NBER WORKING PAPER SERIES

BEQUESTS AND SOCIAL SECURITY WITH UNCERTAIN LIFETIMES

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Working Paper No. 1372

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
June 1984

I thank Olivier Blanchard, Christopher Cavanagh, Stanley Fischer, Benjamin Friedman, Elhanan Helpman, Nobuhiro Kiyotaki, Robert McDonald, James Pesando, Larry Summers, Lars Svensson, Mark Watson, and Jeffrey Wolcowitz for valuable conversations. I also thank the participants in seminars at Boston University, Columbia University, Cornell University, Federal Reserve Board, Harvard University, University of Iowa, M.I.T., University of Montreal, National Bureau of Economic Research, North Carolina State University, the Wharton School and Yale University for their helpful comments. The research reported here is part of the NBER's research program in Economic Fluctuations and project in Government Budget. Any opinions expressed are those of the author and not those of the National Bureau of Economic Research.

NBER Working Paper #1372
June 1984

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ABSTRACT

The fact that consumers do not know in advance the dates at which they will die effects their individual consumption and portfolio decisions. In general, some consumers will end up leaving bequests at death, even if they have no bequest motive, simply because they happen to die at a time when they are holding wealth to provide for their own future consumption. In the model of this paper, consumers who are otherwise identical, die (randomly) at different ages and thus leave bequests of different sizes to their heirs. Therefore, there is intra-cohort variation in wealth and consumption even if all consumers have the same labor income, taxes, and social security benefits. This paper presents explicit steady state distributions for consumption and wealth. The introduction of an actuarially fair social security system reduces steady state private wealth by more than one-for-one so that, even in a fully funded system, national wealth falls. In addition, all central moments of the steady state distributions of consumption and wealth are reduced by actuarially fair social security.

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Consumers do not, in general, know with certainty the date at which they will die. An individual who has accumulated assets to provide for consumption during retirement but then dies prematurely ends up leaving a larger bequest to his heirs than he intended. In this paper we examine the implications of uncertain lifetimes and the consequent "accidental" bequests for the saving-consumption decisions of individuals and for aggregate consumption and savings. We extend the Modigliani-Brumberg (1954) - Samuelson (1958) - Diamond (1965) overlapping generations framework to take account of the fact that a consumer's initial wealth depends on the mortality and bequest history of his ancestors. The fact that the date of death and the size of bequest are random leads to a non-degenerate distribution of wealth and consumption. Therefore, in examining the effects of social security, for instance, we are able to analyze the effects on the variance of consumption within each cohort as well as the effects on the aggregate consumption of each cohort.

The effects of uncertain lifetime on individual consumption behavior were first examined formally in a seminal paper by Yaari (1965). More recently Fischer (1973), Barro and Friedman (1977), Levhari and Mirman (1977), Katz (1979), Kotlikoff and Spivak (1981) and Pelzman and Rousslang (1982) have used Yaari's framework to examine various aspects of consumption and saving behavior in the presence of uncertain lifetimes; however, all of these papers focused on the consumption decision of an individual and ignored the effect of uncertain lifetimes on the bequests received by subsequent generations.[1] As we will show at various points in this paper, changes in the economic environment can have effects on aggregate behavior which differ dramatically from the

1. Kotlikoff and Spivak (1981) focus on the role of the family in providing an (incomplete) annuities market but stop short of a full-scale overlapping generations model in which the distribution of bequests is determined endogenously.

effects on individual behavior because of the endogenous adjustment of bequests. For example, we show that if the net rate of return on capital is equal to the population growth rate, then the introduction of actuarially fair social security will raise the consumption of young consumers, if we hold initial wealth constant; however, allowing for the endogenous adjustment of bequests, we find that in the long run aggregate consumption of the young cohort is invariant to the presence or absence of actuarially fair social security.

Sheshinski and Weiss (1981) have used an overlapping generations model with uncertain lifetimes to examine the effects of social security and to develop an optimal social security system. In their model, all consumers who are born at the same date live exactly the same length of time. Thus there is no intra-cohort variation in bequests, consumption or wealth. However, in the model developed below, all consumers have the same probability of dying but different members of the same generation die at different ages. It is this intra-cohort variation in the time of death which leads to non-degenerate distributions of bequests, wealth and consumption.

Eckstein, Eichenbaum, and Peled (1983a) have developed an overlapping generations model in which consumers with identical ex ante mortality probabilities die at different ages. Since the Eckstein-Eichenbaum-Peled model, which was developed independently of and virtually simultaneously with the model presented below, is so similar to the model here, it is worth commenting on the differences between the two models. First, and most importantly, the Eckstein-Eichenbaum-Peled model has no capital although one could interpret that model as applying to an economy in which the net rate of return on capital is zero (i.e., a costless storage technology). However, as we show below, the effects of social security policy in a stationary economy differ dramatically depending on whether or not the rate of return on capital zero. A second difference between the two models concerns the generality of the instantaneous utility function. In this paper, we assume that the instantaneous utility function displays hyperbolic absolute risk aversion (HARA) whereas Eckstein, Eichenbaum, and Peled use a more general concave utility function. However, the formulation used by Eckstein, Eichenbaum, and Peled is not as general as it might first appear because they must at some point assume that the concavity of the derived saving function is "not too large" (p. 16) without presenting the implied restrictions on the utility function. An advantage of the HARA utility function used here is that it leads to linear decision rules thereby making the analysis easily tractable. A third difference between the models is that the model presented below allows for nonzero rates of time preference and population growth whereas each of these rates is assumed to be zero by Eckstein, Eichenbaum and Peled. Finally, the model presented below is used to analyze the effects of actuarially fair social security whereas Eckstein, Eichenbaum and Peled mention the effects of social security only in the presence of a well-functioning annuity market, in which case social security has no effect, as pointed out later in this paper.[2]

A major finding of this paper is that actuarially fair social security reduces private wealth by more than 100%. That is, the introduction of actuarially fair social security leads to a reduction in total national

2. In a different paper, Eckstein, Eichenbaum and Peled (1983b) examine the effects of social security in the presence of heterogeneous mortality probabilities. However, in that paper, there are private annuity markets so that individuals without explicit bequest motives hold all of their savings in the form of annuities. Thus, there are no bequests in that model. The structure of that model is quite different from the structure of the model presented below in this paper or the model in Eckstein, Eichenbaum, and Peled (1983a).

wealth, even if the social security system is fully funded. If the net rate of return on capital is equal to the population growth rate, then this reduction in national wealth does not reduce steady state sustainable consumption; the aggregate consumption of the young cohort and the aggregate consumption of the old cohort are each invariant to the introduction of actuarially fair social security in this case. However, if the rate of return on capital exceeds the population growth rate, then the reduction in national wealth reduces aggregate sustainable consumption; in this case, the aggregate consumption of the young cohort and the aggregate consumption of the old cohort are each reduced by the introduction of social security. A second major finding is that an increase in the level of actuarially fair social security will uniformly narrow the distributions of consumption and bequests; all central moments of the distributions of consumption and bequests will be reduced.

A consumer's claim to social security benefits can be viewed as an annuity. If the consumer survives until retirement, the annuity pays some specified amount, but if the consumer dies before retirement, the annuity pays zero. Under an actuarially fair social security system, the price that the consumer pays for this annuity (i.e., the social security tax levied on young consumers) is equal to the expected present value of future payoffs. However, consumers would be willing to pay more than the expected present value of future payoffs because the payoffs are positively correlated with future marginal utility of consumption. The annuity has a positive payoff if and only if the consumer survives, thereby having a positive marginal utility of consumption; the annuity has a zero payoff if the consumer dies, in which case wealth has zero marginal utility. Therefore, an actuarially fair increase in the level of social security taxes and benefits will make a young consumer wealthier and hence increase his consumption. Since an increase in the social security tax levied on a young consumer will decrease his disposable income by the amount of the tax and will increase his consumption, it is clear that the saving of the young consumer will be reduced by an amount larger than the tax increase.

The description in the paragraph above is partial equilibrium in nature in that it ignores the effects on bequests and on consumption and savings behavior of subsequent young consumers.[3] To the extent that private saving of young consumers is reduced by a permanent increase in the level of social security taxes and benefits, there will be a reduction in bequests received by subsequent generations. In our model, bequests are received at the beginning of a consumer's life so that the reduction in bequests leads to a reduction in the initial wealth of subsequent young consumers. The effect of this reduction in initial wealth is to mitigate or even reverse the increase in consumption of subsequent young consumers; on the other hand, the effect of reduced initial wealth would tend to reinforce the reduction in saving of subsequent young consumers. Since in our model, the private capital stock is equal to the savings of the young consumers, an actuarially fair permanent increase in social security taxes will reduce the long-run private capital stock by more than one-for-one.

In section I we develop a simple model of individual consumption behavior in the presence of an uncertain lifetime. We assume that consumers are selfish in the sense that they derive no utility from the consumption or utility of their children.[4] In section II we explicitly take account of the fact that

3. Hubbard (1983) also provides a partial equilibrium analysis in which young consumers increase their consumption in response to the introduction of actuarially fair social security.

the unconsumed wealth held by a consumer at the time of his death is passed on to his children. We calculate the steady state distributions of consumption of the young, consumption of the old, and private wealth. Actuarially fair social security is analyzed in section III where we show that national wealth is reduced by the introduction of (fully funded) social security. In addition to analyzing the effects on aggregate consumption of each cohort, we show that the introduction of social security uniformly narrows the steady state distribution of consumption of each cohort. The analysis in section III is confined to steady states, and in section IV we examine behavior along the transition path to the new steady state. It should be noted that social security has an effect in this model only because there is no private market for actuarially fair annuities which would enable consumers effectively to offset the social security system. In section V we examine the effects of the introduction of a private market for actuarially fair annuities. Section VI contains a diagrammatic presentation of many of the results of the paper. Concluding remarks are presented in section VII.

I. Individual Consumption Behavior Under Uncertain Lifetime

Consider an economy with many consumers and a single commodity. This commodity can be either consumed or invested. If one unit of the commodity is invested, it yields R units of the commodity in the following period. Each consumer lives either one period or two periods. A consumer works during the first period of his life earning a fixed labor income Y. Also in the first

4. Fischer (1973) and Sheshinski and Weiss (1981) model consumers as deriving utility from leaving a bequest. This utility is a function of the size of the bequest. Barro (1974) and Drazen (1978), in models without lifetime uncertainty, assume that consumers derive utility from the utility of their children.

period of his life, a consumer consumes an amount c_1 and pays a tax T. At the end of the first period of his life, the consumer has $G \geq 1$ children. There is a probability p that the consumer dies at the end of his first period of life[5] (after having the children). If the consumer survives to the second period of life, he does not work but receives a social security payment S. He then consumes an amount c_2. When a consumer dies (either at the end of period one or period two), any unconsumed wealth is divided equally among his children.

Each consumer in the economy chooses c_1 and c_2 to maximize the following utility function

$$U(c_1) + (1-p)\delta U(c_2) \tag{1}$$

where $0 < \delta \leq 1$ is the inverse of one plus the rate of time preference. This utility function is based on the uncertain lifetime literature in which the discounted utility index for period j is multiplied by the probability of being alive in period j. This formulation is simply the expected value of a state-contingent utility function in which $U(c_j)$ is the utility index contingent on being alive at age j, and the utility index is identically zero contingent on not being alive at age j.[6] Note that according to the utility function in (1), consumers do not care about their children; they derive no utility from leaving bequests.

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5. Although individual consumers face uncertainty about their date of death, there is no aggregate uncertainty; a fraction p of the consumers in each generation dies at the end of the first period of life.

6. It is not necessary that the utility index is equal to zero in the case of death. All that is required is that utility in the state of death does not depend on the level of wealth.

Up to this point it may appear that all consumers are identical: they have identical utility functions, labor income Y, taxes T, childbearing characteristics, probabilities of survival, and, if they survive, identical social security benefits S. However, different consumers receive bequests of different sizes depending on the mortality history of the earlier generations of their families. Let B be the bequest a consumer receives from his parent when he is born.[7] For the moment we take B as given; we will discuss the determination of B later in the paper.

Finally we define W to be the wealth of a consumer at the end of the first period of his life,

$$W = B + Y - T - c_1 \tag{2}$$

If a consumer dies at the end of his first period of life, each of his children receives RW/G as a bequest at the beginning of the following period. If the consumer survives into the second period, he consumes $c_2 = RW + S$, because he derives no utility from leaving a bequest. That is, if the consumer survives into a second period of life, his second-period consumption is

$$c_2 = R[B + Y - T - c_1] + S \tag{3}$$

We can now solve a consumer's first-period consumption decision. Maximizing (1) with respect to c_1, subject to the constraint in (3) yields

$$U'(c_1) = (1-p)R\delta U'(R[B + Y - T - c_1] + S) = (1-p)R\delta U'(c_2) \tag{4}$$

7. If a parent dies after the first period of his life, his child receives a bequest B at the beginning of the first period of the child's life. If a parent lives 2 periods, then as shown below, the child receives no bequest in either period; in this case, of course, the bequest received at birth by the child is zero.

In general, if U( ) is concave, then the optimal first-period consumption is a decreasing function of $(1-p)R\delta$ and an increasing function of $R[B + Y - T] + S$. We will restrict the utility index $U(c)$ to be a member of the HARA family (hyperbolic absolute risk aversion) so that the optimal value of c_1 is a linear function of $R[B + Y - T] + S$, as shown below. A general specification of the HARA class of utility functions is

$$U(c) = \frac{1-\gamma}{\gamma}\left(\frac{\beta c}{1-\gamma} + \eta\right)^{\gamma} \tag{5}$$

subject to the following restrictions: $\gamma \neq 1$; $\beta > 0$; $\frac{\beta c}{1-\gamma} + \eta > 0$; $\eta = 1$ if $\gamma = -\infty$. The HARA family of utility functions includes the following special cases: (1) constant relative risk aversion ($\eta = 0$), which includes logarithmic utility if $\gamma = 0$; (2) constant absolute risk aversion ($\gamma = +\infty$); and (3) quadratic utility ($\gamma = 2$). Differentiating (5) with respect to c yields $U'(c) = \beta(\frac{\beta c}{1-\gamma} + \eta)^{\gamma-1}$ so that the first-order condition in (4) can be written as

$$\left[\frac{\beta c_1}{1-\gamma} + \eta\right]^{\gamma-1} = (1-p)R\delta\left[\frac{\beta(R[B+Y-T-c_1] + S)}{1-\gamma} + \eta\right]^{\gamma-1} \tag{6}$$

Equation (6) can be rearranged to yield the following linear consumption function

$$c_1 = a(R[B + Y - T] + S) + b \tag{7a}$$

$$\text{where} \quad a = [R + [(1-p)R\delta]^{\frac{1}{(1-\gamma)}}]^{-1} \tag{7b}$$

$$b = \frac{(1-\gamma)\eta}{\beta}a[1 - ((1-p)R\delta)^{1/(1-\gamma)}] \tag{7c}$$

The consumption function in (7) is particularly simple in the case in which $U(c)$ has constant relative risk aversion ($\eta = 0$). In this case, since $b = 0$, first-period consumption is proportional to the value of disposable lifetime resources $R[B + Y - T] + S$. Let $\sigma \equiv 1-\gamma$ be the (constant) coefficient of relative risk aversion. Note that if $R = \delta = 1$ (i.e., zero time preference and zero net rate of return on capital), then the fraction of total disposable resources $(B + Y - T + S)$ consumed in the first period of life is $a = [1 + (1-p)^{1/\sigma}]^{-1}$. The greater the coefficient of relative risk aversion, the smaller the fraction of disposable resources consumed in the first period. In the limit as $\sigma \longrightarrow \infty$, a consumer would consume 1/2 of disposable resources in the first period. On the other hand, in the limit as $\sigma \longrightarrow 0$, the consumer would consume all of his disposable resources in the first period.[8]

Using the consumption function (7), we can easily calculate the end-of-first-period wealth and the second-period consumption of the consumer. Combining equations (2) and (7) yields

$$W = (1-aR)(B+Y-T) - aS - b \tag{8}$$

and combining equations (3) and (7) yields

$$c_2 = \frac{1-aR}{a}c_1 - \frac{b}{a} \tag{9}$$

According to (9), the income expansion path relating c_1 and c_2 is linear and positively sloped. (From (7), $a > 0$ and $aR < 1$.) This relation will be useful later when we examine the intra-cohort distributions of c_1 and c_2.

II. Intergenerational Transfers

We have solved the consumer's saving-consumption decision conditional on

8. If the consumer cannot borrow against his (uncertain) future social security benefit S, then $c_1 \longrightarrow B + Y - T$ as $\sigma \longrightarrow 0$. Of course, if $S = 0$, then the consumer will indeed consumer all of his disposable lifetime resources in the first period.

the bequest B received at birth. In this section, we calculate the bequests received by each consumer. The bequest received by a consumer depends on the mortality history of the earlier generations of his family. Specifically, let j be the number of consecutive previous generations in a consumer's family which died at age 1. For example, j = 0 indicates that the consumer's parent lived 2 periods and therefore left no bequest to the consumer. If j = 1, then the consumer's parent died at age 1 leaving a bequest but the consumer's grandparent lived 2 periods leaving no bequest. We index all consumers according to j and observe that for a consumer drawn at random, the probability that he is of type j is $p^j(1-p)$.

Let the superscript j indicate that a variable pertains to a consumer of type j. We will first examine type 0 consumers and then we will examine type j consumers for $j \geq 1$. As indicated above, $B^{(0)} = 0$. The first-period consumption and end-of-first-period wealth of type 0 consumers follow immediately from (7) and (8), respectively,

$$c_1^{(0)} = a(R[Y-T]+S) + b \tag{10}$$

$$W^{(0)} = (1-aR)(Y-T) - aS - b. \tag{11}$$

Note that the sum of $c_1^{(0)}$ and $W^{(0)}$ is equal to after-tax labor income Y-T.[9]

For consumers of type j, $j \geq 1$, consumption and saving behavior depend on the initial bequest, $B^{(j)}$, received at birth. From (7) and (10), we obtain

$$c_1^{(j)} = aRB^{(j)} + c_1^{(0)} \tag{12}$$

9. We assume that S and T are small enough and that the utility function and labor income are such that $W^{(0)} > 0$. Note that if b = 0 (as it would be with constant relative risk aversion), then $W^{(0)} > 0$ provided that S and T are small enough.

and from (8) and (11), we obtain

$$W^{(j)} = (1-aR)B^{(j)} + W^{(0)} \tag{13}$$

In general, if a type j-1 consumer dies after one period, he leaves a bequest of $G^{-1}W^{(j-1)}$ to each of his children (who are type j consumers). The bequest earns a gross rate of return R so that

$$B^{(j)} = (R/G)W^{(j-1)} \tag{14}$$

where, by convention, $W^{(-1)} \equiv 0$. Substituting (14) into (13) yields the first-order linear constant coefficient difference equation

$$W^{(j)} = (1-aR)(R/G)W^{(j-1)} + W^{(0)} \qquad j = 0,1,2,\ldots \tag{15}$$

which has the solution

$$W^{(j)} = W^{(0)} \sum_{i=0}^{j} (1-aR)^{i}(R/G)^{i} \qquad j = 0,1,2,\ldots \tag{16}$$

According to (16), $W^{(j)}$ is an increasing function of j. That is, as we increase the number of previous generations which died early and thus left bequests, we increase $W^{(j)}$. We will assume that $(1-aR)R < G$.[10] Therefore, $W^{(j)}$ approaches $\frac{W^{(0)}}{1-(1-aR)(R/G)}$ as j approaches infinity.

We have now completely solved the model. Given any nonnegative integer j we know that a fraction $(1-p)p^{j}$ of the population is of type j. Then using equations (10)-(12), (14), and (16) it is a simple matter to calculate the

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10. Observe from (7b) that $(1-aR)R = \phi R/(\phi+R)$ where $\phi \equiv [(1-p)R\delta]^{\frac{1}{1-\gamma}} > 0$. Thus, if $\phi \leq G$, then $(1-aR)R < G$. It can be shown that if $(\phi-G)(R-G) < G^{2}$, then $(1-aR)R < G$. Thus, if $\phi > G$ and $R \leq G$, then $(1-aR)R < G$.

consumption, wealth, and bequests received at birth by each type j consumer. Our next step is to summarize the distributions of consumption, wealth and bequests by calculating the values of aggregate first-period consumption, C_1*, aggregate second-period consumption, C_2*, aggregate private wealth W*, and aggregate bequests, B*. Each of these aggregates is expressed on a per capita basis (more precisely, per person in the young generation). For example, aggregate private wealth per capita is defined as

$$W^* = \sum_{j=0}^{\infty} (1-p)p^j W^{(j)} \tag{17}$$

Calculating the aggregate per capita values of both sides of (13) we obtain

$$W^* = (1-aR)B^* + W^{(0)} \tag{18}$$

The aggregate per capita level of bequests received at birth is calculated by recalling that a fraction p of each type of consumer dies early leaving a bequest. Thus the aggregate wealth held by consumers who die young is pW*. Including the accrued interest on this wealth and adjusting for the fact that each generation has G times as many consumers as the previous generation, we obtain

$$B^* = p(R/G)W^* \tag{19}$$

Substituting (19) into (18) yields

$$W^* = \frac{W^{(0)}}{1 - (1-aR)pR/G} \tag{20}$$

Therefore, average per capita wealth is proportional to $W^{(0)}$, the wealth of type 0 consumers.

In order to calculate the aggregate per capita value of first-period consumption, we observe from equation (2) that

$$C_1^* = Y - T + B* - W* \tag{21}$$

Substituting (19) into (21) yields

$$C_1^* = Y - T - (1 - pR/G)W* \tag{22}$$

Before calculating per capita aggregate second-period consumption, recall that our per capita aggregates are calculated as per person in the young cohort. Since only a fraction (1-p) of young consumers survives to the second period of life, and since each generation is only G^{-1} times as large as the succeeding generation, $C_2^* \equiv (1-p)G^{-1}\sum_{j=0}^{\infty}(1-p)p^j c_2^{(j)}$. From the fact that $c_2^{(j)} = RW^{(j)} + S$ we easily obtain

$$C_2^* = (1-p)G^{-1}(RW* + S) \tag{23}$$

To calculate aggregate economy-wide consumption per capita add together (22) and (23) to obtain

$$C_1^* + C_2^* = Y - T + (1-p)G^{-1}S + (R/G - 1)W^* \tag{24}$$

Observe from (24) that aggregate private consumption per capita is equal to the sum of after-tax labor income, Y-T, plus social security payments to the surviving fraction (1-p) of the old cohort, plus the net return on wealth, adjusted for population growth.

A final useful relationship between C_1^* and C_2^* is obtained by calculating the aggregate per capita values of both sides of the income expansion path in (9) to obtain

$$C_2^* = \frac{1-aR}{a}C_1^* - \frac{b}{a} \tag{25}$$

Thus, the aggregate income expansion path relating C_1^* and C_2^* is linear and positively sloped. Thus, in analyzing the steady state effects of changes in the social security parameters S and T, we know from (25) that C_1^* and C_2^* both move in the same direction.

III. The Effects of Actuarially Fair Social Security

In this section we consider the effects on savings and consumption of the introduction of a fully funded actuarially fair social security system. We suppose that the only role of the government is to collect social security taxes from the young and to distribute social security benefits to the old. Thus the taxes T levied on the young are social security taxes. An actuarially fair social security system would levy a tax of $(1-p)R^{-1}$ dollars for each dollar of benefits promised, i.e., $RT = (1-p)S$. Under this system, a consumer contributes $(1-p)R^{-1}S$ to the social security system. He receives S if he survives to the second period of life but receives zero if he dies after one period. Thus the expected present value of the social security benefit is $(1-p)R^{-1}S$ which is equal to the consumer's contribution. Put differently, the social security system runs a balanced account vis-a-vis each generation. The social security system collects taxes from the members of each generation when they are young, invests the tax revenue at a gross rate of return R, and then returns all of the tax revenue with accrued interest to the surviving old members of the generation.

III.A. The Effects of Social Security on Aggregate Consumption and Aggregate Capital Accumulation

In order to study the effects of actuarially fair social security on aggregate consumption, we proceed in three steps. First, we analyze the effects of social security on the saving and consumption behavior of type 0 consumers. Then, we use our results about the effects on $W^{(0)}$ to analyze the effects on the private capital stock and on the total national capital stock. Finally, we use the relations between the national capital stock and aggregate consumption to determine the effects on C_1^* and C_2^*.

To calculate the effects of actuarially fair social security on consumption and saving of young type 0 consumers, we substitute $T = (1-p)R^{-1}S$ into (10) and (11) to obtain

$$c_1^{(0)} = aRY + b + apS \tag{26}$$

$$W^{(0)} = (1-aR)Y - b - T - apS \tag{27}$$

The introduction of actuarially fair social security increases the future value of lifetime resources, $R(B + Y - T) + S$, by $-RT + S = pS$. A consumer who survives to the second period receives a social security payment S which exceeds the value of his contribution with accrued interest, RT, because the surviving members of each generation receive (on a pro rata basis) the taxes-cum-interest contributed by members of their generation who died after one period. The effect of this increase in lifetime resources is to increase $c_1^{(0)}$ by apS. The wealth held at the end of the first period by type 0 consumers is reduced for two reasons: first, disposable resources available in the first period fall by the amount of the tax T; second, the increase in first-period consumption further reduces wealth held at the end of the first period.

In a fully funded social security system, the total national capital stock per capita (measured at the end of a period) is equal to the sum of the aggregate private capital stock per capita, W*, and the per capita capital stock held by the social security system T. Recall from equation (20), that the private capital stock, W*, is proportional to $W^{(0)}$, and that the constant of proportionality does not depend on the parameters of the social security system. Since, from (27), the introduction of social security reduces $W^{(0)}$, it is clear that the aggregate private capital stock is reduced by the introduction of social security. Since $B^* = (pR/G)W^*$, the reduction in the aggregate private capital stock implies an equiproportionate reduction in aggregate bequests per capita.

The effect of actuarially fair social security on the aggregate national capital stock per capita, $W^* + T$, is easily determined by first observing from the definition of end-of-first-period wealth in (2) that

$$W^* + T = Y + B^* - C_1^* \tag{28}$$

Then calculating the aggregate per capita values of both sides of (12) we obtain

$$C_1^* = aRB^* + c_1^{(0)} \tag{29}$$

Substituting (29) into (28) yields

$$W^* + T = Y + (1-aR)B^* - c_1^{(0)} \tag{30}$$

Since we have already shown that the introduction of social security causes B* to fall and $c_1^{(0)}$ to increase, it is clear from (30) that the aggregate national capital stock $W^* + T$ is reduced by the introduction of social security.

Next we examine the effects on aggregate consumption of the introduction of actuarially fair social security. Substituting RT for (1-p)S in (24) gives an expression for aggregate economy-wide consumption per capita

$$C_1^* + C_2^* = Y + ((R/G) - 1)(W^*+T) \qquad (31)$$

Aggregate private consumption is equal to the sum of labor income Y and the net return (adjusted for population growth) on national wealth. Observe that if R=G, so that the net rate of return on capital is equal to the rate of population growth, then the coefficient on national wealth in (31) is zero. In this case, $C_1^* + C_2^*$ is independent of the level of actuarially fair social security taxes and benefits. Furthermore, in view of the aggregate income expansion path in (26), both C_1^* and C_2^* are independent of the level of actuarially fair social security taxes and benefits when R = G. If R > G, then the reduction in aggregate wealth, W* + T, induced by the introduction of social security leads to a reduction in $C_1^* + C_2^*$; in light of (25), C_1^* and C_2^* are each reduced by the introduction of actuarially fair social security. Finally, if R < G, then C_1^* and C_2^* are each increased by the introduction of actuarially fair social security.

III.B. The Effects of Social Security on the Intra-Cohort Distribution of Consumption.

Having analyzed the effects of social security on the aggregate consumption of the young cohort and the aggregate consumption of the old cohort, we now examine the intra-cohort distributions of consumption and wealth. We have already shown (equation (26)) that the first-period consumption of type 0 consumers increases by apS in response to the introduction of social security. Also we have shown that $W^{(0)}$ falls by T + apS when social security is introduced. As a consequence of the fall in $W^{(0)}$, there is a reduction in bequests, $B^{(1)}$, received at birth by type 1 consumers. Indeed, the introduction of social security reduces $B^{(j)}$ for all type j consumers for j = 1,2,3,.... This result follows from the facts that $B^{(j)}$ is proportional to $W^{(0)}$ (see equations (14) and (16)) and that $W^{(0)}$ is reduced by the introduction of social security. Taking account of the induced reduction in bequests, we analyze the intra-cohort distribution of consumption.

The deviation of a type j consumer's first-period consumption from the average level of first-period consumption is easily calculated by subtracting (29) from (12) to obtain

$$c_1^{(j)} - C_1{*} = aR(B^{(j)} - B{*}) \tag{32}$$

Thus, the deviation of $c_1^{(j)}$ from the average consumption of the young cohort is proportional to deviation of $B^{(j)}$ from the average bequest B*. Using equations (14), (16) and (20), equation (32) can be rewritten as[11]

$$c_1^{(j)} - C_1{*} = a(R^2/G)W^{(0)}\left(\sum_{i=0}^{j-1}(1-aR)^i(R/G)^i - \frac{p}{1-p(R/G)(1-aR)}\right) \tag{33}$$

Since the introduction of actuarially fair social security reduces $W^{(0)}$, it is clear from (33) that the (magnitude of the) deviation of type j consumer's first-period consumption from the average first-period consumption is reduced by the introduction of social security. Thus, the distribution is narrowed by the introduction of social security. More precisely, all central moments of the intra-cohort distribution of $c_1^{(j)}$ are reduced by the introduction of social security.

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11. We use the convention that $\sum_{i=0}^{j-1}(1-aR)^i(R/G)^i$ is equal to zero for j = 0.

The effects of the introduction of social security on second-period consumption are easily calculated by observing from (9) that $c_2^{(j)}$ can be expressed as an increasing linear function of $c_1^{(j)}$. Therefore, the narrowing of the distribution of $c_1^{(j)}$ implies that the distribution of $c_2^{(j)}$ is also narrowed by the introduction of social security.

For the case in which $R = G$, it is straightforward to analyze the (steady state) welfare implications of the introduction of social security. In this case, the introduction of actuarially fair social security does not affect the average levels of consumption of the young or of the old as explained in section III.A. However, it narrows the distribution of consumption of each cohort. Therefore, if each consumer has an identical utility function and receives equal weight in the social welfare function, the introduction of social security is welfare-improving. If $R < G$, then the introduction of security raises the average level of consumption and reduces the variance of consumption. Each of these effects increases social welfare. However, if $R > G$, then the introduction of social security reduces average consumption, which tends to reduce welfare, but also reduces the intra-cohort variance of consumption, which tends to raise welfare.

IV. The Transition Path to the New Steady State

In section III we examined the effects on consumption and wealth of the introduction of actuarially fair social security. However, the comparison of regimes with and without social security was actually a comparison of steady states. In particular, we assumed that the social security system had been in effect long enough so that essentially no one received a bequest that included part of the savings of an ancestor who lived in the initial regime without social security. Equivalently, we assumed that each person had at least one ancestor who lived for two periods under the new regime, leaving no bequests and thus severing links to the old regime.

In this section we examine the transition path to the new steady state which accompanies the introduction of an actuarially fair social security system. We show that the introduction of social security reduces the intra-cohort variances of first-period consumption and second-period consumption for every generation (except the first) born under the new social security regime. Also, if $R \leq G$, then the average levels of first-period consumption and second-period consumption of each generation are at least as high under the social security regime as in the absence of social security. In this case, the introduction of social security increases the welfare of every generation born under the social security regime.

Suppose that actuarially fair fully funded social security is introduced at the beginning of period t^*+1. We will assume that since the older cohort (born at time t^*) did not contribute to the social security system they receive no benefits. The young generation pays a tax $T = (1-p)R^{-1}S$ and the survivors will each receive a social security payment of S as discussed in section III. In order to analyze the transition path to the new steady state we introduce the following notational conventions: (1) a double tilda over a variable denotes the value of that variable under the social security system; a single tilda denotes the value that the variable would have had without the introduction of social security; (2) the subscript t^*+m denotes that the variable pertains to a consumer born at the beginning of period t^*+m; without a time subscript, the variable refers to the steady-state value of the variable (i.e., $m = \infty$). Thus, for instance, $\approx{c}^{(j)}_{1,t^*+m}$ is the first-period consumption, under a social security regime, of consumers of type j born at the beginning of period t^*+m.

The effect of social security on the first-period consumption of consumers of type j born at the beginning of period t^*+m is easily obtained from (12) and (14),

$$\tilde{\tilde{c}}^{(j)}_{1,t^*+m} - \tilde{c}^{(j)}_{1,t^*+m} = \tilde{\tilde{c}}^{(0)}_{1} - \tilde{c}^{(0)}_{1} + a(R^2/G)(\tilde{\tilde{W}}^{(j-1)}_{t^*+m-1} - \tilde{W}^{(j-1)}_{t^*+m-1}), \qquad \begin{array}{l} j = 0,1,2,\ldots \\ m = 1,2,3,\ldots \end{array} \tag{34}$$

where we use the convention that $\tilde{\tilde{W}}^{(-1)}_{t^*+m} = \tilde{W}^{(-1)}_{t^*+m} = 0$. We also note that if $m = 1$, then $\tilde{\tilde{W}}^{(j-1)}_{t^*+m-1} - \tilde{W}^{(j-1)}_{t^*+m-1} = 0$ because the wealth of the generation born at time t^* is unchanged by the introduction of social security. Observe from (26) that

$$\tilde{\tilde{c}}^{(0)}_{1} - \tilde{c}^{(0)}_{1} = apS \tag{35}$$

Therefore, the introduction of social security increases by apS the first-period consumption of every consumer born at time t^*+1. Also, from (9), the second-period consumption of each surviving member of this cohort is increased by (1-aR)pS. Thus, the first generation born under the new social security regime is made unambiguously better off. The bequests received by each individual in this cohort are invariant to the introduction of social security, and for a given level of bequests, the introduction of social security increases the consumption possibilities of each consumer.

Now we examine the consumption and wealth of generations born after period t^*+1. A straightforward generalization of (16) yields

$$W^{(j)}_{t^*+m} = \sum_{i=0}^{j} (1-aR)^i (R/G)^i W^{(0)}_{t^*+m-i} \tag{36}$$

where $W^{(0)}_{t^*+m-i}$ is the end-of-first-period wealth of consumers born in period t^*+m-i. Since the social security system first goes into effect in period t^*+1, it follows from (27) that

$$\tilde{\tilde{W}}^{(0)}_{t^*+m-i} - \tilde{W}^{(0)}_{t^*+m-i} = \begin{cases} -(T+apS), & \text{for } i = 0,1,2,\ldots,m-1 \\ 0 \quad\quad , & \text{for } i = m,m+1,m+2,\ldots \end{cases} \tag{37}$$

Then, from (36) and (37) we obtain

$$\tilde{\tilde{W}}^{(j)}_{t^*+m} - \tilde{W}^{(j)}_{t^*+m} = -(T+apS) \sum_{i=0}^{j^*} (1-aR)^i (R/G)^i, \quad \begin{matrix} j = 0,1,2,\ldots \\ m = 1,2,3,\ldots \end{matrix} \tag{38a}$$

$$\text{where} \qquad j^* = \min(j,\ m-1) \tag{38b}$$

Finally, we substitute (35) and (38a) into (34) to obtain

$$\tilde{\tilde{c}}^{(j)}_{1,t^*+m} - \tilde{c}^{(j)}_{1,t^*+m} = apS - a(R^2/G)(T+apS) \sum_{i=0}^{j^*-1} (1-aR)^i (R/G)^i, \tag{39}$$

$$j = 0,1,2,\ldots$$
$$m = 1,2,3,\ldots$$

where we use the convention that $\sum_{i=0}^{-1} (1-aR)^i (R/G)^i = 0$. Observe from (39) that for any given m, $\tilde{\tilde{c}}^{(j)}_{1,t^*+m} - \tilde{c}^{(j)}_{1,t^*+m}$ is non-increasing in j. Moreover, for $j = 1,\ldots, m-1$, the effect on first-period consumption of the introduction of social security is strictly decreasing in j.

We will now show that the introduction of actuarially fair social security reduces the intra-cohort variance of consumption of each generation (except the first) born under the social security regime. We first record three observations:

(i) $\tilde{c}^{(j)}_{1,t^*+m} > \tilde{c}^{(j-1)}_{1,t^*+m}$ $j = 1,2,3,\ldots$

(ii) $\approx{c}$ notation: $\overset{\approx}{c}^{(j)}_{1,t^*+m} > \overset{\approx}{c}^{(j-1)}_{1,t^*+m}$ $j = 1,2,3,\ldots$

(iii) $\overset{\approx}{c}^{(j)}_{1,t^*+m} - \tilde{c}^{(j)}_{1,t^*+m} \leq \overset{\approx}{c}^{(j-1)}_{1,t^*+m} - \tilde{c}^{(j-1)}_{1,t^*+m}$ $j = 1,2,3,\ldots$ with strict inequality for some j if $m \geq 2$.

Observation (i), which follows from (12), (14), and (16), simply states that for a given cohort, first-period consumption is increasing in j in the absence of social security. Observation (ii), which follows from (12), (14), and (36) states that for a given cohort born under the social security regime, first-period consumption is increasing in j. Observation (iii), which follows from (39), states the effect on first-period consumption of the introduction of social security is non-increasing in j, and is strictly decreasing for some j, if $m \geq 2$. According to the lemma below, these three observations imply that the intra-cohort variance of $\overset{\approx}{c}^{(j)}_{1,t^*+m}$ is less than the intra-cohort variance of $\tilde{c}^{(j)}_{1,t^*+m}$.

<u>LEMMA</u>. Consider the discrete random variables x and y and let f_i = prob $(x = x_i)$ = prob $(y = y_i)$, i = 0,1,2.... Suppose that $x_i \geq x_{i-1}$ i = 1,2,3,..., and $y_j \geq y_{j-1}$, j = 1,2,3,..., with strict inequality for some i and some j. Suppose also that $y_i - x_i \leq y_{i-1} - x_{i-1}$ with strict inequality for some i. Then Var (y) < Var (x).

<u>Proof</u>. See Appendix A.

We have shown that the introduction of actuarially fair social security does not affect the intra-cohort variance of first-period consumption of the first generation born under the social security regime; however, it does reduce the intra-cohort variation of first-period consumption of each subsequent generation. Since the second-period consumption of each consumer is a linear function of first-period consumption (equation (9)), it follows that the introduction of social security does not reduce the intra-cohort variance of second-period consumption of the first generation born under the social security regime but does reduce this intra-cohort variance for all subsequent generations.

We have derived unambiguous results about the intra-cohort variance of consumption along the transition path to the new steady state. The effects on the average level of consumption are less clear-cut. We have already shown that for the generation born at the beginning of period t*+1, the average levels of first-period consumption and second-period consumption are increased by the introduction of social security. Also, we have shown that in the new steady state, the average levels of $c_1^{(j)}$ and $c_2^{(j)}$ decrease, increase, or remain unchanged depending on whether R is greater than, less than, or equal to G. Letting $\tilde{\tilde{C}}^*_{1,t^*+m}$ denote the average value of $\tilde{\tilde{c}}^{(j)}_{1,t^*+m}$ and $\tilde{C}^*_{1,t^*+m}$ denote the average value of $\tilde{c}^{(j)}_{1,t^*+m}$, we will show that $\tilde{\tilde{C}}^*_{1,t^*+m} - \tilde{C}^*_{1,t^*+m}$ decreases as m increases. The reason is that as m increases (i.e., as we increase the length of time for which the social security regime has been in effect), there is a decrease in the amount of bequests which represent accumulated saving from generations born before the introduction of social security, when private saving was higher.

The effect of social security on the average first-period consumption of the generation born at the beginning of period t*+m is calculated in Appendix B and is equal to

$$\tilde{\tilde{C}}^*_{1,t^*+m} - \tilde{C}^*_{1,t^*+m} = \frac{apS}{1-p(R/G)(1-aR)}\{1-(R/G) + [1-p(1-aR)](R/G)^m p^{m-1}(1-aR)^{m-1}\} \tag{40}$$

Since we have assumed that (1-aR)pR is less than G, it is clear from (40) that $\tilde{\tilde{C}}^*_{1,t^*+m} - \tilde{C}^*_{1,t^*+m}$ decreases as m increases, as claimed above.

In the case in which R = G, equation (40) implies that $\tilde{\tilde{C}}^*_{1,t^*+m} - \tilde{C}^*_{1,t^*+m}$ is equal to $aSp^m(1-aR)^{m-1}$ which is positive for all finite m. Thus, since the introduction of social security increases the average value and reduces the variance of $c^{(j)}_{1,t^*+m}$ for all finite m, it also (see equation (9)) increases the average value and reduces the variance of $c^{(j)}_{2,t^*+m}$ for all finite m. Therefore, if R = G, the introduction of social security is welfare-improving for every generation born under the new social security regime. More generally, if $R \leq G$, the welfare of every generation (except the current old generation which is unaffected) is improved by the introduction of social security.

The welfare effects of the introduction of social security are less clear-cut in the case in which R > G. Clearly, the welfare of the generation born at time t*+1 is improved because, as explained earlier, the first-period consumption of every consumer in this generation increases by apS (and from equation (9), second-period consumption increases by (1-aR)pS). For all generations born after time t*+1, the introduction of social security reduces the intra-cohort variance of consumption. For sufficiently small m, it follows from (40) that the average level of first-period (and second-period) consumption is increased by the introduction of social security. Thus, for these generations, welfare is increased. The difficulty in our welfare analysis arises for generations born long after t^*+1. If $R > G$, then it follows immediately from (40) that for sufficiently large m, the average first-period (and a fortiori average second-period) consumption of the generation born at time t^*+m is reduced by the introduction of social security. The effect on the welfare of this generation thus depends on whether the welfare-improving effects of reduced variance dominate the welfare-worsening effects of reduced average consumption.

V. Private Annuities

We have examined the effects of actuarially fair social security in which the amount of this annuity held by each individual is determined by the government rather than by the individual. Implicit in this analysis was the assumption that there is no market for private annuities. In this section we will introduce a market for private annuities in which individuals can choose the level of annuities purchased. Because actuarially fair social security is a perfect substitute for actuarially fair private annuities, the introduction of social security has no effect on an economy in which there is already a private annuity market. Thus, we will not consider social security in this section. Rather we will simply focus on the effects of introducing actuarially fair annuities into an economy in which there is no social security.

With the introduction of private annuities, there are now two alternative forms in which a consumer can hold his wealth. As before, he can hold capital directly, earning a gross rate of return R. Alternatively, he can deposit his savings at an annuity company. The annuity company operates by accepting deposits from young consumers and using these deposits to buy capital which earns a gross rate of return R. At the beginning of the following period, the annuity company distributes its holdings (with accumulated interest) to its surviving depositors in proportion to their initial deposits. Thus, each surviving depositor at the annuity company receives $A = \frac{R}{1-p}$ dollars for each dollar initially deposited. As shown by Yaari (1965), consumers who do not have explicit bequest motives will choose to hold all of their wealth in the form of these annuities. Thus, there will be no bequests.

The effect of introducing actuarially fair private annuities is to raise the (gross) rate of return on saving from R to $A = R/(1-p)$. For any given individual, the response to an increased rate of return is to increase second-period consumption. However, the effect on first-period consumption is ambiguous because of the usual conflict of income and substitution effects. We will demonstrate in this section that if $R = \delta = G = 1$ (so that $A = (1-p)^{-1}$), then despite the ambiguity at the individual level, the effect on aggregate first-period consumption is unambiguous. In particular, the introduction of actuarially fair private annuities will reduce aggregate consumption of the young and raise aggregate consumption of the old.

Consumers can, by holding annuities, earn a gross rate of return A on their savings so that $c_2 = A[Y - c_1]$. The maximization problem of the representative consumer[12] is

$$\max_{c_1} U(c_1) + (1-p)\delta U(A(Y - c_1)) \tag{41}$$

12. Since there are no bequests, there is no need to distinguish consumers according to the mortality history of their families.

The first-order condition for this problem is

$$U'(c_1) = (1-p)A\delta U'(A(Y - c_1)) \tag{42}$$

With actuarially fair annuities $(1-p)A = R$. Therefore, since $c_2 = A(Y - c_1)$, equation (42) can be rewritten as

$$U'(\hat{c}_1) = R\delta U'(\hat{c}_2) \tag{43}$$

where we use a circumflex to denote the value of a variable in the presence of a private annuity market. Observe that if U( ) exhibits constant relative risk aversion ($\eta = 0$; $\sigma \equiv 1 - \gamma > 0$), then (43) implies that $(\hat{c}_1/\hat{c}_2)^{-\sigma} = R\delta$, whereas equation (4) implies that in the absence of private annuities, $(c_1^{(j)}/c_2^{(j)})^{-\sigma} = (1-p)R\delta$ for all j. Therefore, it follows that, with constant relative risk aversion, the introduction of private annuities reduces the share of aggregate consumption consumed by the young cohort.

For the remainder of this section we assume that $G = R = \delta = 1$, i.e., that the rate of population growth, the net rate of return on capital and the rate of time preference are all equal to zero. With $R\delta = 1$, (43) implies that $\hat{c}_1 = \hat{c}_2$ for any strictly concave utility function U( ). Since $\hat{c}_2 = A(Y-\hat{c}_1)$ and $A = (1-p)^{-1}$, we obtain $\hat{c}_1 = \hat{c}_2 = \frac{1}{2-p}Y$. Therefore, we obtain

$$\hat{C}_1{*} = \frac{1}{2-p}Y \tag{44a}$$

$$\hat{C}_2{*} = \frac{1-p}{2-p}Y \tag{44b}$$

$$\hat{W}{*} = \frac{1-p}{2-p}Y \tag{44c}$$

In order to calculate aggregate consumption in the absence of annuities, we observe from (4) that when $R = \delta = 1$,

$$U'(c_1^{(j)}) = (1-p)U'(c_2^{(j)}) \tag{45}$$

Since 1-p is less than one, it follows that for any concave utility function U( ),

$$c_1^{(j)} > c_2^{(j)} \qquad j = 0,1,2,... \tag{46}$$

Since $C_1^* = \sum_{j=0}^{\infty}(1-p)p^j c_1^{(j)}$ and $C_2^* = \sum_{j=0}^{\infty}(1-p)^2 p^j c_2^{(j)}$, it follows from (46) that

$$C_2^* < (1-p)C_1^* \tag{47}$$

We can now compare steady state consumption of each cohort with and without private annuities. Because $R = G = 1$, steady state aggregate consumption is not affected by the presence of private annuities: $C_1^* + C_2^* = Y = \hat{C}_1^* + \hat{C}_2^*$. However, from (44a,b) and (47) we see that the older generation consumes a smaller fraction of total output in the absence of annuities than in the presence of annuities. That is, the introduction of actuarially fair annuities will reduce the aggregate consumption of the young and increase the aggregate consumption of the old. This finding is unambiguous despite the fact that for any individual with given initial endowments, the effect on first-period consumption is ambiguous. As will be explained in the next section, the income and substitution effects on first-period consumption work in opposite directions for an individual; however, at the aggregate level, there is no income effect (when $R = G$) so that the substitution effect dominates.

Finally, we examine the effect of the introduction of annuities on private wealth. Our comparison will be limited to the case of constant relative risk aversion and $G = R = \delta = 1$. Even in this simple case, there is an ambiguity. Observe from (7c) that with constant relative risk aversion ($\eta = 0$, $\sigma = 1 - \gamma > 0$), $b = 0$; from (7b), we see that with $R = \delta = 1$, $a = [1 + (1-p)^{\frac{1}{\sigma}}]^{-1}$. Substituting these values of a and b into the expression for $W^{(0)}$, equation (11), and substituting the result into (20) yields the level of wealth in the absence of private annuities

$$W^* = \frac{Y}{1-p + (1-p)^{-\frac{1}{\sigma}}} \tag{48}$$

To calculate the effect on private wealth of the introduction of private annuities we subtract (48) from (44c) to obtain

$$\hat{W}^* - W^* = \frac{W^*}{2-p}\{(1-p)^{\frac{\sigma-1}{\sigma}} - (1+p-p^2)\} \tag{49}$$

Observe that $\hat{W}^* - W^*$ has the same sign as the term in curly brackets. For example, with logarithmic utility ($\sigma = 1$), the term in curly brackets is equal to $-p + p^2 < 0$ so that $\hat{W}^* < W^*$. There is some value of the coefficient of relative risk aversion σ less than 1 such that $\hat{W}^* = W^*$. Using (49) it can be shown that

$$\hat{W}^* \underset{>}{\overset{<}{=}} W^* \quad \text{as} \quad \sigma \underset{<}{\overset{>}{=}} \bar{\sigma} \tag{50a}$$

$$\text{where} \quad \bar{\sigma} \equiv [1 - \frac{\ln(1+p-p^2)}{\ln(1-p)}]^{-1} < 1. \tag{50b}$$

To summarize, we have shown that the introduction of private annuities reduces the share of aggregate consumption consumed by the young cohort if either of the following two conditions is met: (a) the utility function U( ) exhibits constant relative risk aversion; or (b) $G = R = \delta = 1$. The direction of the effect of private annuities on the level of private wealth is harder to determine. Even if we make both assumptions (a) and (b), the introduction of private annuities can either increase or decrease private wealth, depending on the coefficient of relative risk aversion.

VI. A Diagrammatic Presentation

In this section we present a simple diagrammatic illustration of the effects of introducing social security and of introducing private annuities. We limit our analysis to the case in which $R = G = \delta = 1$. By restricting the net return on capital and the population growth rate to equal zero, we essentially remove any aggregate income effects associated with the introduction of either actuarially fair social security or actuarially fair private annuities. More precisely, if $R = G = 1$, then steady state aggregate consumption is equal to Y regardless of the level of actuarially fair social security (see equation (31)) and regardless of the presence of actuarially fair private annuities (see equations (44a,b)). In the case with $R = G = \delta = 1$, there are particularly sharp distinctions between the responses of individual behavior and of aggregate behavior to various changes. For example, for an individual with a given endowment, the introduction of social security will raise both first-period and second-period consumption; however, the introduction of social security has no effect on the (steady state) level of aggregate consumption of either the young or the old. As another example, the introduction of private annuities has an ambiguous effect on individual consumption behavior, but the effect on aggregate consumption is unambiguous. The difference between the analyses of individual effects and of aggregate effects is that the aggregate analysis takes account of the endogenous adjustment of bequests. Although the introduction of either social security or private annuities has a positive income effect from the viewpoint of the individual, the endogenous adjustment of bequests offsets any income effect in the aggregate analysis. The fact that neither social security nor private annuities affects aggregate income essentially imposes a zero income effect in the aggregate analysis.

First consider the introduction of social security as illustrated in figure 1. In the absence of social security or private annuities the individual's budget line is given by JJ' which has a slope of -1 and has intercepts equal to B+Y. The consumer chooses point E_1 in the absence of social security or private annuities. The dashed line JFH has slope equal to $-A = -1/(1-p)$. In the presence of actuarially fair social security, which levies a tax of T on the young and pays a benefit $S = AT$ to the old, the consumer's budget line is DFG. Provided that the social security benefit S is smaller than second-period consumption at E_1, the consumer moves to a point such as E_2. The introduction of social security has a positive income effect because $S > T$ but has no substitution effect. The positive income effect leads to an increase in first-period consumption and hence reduces the size of the bequest if the consumer dies after one period. In the aggregate, the negative income effect on subsequent generations resulting from the lower bequests received by these generations exactly offsets the positive income effect arising from the fact that $S > T$. Thus, C_1^* and C_2^* are invariant to the presence or absence of social security.

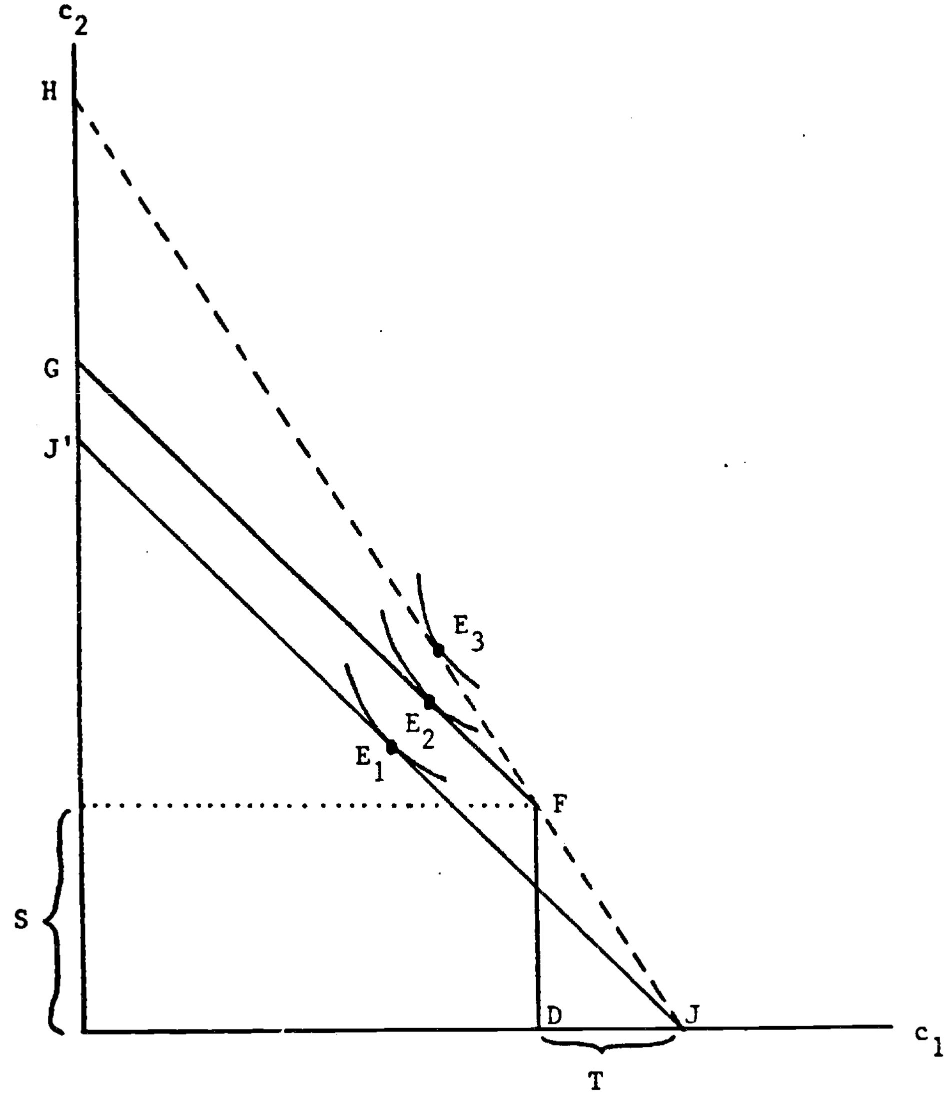

Figure 1

Now consider the introduction of private annuities which shifts the budget line from JJ' to the dashed line JFH. The consumption point shifts from E_1 to E_3. The income effect tends to increase consumption in both periods whereas the substitution effect tends to decrease c_1 and increase c_2. The total effect on c_2 is thus unambiguously positive but the effect on c_1 is ambiguous. In the presence of private annuities, all consumers will hold their savings in the form of these annuities and there will be no bequests. The elimination of bequests imposes a negative income effect on all subsequent consumers except for type 0 consumers. In the aggregate, the negative effect arising from the elimination of bequests exactly offsets the positive income effect arising from the availability of private annuities as illustrated in figure 1. Thus, in the aggregate, only the substitution effect is operative so that aggregate first-period consumption falls and aggregate second-period consumption rises.

From the analysis in this section it is clear that the distinction between social security and private annuities is not that the claims to future social security benefits are liabilities of the government and the claims to future annuity payments are liabilities of private firms. Rather, the distinction is that the social security system we have examined is compulsory and has a fixed level of participation whereas individuals can choose the amount of privately marketed annuities to hold. As explained above, the compulsory social security system has no substitution effect, i.e., it does not affect the intertemporal marginal rate of transformation for an individual consumer. However, the introduction of a market for annuities allows the consumer to choose the level of annuities in his portfolio, and hence affects the intertemporal marginal rate of transformation.

VII. Concluding Remarks

We have developed an overlapping generations model with random lifetimes which induce random intergenerational transfers in the form of bequests. This model adds considerable richness to the standard overlapping generations framework because it generates intra-cohort distributions of wealth and consumption. Therefore, the model can be used to analyze the effect of policies on the distribution of consumption and wealth within cohorts as well as the effects on the aggregate consumption and wealth of each cohort.

The model developed in the paper has been used to analyze the effects on wealth and consumption of changes in the level of actuarially fair social security. We showed that increasing the level of social security reduces national wealth (even if the social security system is fully funded) and uniformly narrows the distribution of consumption within each cohort. The reduction in national wealth leads to a decrease, an increase, or no change in the steady state consumption of each cohort depending on whether the net rate of return on capital is greater than, less than, or equal to the rate of population growth.

Appendix A

Proof of Lemma

Observe that Var $(y) = \sum_{i=0}^{\infty} f_i(y_i-\bar{y})$ where a bar over a variable denotes the expectation of that variable. Define $z_i \equiv y_i - x_i$ and observe that

$$\mathrm{Var}(y) = \sum_{i=0}^{\infty} f_i(x_i-\bar{x})^2 + \sum_{i=0}^{\infty} f_i(z_i-\bar{z})^2 + 2\sum_{i=0}^{\infty} f_i(x_i-\bar{x})(z_i-\bar{z}) \tag{A1}$$

Since $\mathrm{Var}(x) = \sum_{i=0}^{\infty} f_i(x_i-\bar{x})^2$, we have

$$\begin{aligned}\mathrm{Var}(y) - \mathrm{Var}(x) &= \sum_{i=0}^{\infty} f_i(z_i-\bar{z})[z_i-\bar{z} + 2(x_i-\bar{x})] \\ &= \sum_{i=0}^{\infty} f_i(z_i-\bar{z})(x_i+y_i-(\bar{x}+\bar{y}))\end{aligned} \tag{A2}$$

Therefore $\mathrm{Var}(y) - \mathrm{Var}(x) = \mathrm{Cov}(z, x+y)$. Since x_i+y_i is nondecreasing in i and z is nonincreasing in i, $\mathrm{Cov}(z, x+y) < 0$. Therefore $\mathrm{Var}(y) < \mathrm{Var}(x)$.

q.e.d.

Appendix B

In this appendix we calculate the effect on the aggregate first-period consumption of the generation born in period t^*+m of the introduction in period t^*+1 of actuarially fair social security. That is, we derive equation (40) in the text.

It will be useful to define x as

$$x \equiv (1-aR)(R/G) \tag{B1}$$

Under actuarially fair social security, $RT = (1-p)S$ so that

$$(R^2/G)(T+apS) = (R/G)(1-p+apR)S = ((R/G)-px)S \tag{B2}$$

Substituting (B1) and (B2) into (39) yields

$$\tilde{\tilde{c}}^{(j)}_{1,t^*+m} - \tilde{c}^{(j)}_{1,t^*+m} = aS\{p - [(R/G) - px]\sum_{i=0}^{j^*-1} x^i\} \tag{B3}$$

As a step toward calculating the average value of each side of (B3), we first calculate

$$\sum_{j=0}^{\infty}(1-p)p^j\sum_{i=0}^{j^*-1}x^i = (1-p)\sum_{j=0}^{\infty}p^j\frac{1-x^{j^*}}{1-x} \tag{B4}$$

Recalling that $j^* = \min(j, m-1)$, (B4) can be rearranged to yield

$$\sum_{j=0}^{\infty}(1-p)p^j\sum_{i=0}^{j^*-1}x^i = \frac{1-p}{1-x}[\frac{1}{1-p} - \sum_{j=0}^{m-1}p^j x^j - \sum_{j=m}^{\infty}p^j x^{m-1}] \tag{B5}$$

Calculating the sums on the right hand side of (B5) yields

$$\sum_{j=0}^{\infty}(1-p)p^{j}\sum_{i=0}^{j^*-1}x^{i} = \frac{1-p}{1-x}\left[\frac{1-p^{m}x^{m-1}}{1-p} - \frac{1-p^{m}x^{m}}{1-px}\right] \tag{B6}$$

which can be simplified to yield

$$\sum_{j=0}^{\infty}(1-p)p^{j}\sum_{i=0}^{j^*-1}x^{i} = \frac{p}{1-px}(1 - p^{m-1}x^{m-1}). \tag{B7}$$

Now calculate the average value of each side of (B3) and use (B7) to obtain

$$\tilde{\tilde{C}}^{*}_{1,t^*+m} - \tilde{C}^{*}_{1,t^*+m} = apS\{1 - \frac{(R/G)-px}{1-px}(1-p^{m-1}x^{m-1})\} \tag{B8}$$

Rearranging (B8) yields

$$\tilde{\tilde{C}}^{*}_{1,t^*+m} - \tilde{C}^{*}_{1,t^*+m} = \frac{apS}{1-px}\{1 - (R/G) + [(R/G)-px]p^{m-1}x^{m-1}\} \tag{B9}$$

Recognizing that

$$[(R/G) - px]p^{m-1}x^{m-1} = [1 - p(1-aR)]p^{m-1}(1-aR)^{m-1}(R/G)^{m} \tag{B10}$$

then yields equation (40) in the text.

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