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ABSTRACT

This paper presents a dynamic equilibrium model of bond markets, in which two groups of agents hold heterogeneous expectations about future economic conditions. Our model shows that heterogeneous expectations can not only lead to speculative trading, but can also help resolve several challenges to standard representative-agent models of the yield curve. First, the relative wealth fluctuation between the two groups of agents caused by their speculative positions amplifies bond yield volatility, thus providing an explanation for the "excessive volatility puzzle" of bond yields. In addition, the fluctuation in the two groups' expectations and relative wealth also generates time-varying risk premia, which in turn can help explain the failure of the expectation hypothesis. These implications, essentially induced by trading between agents, highlight the importance of incorporating heterogeneous expectations into economic analysis of bond markets.

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1 Introduction

Following the seminal work of Vasicek (1977) and Cox, Ingersoll and Ross (1985), most academic studies in the economics and finance literature use representative-agent models to analyze yield curve dynamics. These models typically derive bond pricing formulas based on a representative agent's risk preferences and belief processes. This approach leads to tractable parametric yield curve models that researchers can directly apply to data, e.g., Duffie and Kan (1996) and Dai and Singleton (2001). Despite their recent success in capturing certain dynamics of the yield curve,[1] representative-agent models have limitations that prevent them from addressing several other aspects of bond markets, such as trading and liquidity, because these models do not involve interactions among heterogeneous agents. In this paper, we aim to provide an equilibrium model of bond markets, in which heterogeneous agents trade with each other.

We allow agents to hold heterogeneous expectations of future economic conditions,[2] and then study the bond market dynamics resulting from the trading among these agents. Our model builds on the equilibrium framework of Cox, Ingersoll and Ross (1985) with log-utility agents and a constant-return-to-scale risky investment technology. Unlike their model, we assume that there are two groups of agents using different learning models to infer the values of an unobservable variable that determines the long-run returns of the risky technology. Because of the difference in the learning processes, the two groups of agents hold heterogeneous expectations about future interest rates. Heterogeneous expectations motivate agents to take speculative positions against each other in the bond markets, and market clearing conditions determine equilibrium bond prices. We manage to solve this equilibrium in a closed form. In particular, we derive that the price of a bond is a wealth weighted average of bond prices in homogeneous economies, in each of which only one type of agent is present. We also

[1] See Dai and Singleton (2003) and Piazzesi (2003) for recent reviews of this literature.

[2] There is ample evidence supporting the existence of heterogeneous expectations among agents. Mankiw, Reis and Wolfers (2004) find that the interquartile range among professional economists' inflation expectations, as shown in the Livingston Survey and the Survey of Professional Forecasters, varies from above 2% in the early 1980s to around 0.5% in the early 2000s. Swanson (2005) finds that in the Blue Chip Economic Indicators survey of major U.S. corporations and financial institutions between 1991 and 2004, the difference between the 90th and 10th percentile forecasts of next-quarter real US GDP growth rate fluctuates between 1.5% and 5%, and the 90th and 10th percentile forecasts of four-quarter-behind 3-month Treasury bill rate fluctuates between 0.8% and 2.2%.

obtain similar results for nominal bond pricing when we extend the model to incorporate heterogeneous expectations about future inflation. By analyzing this equilibrium, our model shows that agents' heterogeneous expectations provide implications for the joint dynamics of trading volume, bond yield volatility, market liquidity, time-varying risk premia, and the yield curve.

A direct implication of our model is that trading volume increases as the difference between agents' beliefs widens. A higher belief dispersion causes agents to take larger speculative positions against each other. As a result, they are more exposed to random shocks and have to trade more to rebalance their portfolios after a shock.

Incorporating heterogeneous expectations and the resulting speculative trading into our model helps resolve several challenges encountered by standard representative-agent models. Because aggregate consumption is rather smooth, standard representative-agent models have difficulties in generating the large bond yield volatility and highly variable risk premia observed in actual data. To this end, our model shows that the relative wealth fluctuation caused by agents' speculative positions against each other amplifies bond yield volatility. Since agents who are more optimistic about future interest rates bet on rates rising against more pessimistic agents, any positive news about future rates would cause wealth to flow from the pessimistic agents to the optimistic agents, giving the optimistic belief a larger weight in determining equilibrium bond yields. The relative-wealth fluctuation thus amplifies the effect of the initial news on bond yields. Our calibration exercise shows that this mechanism can cause a significant amount of volatility amplification even with a modest amount of belief dispersion. This volatility amplification effect thus helps explain the "excess volatility puzzle" documented by Shiller (1979), Gurkaynak, Sack and Swanson (2005), and Piazzesi and Schneider (2006a). These studies find that long-term yields appear to be too volatile relative to the levels implied by standard representative-agent models.

Agents' belief and wealth fluctuation can also cause the equilibrium risk premia to change over time. From the view point of an econometrician who uses an objective learning process to evaluate this equilibrium, the market price of risk (risk premium per unit of risk) associated with the information shocks about future interest rates is proportional to the difference between agents' wealth weighted average belief and the econometrician's belief.

While agents' belief fluctuation directly affects these risk premia, it is important to note that agents' relative wealth fluctuation can lead to time-varying risk premia even without any belief fluctuation. The intuition is as follows. Suppose that the beliefs of the optimistic group and the pessimistic group both stay constant over time and their average is exactly that of the econometrician, which also stays constant. If the two groups have equal wealth and so the wealth weighted average belief is the same as the econometrician's belief, then the current risk premia associated with the information shocks are exactly zero. However, after a positive shock hits the market, the optimistic group would profit from the pessimistic group through their existing positions against each other. As a result, the optimistic group's belief would carry a greater weight in the market, causing the two groups' wealth weighted average belief to rise above the econometrician's belief and the risk premia to fluctuate.

The time-varying risk premia generated by our model shed some light on the failure of the expectations hypothesis, one of the classic theories of the yield curve dating at least back as Fisher (1896), Hicks (1939), and Lutz (1940). According to Lutz (1940, pp. 37), "An owner of funds will go into the long (term bond) market if he thinks the return he can make there over the time for which he has funds available will be above the return he can make in the short (rate) market over the same time, and vice versa." To make the fund owner, a representative agent in the bond market, indifferent about investing in a long-term bond or the short rate, this argument implies that when the spread between the long rate and short rate is large, the long rate tends to rise further (or the long bond price tends to fall). However, empirical studies, e.g., Fama and Bliss (1987), Campbell and Shiller (1991), and Cochrane and Piazzesi (2005), reject this prediction by finding that long rates tend to fall when their spreads relative to the short rate are high. This finding is often attributed to time-varying risk premia, but their sources remain elusive. Our model proposes a mechanism through agents' belief and wealth fluctuation, as the resulting risk premia are negatively correlated with the yield spread between long and short rates. Our calibration exercise also demonstrates that, with reasonable parameters, this mechanism is able to generate enough time variation in risk premia to explain the observed empirical finding.

By highlighting the effects caused by trading among agents with heterogeneous expectations, our analysis also cautions against a widespread practice of interpreting a representative

agent's belief process as the outcome of an actual agent's learning process. In fact, we could replicate the equilibrium price dynamics in our model by constructing a representative agent who always holds the wealth weighted average belief of the two groups. This exercise suggests that the change in the representative agent's belief not only responds to the two groups' belief fluctuation, but also to their relative wealth fluctuation. As a result, the constructed representative agent's belief process is inconsistent with a realistic Bayesian learning process, because the former compromises the effects caused by trading between the two groups of agents with heterogeneous expectations.

In summary, our model provides a tractable but non-affine yield curve structure, which simultaneously embeds stochastic volatility and time-varying risk premia. These features, as emphasized by Dai and Singleton (2003) and Duffee (2002), are crucial for capturing the yield curve dynamics. Our model also generates several testable predictions. First, higher belief dispersion increases bond market trading volume. Second, higher belief dispersion increases bond yield volatility and reduces bond market liquidity. Third, in an economy or a time period with more belief dispersion among agents, spread between long-term bond yield and short rate has a stronger predictive power for future yield changes. Finally, higher belief dispersion reduces bond yields, especially for bonds with longer maturities.

Our model complements the earlier equilibrium models with heterogeneous beliefs, e.g., Detemple and Murthy (1994) and Basak (2000, 2005). These models study the effects of heterogeneous beliefs on stock returns and short rates, but not on the yield curve dynamics. In addition, they do not analyze the effects of heterogeneous beliefs on volatility amplification and time-varying risk premia. Our model differs in emphasis from Dumas (1989) and Wang (1996), which provide models to analyze the effects of agents' preference heterogeneity on their wealth distribution and the yield curve. These papers also do not address volatility amplification and time-varying risk premia generated by trading between heterogeneous agents. Finally, Dumas, Kurshev and Uppal (2005) analyze the effects of some agents' irrational beliefs on market volatility and equity premium. Their study especially focuses on the rational investors' optimal responses to the presence of irrational investors in a general equilibrium set-up. In contrast, our model focuses on the impacts of agents' belief dispersion on the yield curve dynamics, especially highlighting the role of agents' relative wealth fluctuation.

The rest of the paper is organized as follows. Section 2 presents the model and derives the equilibrium. Section 3 analyzes the effects of agents' heterogeneous expectations on bond market dynamics. Section 4 reconciles our model with standard representative-agent models and Section 5 provides a calibration exercise of our model. Finally, Section 6 concludes the paper. We provide all the technical proofs in Appendix A and an extension of our model in Appendix B.

2 The Model

Our model adopts the equilibrium framework of Cox, Ingersoll and Ross (1985) with log-utility agents and a constant-return-to-scale risky investment technology. Unlike their model, ours assumes that agents cannot directly observe a random variable that determines future returns of the risky technology, and that agents have to infer its value. There are two groups of agents holding heterogeneous expectations regarding this variable. Because of this belief dispersion, agents speculate in capital markets. We study a competitive equilibrium, in which each agent optimizes consumption and investment decisions based on his own expectation. Market clearing conditions determine the equilibrium short rate and bond prices. In the main text of the paper, we focus on a model without inflation. In Appendix B, we obtain similar results by extending the model to price nominal bonds.

2.1 The economy

We consider an economy with only one constant-return-to-scale technology. The return of the technology follows a diffusion process:

$$\frac{dI_t}{I_t} = f_t dt + \sigma_I dZ_I(t), \tag{1}$$

where f_t is the expected instantaneous return, σ_I is a volatility parameter, and $Z_I(t)$ is a standard Brownian motion.

The expected instantaneous return from the risky technology, f_t, follows another linear diffusion process:

$$df_t = -\lambda_f (f_t - l_t)dt + \sigma_f dZ_f(t), \tag{2}$$

where λ_f is a constant governing the mean reverting speed of f_t, l_t represents a moving long-run mean of the risky technology's expected return, σ_f is a volatility parameter, and $Z_f(t)$ is a standard Brownian motion independent of $Z_I(t)$. As we will show later, the expected instantaneous return of the technology f_t, after adjusted for risk, determines the equilibrium short rate, because this risky technology represents an alternative investment to investing in the short term bond.

The long-run mean l_t is unobservable and follows an *Ornstein-Uhlenbeck* process:

$$dl_t = -\lambda_l(l_t - \bar{l})dt + \sigma_l dZ_l(t), \tag{3}$$

where λ_l is a parameter governing the mean-reverting speed of l_t, $\bar{l}$ the long-run mean of l_t, σ_l a volatility parameter, and $Z_l(t)$ a standard Brownian motion independent of $Z_I(t)$ and $Z_f(t)$. Since l_t is the level, to which f_t mean-reverts, it determines future short rates. As we will later show, agents' disagreement about l_t thus leads to heterogeneous expectations about future short rates.

2.2 Agents' heterogeneous expectations

The existing economics and finance literature has pointed out several sources of heterogeneous expectations. First, Harris and Raviv (1993), Detemple and Murthy (1994), Morris (1996) and Basak (2000) assume that agents hold heterogeneous prior beliefs about unobservable economic variables. In these models, agents continue to disagree with each other even after they update their beliefs using identical information and the difference in their beliefs deterministically converges to zero. Second, Kurz (1994) argues that limited data make it difficult for rational agents to identify the correct model of the economy from alternative ones. As a result, model uncertainty could cause agents to use different learning models and therefore to possess heterogeneous beliefs. Third, consistent with a broader interpretation of heterogeneous priors and the model uncertainty argument, Scheinkman and Xiong (2003) and Dumas, Kurshev and Uppal (2005) assume that agents use different model parameters in their learning processes. As a result, agents could react differently to the same information, and the difference in their posterior beliefs follows stationary processes.

Following the last approach, we also assume that agents use different model parameters in their learning processes. Since these parameters are part of their model for the whole

economy, they do not update these parameters, instead they use them as the basis for their learning processes about unobservable economic variables such as the long-run mean of risky technology returns. This approach is tractable and generates stationary processes for differences in agents' expectations.

We now discuss agents' expectations about future risky technology returns. In addition to observing f_t, agents also receive two public signals, S_1 and S_2, about the unobservable long-run mean of the risky technology's return l_t. These two signals follow the following processes:

$$dS_1(t) = l_t dt + \sigma_s dZ_{s1}(t), \tag{4}$$

$$dS_2(t) = l_t dt + \sigma_s dZ_{s2}(t), \tag{5}$$

where $Z_{s1}(t)$ and $Z_{s2}(t)$ are standard Brownian motions that are independent of each other and all other Brownian motions. For symmetry, we assume that these two signals share the same noise volatility parameter σ_s.

We assume that agents are divided in two groups and differ in their perceptions of the signal processes. Specifically, agents in group 1 believe that S_1 evolves according to

$$dS_1(t) = l_t dt + \sigma_s \left[\phi_l dZ_l(t) + \sqrt{1-\phi_l^2} dZ_{s1}(t) \right]. \tag{6}$$

Although this process has the same instantaneous volatility as the actual process in equation (4), group-1 agents believe that $\phi_l \in [0,1]$ fraction of the innovations to dS_1 comes from dZ_l, the fundamental innovation to dl_t itself. Thus, group-1 agents under-estimate the noise in S_1. The parameter ϕ_l measures the degree of this noise under-estimation.[3] On the other hand, group-1 agents perceive that S_2 evolves according to the actual process in equation (5).

Similarly, we assume that group-2 agents perceive S_1 in the actual process in equation (4) and that they believe that S_2 evolves according to

$$dS_2(t) = l_t dt + \sigma_s \left[\phi_l dZ_l(t) + \sqrt{1-\phi_l^2} dZ_{s2}(t) \right]. \tag{7}$$

In the same way that group-1 agents perceive S_1, group-2 agents incorrectly believe that ϕ_l fraction of the innovation to S_2 comes from dZ_l, and thus under-estimate the noise in S_2. For

[3]Since group-1 agents have the correct volatility parameter of dS_1 itself and the fundamental innovation dZ_l is not observable, the value of ϕ_l cannot be directly inferred from the quadratic variation of dS_1 and a precise estimation would require a long series of data.

the sake of symmetry, the degree of group-2 agents' noise under-estimation, ϕ_l, is the same as that of group-1 agents.

In summary, group-1 agents believe that these signals evolve according to equations (6) and (5); while group-2 agents believe that these signals evolve according to equations (4) and (7). Agents in each group make their economic decisions based on their own model about the signals. We further assume that although agents in one group are aware of the model used by the other group, they agree to disagree about the differences between their models. The market equilibrium is thus determined by the interaction of the two groups of agents. To evaluate the dynamics of this equilibrium, we will stand from the perspective of an econometrician who believes the signals follow the actual processes in equations (4) and (5). We will focus on the learning processes of the two groups in this section and derive the econometrician's in a later section.

Agents' information set at time t about l_t includes $\{f_\tau, S_1(\tau), S_2(\tau)\}_{\tau=0}^{t}$. We assume that agents' prior beliefs about l_t have a Gaussian distribution. Since their information flow also follows Gaussian processes, their posterior beliefs must likewise be Gaussian. The difference in agents' perceptions about the signal processes would cause the mean of their posterior beliefs to differ; however, because of their symmetry, they would still share the same posterior variance. According to the standard results in linear filtering, e.g., Theorem 12.7 of Liptser and Shiryaev (1977), agents' belief variance converges to a stationary level at an exponential rate. For our analysis, we will focus on the stationary equilibrium, in which the belief variance of agents in both groups has already reached its stationary level $\bar{\gamma}_l$, which is the positive root to the following quadratic equation of γ:

$$\left(\frac{\lambda_f^2}{\sigma_f^2}+\frac{2}{\sigma_s^2}\right)\gamma^2+2\left(\lambda_l+\frac{\phi_l\sigma_l}{\sigma_s}\right)\gamma-\left(1-\phi_l^2\right)\sigma_l^2=0.$$

We denote group-i agents' posterior distribution about l_t at time t by

$$l_t|\left\{f_\tau, S_1(\tau), S_2(\tau)\right\}_{\tau=0}^{t} \sim N\left(\hat{l}_t^i, \bar{\gamma}_l\right), \qquad i\in\{1,2\},$$

where $\hat{l}_t^i$ is the mean of group-i agents' posterior distribution. We will refer to $\hat{l}_t^i$ as their belief hereafter.

Theorem 12.7 of Liptser and Shiryaev (1977) also provides that $\hat{l}_t^i$ is determined by

$$d\hat{l}_t^i = -\lambda_l(\hat{l}_t^i - \bar{l})dt + \lambda_f \sigma_f^{-1}\bar{\gamma}_l d\widehat{Z}_f^i(t) + \sigma_s^{-1}\left(\bar{\gamma}_l + \phi_l\sigma_s\sigma_l\right)d\widehat{Z}_{si}^i(t) + \sigma_s^{-1}\bar{\gamma}_l d\widehat{Z}_{sj}^i(t) \tag{8}$$

where $j \in \{1,2\}$ and $j \neq i$. $d\widehat{Z}_f^i$, $d\widehat{Z}_{si}^i$ and $d\widehat{Z}_{sj}^i$ are "surprises" in the three sources of information to group-i agents:

$$d\widehat{Z}_f^i = \frac{1}{\sigma_f}\left[df_t + \lambda_f(f_t - \hat{l}_t^i)dt\right], \tag{9}$$

$$d\widehat{Z}_{si}^i = \frac{1}{\sigma_s}\left[dS_i(t) - \hat{l}_t^i dt\right], \tag{10}$$

$$d\widehat{Z}_{sj}^i = \frac{1}{\sigma_s}\left[dS_j(t) - \hat{l}_t^i dt\right]. \tag{11}$$

Note that $\widehat{Z}_f^i$, $\widehat{Z}_{si}^i$ and $\widehat{Z}_{sj}^i$ are independent standard Brownian motions in group-i agents' probability measure. Equation (8) shows that under-estimation of noise in signal S_1 causes group-1 agents to "over-react" to $d\widehat{Z}_{s1}^1(t)$, the surprise in dS_1. Similarly, group-2 agents over-react to the surprise in dS_2. As a result, these two groups hold different beliefs about l_t.

In group-i agents' probability measure, f_t, $S_1(t)$ and $S_2(t)$ follow

$$df_t = -\lambda_f(f_t - \hat{l}_t^i)dt + \sigma_f d\hat{Z}_f^i(t), \tag{12}$$

$$dS_1 = \hat{l}_t^i dt + \sigma_s d\hat{Z}_{s1}^i(t), \tag{13}$$

$$dS_2 = \hat{l}_t^i dt + \sigma_s d\hat{Z}_{s2}^i(t). \tag{14}$$

Thus, the difference in agents' beliefs about l_t translates into different views about the dynamics of these variables and, subsequently, into different expectations of future short rates.

2.3 Capital markets

The difference in agents' beliefs causes speculative trading among them. Agents who are more optimistic about l_t would bet on interest rates going up against more pessimistic agents. Note that, in each group's measure, there are four types of random shocks. For group-i agents, the shocks are dZ_I, $d\hat{Z}_f^i$, $d\hat{Z}_{s1}^i$, and $d\hat{Z}_{s2}^i$. Thus, the markets are complete if agents can trade a risk free asset and four risky assets that span these four sources of random shocks. In reality, bond markets offer many securities, such as bonds with different maturities, for agents to construct their bets and to complete the markets. As a result, we analyze agents'

investment and consumption decisions, as well as their valuations of financial securities, in a complete-markets equilibrium.

We introduce a zero-net-supply risk free asset and three zero-net-supply risky financial securities in the capital markets, in addition to the risky production technology.[4] At time t, the risk free asset offers a short rate r_t. The rate is determined endogenously in the equilibrium. The three risky financial securities offer the following return processes:

$$\frac{dp_f}{p_f} = \mu_f(t)dt + df_t, \tag{15}$$

$$\frac{dp_{s1}}{p_{s1}} = \mu_{s1}(t)dt + dS_1(t), \tag{16}$$

$$\frac{dp_{s2}}{p_{s2}} = \mu_{s2}(t)dt + dS_2(t). \tag{17}$$

We refer to these securities as security f, security S_1, and security S_2, respectively. Like futures contracts, these securities are continuously marked to the fluctuations of df_t, $dS_1(t)$, and $dS_2(t)$, respectively. Since agents hold different views about the underlying innovation processes of these securities, they disagree about their expected returns. As a result, some agents want to take long positions, while others want to take short positions. Through trading, the contract terms $\mu_f(t)$, $\mu_{s1}(t)$, and $\mu_{s2}(t)$ are continuously determined so that the aggregate demand for each of the securities is zero at any instant. We could also view these financial securities as synthetic positions constructed by dynamically trading bonds. We choose to introduce these securities instead of specific bonds to simplify notation, and our specific choice of securities does not affect the equilibrium in complete markets.

To further simplify notation, we put the return processes of securities f, S_1, and S_2 in a column vector:

$$d\vec{R}_t = \left(\frac{dp_f}{p_f}, \frac{dp_{s1}}{p_{s1}}, \frac{dp_{s2}}{p_{s2}}\right)',$$

where $'$ is the transpose operator. By substituting equations (12), (13), and (14) into the return processes of the risky securities, we can rewrite them in group-i agents' probability measure as

$$d\vec{R}_t = \vec{\mu}_t^i dt + \Sigma \cdot d\vec{Z}^i(t),$$

[4]We also allow agents to short-sell the risky technology. This can be implemented by offering a derivative contract on the return of the technology. The market clearing conditions, however, require that agents in aggregate hold a long position in the risky technology.

where the vector of expected returns is given by

$$\vec{\mu}_t^i = \begin{pmatrix} \hat{\mu}_f^i(t) \\ \hat{\mu}_{s1}^i(t) \\ \hat{\mu}_{s2}^i(t) \end{pmatrix} = \begin{pmatrix} \mu_f(t) - \lambda_f(f_t - \hat{l}_t^i) \\ \mu_{s1}(t) + \hat{l}_t^i \\ \mu_{s2}(t) + \hat{l}_t^i \end{pmatrix}, \tag{18}$$

and the volatility matrix Σ and the diffusion vector $d\vec{Z}^i(t)$ are given by

$$\Sigma = \begin{pmatrix} \sigma_f & & \\ & \sigma_s & \\ & & \sigma_s \end{pmatrix} \quad \text{and} \quad d\vec{Z}^i(t) = \begin{pmatrix} d\hat{Z}_f^i(t) \\ d\hat{Z}_{s1}^i(t) \\ d\hat{Z}_{s2}^i(t) \end{pmatrix}.$$

We assume that all agents have an identical logarithmic preference. Agents in group i maximize their lifetime utility from consumption by investing in all available securities according to their beliefs:

$$\max_{\{c_t^i, x_I^i, \vec{X}^i\}} E^i \int_0^\infty e^{-\beta t} u(c_t^i) dt,$$

where E^i is the expectation operator under their probability measure, β is their time-preference parameter, and

$$u(c_t^i) = \log(c_t^i)$$

is their utility function from consumption. Agents can choose their consumption c_t^i, the fraction of their wealth invested in the risky technology x_I^i, and the fractions of their wealth invested in the three financial securities:

$$\vec{X}^i = \left(x_f^i, x_{s1}^i, x_{s2}^i\right)'$$

with each component of $\vec{X}^i$ corresponding to the fraction of wealth invested in securities f, S_1, and S_2.

Given group-i agents' investment and consumption strategies, their wealth process follows

$$\frac{dW_t^i}{W_t^i} = \left[r_t - c_t^i/W_t^i + x_I^i\left(f_t - r_t\right) + \vec{X}^i \cdot \left(\vec{\mu}_t^i - r_t\right)\right] dt + \vec{X}^i \cdot \Sigma \cdot d\vec{Z}^i(t) + x_I^i dZ_I(t). \tag{19}$$

We can solve these agents' consumption and investment problems using the standard dynamic programming approach developed by Merton (1971). The results for logarithmic utility are well known. Agents always consume wealth at a constant rate equal to their time preference parameter:

$$c_t^i = \beta W_t^i,$$

and they invest in risky assets according to the assets' instantaneous risk-return tradeoff – the ratio between expected excess return and return variance:

$$x_I^i = \frac{f_t - r_t}{\sigma_I^2} \quad \text{and} \quad \vec{X}^i = \left(\vec{\mu}_t^i - r_t\right)' \cdot \Sigma^{-2}. \tag{20}$$

2.4 Equilibrium asset prices

We adopt a standard definition of competitive equilibrium. In the equilibrium, each agent chooses optimal consumption and investment decisions in accordance with his expectations and all markets clear. Market clearing conditions ensure: 1) the aggregate investment to the risk free asset is zero; 2) the aggregate investment to each of the risky securities f, S_1, and S_2 is also zero; and 3) the aggregate investment to the risky technology is equal to the total wealth in the economy. We describe the equilibrium in the following theorem, and provide the proof in Appendix A.1.

Theorem 1 *In equilibrium, the real short rate is*

$$r_t = f_t - \sigma_I^2. \tag{21}$$

Let ω_t^i is the wealth share of group-i agents in the economy:

$$\omega_t^i \equiv \frac{W_t^i}{W_t}, \qquad W_t \equiv \sum_{i=1}^{2} W_t^i.$$

Then, the contract terms $\mu_f(t)$, $\mu_{s1}(t)$, and $\mu_{s2}(t)$ of the risky securities are determined by

$$\mu_f = r_t + \lambda_f f_t - \lambda_f \sum_{i=1}^{2} \omega_t^i \hat{l}_t^i, \tag{22}$$

$$\mu_{s1} = r_t - \sum_{i=1}^{2} \omega_t^i \hat{l}_t^i, \tag{23}$$

$$\mu_{s2} = r_t - \sum_{i=1}^{2} \omega_t^i \hat{l}_t^i. \tag{24}$$

The aggregate wealth in the economy fluctuates according to

$$\frac{dW_t}{W_t} = (f_t - \beta)\, dt + \sigma_I dZ_I(t). \tag{25}$$

This theorem shows that the short rate is the expected instantaneous return of the risky technology adjusted for risk (equation (21)) This is because agents would demand a higher return from lending out capital when the expected return from the alternative option of investing in the risky technology is higher. Equations (22)-(24) provide the contract terms of the three financial securities. Each of these terms is determined by the short rate, r_t, minus the wealth weighted average of agents' beliefs about the drift rate of the corresponding security's underlying factor. Equation (25) shows that the aggregate wealth in the economy grows at a rate determined by the return from the risky technology, $f_t dt + \sigma_I dZ_I(t)$, minus agents' consumption rate, βdt. This is because the risky technology is the only storage technology in the economy.

We denote the wealth ratio between agents in groups 1 and 2 by $\eta_t \equiv \frac{W_t^1}{W_t^2}$, and the difference in beliefs by $g_l(t) \equiv \hat{l}_t^1 - \hat{l}_t^2$. Then we will refer to the absolute value of $g_l(t)$, $|g_l(t)|$, as the belief dispersion. The following proposition characterizes the dynamics of the wealth ratio, with the proof in Appendix A.2.

Proposition 1 *The wealth ratio process η_t evolves in group-2 agents' probability measure according to*

$$\frac{d\eta_t}{\eta_t} = g_l \left[\frac{\lambda_f}{\sigma_f} d\hat{Z}_f^2(t) + \frac{1}{\sigma_s} d\hat{Z}_{s1}^2(t) + \frac{1}{\sigma_s} d\hat{Z}_{s2}^2(t) \right]. \tag{26}$$

Moreover, if X_T is a random variable to be realized at time $T > t$ and $E^1[X_T] < \infty$, then group-1 agents' expectation of X_T at time t can be transformed into group-2 agents' expectation through the wealth ratio process between the two groups:

$$E_t^1 [X_T] = E_t^2 \left[\frac{\eta_T}{\eta_t} X_T \right].$$

Equation (26) shows that the volatility of the wealth ratio is proportional to belief dispersion $|g_l(t)|$. Intuitively, higher belief dispersion induce agents to take more aggressive speculative and so their wealth ratio becomes more volatile. Proposition 1 also shows that the wealth ratio process between agents in groups 1 and 2 acts as the Randon-Nikodyn derivative of group-1 agents' probability measure with respect to group-2 agents' measure. The intuition is as follows. If group-1 agents assign a higher probability to a future state than group-2 agents, it is natural for these agents to trade in such a way that the wealth

ratio between them, W^1/W^2, is also higher in that state. Proposition 1 implies that, as a consequence of logarithmic preference, the ratio of probabilities assigned by these groups to different states is perfectly correlated with their wealth ratio. This result allows us to derive a simple asset pricing formula in the heterogeneous economy. It is also important to note that no single group would be able to drive out the other one and eventually dominate the market. Due to the symmetric structure in the two groups' learning models, the wealth distribution is stationary.[5]

The property of the two groups' wealth ratio process in Proposition 1 leads to a simple expression of asset prices in the heterogeneous economy, as shown in the following theorem. We provide the proof in Appendix A.3.

Theorem 2 *In a heterogeneous economy with two groups of agents, the price of an asset, which provides a single payoff X_T at time T, is given by*

$$P_t = \omega_t^1 P_t^1 + \omega_t^2 P_t^2,$$

where P_t^i is the value of the asset in a homogeneous economy, whereby only group-i agents are present.

Theorem 2 shows that the price of an asset is the wealth weighted average of each group's valuation of the asset in a corresponding homogeneous economy. This result allows us to derive asset prices in a heterogeneous economy using prices in homogeneous economies. Thus, asset pricing is remarkably simple even in a complex environment with heterogeneous agents. While this result depends on agents' logarithmic preference and linear risky technology, it is independent of the specific information structure in our model. Detemple and Murthy (1994) provide a similar result in a model with heterogeneous prior beliefs.

2.5 Bond pricing with homogeneous agents

Theorem 2 allows us to express the price of a bond as the wealth weighted average of each group's bond valuation in a homogeneous economy. Thus, before analyzing the effects of

[5]See Kogan, Ross, Wang, Westerfield (2004) and Yan (2005) for more discussions on the possibility that one group might dominate in the long run when it has a more accurate belief process than the other group.

agents' heterogeneous expectations on bond markets, we first derive bond prices in homogeneous economies in the following proposition 2, with a proof in Appendix A.4.

Proposition 2 *In a homogeneous economy with only group-i agents, the price of a zero-coupon bond with a maturity τ is determined by*

$$B^H\left(\tau, f_t, \hat{l}_t^i\right) = e^{-a_f(\tau)f_t - a_l(\tau)\hat{l}_t^i - b(\tau)},$$

where

$$\begin{aligned} a_f(\tau) &= \frac{1}{\lambda_f}\left(1 - e^{-\lambda_f \tau}\right), && (27) \\ a_l(\tau) &= \frac{1}{\lambda_l}\left(1 - e^{-\lambda_l \tau}\right) + \frac{1}{\lambda_f - \lambda_l}\left(e^{-\lambda_f \tau} - e^{-\lambda_l \tau}\right), && (28) \\ b(\tau) &= \int_0^\tau \left[\lambda_l \bar{l} a_l(s) - \frac{1}{2}\sigma_f^2 a_f(s)^2 - \frac{1}{2}\left(\sigma_l^2 - 2\lambda_l \bar{\gamma}_l\right) a_l(s)^2 - \lambda_f \bar{\gamma}_l a_f(s) a_l(s) - \sigma_I^2\right] ds. \end{aligned}$$

Proposition 2 implies that the yield of a τ-year bond in a homogeneous economy

$$Y^H\left(\tau, f_t, \hat{l}_t^i\right) = -\frac{1}{\tau}\log\left(B^H\right) = \frac{a_f(\tau)}{\tau} f_t + \frac{a_l(\tau)}{\tau}\hat{l}_t^i + \frac{b(\tau)}{\tau}$$

is a linear function of two fundamental factors: f_t and $\hat{l}_t^i$. This specific form belongs to the general affine structure proposed by Duffie and Kan (1996).

The loading on f_t, $a_f(\tau)/\tau$, has a value of 1 when the bond maturity τ is zero and monotonically decreases to zero as the maturity increases, suggesting that short-term yields are more exposed to fluctuations in f_t. The intuition of this pattern is as follows. f_t is the expected instantaneous return from the risky technology, which can serve as a close substitute for investing in short-term bonds. As a result, the fluctuation in f_t has a greater impact on short-term yields. As bond maturity increases, the impact of f_t becomes smaller.

Agents' belief about l_t determines their expectation of future returns from the risky technology, because l_t is the level to which f_t mean-reverts. In the case with mean-reversion ($\lambda_l > 0$), the loading of the bond yield on $\hat{l}_t^i$, $a_l(\tau)/\tau$, has a humped shape. As the bond maturity increases from 0 to an intermediate value, $a_l(\tau)/\tau$ increases, suggesting that agents' expectation has a greater impact on longer term yields. As the bond maturity increases further, $a_l(\tau)/\tau$ drops. This is caused by the mean reversion of l_t, which causes any shock

to l_t to eventually die out. This force causes the yields of very long-term bonds to have low exposure to agents' belief about l_t. In the case where mean reversion is no present ($\lambda_l = 0$), the factor loading $a_l(\tau)/\tau$ is a monotonically increasing function of bond maturity.

3 Effects of Heterogeneous Expectations

In this section, we discuss the effects of agents' heterogeneous expectations on bond markets. We combine Proposition 2 with Theorem 2 to express the price of a τ-year zero-coupon bond at time t as

$$B_t = \omega_t^1 B^H\left(\tau, f_t, \hat{l}_t^1\right) + \omega_t^2 B^H\left(\tau, f_t, \hat{l}_t^2\right), \tag{29}$$

where ω_t^1 and ω_t^2 are the two groups' wealth shares in the economy, and $B^H\left(\tau, f_t, \hat{l}_t^i\right)$, given in Proposition 2, is the bond price in a homogeneous economy wherein only group-i agents are present. The implied bond yield is

$$\begin{aligned} Y_t(\tau) &= -\frac{1}{\tau}\log\left(B_t\right) \\ &= \frac{a_f(\tau)}{\tau} f_t + \frac{b(\tau)}{\tau} - \frac{1}{\tau}\log\left[\omega_t^1 e^{-a_l(\tau)\hat{l}_t^1} + \omega_t^2 e^{-a_l(\tau)\hat{l}_t^2}\right]. \end{aligned}$$

Note that Y_t is not a linear function of agents' beliefs $\hat{l}_t^1$ and $\hat{l}_t^2$. That is, bond yields in this heterogeneous economy have a non-affine structure. This structure derives from the market aggregation of agents' heterogeneous valuations of the bond. This structure serves as the basis for our analysis of the effects of heterogeneous expectations. Note that this structure still holds for nominal bond pricing, as illustrated in Appendix B.

3.1 Trading volume

Heterogeneous expectations cause agents to take speculative positions against each other in bond markets. These speculative positions can cause fluctuations in agents' wealth upon the arrivals of random shocks. As a result, agents trade with each other to rebalance their positions. Intuitively, when belief dispersion increases, the size of their speculative positions becomes larger. This in turn leads to a higher volatility of agents' wealth and therefore a larger trading volume in the bond markets. We use the volatility of one group's position changes as a measure of trading volume. This measure corresponds to the conventional volume

measure in a discrete-time set up. We summarize the effect of agents' belief dispersion on trading volume in Proposition 3, and provide a formal derivation and further discussion on our volume measure in Appendix A.5.

Proposition 3 *Trading volume (fluctuation in agents' speculative positions) increases with the belief dispersion between the two groups of investors.*

There is now a growing literature analyzing trading volume caused by heterogeneous beliefs, e.g., Harris and Raviv (1993) and Scheinkman and Xiong (2003). While these models demonstrate that heterogeneous beliefs lead to trading, trading typically occurs when agents' beliefs flip, that is, g_l changes its sign. Thus, trading volume of this type only increases with the frequency that agents' beliefs flip. Our model adds to this literature by showing that even without agents' beliefs flipping, the wealth fluctuation caused by their speculative positions already leads to trading.

3.2 Volatility amplification

The wealth fluctuation caused by agents' speculative positions against each other not only leads to trading in bond markets, but also amplifies bond yield volatility. Loosely speaking, bond yields are determined by agents' wealth weighted average belief about future interest rates. Since agents who are more optimistic about future rates bet on these rates rising against more pessimistic agents, any positive news about future rates would cause wealth to flow from pessimistic agents to optimistic agents, making the optimistic belief carry a greater weight in bond yields. The relative-wealth fluctuation thus amplifies the impact of the initial news on bond yields. As a result, a higher belief dispersion increases the relative-wealth fluctuation and so increases bond yield volatility. We summarize this intuition in the following proposition, and provide a formal proof in Appendix A.6.

Proposition 4 *Bond yield volatility increases with belief dispersion.*

This volatility amplification mechanism can help explain the "excess volatility puzzle" for bond yields. Shiller (1979) shows that the observed bond yield volatility exceeds the upper limits implied by the expectations hypothesis and the observed persistence in short

rates. Gurkaynak, Sack and Swanson (2005) also document that bond yields exhibit excess sensitivity to particular shocks, such as macroeconomic announcements. Furthermore, Piazzesi and Schneider (2006a) find that by estimating a representative agent asset pricing model with recursive utility preferences and exogenous consumption growth and inflation, the model predicts less volatility for long yields relative to short yields. Relating to this literature, Proposition 4 shows that extending standard representative-agent models with heterogeneous expectations can help account for the observed high bond yield volatility. In Section 5, we provide a calibration exercise to illustrate the magnitude of this mechanism.

Through the volatility amplification effect, heterogeneous expectations could also shed some light on the time variation of market liquidity in bond markets. Let's consider the following thought experiment. Suppose that agents in one group suffer a liquidity shock and need to sell a fraction of their positions. The resulting price impact is a commonly used measure of liquidity. Since this selling would suppress prices and reduce these agents' wealth, the initial price impact of these sales would be further amplified by the change in these agents' wealth relative the other group. As a result, if there exists a larger belief dispersion among the two groups (or if agents' existing positions are larger), the amplification effect is stronger and the net price impact of one group's liquidity selling will be larger, causing the market liquidity to be lower.[6]

3.3 Time-varying risk premia

Fluctuations in agents' belief dispersion and relative wealth also cause risk premia in the economy to vary over time. To examine risk premia, we analyze the dynamics of the stochastic discount factor from the perspective of an econometrician who uses the actual signal processes in forming his expectations. We derive this econometrician's learning process and stochastic discount factor in Appendix A.7 and summarize the result in the following Proposition.

[6]A similar wealth amplification mechanism has been employed by Xiong (2001) to explain the observed high volatility and low liquidity during the crisis period of the hedge fund Long-Term Capital Management in the late summer of 1998. His model shows that the wealth fluctuation of some highly leveraged market participants can lead to large price reactions to liquidity shocks. By explicitly relating the magnitude of this wealth amplification effect to agents' belief dispersion, our model demonstrates fluctuation in agents' belief dispersion as a source of time-varying volatility and liquidity.

Proposition 5 *From the view point of an econometrician who holds the objective probability measure, the stochastic discount factor has the following process*

$$\frac{dM_t}{M_t} = -(f_t - \sigma_I^2)dt - \sigma_I dZ_I - \left(\hat{l}_t^R - \sum_{i=1}^{2} \omega_t^i \hat{l}_t^i\right)\left(\frac{\lambda_f}{\sigma_f} dZ_f^R + \frac{1}{\sigma_s} dZ_{s1}^R + \frac{1}{\sigma_s} dZ_{s2}^R\right),$$

where $\hat{l}_t^R$ is the econometrician's belief about l_t, and dZ_f^R, dZ_{s1}^R, and dZ_{s2}^R, defined in equations (39)-(41), are independent information shocks in the econometrician's probability measure.

Proposition 5 shows that from the view point of the econometrician the market price of risk (risk premium per unit of risk) for the aggregate production shock dZ_I is σ_I. The market prices of risk for the three information shocks related to l_t (dZ_f^R, dZ_{s1}^R, and dZ_{s2}^R) are proportional to $\hat{l}_t^R - \sum_{i=1}^{2} \omega_t^i \hat{l}_t^i$, the difference between the econometrician's belief about l_t and the wealth weighted average belief of group-1 and group-2 agents. If the two groups' wealth weighted average belief about l_t happens to equal the econometrician's, the instantaneous risk premia for the information shocks are zero. However, as the two groups' beliefs and their relative wealth change over time, these risk premia also fluctuate.

It is simple to see how agents' belief fluctuation affects risk premia. When all agents become more optimistic about l_t than the econometrician, the current bond price would appear low to the econometrician. As a result, the econometrician expects a high bond return going forward, or equivalently, he perceives positive risk premia associated with the information shocks about l_t. It is important to note that agents' relative wealth fluctuation could lead to time-varying risk premia even without any belief fluctuation. The intuition works as follows. Suppose that the beliefs of the optimistic group and the pessimistic group both stay constant over time and their average is exactly that of the econometrician, which also stays constant. If the two groups have equal wealth, and therefore the difference between their wealth weighted average belief and the econometrician's belief is zero, then the current risk premia associated with the information shocks are exactly zero. However, after a positive shock hits the market, the optimistic group would profit from the pessimistic group through their existing positions against each other. As a result, the optimistic group's belief would carry a greater weight in the market, causing the two groups' wealth weighted average belief to rise above the econometrician's belief and the risk premia to become positive. Thus,

as long as the two groups hold heterogeneous expectations, the relative wealth fluctuation caused by their speculative positions could generate time-varying risk premia even without any fluctuation in their beliefs.

The time variation of risk premia in our model can help explain the failure of the expectations hypothesis. The expectations hypothesis posits that a representative agent in the bond market should be indifferent about the choice to invest his money in a long-term bond or in the short rate over the same period. A direct implication of this argument is that when the spread between the long rate and short rate is large, the long rate tends to rise further (or the long bond price tends to fall), because otherwise the representative agent could not be indifferent about the investment choice between the long-term bond and the short rate. Despite its intuitive appeal, this prediction is rejected by many empirical studies, e.g., Fama and Bliss (1987), Campbell and Shiller (1991) and, more recently, Cochrane and Piazzesi (2005). By regressing the monthly change of the yield of a zero coupon bond onto the spread between the bond yield and one-month short rate, Campbell and Shiller (1991) find negative coefficients for bonds with maturities ranging from 3 months to 10 years.

The literature often attributes the failure of the expectations hypothesis to time-varying risk premia. Dai and Singleton (2002) find that certain classes of affine term structure models with time-varying risk premia are able to match the aforementioned bond yield regression results. However, the economic determinants of the time-varying risk premia still remain elusive. Some studies, e.g., Wachter (2006) and Dai (2003), argue for the time-varying risk preference of the representative agent, while our model proposes a new mechanism based on agents' heterogeneous expectations. The intuition is quite simple. When agents' wealth weighted belief about l_t is high relative to the econometrician's, the yield spread between a long-term bond yield and the short rate tends to be large. Proposition 5 provides that the risk premia associated with the information shocks on l_t are negative in this case. Since the long-term bond price loads negatively on these shocks (bond prices are inversely related to l_t), the expected bond return from the econometrician's view point is high, or equivalently, the bond yield is expected to fall. Thus, the time-varying risk premia in our model lead to a negative relationship between the yield spread and future bond yield changes. In Section 5, we provide a simulation exercise to show that, with reasonable parameter values, this mechanism can

generate bond yield regression coefficients close to those obtained in empirical studies.

3.4 Convex price aggregation

Aggregating agents' heterogeneous bond valuations also directly affects the levels of equilibrium bond prices. Proposition 2 shows that the price of a bond in a homogeneous economy is a convex function of agents' beliefs about l_t:

$$B^H\left(\tau, f_t, \hat{l}_t^i\right) \sim e^{-a_l(\tau)\hat{l}_t^i}.$$

This property is a natural outcome of the fact that the bond price is a convex function of the bond yield. Since the price of the bond in a heterogeneous economy is a wealth weighted average of each group's bond valuation in the corresponding homogeneous economy, Jensen's inequality implies that agents' belief dispersion would increase the bond price.[7] We state this effect in Proposition 6, with the proof in Appendix A.8.

Proposition 6 *Bond prices increase in belief dispersion. Furthermore, the price increases are larger for bonds with longer maturities.*

It is important to note that the effect of belief dispersion on bond prices does not rely on short-sales constraints. The existing literature, e.g., Miller (1977), Harrison and Kreps (1978), Morris (1996), Chen, Hong and Stein (2002) and Scheinkman and Xiong (2003), has shown that when short-sales of assets are prohibited or costly, investors' heterogeneous beliefs would cause asset overvaluation because asset prices are determined by optimists' beliefs with pessimists sitting on the sideline. Our model shows that even without short-sales constraints, heterogeneous beliefs could still increase bond prices through the aggregation of agents' (convex) bond valuations.[8] We have also examined various numerical examples and find that, while this effect is small when agents' beliefs are close to each other, it could become large when agents' belief dispersion is great.[9]

[7] Note that even though agents' belief dispersion increases bond prices, shorting bonds does not provide an arbitrage profit. This is because that bond prices fluctuate randomly before maturities and the interim price volatility is particularly high when belief dispersion is larger, as shown in Proposition 4.

[8] Yan (2006) analyzes a similar mechanism on the aggregation of noise trading.

[9] To save space, we do not report these examples in the paper, but they are available upon request.

There is some evidence supporting the effect of heterogeneous expectations on bond prices. Bomberger and Frazer (1981) examine the relationship between long-term interest rates and dispersion of inflation forecasts in the Livingston survey data. They find that the 3 to 5-year rate and 10-year rate are both negatively related to the dispersion in inflation forecasts. Their result implies that belief dispersion increases bond prices, thus is consistent with our model.

4 Reconciling with Representative-Agent Models

Standard results suggest that we can construct a representative agent to replicate price dynamics in a complete-markets equilibrium with heterogeneous agents. Does this mean that we can simply focus on the representative agent's belief process and ignore the heterogeneity between agents? This section explains why the answer is no.

We could construct a representative agent model to replicate the above equilibrium.[10] If we restrict the representative agent to having the same logarithmic preference as the group-1 and group-2 agents, we obtain the same equilibrium as before by "twisting" the representative agent's belief, as summarized in the following proposition with a proof in Appendix A.9.

Proposition 7 *Suppose that we want to construct a representative agent model to replicate the equilibrium in Section 2, and that the representative agent has the same logarithmic preference as agents in the heterogeneous economy. Then, at any point of time, the representative agent's belief about* l_t, $\hat{l}_t^A$, *has to be the wealth weighted average belief of group*-1 *and group*-2 *agents:*

$$\hat{l}_t^A = \omega_t^1 \hat{l}_t^1 + \omega_t^2 \hat{l}_t^2. \tag{30}$$

It is important to stress that the representative agent's belief must equal the wealth weighted average belief not only at one point of time, but also at all future points. Thus, over time, the representative agent's belief would change in response to not only the belief fluctuation of each group, but also to the relative wealth fluctuation caused by trading between the two groups. Note that the relative wealth fluctuation can be affected by some factors, which are *unrelated* to l_t. In this case, although any rational Bayesian investor's belief

[10] See Jouini and Napp (2005) for a recent discussion on the existence of a "consensus" belief for the representative agent in markets with agents holding heterogeneous beliefs.

about l_t should not respond to these factors, Proposition 7 suggests that the representative agent's belief would respond to these unrelated factors because they affect the relative wealth distribution.[11]

Another issue concerns the interpretation of the uncertainty faced by this constructed representative agent. Intuitively, in the presence of agents with heterogeneous expectations, there are two distinct concepts: belief dispersion and uncertainty. Belief dispersion captures the interpersonal variation in expectations, while uncertainty represents the intrapersonal variation.[12] However, these two concepts collapse into one when we construct the representative agent. We illustrate this point by applying Ito's lemma to equation (30) to derive the representative agents' belief dynamics from the econometrician's view point:

$$\begin{aligned} d\hat{l}_t^A &= -\lambda_l\left(\hat{l}_t^R-\bar{l}\right)dt-\left[\lambda_l+\left(\frac{\lambda_f^2}{\sigma_f^2}+\frac{2}{\sigma_s^2}\right)\left(\bar{\gamma}_l+\frac{g_l^2\eta_t}{(\eta_t+1)^2}\right)+\frac{\phi_l\sigma_l}{\sigma_s}\right]\left(\hat{l}_t^A-\hat{l}_t^R\right)dt \\ &\quad+\frac{\lambda_f}{\sigma_f}\left(\bar{\gamma}_l+\frac{g_l^2\eta_t}{(\eta_t+1)^2}\right)d\hat{Z}_f^R+\frac{1}{\sigma_s}\left(\bar{\gamma}_l+\frac{g_l^2\eta_t}{(\eta_t+1)^2}+\frac{\eta_t}{\eta_t+1}\phi_l\sigma_l\sigma_s\right)d\hat{Z}_{s1}^R \\ &\quad+\frac{1}{\sigma_s}\left(\bar{\gamma}_l+\frac{g_l^2\eta_t}{(\eta+1)^2}+\frac{1}{\eta_t+1}\phi_l\sigma_l\sigma_s\right)d\hat{Z}_{s2}^R \end{aligned} \tag{31}$$

where $d\hat{Z}_f^R$, $d\hat{Z}_{s1}^R$, and $d\hat{Z}_{s2}^R$ are independent information shocks to the econometrician.

One fact we extract from the above formula is that the uncertainty faced by this representative agent reflects both individual investors' uncertainty and the belief dispersion among them. More precisely, we can compare the representative agent's belief process with that of an individual agent in equation (8). The representative agent's response coefficient to the information shock in df_t is $\frac{\lambda_f}{\sigma_f}\left(\bar{\gamma}_l+\frac{g_l^2\eta_t}{(\eta_t+1)^2}\right)$, while a group-$i$ agent's response coefficient is $\frac{\lambda_f}{\sigma_f}\bar{\gamma}_l$. The difference in these two coefficients suggests that we can interpret $\bar{\gamma}_l+\frac{g_l^2\eta_t}{(\eta_t+1)^2}$ as the

[11] For example, when we extend our model to incorporate price inflation and agents' speculation about future inflation rates in Appendix B, the two groups' relative wealth also fluctuates with shocks related to future inflation rates. In this case, the representative agent's belief about future technology returns would also respond to these inflation related shocks, even though they contain no information about future technology returns.

[12] Belief dispersion is often taken for granted as a symptom of greater uncertainty. However, these are two distinct concepts. However, Zarnowitz and Lambros (1987) clarify this conceptual difference, and empirically examine it using survey data from the Survey of Professional Forecasters. Since this survey also asks respondents to supplement their point estimates with estimates of the probability that GDP and the implicit price deflator will fall into various ranges, Zarnowitz and Lambros measure uncertainty from these probability estimates. By comparing the uncertainty measure with measures of interpersonal forecast dispersion, they find only weak evidence that uncertainty and belief dispersion are positively correlated.

representative agent's uncertainty level. Hence, the representative agent's uncertainty reflects both individual investors' uncertainty $\bar{\gamma}$, and the belief dispersion among them $|g_l|$. When the belief dispersion increases, the representative agent would act as if there was greater uncertainty even though the uncertainty faced by each agent remains unchanged. The same effect also exists in the representative agents' response coefficients to the information shocks in dS_1 and dS_2.

In summary, we could construct a representative agent model to replicate the model with heterogeneous expectations. However, this does not mean that the effects of heterogeneous expectations are not important. First, one has to be cautious when interpreting the constructed representative agent's belief process as representing a realistic learning process. In the presence of heterogenous beliefs, the representative agent's belief about a fundamental variable has to respond to informationally *irrelevant* factors. Moreover, the "uncertainty" faced by this representative agent reflects both individual investors' uncertainty and the belief dispersion among them. Second, incorporating agents' heterogeneous expectations and the resulting speculation and wealth fluctuation can help resolve several challenges to standard representative-agent models, such as large bond yield volatility and time-varying risk premia, as we discussed in the previous section.

5 Calibration

In this section, we illustrate the impact of agents' heterogeneous expectations on bond markets by simulating 50 years of monthly bond yield data based on a set of calibrated model parameters. In particular, we highlight the magnitudes of the volatility amplification effect and of the bond-yield regression result.

In our model, the short rate process is independent of agents' heterogeneous expectations of future rates. Applying Ito's lemma to the short rate ($r_t = f_t - \sigma_I^2$) provides that

$$dr_t = -\lambda_f[r_t - (l_t - \sigma_I^2)]dt + \sigma_f dZ_f.$$

The short rate mean-reverts to a time-varying long-run mean $l_t - \sigma_I^2$. Baldduzzi, Das and Foresi (1998) and Fama (2006) estimate two-factor interest rate models with the same structure as described above. Baldduzzi, Das and Foresi find that the long-run mean of the short

rate moves slowly with a mean-reversion parameter as low as 0.03. Fama argues that this long-run mean process might be nonstationary or have a mean reversion parameter close to zero. Since the mean-reversion parameter of this long-run mean process corresponds to λ_l, we choose λ_l to be 0.02, as a compromise between these two studies. This number implies that it takes $\log(2)/\lambda_l = 34.66$ years for the effect of a shock to the long-run mean of the short rate to die out by half. Baldduzzi, Das and Foresi also show that the mean-reversion parameter of the short rate (λ_f in our model) ranges from 0.2 to 3 in different sample periods between 1952 and 1993. Thus, we choose a value of 1.5 for λ_f. This number implies that it takes $\log(2)/\lambda_f = 0.46$ year for the difference between the short rate and its long-run mean to converge by half.[13]

We choose $\sigma_f = 1.2\%$ to match the short rate volatility in the data, and set $\sigma_l = 1.5\%$ so that the implied volatility of each agent's belief about l_t is 0.44% per month, in the middle of the range from 0.1% to 0.6% estimated by Balduzzi, Das, and Foresi (1998). Furthermore, since σ_I measures agents' aggregate consumption volatility (Theorem 1), we choose $\sigma_I = 2\%$ to match the aggregate consumption volatility in the data.[14]

Parameters ϕ and σ_s directly affect the amount of belief dispersion between the two groups. We choose $\phi_l = 0.7$ and $\sigma_s = 5\%$ to generate some modest belief dispersion: In our simulated data, the average dispersion between the two groups, $|g_l|$, is only 0.67%. This amount is rather modest compared with the typical dispersion in survey forecasts of future inflation and GDP growth rates (see footnote 2 for examples). We choose the following initial conditions for our simulation. The two groups have an equal wealth share at $t = 0$, i.e., $\eta_0 = 1$; Both f_0 and l_0 start with their steady state value $\bar{l}$ and the two groups also share an identical prior belief equal to the steady value $\bar{l}$: $f_0 = l_0 = \hat{l}_0^1 = \hat{l}_0^2 = \bar{l} = 5\%$. All the model parameters are summarized below:

$$\begin{aligned} &\lambda_l = 0.02,\ \lambda_f = 1.5,\ \sigma_f = 1.2\%,\ \sigma_l = 1.5\%,\ \sigma_I = 2\%, \\ &\phi_l = 0.7,\ \sigma_s = 5\%,\ \eta_0 = 1,\ f_0 = l_0 = \hat{l}_0^1 = \hat{l}_0^2 = \bar{l} = 5\%. \end{aligned} \tag{32}$$

[13] Note that these two mean-reversion parameters are important for agents' belief dispersion effect. Intuitively, a larger λ_l parameter causes l_t to revert faster to its mean, therefore making agents' belief dispersion about l_t less important for bond prices; while a larger λ_f parameter causes f_t to revert faster to l_t, therefore making agents' belief dispersion about l_t more important for bond prices.

[14] One could also choose σ_I to match the volatility of the aggregate production. This would have little or no impact on the volatility amplification effect and the bond-yield regression result.

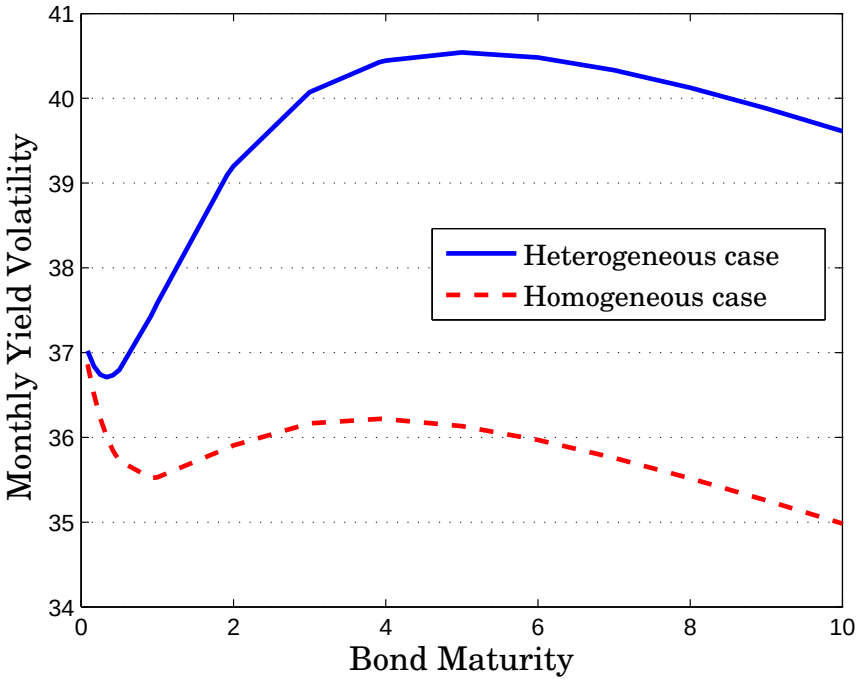

Figure 1: The term structure of bond yield volatility. This figure is based on a simulation exercise using 50 years of bond yields, using parameters specified in equation (32). The volatility is monthly volatility measured in basis points. The solid line plots the volatility curve in a heterogeneous economy with two groups of agents holding different beliefs, while the dash line plots the volatility curve in a homogeneous economy with a representative agent holding the *equal* weighted average belief of the two groups in the heterogeneous economy.

Based on these model parameters, we simulate a heterogeneous economy with two groups of agents, as described in our model, for 50 years at a daily interval.[15] The length of 50 years roughly matches the sample duration used in most empirical studies of the yield curve. We extract bond yields for various maturities at the end of each month. The solid line in Figure 1 plots the monthly bond yield volatility for different maturities from zero to 10 years. As the maturity increases from zero to five year, the yield volatility increases from 37 to above 40 basis points. As the maturity further increases, the yield volatility then starts to fall slightly. The magnitude and shape of this volatility curve is similar to those estimated in Piassezi (2005) and Dai and Singleton (2003).

[15] We also perform the simulation 1000 times and the average results across the 1000 paths are similar those reported here.

To further illustrate the volatility amplification effect discussed in Section 3.2, we compute the volatility curve in a homogeneous economy in which the representative agent holds the equal weighted average belief of the two groups in the simulated heterogeneous economy. Note that the representative agent's belief reflect the changes in the two groups' beliefs, but not their relative wealth fluctuation. As a result, the volatility curve in the homogeneous economy does not capture the volatility amplification effect caused by the two groups' relative wealth fluctuation. The dashed line in Figure 1 plots the volatility curve in the homogeneous economy. While this line maintains a similar shape as the solid line, it is always below the solid line. The difference between the solid and dashed lines measures the volatility amplification effect. This effect is small at short maturities, but increases dramatically from zero to near 5 basis points per month as maturity increases from zero to 5 years. This number shows that the volatility amplification caused by agents' relative wealth fluctuation could be economically meaningful even for a modest amount of belief dispersion.

Based on the simulated bond yield data in both the heterogeneous and homogeneous economies, we further regress the change over the next month in the yield of a $n-$month zero coupon bond onto the yield spread between the yield and 1-month rate:

$$Y_{t+1}(n-1) - Y_t(n) = \alpha_n + \beta_n \frac{Y_t(n) - Y_t(1)}{n-1}, \tag{33}$$

where $Y_t(n)$ is the n-month yield at month t, α_n is the regression constant, and β_n is the regression coefficient. This regression is directly motivated by the expectations hypothesis and has been examined by numerous empirical studies, e.g., Campbell and Shiller (1991). Intuitively, $\frac{Y_t(n)-Y_t(1)}{n-1}$ represents the excess yield from holding the n-month bond over the 1-month rate each month. The expectations hypothesis suggests that a representative investor in the bond markets must be indifferent about investing in the bond or in the short rate, thus he must expect the bond price to fall or the bond yield to rise by the same amount over the next month:

$$\frac{Y_t(n) - Y_t(1)}{n-1} = E_t[Y_{t+1}(n-1) - Y_t(n)].$$

Hence, the expectations hypothesis provides a null hypothesis for the regression exercise:

$$\beta_n = 1.$$

n	2	3	6	12	24	48	120
	A. Results from Campbell-Lo-MacKinlay						
β_n	0.003	−0.145	−0.835	−1.435	−1.448	−2.262	−4.226
s.e.	(0.191)	(0.282)	(0.442)	(0.599)	(1.004)	(1.458)	(2.076)
	B. Results from our simulation (heterogeneous economy)						
β_n	−0.644	−0.678	−0.786	−1.028	−1.574	−2.693	−4.644
s.e.	(0.359)	(0.372)	(0.417)	(0.524)	(0.777)	(1.321)	(2.764)
	C. Results from our simulation (homogeneous economy)						
β_n	−0.325	−0.344	−0.406	−0.546	−0.859	−1.440	−1.932
s.e.	(0.416)	(0.429)	(0.473)	(0.577)	(0.820)	(1.336)	(2.595)

Table 1: The coefficients of yield change regressions. This table reports the β_n coefficients and their standard errors of regressions in equation (33) for bond maturities of 2 months, 3 months, 6 months, 12 months, 24 months, 48 months and 120 months. Panel A is taken from Table 10.3 of Campbell, Lo and MacKinlay (1997), which uses U.S. treasury bond yield data from 1952-1991. Panel B uses the 50-year bond yield data extracted from our simulation of a heterogeneous economy with two groups of agents holding different beliefs. Panel C uses the yield data constructed from a homogeneous economy with a representative agent holding the *equal* weighted average belief of the two groups in the heterogeneous economy.

Table 1 reports the regression coefficients. Panel A takes the empirical results from Table 10.3 of Campbell, Lo and MacKinlay (1997), which uses 40 years of U.S. treasury bond yield data from 1952-1991. The panel shows that β_n starts with a value of 0.003 for 2-month yield, and then monotonically decreases as the bond maturity increases to 3 months, 6 months, 1 year, 2 years, 4 years and 10 years. β_n eventually takes a value of -4.226 for 10-year yield. All these coefficients are significantly different from 1 (the null), and the coefficient of 10-year yield is significantly negative. Taken together, these regression coefficients clearly reject the expectations hypothesis. Panel B reports the regression results using the 50-year bond yield data from our simulation of the heterogeneous economy. The regression coefficient decreases monotonically from -0.644 to -4.644 as the bond maturity increases from 2 months to 10 years, with a similar trend and magnitude to that in Panel A. Therefore, extending a standard asset pricing model with heterogeneous expectations offers a potential explanation for the failure of the expectations hypothesis in actual data.

To further examine the source of this result, in Panel C, we report the regression results

using bond yield data constructed from the homogeneous economy in which the representative agent holds the equal weighted average belief. In this case, the regression coefficient ranges from -0.325 to -1.932 when the bond maturity increases from 2 months to 10 years. The magnitude of the difference in the corresponding coefficients in Panels B and C is substantial. This difference also becomes statistically significant when we increase the length of the simulation period. As we discussed before, the simulated homogeneous economy fails to account for the relative wealth fluctuation between the two groups in the simulated heterogeneous economy. The difference in the regression coefficients between the two economies shows that the relative wealth fluctuation between the two groups plays an important role in generating the regression result in Panel B.

6 Conclusion and Further Discussion

This paper provides a dynamic equilibrium model of bond markets, in which two groups of agents hold heterogeneous expectations about future economic conditions. Heterogeneous expectations motivate agents to take speculative positions against each other. We are able to solve the equilibrium in a closed form. In particular, the price of a bond equals the wealth weighted average of bond prices in homogeneous economies, in each of which only one group of agents is present. Our model shows that heterogeneous expectations can not only lead to speculative trading, but can also help resolve several challenges facing standard representative-agent models of the yield curve. First, the relative wealth fluctuation between the two groups of agents caused by their speculative positions amplifies bond yield volatility, thus providing an explanation to the "excessive volatility puzzle" in the bond markets. In addition, the fluctuation in the two groups' expectations and relative wealth also generates time-varying risk premia, which in turn helps explain the failure of the expectation hypothesis. These implications, essentially induced by trading between agents, highlight the importance of incorporating heterogeneous expectations into economic analysis of bond markets.

Our model also provides a tractable but non-affine yield curve structure, which is ready for econometric estimation. This structure simultaneously embeds stochastic volatility and time-varying risk premia. Both features, as emphasized by Dai and Singleton (2003) and Duffee (2002), are crucial for capturing the actual yield curve dynamics. Our model provides

several testable implications of belief dispersion. First, higher belief dispersion increases bond market trading volume. Second, higher belief dispersion increases bond yield volatility and reduces bond market liquidity. Third, in an economy or a time period with greater belief dispersion among agents, the spread between long-term bond yield and short rate has a stronger predictive power for future yield changes. Finally, higher belief dispersion reduces bond yields, especially for bonds with longer maturities and when belief dispersion is large.

There is now a burgeoning empirical literature analyzing the effects of agents' heterogeneous beliefs on stock markets (see Hong and Stein (2006) for a recent survey of these studies). There is, however, little effort analyzing the effects on bond markets. As we discussed in the introduction, a substantial amount of belief dispersion exists in various surveys of agents' expectations of future economic conditions. These survey data invite future studies of the impacts of belief dispersion on bond markets.

The existence of heterogeneous expectations also has potential implications for monetary policies. Usually, a monetary authority only directly controls the overnight interest rate. For the overnight interest rate to affect long term interest rates and other prices, the links rely almost entirely on market expectations for the future course of short-term rates. Many monetary economists have pointed out the importance of managing market expectations in monetary policies, e.g., Blinder (1998) and Bernanke (2004). Consistent with this view, our model highlights that dispersion in market expectations can directly affect long-term interest rates and increase their volatility. Heterogeneous expectations could also affect the broad economy. For instance, a recent study by Piazzesi and Schneider (2006b) argues that, due to collateral constraints, households' heterogeneous expectations about future inflation rates can lead to housing booms. Thus, if the objective of monetary authorities is to stabilize prices, our model suggests that they should pay close attention to the dispersion in market expectations, and reduce this dispersion in their capacity.

A Some Proofs

A.1 Proof of Theorem 1

The market clearing conditions require that the aggregate investment to the risky technology is equal to the total wealth in the economy:

$$\sum_{i=1}^{2} x_I^i(t) W_t^i = W_t.$$

By substituting agents' investment strategy in equation (20) and dividing both sides by W_t, we obtain that

$$\frac{f_t - r_t}{\sigma_I^2} \sum_{i=1}^{2} \omega_t^i = 1.$$

Since $\sum_{i=1}^{2} \omega_t^i = 1$, we have that $r_t = f_t - \sigma_I^2$.

The market clearing conditions also require that the aggregate investment to the security f is zero:

$$\sum_{i=1}^{2} x_f^i(t) W_t^i = 0.$$

By substituting agents' investment strategy in equation (20) and dividing both sides by W_t, we obtain that

$$\sum_{i=1}^{2} \omega_t^i \frac{\hat{\mu}_f^i - r_t}{\sigma_f^2} = \sum_{i=1}^{2} \omega_t^i \frac{\mu_f(t) - \lambda_f(f_t - \hat{l}_t^i) - r_t}{\sigma_f^2} = 0.$$

Thus,

$$\mu_f(t) = r_t + \sum_{i=1}^{2} \omega_t^i \lambda_f (f_t - \hat{l}_t^i) = r_t + \lambda_f f_t - \lambda_f \sum_{i=1}^{2} \omega_t^i \hat{l}_t^i.$$

Following a similar procedure, we can also derive $\mu_{s1}(t)$ and $\mu_{s2}(t)$.

Since the risky technology is the only storage tool in the economy and every agent consumes a fraction β of his wealth, the aggregate wealth fluctuates according to

$$\frac{dW_t}{W_t} = dI_t/I_t - \beta dt = (f_t - \beta)\, dt + \sigma_I dZ_I(t).$$

A.2 Proof of Proposition 1

Under group-2 agents' probability measure, applying Ito's lemma to η_t we obtain

$$\frac{d\eta_t}{\eta_t} = \frac{dW_t^1}{W_t^1} - \frac{dW_t^2}{W_t^2} + \left(\frac{dW_t^2}{W_t^2}\right)^2 - \left(\frac{dW_t^1}{W_t^1}\right)\left(\frac{dW_t^2}{W_t^2}\right). \tag{34}$$

By substituting group-2 agents' consumption and investment strategies into equation (19), we obtain their wealth process:

$$\frac{dW_t^2}{W_t^2} = \left[r_t - \beta + \left(\frac{f_t - r_t}{\sigma_I} \right)^2 + \left(\vec{\mu}_t^2 - r_t \right)' \cdot \Sigma^{-2} \cdot \left(\vec{\mu}_t^2 - r_t \right) \right] dt + \left(\vec{\mu}_t^2 - r_t \right)' \cdot \Sigma^{-1} \cdot d\vec{Z}^2(t) .$$

By substituting group-1 agents' consumption and investment strategies into equation (19) and expressing the security returns in group-2 agents' probability measure, we obtain group-1 agents' wealth process as

$$\frac{dW_t^1}{W_t^1} = \left[r_t - \beta + \left(\frac{f_t - r_t}{\sigma_I} \right)^2 + \left(\vec{\mu}_t^1 - r_t \right)' \cdot \Sigma^{-2} \cdot \left(\vec{\mu}_t^2 - r_t \right) \right] dt + \left(\vec{\mu}_t^1 - r_t \right)' \cdot \Sigma^{-1} \cdot d\vec{Z}^2(t) .$$

By substituting $\frac{dW_t^2}{W_t^2}$ and $\frac{dW_t^1}{W_t^1}$ into equation (34), we obtain

$$\begin{aligned} \frac{d\eta_t}{\eta_t} &= \left[\left(\vec{\mu}_t^1 - \vec{\mu}_t^2 \right)' \cdot \Sigma^{-2} \cdot \left(\vec{\mu}_t^2 - r_t \right) \right] dt + \left(\vec{\mu}_t^1 - \vec{\mu}_t^2 \right)' \cdot \Sigma^{-1} \cdot d\vec{Z}^2(t) \\ &\quad + \left[\left(\vec{\mu}_t^2 - r_t \right)' \cdot \Sigma^2 \cdot \left(\vec{\mu}_t^2 - r_t \right) - \left(\vec{\mu}_t^1 - r_t \right)' \cdot \Sigma^2 \cdot \left(\vec{\mu}_t^2 - r_t \right) \right] dt \\ &= \left(\vec{\mu}_t^1 - \vec{\mu}_t^2 \right)' \cdot \Sigma^{-1} \cdot d\vec{Z}^2(t) . \end{aligned}$$

Equation (18) implies that

$$\left(\vec{\mu}_t^1 - \vec{\mu}_t^2 \right)' = (\lambda_f g_l, g_l, g_l) .$$

Then, by substituting $\left(\vec{\mu}_t^1 - \vec{\mu}_t^2 \right)'$ and Σ^{-1} into $\frac{d\eta_t}{\eta_t}$ above, we obtain equation (26).

For any random variable X_T with $E^1[X_T] < \infty$, we can define $Y_T = \frac{W_T^1}{W_t^1} X_T$. Suppose there is a financial security which is a claim to the cash flow Y_T. Then group-1 agents' valuation for this security is

$$E_t^1 \left[\frac{u'(c_T^1)}{u'(c_T^1)} Y_T \right] = E_t^1 \left[\frac{c_t^1}{c_T^1} Y_T \right] = E_t^1 \left[\frac{W_t^1}{W_T^1} Y_T \right] = E_t^1 \left[X_T \right] ,$$

where the second equality follows from these agents' consumption rule $c_t^1 = \beta W_t^1$. Similarly, investor 2's valuation for this security is

$$E_t^2 \left[\frac{u'(c_T^2)}{u'(c_T^2)} Y_T \right] = E_t^2 \left[\frac{c_t^2}{c_T^2} Y_T \right] = E_t^2 \left[\frac{W_t^2}{W_T^2} Y_T \right] = E_t^2 \left[\frac{\eta_T}{\eta_t} X_T \right] .$$

In the absence of arbitrage, group-1 and group-2 agents should have the same valuation:

$$E_t^1 \left[X_T \right] = E_t^2 \left[\frac{\eta_T}{\eta_t} X_T \right] .$$

A.3 Proof of Theorem 2

To derive asset prices, we start with agents' stochastic discount factor. When agents are homogeneous, they share the same stochastic discount factor, which is determined by their marginal utility of consumption. With a logarithmic preference, agents consume a fixed fraction of their wealth and the stochastic discount factor is inversely related to their aggregate wealth. More specifically, the stochastic discount factor, which we denote by M_t^H, is

$$\frac{M_t^H}{M_0^H} = e^{-\beta t}\frac{u'(c_t)}{u'(c_0)} = e^{-\beta t}\frac{c_0}{c_t} = e^{-\beta t}\frac{W_0}{W_t}. \tag{35}$$

When agents have heterogeneous beliefs about the probabilities of future states, they have different stochastic discount factors. However, in the absence of arbitrage, they have to share the same security valuations. For our derivation, we will use the probability measure and the stochastic discount factor of group-2 agents. Group-2 agents' consumption is

$$c_t^2 = \beta W_t^2 = \frac{\omega_t^2}{\sum_{i=1}^2 \omega_t^2}\beta W_t = \frac{1}{\eta_t + 1}\beta W_t.$$

The implied stochastic discount factor is

$$\begin{aligned}
\frac{M_t}{M_0} &= e^{-\beta t}\frac{u'(c_t^2)}{u'(c_0^2)} = e^{-\beta t}\frac{c_0^2}{c_t^2} = e^{-\beta t}\frac{W_0}{W_t}\frac{\eta_t + 1}{\eta_0 + 1} \\
&= e^{-\beta t}\frac{W_0}{W_t}\left(\frac{\eta_t}{\eta_0 + 1} + \frac{1}{\eta_0 + 1}\right) \\
&= e^{-\beta t}\frac{W_0}{W_t}\left(\omega_0^1\frac{\eta_t}{\eta_0} + \omega_0^2\right) \\
&= \left(\omega_0^1\frac{\eta_t}{\eta_0} + \omega_0^2\right)\frac{M_t^H}{M_0^H}.
\end{aligned}$$

Thus, at time t, the price of a financial security that pays off X_T at time T is

$$\begin{aligned}
P_t &= E_t^2\left[\frac{M_T}{M_t}X_T\right] \\
&= E_t^2\left[e^{-\beta(T-t)}\frac{W_t}{W_T}\left(\omega_t^1\frac{\eta_T}{\eta_t} + \omega_t^2\right)X_T\right] \\
&= \omega_t^1 E_t^2\left[e^{-\beta(T-t)}\frac{W_t}{W_T}\frac{\eta_T}{\eta_t}X_T\right] + \omega_t^2 E_t^2\left[e^{-\beta(T-t)}\frac{W_t}{W_T}X_T\right].
\end{aligned}$$

Since $\frac{\eta_T}{\eta_t}$ is the Randon-Nikodyn derivative of group-1 agents' probability measure with respect to the measure of group-2 agents (Proposition 1),

$$E_t^2\left[e^{-\beta(T-t)}\frac{W_t}{W_T}\frac{\eta_T}{\eta_t}X_T\right] = E_t^1\left[e^{-\beta(T-t)}\frac{W_t}{W_T}X_T\right].$$

Thus,

$$\begin{aligned} P_t &= \omega_t^1 E_t^1\left[e^{-\beta(T-t)}\frac{W_t}{W_T}X_T\right] + \omega_t^2 E_t^2\left[e^{-\beta(T-t)}\frac{W_t}{W_T}X_T\right] \\ &= \omega_t^1 E_t^1\left[\frac{M_T^H}{M_t^H}X_T\right] + \omega_t^2 E_t^2\left[\frac{M_T^H}{M_t^H}X_T\right], \end{aligned}$$

where $E_t^i\left[\frac{M_T^H}{M_t^H}X_T\right]$ is the price of the security in a homogeneous economy where only group-i agents are present.

A.4 Proof of Proposition 2

The price of the bond in a homogeneous economy has the following function form:

$$B_t^i = B^H\left(\tau, f_t, \hat{l}_t^i\right). \tag{36}$$

The bond's return has to satisfy the following relationship with the stochastic discount factor in the homogeneous economy:

$$E_t^i\left(\frac{dB^H}{B^H}\right) + E_t^i\left(\frac{dM_t^H}{M_t^H}\right) + E_t^i\left(\frac{dB^H}{B^H}\frac{dM_t^H}{M_t^H}\right) = 0. \tag{37}$$

Applying Ito's lemma to equations (35) and (36) provides

$$\frac{dM_t^H}{M_t^H} = \left(-f_t + \sigma_I^2\right)dt - \sigma_I dZ_I,$$

and

$$\begin{aligned} \frac{dB^H}{B^H} &= \left\{-\frac{B_\tau^H}{B^H} - \lambda_f(f_t - \hat{l}_t^i)\frac{B_f^H}{B^H} - \lambda_l(\hat{l}_t^i - \bar{l})\frac{B_l^H}{B^H} + \frac{1}{2}\sigma_f^2\frac{B_{ff}^H}{B^H}\right. \\ &\quad \left. + \frac{1}{2}\left[\sigma_l^2 - 2\lambda_l\bar{\gamma}_l\right]\frac{B_{ll}^H}{B^H} + \lambda_f\bar{\gamma}_l\frac{B_{fl}^H}{B^H}\right\}dt + \left[\rho_\theta\sigma_\theta\frac{B_\theta^H}{B^H} + \rho_\pi\sigma_\pi\frac{B_\pi^H}{B^H}\right]dZ_I(t) \\ &\quad + \left(\sigma_f\frac{B_f^H}{B^H} + \frac{\lambda_f\bar{\gamma}_l}{\sigma_f}\frac{B_l^H}{B^H}\right)d\widehat{Z}_f^i(t) + \left(\frac{\bar{\gamma}_l}{\sigma_s} + \phi_l\sigma_l\right)\frac{B_l^H}{B^H}d\widehat{Z}_{s1}^i + \frac{\bar{\gamma}_l}{\sigma_s}\frac{B_l^H}{B^H}d\widehat{Z}_{s2}^i. \end{aligned}$$

By substituting $\frac{dB^H}{B^H}$ and $\frac{dM_t^H}{M_t^H}$ into equation (37), we obtain the following equation:

$$0 = -\frac{B_\tau^H}{B^H} - \lambda_f(f_t - \hat{l}_t^i)\frac{B_f^H}{B^H} - \lambda_l(\hat{l}_t^i - \bar{l})\frac{B_l^H}{B^H} + \frac{1}{2}\sigma_f^2\frac{B_{ff}^H}{B^H} + \frac{1}{2}\left(\sigma_l^2 - 2\lambda_l\bar{\gamma}_l\right)\frac{B_{ll}^H}{B^H} + \lambda_f\bar{\gamma}_l\frac{B_{fl}^H}{B^H} - f_t + \sigma_I^2 \tag{38}$$

We conjecture the following solution

$$B^H\left(\tau, f_t, \hat{l}_t^i\right) = e^{-a_f(\tau)f_t - a_l(\tau)\hat{l}_t^i - b(\tau)}.$$

By substituting the conjectured solution into the differential equation in (38) and collecting common terms, we obtain the following algebra equation:

$$\begin{aligned} 0 &= \left[a_f'(\tau) + \lambda_f a_f(\tau) - 1\right] f_t + \left[a_l'(\tau) - \lambda_f a_f(\tau) + \lambda_l a_l(\tau)\right] \hat{l}_t^i \\ &\quad + [b'(\tau) - \lambda_l \bar{l} a_l(\tau) + \frac{1}{2}\sigma_f^2 a_f(\tau)^2 + \frac{1}{2}\left(\sigma_l^2 - 2\lambda_l\bar{\gamma}_l\right) a_l(\tau)^2 + \lambda_f\bar{\gamma}_l a_f(\tau)a_l(\tau) + \sigma_I^2]. \end{aligned}$$

Since this equation has to hold for any values of f_t and $\hat{l}_t^i$, their coefficients must be zero. Thus, $a_f(\tau)$, $a_l(\tau)$, and $b(\tau)$ satisfy the following differential equations

$$\begin{aligned} &a_f'(\tau) + \lambda_f a_f(\tau) - 1 = 0, \\ &a_l'(\tau) - \lambda_f a_f(\tau) + \lambda_l a_l(\tau) = 0, \\ &b'(\tau) - \lambda_l \bar{l} a_l(\tau) + \frac{1}{2}\sigma_f^2 a_f(\tau)^2 + \frac{1}{2}\left(\sigma_l^2 - 2\lambda_l\bar{\gamma}_l\right) a_l(\tau)^2 + \lambda_f\bar{\gamma}_l a_f(\tau)a_l(\tau) + \sigma_I^2 = 0, \end{aligned}$$

subject to the boundary conditions

$$a_f(0) = a_l(0) = b(0) = 0.$$

Solving these equations provides the bond price formula given in Proposition 2.

A.5 Proof of Proposition 3

Agents' belief dispersion about l_t leads to speculative positions in risky securities f, S_1 and S_2. We can directly compute group-2 agents' positions in these securities. Equation (20) shows that their position in security f is

$$n_f(t) = W_t^2 \frac{\hat{\mu}_f^2(t) - r_t}{\sigma_f^2} = W_t \omega_t^2 \frac{\mu_f(t) - \lambda_f\left(f_t - \hat{l}_t^2\right) - r_t}{\sigma_f^2}.$$

By substituting in $\mu_f(t)$ from Theorem 1, we obtain that

$$n_f(t) = \frac{\lambda_f}{\sigma_f^2} W_t \frac{\eta_t}{(\eta_t + 1)^2} g_l(t).$$

Similarly, we can derive group-2 agents' positions in securities S_1 and S_2:

$$n_{s1}(t) = n_{s2}(t) = \frac{1}{\sigma_s^2} W_t \frac{\eta_t}{(\eta_t + 1)^2} g_l(t).$$

Note that group-2 agents' positions in all these securities are proportional to the same random variable $W_t \frac{\eta_t}{(\eta_t+1)^2} g_l(t)$. This implies that these positions have the same time-series properties. As the belief dispersion $|g_l(t)|$ widens, group-2 agents take larger positions in securities f, S_1 and S_2.

Since group-2 agents have to trade with group-1 agents to change their positions, the absolute values of the changes in group-2 agents' positions determine trading volume in the bond markets. In our model, the changes in agents' positions follow diffusion processes. It is well known that diffusion processes have infinite variation over a given time interval. However, since actual trading occurs in discrete time, it is reasonable to analyze trading volume through the change in agents' positions across a finite time interval. Since the absolute value of a realized position change across a finite but small interval is finite and on average increases with the volatility of the position change, this motivates us to use the volatility as a measure of trading volume.

Here, we examine the change in group-2 agents' position in security f, $dn_f(t)$, whose diffusion terms are

$$\frac{\lambda_f}{\sigma_f^2}\left[\frac{\eta_t}{(\eta_t+1)^2} g_l(t)\, dW_t - W_t g_l(t)\frac{\eta_t - 1}{(\eta_t+1)^3} d\eta_t + W_t \frac{\eta_t}{(\eta_t+1)^2} dg_l(t)\right].$$

The fluctuation in the position is determined by the fluctuations in the aggregate wealth, in the wealth ratio between the two groups, and in the difference in agents' beliefs.

By deriving the diffusion processes of dW_t, $d\eta_t$ and $dg_l(t)$, and substituting them into the equation above, we can derive the variance of the position change as

$$\begin{aligned} Var\left[dn_f(t)\right] = & \left[\frac{\lambda_f^2\sigma_I^2}{\sigma_f^4}(W_t)^2\frac{\eta_t^2}{(\eta_t+1)^4} g_l^2(t) + \frac{\lambda_f^4}{\sigma_f^6}(W_t)^2\frac{(\eta_t-1)^2}{(\eta_t+1)^6} g_l^4(t)\right. \\ & \left. + \frac{2\lambda_f^2}{\sigma_s^2\sigma_f^4}(W_t)^2\frac{(\eta_t-1)^2}{(\eta_t+1)^6} g_l^4(t) + \frac{2\lambda_f^2\phi_l^2\sigma_l^2}{\sigma_f^4}(W_t)^2\frac{\eta_t^2}{(\eta_t+1)^4}\right] dt \end{aligned}$$

It is direct to see that the variance of the position change increases with $g_l^2(t)$. Thus, trading volume of security f increases with agents' belief dispersion.

A.6 Proof of Proposition 4

By the definition of bond yield $Y_t(\tau) = -\frac{1}{\tau}\log(B_t)$, its volatility is proportional to that of the bond return:

$$Vol[dY(\tau)_t] = \frac{1}{\tau} Vol(dB_t/B_t).$$

Applying Ito's lemma to equation (29) provides the following diffusion terms of $\frac{dB_t}{B_t}$:

$$
\begin{aligned}
&-\left[a_f(\tau)\sigma_f + a_l(\tau)\lambda_f\sigma_f^{-1}\bar{\gamma}_l - \frac{\lambda_f}{\sigma_f}\frac{\eta_t}{(\eta_t+1)}g_l(t)\frac{e^{-a_l(\tau)g_l(t)/2} - e^{a_l(\tau)g_l(t)/2}}{\eta_t e^{-a_l(\tau)g_l(t)/2} + e^{a_l(\tau)g_l(t)/2}}\right] d\hat{Z}_f^R \\
&-\left[a_l(\tau)\left(\sigma_s^{-1}\bar{\gamma}_l + \frac{\phi_l\sigma_l}{2}\right) - \frac{1}{\sigma_s}\frac{\eta_t}{(\eta_t+1)}g_l(t)\frac{e^{-a_l(\tau)g_l(t)/2} - e^{a_l(\tau)g_l(t)/2}}{\eta_t e^{-a_l(\tau)g_l(t)/2} + e^{a_l(\tau)g_l(t)/2}}\right]\left(d\hat{Z}_{s1}^R + d\hat{Z}_{s2}^R\right) \\
&-\phi_l\sigma_l\frac{a_l(\tau)}{2}\frac{\eta_t e^{-a_l(\tau)g_l(t)/2} - e^{a_l(\tau)g_l(t)/2}}{\eta_t e^{-a_l(\tau)g_l(t)/2} + e^{a_l(\tau)g_l(t)/2}}\left(d\widehat{Z}_{s1}^R - d\widehat{Z}_{s2}^R\right).
\end{aligned}
$$

Since the diffusion term in each row is independent to each other, we obtain

$$
\begin{aligned}
\left(\frac{dB_t}{B_t}\right)^2 &= \left[a_f(\tau)\sigma_f + a_l(\tau)\lambda_f\sigma_f^{-1}\bar{\gamma}_l + \frac{\lambda_f}{\sigma_f}\frac{\eta_t}{(\eta_t+1)}K_1(g_l)\right]^2 dt \\
&\quad +2\left[a_l(\tau)\left(\sigma_s^{-1}\bar{\gamma}_l + \frac{\phi_l\sigma_l}{2}\right) + \frac{1}{\sigma_s}\frac{\eta_t}{(\eta_t+1)}K_1(g_l)\right]^2 dt + \frac{\phi_l^2\sigma_l^2}{2}a_l^2(\tau)K_2(g_l)dt
\end{aligned}
$$

where

$$
K_1(g_l) = -g_l(t)\frac{e^{-a_l(\tau)g_l(t)/2} - e^{a_l(\tau)g_l(t)/2}}{\eta_t e^{-a_l(\tau)g_l(t)/2} + e^{a_l(\tau)g_l(t)/2}},
$$

and

$$
K_2(g_l) = \left[\frac{\eta_t e^{-a_l(\tau)g_l(t)/2} - e^{a_l(\tau)g_l(t)/2}}{\eta_t e^{-a_l(\tau)g_l(t)/2} + e^{a_l(\tau)g_l(t)/2}}\right]^2.
$$

Direct derivations of K_1 and K_2 provide that both of them increase as $|g_l|$ increases. Thus, the conditional variance of the bond return increases in the belief dispersion.

A.7 Proof of Proposition 5

We first derive the learning processes of an econometrician who uses the objective probability measure. We assume that the econometrician's belief distribution about l_t at time t is Gaussian and denote it by

$$
l_t|\left\{f_\tau, S_1(\tau), S_2(\tau)\right\}_{\tau=0}^t \sim N\left(\hat{l}_t^R, \bar{\gamma}_l^R\right),
$$

where $\hat{l}_t^R$ is the mean and $\bar{\gamma}_l^R$ is the stationary variance level. Since the econometrician knows the objective signal processes in equations (4) and (5), $\bar{\gamma}_l^R$ is the positive root to the following quadratic equation of γ:

$$
\left(\frac{\lambda_f^2}{\sigma_f^2} + \frac{2}{\sigma_s^2}\right)\gamma^2 + 2\lambda_l\gamma - \sigma_l^2 = 0,
$$

and the mean of his belief distribution follows

$$
d\hat{l}_t^R = -\lambda_l(\hat{l}_t^R - \bar{l})dt + \lambda_f\sigma_{fl}^{-1}\bar{\gamma}_l^R d\widehat{Z}_f^R + \sigma_s^{-1}\bar{\gamma}_l^R d\widehat{Z}_{s1}^R + \sigma_s^{-1}\bar{\gamma}_l^R d\widehat{Z}_{s2}^R,
$$

where

$$d\widehat{Z}_f^R = \frac{1}{\sigma_f}\left[df_t + \lambda_f(f_t - \hat{l}_t^R)dt\right], \tag{39}$$

$$d\widehat{Z}_{s1}^R = \frac{1}{\sigma_s}\left[dS_1(t) - \hat{l}_t^R dt\right], \tag{40}$$

$$d\widehat{Z}_{s2}^R = \frac{1}{\sigma_s}\left[dS_2(t) - \hat{l}_t^R dt\right], \tag{41}$$

are the surprises in the three corresponding sources of information. These surprises are independent standard Brownian motions in the econometrician's probability measure.

Standard results in asset pricing, e.g., Duffie (1996) and Cochrane (2001), state that the stochastic discount factor's drift rate is the negative of the risk free rate and its loading on each source of shock is the negative of the corresponding risk premium term in the equilibrium. Thus, to derive the stochastic discount factor process from the view point of the econometrician, we only need to compute the Shape ratios of the risky technology and the three financial securities in his probability measure.

According to equation (1), the Sharpe ratio corresponding to dZ_I is σ_I. To compute the Sharpe ratio of security f, we first express the f_t process in the econometrician's measure, using equation (39):

$$df_t = -\lambda_f\left(f_t - \hat{l}_t^R\right)dt + \sigma_f dZ_f^R.$$

By substituting this equation and equation (22) into equation (15), we obtain that

$$\begin{aligned}\frac{dp_f}{p_f} &= \left(r_t + \lambda_f f_t - \lambda_f \sum_{i=1}^{2}\omega_t^i \hat{l}_t^i\right)dt - \lambda_f\left(f_t - \hat{l}_t^R\right)dt + \sigma_f dZ_f^R \\ &= \left[r_t - \lambda_f\left(\sum_{i=1}^{2}\omega_t^i \hat{l}_t^i - \hat{l}_t^R\right)\right]dt + \sigma_f dZ_f^R.\end{aligned}$$

Thus, the Sharpe ratio corresponding to shock dZ_f^R is $-\frac{\lambda_f}{\sigma_f}\left(\sum_{i=1}^{2}\omega_t^i \hat{l}_t^i - \hat{l}_t^R\right)$. Following a similar procedure as above, we can derive the Sharpe ratios corresponding to dZ_{s1}^R and dZ_{s2}^R as $-\frac{1}{\sigma_s}\left(\sum_{i=1}^{2}\omega_t^i \hat{l}_t^i - \hat{l}_t^R\right)$. By combining these risk premium terms, we obtain the process of the stochastic discount factor given in Proposition 5.

A.8 Proof of Proposition 6

We define agents' wealth weighted average belief about l_t as

$$\hat{l}_t^A = \frac{\eta_t}{\eta_t + 1}\hat{l}_t^1 + \frac{1}{\eta_t + 1}\hat{l}_t^2.$$

Then, we can rewrite the bond price of a τ-year zero-coupon bond as

$$B_t = e^{-a_f(\tau)f_t - a_l(\tau)\hat{l}_t^A - b(\tau)} \left[\frac{\eta_t}{\eta_t + 1} e^{-\frac{1}{\eta+1}a_l(\tau)g_l(t)} + \frac{1}{1+\eta_t} e^{\frac{\eta}{\eta+1}a_l(\tau)g_l(t)} \right].$$

We define the expression in the bracket above as

$$K(g_l) = \frac{\eta_t}{\eta_t + 1} e^{-\frac{1}{\eta+1}a_l(\tau)g_l} + \frac{1}{1+\eta_t} e^{\frac{\eta}{\eta+1}a_l(\tau)g_l}$$

Direct differentiation provides that

$$K'(g_t) = -\frac{\eta_t}{(\eta_t+1)^2} a_l(\tau) \left[e^{-\frac{1}{\eta+1}a_l(\tau)g_l} - e^{\frac{\eta}{\eta+1}a_l(\tau)g_l} \right]$$

which is positive for $g_l > 0$ and is negative for $g_l < 0$. Thus, $K(g_l)$ increases as the difference in agents' beliefs widens. This result in turn provides that after controlling for agents' wealth weighted average belief $\hat{l}_t^A$, the bond price increases with agents' belief dispersion.

Furthermore, it is direct to see that the magnitude of $K'(g_t)$ increases monotonically with respect to $a_l(\tau)$. In addition, $a_l(\tau)$ increases with the bond maturity τ. Thus, the increase in bond price in response to the widening in g_l is larger for bonds with longer maturities.

A.9 Proof of Proposition 7

To replicate the price dynamics in the heterogeneous-agent economy, we need to make the representative agent's stochastic discount factor is the same as group-2 agents' after adjusting for the difference in their probability measures. That is, the representative agent's marginal utility have the following property in any future state:

$$u'(c_t^2) = \eta_t^A u'(c_t^A),$$

where $u'(c_t^2)$ is group-2 agents' marginal utility from consumption, $u'(c_t^A)$ is the representative agent's marginal utility, and η_t^A is the change of measure from the representative agent's measure to group-2 agents' measure. Therefore, η_t^A has the following property

$$\frac{d\eta_t^A}{\eta_t^A} = \left(\vec{\mu}_t^A - \vec{\mu}_t^2\right)' \Sigma^{-1} d\vec{Z}^2(t), \tag{42}$$

where $\vec{\mu}_t^A$ is the representative agent's expected returns

$$\vec{\mu}_t^A = \begin{pmatrix} \mu_f(t) - \lambda_f(f_t - \hat{l}_t^A) \\ \mu_{s1}(t) + \hat{l}_t^A \\ \mu_{s2}(t) + \hat{l}_t^A \end{pmatrix}.$$

Note that agents with a logarithmic preference always consume a fixed fraction of their wealth over time: $c^2 = \beta W_t^2$ and $c^A = \beta(W_t^1 + W_t^2)$. Thus, we can derive the difference in the probability measures of group-2 agents and the representative agents:

$$\eta_t^A = \frac{c_t^A}{c_t^2} = \frac{W_t^1 + W_t^2}{W_t^2} = \eta_t + 1.$$

This further implies that

$$d\eta_t^A = d\eta_t,$$

and

$$\frac{d\eta_t^A}{\eta_t^A} = \frac{\eta_t}{\eta_t^A}\frac{d\eta_t}{\eta_t} = \frac{\eta_t}{1+\eta_t}\frac{d\eta_t}{\eta_t}.$$

Substituting in the dynamics of $\frac{d\eta_t}{\eta_t}$ (Proposition 1), we obtain

$$\frac{d\eta_t^A}{\eta_t^A} = \left(\lambda_f \frac{\eta_t}{1+\eta_t} g_l \quad \frac{\eta_t}{1+\eta_t} g_l \quad \frac{\eta_t}{1+\eta_t} g_l \cdot \right) \Sigma^{-1} d\vec{Z}^2(t). \tag{43}$$

Comparing (42) and (43), we obtain

$$\hat{l}_t^A = \hat{l}_t^2 + \frac{\eta_t}{1+\eta_t} g_l = \frac{\eta_t}{1+\eta_t}\hat{l}_t^1 + \frac{1}{1+\eta_t}\hat{l}_t^2.$$

Note that $\frac{\eta_t}{1+\eta_t}$ and $\frac{1}{1+\eta_t}$ are the wealth shares of group-1 and group-2 agents.

B An Extension with Price Inflation

The model presented in the main context has only the real side of the economy, i.e., agents' consumptions and wealth are all measured in units of consumption good. In this section, we extend the model with price inflation and agents' heterogeneous expectations of future inflation rates. In particular, we assume that the dollar price of one unit of consumption good, p_t, changes over time according to an exogenous process:

$$\frac{dp_t}{p_t} = \pi_t dt, \tag{44}$$

where π_t is the inflation rate.

B.1 Time-varying inflation rate

The inflation rate π_t fluctuates according to a linear diffusion process:

$$d\pi_t = -\lambda_\pi(\pi_t - \theta_t)dt + \sigma_\pi \left[\sqrt{1-\rho_\pi^2} dZ_\pi(t) + \rho_\pi dZ_I(t)\right], \tag{45}$$

where λ_π is a parameter governing the speed that π_t reverts to its long-run mean θ_t, and σ_π is the volatility of π_t. The innovation to π_t has a correlation of ρ_π with $dZ_I(t)$, the innovation to the production growth. $Z_\pi(t)$ is a standard Brownian motion independent of other shocks in the economy. The long-run mean of the inflation rate, θ_t, is unobservable and follows an *Ornstein-Uhlenbeck* process:

$$d\theta_t = -\lambda_\theta(\theta_t - \bar{\theta})dt + \sigma_\theta \left[\sqrt{1-\rho_\theta^2}dZ_\theta(t) + \rho_\theta dZ_I(t)\right], \tag{46}$$

where λ_θ is a parameter governing the speed at which θ_t reverts to its long-run mean $\bar{\theta}$, and σ_θ is the volatility of θ_t. The innovation to θ_t has a correlation of ρ_θ with the innovation to the production growth. $Z_\theta(t)$ is a standard Brownian motion independent of other shocks in the economy. Note that θ_t determines future inflation rates. Intuitively, we can interpret θ_t as the central bank's inflation target, which is not directly observable. We introduce correlations between the innovations to the inflation related variables and the innovation to the production growth because such correlations are useful in capturing a positive slope of the yield curve, as suggested by Piazzesi and Schneider (2006a).

B.2 Agents' expectation of future inflation rates

Next, we discuss agents' expectations about future inflation rates. In addition to observing π_t, we assume that agents also receive two public signals $\Psi_1(t)$ and $\Psi_2(t)$ about the long-run mean of inflation rates θ_t. These signals have the following processes:

$$d\Psi_1(t) = \theta_t dt + \sigma_\Psi dZ_{\Psi 1}(t), \tag{47}$$

$$d\Psi_2(t) = \theta_t dt + \sigma_\Psi dZ_{\Psi 2}(t). \tag{48}$$

These two signals have the same noise volatility parameter σ_Ψ, but are subjective to independent noise, $Z_{\Psi 1}(t)$ and $Z_{\Psi 2}(t)$.

Similar to the setup of the real side, we assume that agents in the two groups have different perceptions about these two signal processes. Agents in group 1 believe that the innovation to $d\Psi_1(t)$ is partially correlated with the shock to $d\theta_t$, thus under-estimate the amount of noise in the signal. More specifically, they believe that $\Psi_1(t)$ has the following process:

$$d\Psi_1(t) = \theta_t dt + \sigma_\Psi \left[\phi_\theta dZ_\theta(t) + \sqrt{1-\phi_\theta^2}dZ_{\Psi 1}(t)\right], \tag{49}$$

where parameter $\phi_\theta \in [0,1]$ measures the fraction of the innovation to $d\Psi_1(t)$ that comes from $dZ_\theta(t)$. Symmetrically, group-2 agents under-estimate the noise in signal $\Psi_2(t)$, i.e.,

they believe that ϕ_θ fraction of the innovation to $d\Psi_2(t)$ comes from $dZ_\theta(t)$:

$$d\Psi_2(t) = \theta_t dt + \sigma_\Psi \left[\phi_\theta dZ_\theta(t) + \sqrt{1-\phi_\theta^2} dZ_{\Psi 2}(t) \right]. \tag{50}$$

In summary, while the two signals $\Psi_1(t)$ and $\Psi_2(t)$ follow processes (47) and (48) in the econometrician's mind, group-1 agents use equations (49) and (48), and group-2 agents use equations (47) and (50).

Agents' information set at time t about θ_t includes $\{\pi_\tau, \Psi_1(\tau), \Psi_2(\tau)\}_{\tau=0}^t$. Again, we assume that agents have Gaussian prior distributions about θ_t. As a result, their posterior distributions are also Gaussian. We focus on the stationary equilibrium, in which agents' belief variance has already reached its stationary level $\bar{\gamma}_\theta$, which is the positive root to the following quadratic equation of γ:

$$\left[\frac{\lambda_\pi^2}{(1-\rho_\pi^2)\sigma_\pi^2} + \frac{2}{\sigma_\Psi^2} \right] \gamma^2 + 2 \left(\lambda_\theta + \frac{\phi_\theta \sigma_\theta \sqrt{1-\rho_\theta^2}}{\sigma_\Psi} \right) \gamma - \left(1-\rho_\theta^2\right)\left(1-\phi_\theta^2\right)\sigma_\theta^2 = 0.$$

We denote $\hat{\theta}_t^i$ as the mean of group-i agents' posterior distribution about θ_t at time t ($i \in \{1,2\}$). The mean is determined by

$$\begin{aligned} d\hat{\theta}_t^i &= -\lambda_\theta(\hat{\theta}_t^i - \bar{\theta})dt + \rho_\theta \sigma_\theta dZ_I + \frac{1}{\sqrt{1-\rho_\pi^2}} \lambda_\pi \sigma_\pi^{-1} \bar{\gamma}_\theta d\widehat{Z}_\pi^i \\ &\quad + \sigma_\Psi^{-1} \left(\bar{\gamma}_\theta + \phi_\theta \sigma_\Psi \sigma_\theta \sqrt{1-\rho_\theta^2} \right) d\widehat{Z}_{\Psi i}^i + \sigma_\Psi^{-1} \bar{\gamma}_\theta d\widehat{Z}_{\Psi j}^i \end{aligned} \tag{51}$$

where $j \in \{0,1\}$ and $j \neq i$. $d\widehat{Z}_\pi^i$, $d\widehat{Z}_{\Psi i}^i$ and $d\widehat{Z}_{\Psi j}^i$ are "surprises" in the three sources of information:

$$\begin{aligned} d\widehat{Z}_\pi^i &= \frac{1}{\sigma_\pi \sqrt{1-\rho_\pi^2}} \left[d\pi_t - \rho_\pi \sigma_\pi dZ_I(t) + \lambda_\pi(\pi_t - \hat{\theta}_t^i)dt \right], \\ d\widehat{Z}_{\Psi i}^i &= \frac{1}{\sigma_\Psi} \left[d\Psi_i(t) - \hat{\theta}_t^i dt \right], \\ d\widehat{Z}_{\Psi j}^i &= \frac{1}{\sigma_\Psi} \left[d\Psi_j(t) - \hat{\theta}_t^i dt \right]. \end{aligned}$$

Note that $\widehat{Z}_\pi^i$, $\widehat{Z}_{\Psi 1}^i$ and $\widehat{Z}_{\Psi 2}^i$ are independent standard Brownian motions in group-i agents' probability measure. Group-i agents "over-react" to $d\widehat{Z}_{\Psi i}^i$, the surprise in signal Ψ_i, and thus have a different belief from group-j agents.

In group-i agents' probability measure, variables π_t, $\Psi_1(t)$ and $\Psi_2(t)$ follow

$$d\pi_t = -\lambda_\pi(\pi_t - \hat{\theta}_t^i)dt + \rho_\pi \sigma_\pi dZ_I(t) + \sigma_\pi \sqrt{1-\rho_\pi^2} d\widehat{Z}_\pi^i, \tag{52}$$

$$d\Psi_1 = \hat{\theta}_t^i dt + \sigma_\Psi d\hat{Z}_{\Psi 1}^i(t), \tag{53}$$

$$d\Psi_2 = \hat{\theta}_t^i dt + \sigma_\Psi d\hat{Z}_{\Psi 2}^i(t), \tag{54}$$

Thus, the difference in agents' beliefs about θ_t translates into different views about the dynamics of these variables and, subsequently, into different expectations of future inflation rates.

B.3 Speculation about future inflation

Agents' disagreement about future inflation rates would lead agents to additional speculative positions against each other. To complete the markets, we introduce three more financial securities to facilitate agents' trading need. For our analysis, we first derive the price processes and agents' wealth processes in units of real consumption good, and then derive prices of nominal bonds by adjusting for price inflation. The price processes of the three additional financial securities are

$$\frac{dp_\pi}{p_\pi} = \mu_\pi(t)dt + \left(d\pi_t - \rho_\pi\sigma_\pi dZ_I\right), \tag{55}$$

$$\frac{dp_{\Psi 1}}{p_{\Psi 1}} = \mu_{\Psi 1}(t)dt + d\Psi_1\left(t\right), \tag{56}$$

$$\frac{dp_{\Psi 2}}{p_{\Psi 2}} = \mu_{\Psi 2}(t)dt + d\Psi_2\left(t\right). \tag{57}$$

These securities are continuously marked to the fluctuations of $d\pi_t - \rho_\pi\sigma_\pi dZ_I$, $d\Psi_1\left(t\right)$ and $d\Psi_2\left(t\right)$, respectively. We call them security π, security Ψ_1, and security Ψ_2, respectively. Their contract terms $\mu_\pi(t)$, $\mu_{\Psi 1}(t)$, and $\mu_{\Psi 2}(t)$ are continuously determined so that the aggregate demand for each of the securities is zero at any instant. Now agents can trade in six securities instead of three. We can repeat our derivation of the equilibrium in Section 2.4, by solving each agent's optimal position in each security and then imposing the market clearing condition for each security. To save space, we skip the proof and summarize the results in the following theorem.

Theorem 3 *In equilibrium, the real short rate is the same as the short rate provided in Theorem 1. The contract terms $\mu_f(t)$, $\mu_{s1}(t)$, $\mu_{s2}(t)$ of securities f, S_1 and S_2 remain the same as the corresponding ones in Theorem 1, while the contract terms $\mu_\pi(t)$, $\mu_{\Psi 1}(t)$, and*

$\mu_{\Psi 2}(t)$ of securities π, Ψ_1, and Ψ_2 are determined by

$$\mu_\pi = r_t + \sum_{i=1}^{2} \omega_t^i \lambda_\pi (\pi_t - \hat{\theta}_t^i), \tag{58}$$

$$\mu_{\Psi 1} = r_t - \sum_{i=1}^{2} \omega_t^i \hat{\theta}_t^i, \tag{59}$$

$$\mu_{\Psi 2} = r_t - \sum_{i=1}^{2} \omega_t^i \hat{\theta}_t^i. \tag{60}$$

The aggregate wealth in the economy fluctuates in the same way as given in Theorem 1.

We also repeat our derivation of the wealth ratio process between agents in groups 1 and 2, as in the following proposition.

Proposition 8 *If we denote the belief dispersion between agents in groups 1 and 2 about l_t and θ_t by*

$$g_l(t) \equiv \hat{l}_t^1 - \hat{l}_t^2, \quad g_\theta(t) \equiv \hat{\theta}_t^1 - \hat{\theta}_t^2,$$

then the wealth ratio process η_t evolves in group-2 agents' probability measure according to

$$\begin{aligned} \frac{d\eta_t}{\eta_t} &= g_l \left[\frac{\lambda_f}{\sigma_f} d\hat{Z}_f^2(t) + \frac{1}{\sigma_s} d\hat{Z}_{s1}^2(t) + \frac{1}{\sigma_s} d\hat{Z}_{s2}^2(t) \right] \\ &\quad + g_\theta \left[\frac{\lambda_\pi}{\sigma_\pi \sqrt{1-\rho_\pi^2}} d\hat{Z}_\pi^2(t) + \frac{1}{\sigma_\Psi} d\hat{Z}_{\Psi 1}^2(t) + \frac{1}{\sigma_\Psi} d\hat{Z}_{\Psi 2}^2(t) \right] \end{aligned} \tag{61}$$

Given agents' disagreement about future inflation rates, their wealth ratio now also fluctuates with shocks to the monetary side of the economy, such as $d\hat{Z}_\pi^2(t)$, $d\hat{Z}_{\Psi 1}^2(t)$ and $d\hat{Z}_{\Psi 2}^2(t)$. More importantly, Proposition 8 shows that both agents' belief dispersion about future technology returns ($|g_l|$) and inflation rates ($|g_\theta|$) increase the volatility of the wealth ratio.

With the updated wealth ratio process in Proposition 8, we can show that the wealth ratio can still serve as the Randon-Nikodyn derivative of group-1 agents' probability measure with respect to group-2 agents' measure. Furthermore, we can also show that Theorem 2 still holds for asset prices measured in units of real consumption good.

B.4 Pricing nominal bonds

Proposition 9 *At time t, the dollar price of a nominal bond, which pays off one dollar at time T, is given by*

$$B_t = \omega_t^1 B_t^1 + \omega_t^2 B_t^2,$$

where

$$B_t^i = E_t^i \left[\frac{M_T^H}{M_t^H} \frac{p_t}{p_T} \right]$$

is the dollar value of the bond in a homogeneous economy, whereby only group-i agents are present.

Proof. One dollar at time T can buy $\frac{1}{p_T}$ units of real consumption. Thus, its value in real term at time t, from the perspective of group-2 agents, is $E_t^2 \left[\frac{M_T}{M_t} \frac{1}{p_T} \right]$, which corresponds to $p_t E_t^2 \left[\frac{M_T}{M_t} \frac{1}{p_T} \right]$ in dollars. Although this value is derived from group-2 agents' perspective, group-1 agents must share the same valuation to avoid arbitrage. Therefore, the dollar price of the nominal bond is

$$B_t = E_t^2 \left[\frac{M_T}{M_t} \frac{p_t}{p_T} \right].$$

Using the expansion of group-2 agents' stochastic discount factor given in Theorem 2, we obtain that

$$\begin{aligned} B_t &= E_t^2 \left[\frac{M_T}{M_t} \frac{p_t}{p_T} \right] = E_t^2 \left[\left(\omega_t^1 \frac{\eta_T}{\eta_t} + \omega_t^2 \right) \frac{M_T^H}{M_t^H} \frac{p_t}{p_T} \right] \\ &= \omega_t^1 E_t^2 \left[\frac{\eta_T}{\eta_t} \frac{M_T^H}{M_t^H} \frac{p_t}{p_T} \right] + \omega_t^2 E_t^2 \left[\frac{M_T^H}{M_t^H} \frac{p_t}{p_T} \right] \\ &= \omega_t^1 E_t^1 \left[\frac{M_T^H}{M_t^H} \frac{p_t}{p_T} \right] + \omega_t^2 E_t^2 \left[\frac{M_T^H}{M_t^H} \frac{p_t}{p_T} \right]. \end{aligned}$$

Note that in deriving the last equation, we use the fact that $\frac{\eta_T}{\eta_t}$ is the Randon-Nikodyn derivative of group-1 agents' probability measure with respect to the measure of group-2 agents. ■

Proposition 9 allows us to express the dollar price of a nominal bond as a wealth weighted average of each group's bond valuation in a homogeneous economy, just like how we analyzed prices of real bonds in the main text. The following proposition provides the price of a nominal bond in a homogeneous economy.

Proposition 10 *In a homogeneous economy with only group-i agents, the dollar price of a nominal bond with a maturity τ is determined by*

$$B^H \left(\tau, f_t, \hat{l}_t^i, \pi_t, \hat{\theta}_t^i \right) = e^{-a_f(\tau) f_t - a_l(\tau) \hat{l}_t^i - a_\pi(\tau) \pi_t - a_\theta(\tau) \hat{\theta}_t^i - b(\tau)}$$

where $a_f(\tau)$ and $a_l(\tau)$ are given in equations 27 and 28, and

$$a_\pi(\tau) = \frac{1}{\lambda_\pi}\left(1-e^{-\lambda_\pi\tau}\right), \tag{62}$$

$$a_\theta(\tau) = \frac{1}{\lambda_\theta}\left(1-e^{-\lambda_\theta\tau}\right)+\frac{1}{\lambda_\pi-\lambda_\theta}\left(e^{-\lambda_\pi\tau}-e^{-\lambda_\theta\tau}\right), \tag{63}$$

$$b(\tau) = \int_0^\tau F(s)ds, \tag{64}$$

with

$$\begin{aligned}F(s) = {} & \lambda_l\bar{l}a_l(s)-\frac{1}{2}\sigma_f^2a_f(s)^2-\frac{1}{2}\left(\sigma_l^2-2\lambda_l\bar{\gamma}_l\right)a_l(s)^2-\lambda_f\bar{\gamma}_la_f(s)a_l(s)+\lambda_\theta\bar{\theta}a_\theta(s)\\ & -\frac{1}{2}\sigma_\pi^2a_\pi(s)^2-\frac{1}{2}\left(\sigma_\theta^2-2\lambda_\theta\bar{\gamma}_\theta\right)a_\theta(s)^2-\left(\lambda_\pi\bar{\gamma}_\theta+\rho_\theta\rho_\pi\sigma_\theta\sigma_\pi\right)a_\pi(s)a_\theta(s)\\ & -\rho_\theta\sigma_\theta\sigma_Ia_\theta(s)-\rho_\pi\sigma_\pi\sigma_Ia_\pi(s)-\sigma_I^2.\end{aligned}$$

Proof. Here we provide a sketch of the proof. The dollar price of the nominal bond in a homogeneous economy has the following function form:

$$B_t^i = B^H\left(\tau,f_t,\hat{l}_t^i,\pi_t,\hat{\theta}_t^i\right).$$

To derive this function, we first convert the price into real term: $y_t=\frac{B_t^i}{p_t}$. Thus, dy_t/y_t is the bond's real return. The real return has to satisfy the following relationship with the stochastic discount factor in the homogeneous economy:

$$E_t^i\left(\frac{dy_t}{y_t}\right)+E_t^i\left(\frac{dM_t^H}{M_t^H}\right)+E_t^i\left(\frac{dy_t}{y_t}.\frac{dM_t^H}{M_t^H}\right)=0.$$

By applying Ito's lemma to $\frac{dy_t}{y_t}$and $\frac{dM_t^H}{M_t^H}$ and substituting these terms back into the equation above, we obtain the following differential equation for $B^H\left(\tau,f_t,\hat{l}_t^i,\pi_t,\hat{\theta}_t^i\right)$:

$$\begin{aligned}0 = {} & -\frac{B_\tau^H}{B^H}-\lambda_f(f_t-\hat{l}_t^i)\frac{B_f^H}{B^H}-\lambda_l(\hat{l}_t^i-\bar{l})\frac{B_l^H}{B^H}-\lambda_\pi(\pi_t-\hat{\theta}_t^i)\frac{B_\pi^H}{B^H}-\lambda_\theta(\hat{\theta}_t^i-\bar{\theta})\frac{B_\theta^H}{B^H}+\frac{1}{2}\sigma_f^2\frac{B_{ff}^H}{B^H}\\ & +\frac{1}{2}\left(\sigma_l^2-2\lambda_l\bar{\gamma}_l\right)\frac{B_{ll}^H}{B^H}+\lambda_f\bar{\gamma}_l\frac{B_{fl}^H}{B^H}+\frac{1}{2}\sigma_\pi^2\frac{B_{\pi\pi}^H}{B^H}+\left(\lambda_\pi\bar{\gamma}_\theta+\rho_\theta\rho_\pi\sigma_\theta\sigma_\pi\right)\frac{B_{\pi\theta}^H}{B^H}\\ & +\frac{1}{2}\left(\sigma_\theta^2-2\lambda_\theta\bar{\gamma}_\theta\right)\frac{B_{\theta\theta}^H}{B^H}-\pi_t-f_t+\sigma_I^2-\sigma_I\left(\rho_\theta\sigma_\theta\frac{B_\theta^H}{B^H}+\rho_\pi\sigma_\pi\frac{B_\pi^H}{B^H}\right).\end{aligned}$$

Then, by conjecturing the following solution

$$B^H\left(\tau,f_t,\hat{l}_t^i,\pi_t,\hat{\theta}_t^i\right)=e^{-a_f(\tau)f_t-a_l(\tau)\hat{l}_t^i-a_\pi(\tau)\pi_t-a_\theta(\tau)\hat{\theta}_t^i-b(\tau)},$$

and substituting this function back into the differential equation, we obtain the following ordinary differential equations for $a_f(\tau)$, $a_l(\tau)$, $a_\pi(\tau)$, $a_\theta(\tau)$, and $b(\tau)$:

$$
\begin{aligned}
&a_f'(\tau) + \lambda_f a_f(\tau) - 1 = 0,\\
&a_l'(\tau) - \lambda_f a_f(\tau) + \lambda_l a_l(\tau) = 0,\\
&a_\pi'(\tau) + \lambda_\pi a_\pi(\tau) - 1 = 0,\\
&a_\theta'(\tau) - \lambda_\pi a_\pi(\tau) + \lambda_\theta a_\theta(\tau) = 0,\\
&b'(\tau) - \lambda_l \bar{l} a_l(\tau) - \lambda_\theta \bar{\theta} a_\theta(\tau) + \frac{1}{2}\sigma_f^2 a_f(\tau)^2 + \frac{1}{2}\left(\sigma_l^2 - 2\lambda_l \bar{\gamma}_l\right) a_l(\tau)^2 + \lambda_f \bar{\gamma}_l a_f(\tau) a_l(\tau)\\
&\qquad + \frac{1}{2}\sigma_\pi^2 a_\pi(\tau)^2 + \frac{1}{2}\left(\sigma_\theta^2 - 2\lambda_\theta \bar{\gamma}_\theta\right) a_\theta(\tau)^2 + \left(\lambda_\pi \bar{\gamma}_\theta + \rho_\theta \rho_\pi \sigma_\theta \sigma_\pi\right) a_\pi(\tau) a_\theta(\tau)\\
&\qquad + \sigma_I^2 + \rho_\theta \sigma_\theta \sigma_I a_\theta(\tau) + \rho_\pi \sigma_\pi \sigma_I a_\pi(\tau) = 0.
\end{aligned}
$$

By further imposing the boundary conditions

$$a_f(0) = a_l(0) = a_\pi(0) = a_\theta(0) = b(0) = 0,$$

we obtain the bond price formula given in Proposition 10. ■

By combining Propositions 9 and 10, we can expand the price of a nominal bond in a similar way as in equation (29). The only difference is that there are now two sources of belief dispersion, one about future technology returns and the other about future inflation rates. Other than this feature, the basic structure of the bond pricing formula is the same. We are also able to derive, based on the extended model, similar effects of agents' belief dispersion as those discussed in Section 3.

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