e_T = (1-F)^{-1}(D+G\Gamma)x_T = H_T x_T$$

Then if we assume: $e_{t+1} = H_{t+1}x_{t+1}$,

$$\text{(A41)} \qquad e_t = (F-H_{t+1}B)^{-1}[H_{t+1}(A+C\Gamma) - (D+G\Gamma)]x_t$$

the recursion is thus simply

$$\text{(A42)} \qquad H_t = (F-H_{t+1}B)^{-1}[H_{t+1}(A+C\Gamma) - (D+G\Gamma)]$$

which, starting with H_T, has a stationary solution for values of the parameters such that the transition matrix in (A39) has only one eigenvalue greater than unity. More precisely:

$$\lim_{T \to \infty} H_0 = -C_{22}^{-1} C_{21}$$

where C_{22} and C_{21} are submatrixes of C, the matrix of row eigenvectors of the transition matrix defined by

$$C = \begin{bmatrix} \underset{n \times n}{C_{11}} & \underset{n \times 1}{C_{12}} \\ \underset{1 \times n}{C_{21}} & \underset{1 \times 1}{C_{22}} \end{bmatrix}$$

Footnotes

1. See, for example, W. Nordhaus, "The Political Business Cycle," <u>Review of Economic Studies</u> 42 (1975), pp. 169-190.

2.

$$A = \begin{bmatrix} 1+\lambda-(\psi+\theta)\Delta^{-1}(\delta+\sigma\rho-\sigma\lambda) & -1-(\psi+\theta)\sigma\Delta^{-1} & -\theta & 1-\lambda+(\psi+\theta)\Delta^{-1}[\delta+\sigma(1-\lambda)] \\ \lambda & 0 & 0 & 1-\lambda \\ -(\delta+\sigma\rho-\sigma\lambda)\Delta^{-1} & -\sigma\Delta^{-1} & 0 & [\delta+\sigma(1-\lambda)]\Delta^{-1} \\ \rho-\mu\Delta^{-1}(\delta+\sigma\rho-\sigma\lambda) & -\sigma\mu\Delta^{-1} & 0 & 1+\mu\Delta^{-1}[\delta+\sigma(1-\lambda)] \end{bmatrix}$$

$$B = \begin{bmatrix} \sigma\rho(\mu+\theta)\Delta^{-1} \\ 0 \\ \sigma\rho\Delta^{-1} \\ -\rho+\mu\Delta^{-1}\sigma\beta \end{bmatrix}$$

$$C = \begin{bmatrix} 1+\lambda-(\psi+\theta)\Delta^{-1}[\delta+\sigma(1-\lambda)] & 0 & \gamma(\psi+\theta)\Delta^{-1} \\ 1-\lambda & 0 & 0 \\ [\delta+\sigma(1-\lambda]\Delta^{-1} & 0 & \gamma\Delta^{-1} \\ [\delta+\sigma(1-\lambda)]\mu\Delta^{-1} & -1 & \gamma\mu\Delta^{-1} \end{bmatrix}$$

where $\Delta = [1+\sigma(\mu-\psi-\theta)]^{-1}$

3. Using the notation of the appendix, it is readily checked that if $\sigma = 0$, C and N_1 in (A1) are null matrixes, and G in (A1) is equal to $-\rho$. This implies that the money stock has no direct effect on either the state variables or on

the government's targets: output and inflation. Thus the first-order condition (A25) reduces to $-\beta\rho\mu_{t+1} = 0$.

4. This point is easily proved by considering the following change of variables:

$$(\pi_t, \mu_{1t}, \mu_{2t}) \Rightarrow (\pi_t, \mu_{1t}, \zeta_t)$$

where $\zeta_t = \mu_{1t} + \mu_{1t}$

The differential system (38) becomes:

$$\begin{bmatrix} \mu_{1t+1} \\ \pi_{t+1} \\ \zeta_{t+1} \end{bmatrix} \quad \begin{bmatrix} 1/\beta + \phi\psi^2 & \phi^2\psi\alpha(1-\gamma)-\phi & -1/\beta \\ -\psi^2 & 1-\psi\alpha(1-\gamma)\phi & -\psi\alpha(1-\gamma) \\ 0 & 0 & 1/\beta \end{bmatrix} \begin{bmatrix} \mu_{1t+1} \\ \pi_t \\ \zeta_t \end{bmatrix}$$

This system is saddle point stable under the conditions discussed in the text and has one stable root λ_1^N and two unstable roots λ_2^N and $1/\beta$. One variable π_r is backward looking while μ_{1t} and ζ_t are forward looking. Given that $1/\beta > 1$, it is clear from the third equation that along the stable path ζ_t must always be equal to zero, so that $\mu_{1t} = -\mu_{2t}$ for all t.

5. The roots of the system can be found by solving the characteristic equation:

$$\lambda^2 - (\omega + 1/\beta + \phi\psi^2)\lambda + (1/\beta)\omega = 0, \text{ where } \omega = [1 - \psi\alpha(1-\gamma)\phi]$$

We assume $\omega > 0$. To show that there is exactly one stable root $0 < \lambda_1^N < 1$ and one unstable root $1 < \lambda_2^N$, observe the values of the characteristic equation $C(\lambda)$ at $\lambda = 0$ and $\lambda = 1$. $C(0) = (1/\beta)\omega > 0$ and $C(1) = -\phi\psi^2 - [1/\beta - 1]\psi\alpha(1-\gamma)\phi < 0$. Also, for $\lambda >> 1$, $C(\lambda) > 0$. Thus, there is exactly one root between 0 and 1, and one root exceeding 1.

The stable root is

$$\lambda_1^N = (\omega + 1/\beta + \phi\psi)/2 - (1/2)[(\omega + 1/\beta + \phi\psi^2)^2 - 4\omega/\beta]^{1/2}.$$

The unstable root is:

$$\lambda_2^N = (\omega + 1/\beta + \phi\psi)/2 + (1/2)[(\omega + 1/\beta + \phi\psi^2)^2 - 4\omega/\beta]^{1/2}.$$

6. The roots for the cooperative case can be found by setting $\alpha = 0$ (i.e. $\omega = 1$) in the equations for the roots derived in Footnote 5.

The stable root is

$$\lambda_1^C = (1/2)(1 + 1/\beta + \phi\psi^2) - (1/2)[(1 + 1/\beta + \phi\psi^2)^2 - 4/\beta]^{1/2}.$$

The unstable root is

$$\lambda_2^C = (1/2)(1 + 1/\beta + \phi\psi^2) + (1/2)[(1 + 1/\beta + \phi\psi^2)^2 - 4/\beta]^{1/2}.$$

7. It was shown in footnote 6 that $\lambda_1^C = \lambda_1^N$ when $\alpha = 0$. To prove that $\lambda_1^N > \lambda_1^C$ for $\alpha > 0$, we need only show that $\partial(\lambda_1^C - \lambda_1^N)/\partial\alpha > 0$ for all α. We know that $\partial(\lambda_1^C)/\partial\alpha = 0$. Consider $\partial(\lambda_1^N)/\partial\alpha$

$$\partial(\lambda_1^N)/\partial\alpha = (1/2)\psi(1-\gamma)\phi[-1 + (\omega - 1/\beta + \phi\psi^2)\{(\omega + 1/\beta + \phi\psi^2)^2 - 4\omega/\beta\}^{-1/2}]$$

We want to prove that the last expression is negative. We know $-4\phi\psi^2/\beta < 0$. Therefore,

$$-4/\beta(\phi\psi^2 + \omega + 1/\beta) + 4/\beta^2 + (\omega + 1/\beta + \phi\psi^2)^2 < (\omega + 1/\beta + \phi\psi^2)^2 - 4\omega/\beta,$$

or

$$(\omega - 1/\beta + \phi\psi^2)^2 < (\psi + 1/\beta + \phi\psi^2)^2 - 4\omega/\beta.$$

Taking the square root of both sides and dividing gives

$$(\omega - 1/\beta + \phi\psi^2)\{(\omega + 1/\beta + \phi\psi^2)^2 - 4\omega/\beta\}^{-1/2} < 1$$

Substituting into the expression for $\partial(\lambda_1^N)/\partial\alpha$, we see

$\partial(\lambda_1^N)/\partial\alpha < 0$ for all α.

Thus $\partial(\lambda_1^C-\lambda_1^N)/\partial\alpha > 0$ for all α.

8. Note that in the paper the notation is slightly different, with μ_{4t} being the co-state variable corresponding to the exchange rate in the one-country case.

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