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ABSTRACT

We characterize equilibria with endogenous debt constraints for a general equilibrium economy with limited commitment in which the only consequence of default is losing the ability to borrow in future periods. First, we show that equilibrium debt limits must satisfy a simple condition that allows agents to exactly roll over existing debt period by period. Second, we provide an equivalence result, whereby the resulting set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations that are sustained with unbacked public debt or rational bubbles; for the latter, there exist well known existence and characterization results. In contrast to the classic result by Bulow and Rogoff (AER 1989), positive levels of debt are sustainable in our environment because the interest rate is sufficiently low to provide repayment incentives.

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1 Introduction

This paper addresses the classic question whether debt can be sustained purely by a reputation mechanism. Suppose that the only punishment imposed on a borrower who defaults on his obligations is that he will not be able to borrow again in the future. A seminal result in Bulow and Rogoff (1989a - henceforth BR) claims that, under this type of punishment, debt is unsustainable. They analyze the case of a small open economy, borrowing at a given positive world interest rate. In that environment, if the country ever borrows a positive amount, it will eventually reach a point where it is strictly better off defaulting and financing all future consumption with positive asset positions, out of a ‘savings’ account.¹

This result has sparked a rich literature on reputational mechanisms for sustaining debt. Some of these contributions have augmented Bulow and Rogoff’s (1989a) framework to sustain debt by non-competitive mechanisms, such as reduction of trade, loss of trade credit, or other non-financial sanctions (Bulow and Rogoff 1989b), collusion among non-competitive lenders (Kletzer and Wright 2000, Wright 2001), loss of reputation in other dimensions (Cole and Kehoe 1998), time inconsistency in the borrower’s preferences (Gul and Pesendorfer 2003, Amador 2003), or reduced access to state-contingent securities (Pesendorfer 1992, Thomas 1992, Grossman and Han 1999). A separate branch of the literature has studied markets with stronger consequences of default, such as outright exclusion from markets into autarky (Eaton and Gersovitz 1981, Kehoe and Levine 1993, and Kocherlakota 1996), or loss of productive collateral (Lustig 2004). One appealing feature of this latter class of models is the endogenous determination of debt limits in general equilibrium, so as to provide proper incentives to honor existing outstanding debt (Alvarez and Jermann 2000).

In this paper, we go back to the original Bulow and Rogoff (1989a) setup, but frame the problem in a general equilibrium model with endogenous debt limits. We consider a symmetric environment in which *all* agents have limited commitment, and default is punished only by the exclusion from future borrowing. We show that positive amounts of debt *are* sustainable in equilibrium.

Key to our analysis is that, when all the agents have limited commitment, the equilibrium *interest rate* adjusts endogenously so as to ensure that agents repay their debt. Reputational

¹Related results appear in Chari and Kehoe (1993) and in Krueger and Uhlig (2005). Chari and Kehoe consider government debt in a model with distortive taxes and lack of commitment by the government, but not the households. Krueger and Uhlig analyze competitive risk-sharing contracts with one-sided commitment by the insurers, and show that such contracts never allow the insured to incur debt. Both papers have in common with each other and with BR the assumption of one-sided commitment and access to savings at competitive market rates after a default.

incentives for debt repayments thus rely not only on the amount of credit to which agents have access in future periods, but, perhaps more importantly, on the interest rate at which this credit is made available.

Our main argument can be split in two steps. First, we show that incentives for default disappear if the interest rate is sufficient low. Second, we show that interest rates low enough to be consistent with repayment can emerge in equilibrium in an economy where no agent can commit to repay.

To illustrate these results, we first present a simple deterministic example where positive borrowing is sustained in equilibrium. In the example, private debt is self-enforcing as long as the equilibrium interest rate is smaller than or equal to the growth rate of debt limits, which equals the growth rate of aggregate endowments in steady-state. In the rest of the paper, we give a full characterization of the conditions under which private debt is sustainable.

For the general analysis, we consider a stochastic endowment economy with sequential trade in complete contingent securities markets. Agents may issue securities up to a state-contingent limit. If they default, they are denied credit in all future periods. The equilibrium debt limits are determined endogenously as the largest possible limits such that repayment is always individually rational. Our first general result (Theorem 1) states that debt limits are self-enforcing if and only if they allow all individuals to exactly finance outstanding obligations by issuing new claims. In a deterministic environment, this is satisfied if and only if they grow at a rate equal to or larger than the real rate of interest.

Our second main result (Theorem 2) establishes conditions for the existence of an equilibrium with self-enforcing debt and gives a characterization of sustainable equilibrium allocations, by means of an equivalence result. Consider an alternative environment with no private debt, but where we allow a government to issue state-contingent debt that is not backed by any fiscal revenue, i.e., where the government must finance all existing claims by issuing new debt. This unbacked public debt has the feature of a rational bubble (Tirole 1982); in a deterministic environment, it can be reinterpreted as fiat money. We show that any equilibrium allocation of the economy with self-enforcing private debt can also be sustained as an equilibrium allocation of the economy with unbacked public debt, and vice versa. Since there exist well known conditions for the sustainability of positive levels of unbacked public debt, or the existence of rational bubbles or fiat money more generally (see Santos and Woodford 1997 for a general analysis), these conditions also characterize the sustainability of positive levels of private debt in a general equilibrium Bulow-Rogoff economy.

Related Literature: First and foremost, our paper presents an answer to the theoretical puzzle

posed by the no-lending result of BR. Contrary to much of the literature that sought to overturn this result, we show that stronger enforcement power such as complete market exclusion, collateral, or other non-competitive mechanisms, is not necessary to sustain debt, once interest rates can adjust to account for the common lack of commitment.² A key assumption in BR is that the net present value of a borrower’s life-time endowments, when discounted at market prices, is finite, and that this life-time endowment value gives an upper bound for the borrower’s outstanding debt. In our model, endowments and consumption allocations are no longer finite valued at the resulting equilibrium prices. In light of this result, BR’s partial equilibrium assumption seems unwarranted, since it exactly rules out the debt contracts that emerge endogenously in equilibrium.³

Our paper also builds on the literature on endogenous debt constraints (Kehoe and Levine 1993, Alvarez and Jermann 2000) by developing a similar theory in a model in which borrowers face future denial of credit as the only consequence of a default. It further draws a connection between models of endogenous debt limits and rational asset pricing bubbles (Tirole 1982). That rational bubbles may exist in models with borrowing constraints *à la* Bewley (1980) was recognized in Scheinkman and Weiss (1986), Kocherlakota (1992) and Santos and Woodford (1997). By establishing an equivalence result between trade in self-enforcing private debt in a Bulow-Rogoff environment and a specific form of a rational bubble (i.e., as unbacked public debt), we exploit existing results on the sustainability of such bubbles to arrive at a characterization of when private debt is sustainable. In fact, one interpretation of our equivalence result is that rational bubbles can be competitively supplied by the market, in the form of circulating private debt.

Finally, our results have implications for applications of intertemporal general equilibrium models to sovereign debt and international capital flows, risk-sharing and consumption smoothing or to monetary theory, among others. We defer a detailed discussion of these applications and other extensions to Section 5, after we have established our main results.

In Section 2, we describe our general model and define competitive equilibria with self-enforcing private debt and unbacked public debt. In Section 3, we illustrate our main results in a simple deterministic example. In Section 4, we establish our two main theorems, and discuss the intuition

²The idea that debt is sustainable once *all* agents have limited commitment also appears in Cole and Kehoe (1995). However, they consider a game-theoretic environment in which interest rates are exogenous, and debt is sustained by means of a trigger-strategy equilibrium, in which agents revert to a no-lending equilibrium after a default by any market participant.

³In the working-paper version of their paper, Bulow and Rogoff (1988, p. 5) remark that this assumption rules out “Ponzi’-type reputational contracts.” These are precisely the type of contracts that emerge in general equilibrium. Mohr (1991) derives a similar Ponzi-type condition in a two-period overlapping generations model.

behind our characterization of self-enforcing debt constraints. Section 5 discusses extensions and applications of our results. Proofs omitted from the text are in the appendix.

2 The Model

Uncertainty: Consider a discrete-time, infinite-horizon endowment economy. At each date $t \in \{0, 1, 2, \dots\}$, there exists a finite set S^t of publicly observable events s^t , which are partially ordered into an event tree \mathcal{S} . Each event s^t has a unique predecessor $\sigma(s^t) \in S^{t-1}$, and is followed by a positive, finite number of events $s^{t+1} \in S^{t+1}$, s.t. $s^t = \sigma(s^{t+1})$. There exists a unique initial event dated by $t = 0$ and denoted s^0 . Event $s^{t+\tau}$ is said to follow event s^t (denoted $s^{t+\tau} \succ s^t$) if $\sigma(\tau)(s^{t+\tau}) = s^t$. The set $\mathcal{S}(s^t) = \{s^{t+\tau} : s^{t+\tau} \succ s^t\}$ denotes the set of all events that follow s^t .

Let \mathcal{S}^∞ denote the set of infinite sequences of events $s^\infty = \{s^0, s^1, \dots\}$ s.t. $s^t \in S^t$, and $s^t = \sigma(s^{t+1})$ for all t . At date 0, nature draws $s^\infty \in \mathcal{S}^\infty$. At each date $t > 0$, s^t is publicly revealed to all agents. We let $\pi(s^t)$ denote the unconditional probability that event s^t is observed, and assume that $\pi(s^t) > 0$ for all $s^t \in \mathcal{S}$. For all $s^{t+\tau} \succ s^t$, $\pi(s^{t+\tau}|s^t) = \pi(s^{t+\tau})/\pi(s^t)$ denotes the conditional probability of $s^{t+\tau}$, given s^t .

Preferences and endowments: At each event s^t , there is a single non-storable consumption good. There is a finite number J of consumer types, each represented by a unit measure of agents, and indexed by j . Each consumer type is characterized by a sequence of endowments of the consumption good, $\{y^j(s^t)\}_{s^t \in \mathcal{S}} \in \mathbb{R}_+^\infty$. Preferences over consumption sequences $C \equiv \{c(s^t)\}_{s^t \in \mathcal{S}} \in \mathbb{R}_+^\infty$ are represented by a lifetime utility functional $U(C)$, which is defined as

$$U(C) = \sum_{s^t \in \mathcal{S}} \beta^t \pi(s^t) u(c(s^t)) \quad (1)$$

where $\beta \in (0, 1)$ and $u(\cdot)$ is strictly increasing, convex, and bounded above.

Self-enforcing private debt: At each history s^t , agents may issue contingent claims, which promise to pay one unit of consumption in period $t+1$, contingent on the occurrence of event $s^{t+1} \succ s^t$, in exchange for current consumption goods. These claims are traded in complete sequential Walrasian markets. If the promises are always fulfilled (as they will be in equilibrium), individual promises issued by different agents are perfect substitutes and are equivalent to a state-contingent one-period bond. Denote by $q(s^{t+1}|s^t)$ the price of such a bond at event s^t , or, equivalently, the price at s^t of consumption at event $s^{t+1} \succ s^t$. Using consumption at s^0 as the numeraire and setting $p(s^0) = 1$, we let $p(s^t)$ denote the period 0 price of consumption at s^t . $p(s^t)$ is recursively defined by $p(s^t) = q(s^t|\sigma(s^t)) \cdot p(\sigma(s^t))$, for all $s^t \in \mathcal{S}$.

An agent's asset profile is defined as a function $a^j : \mathcal{S} \rightarrow \mathbb{R}$, where $a^j(s^t)$ denotes the net financial position at s^t of an agent of type j , that is, the amount of promises due to him in s^t , net of the amount of promises issued by him; this position is determined by trade at history $\sigma(s^t)$. If the agent does not default, his consumption at s^t , denoted $c^j(s^t)$, must satisfy the flow budget constraint

$$c^j(s^t) \leq y^j(s^t) + a^j(s^t) - \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} a^j(s^{t+1}) \quad (2)$$

If agents had the ability to fully commit to their promises, they would be able to smooth all type-specific endowment fluctuations. In our model, agents lack this type of commitment: at any date s^t , agents can simply refuse to honor their past promises and default. Agents will then fulfill their obligations only if it is in their best interest to do so, and the incentives for repayment depend on the consequences of default. We assume that, if an agent defaults, this fact becomes common knowledge and the agent loses the ability to issue claims in all future periods. Creditors can seize any financial assets he might hold at the moment of default (i.e., his holdings of other agents' claims). However, creditors are unable to seize any of his current or future endowments $\{y^j(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}}$, nor are they able to seize the financial claims he will accumulate in future periods. In summary, after a default, an agent retains his ability to purchase claims but he loses his privilege to issue claims, and he starts with a net financial position of 0.⁴

The amount of claims that an agent issues are publicly observable. To provide repayment incentives, at each event s^t , each agent faces an upper bound $-\phi^j(s^{t+1})$ on the amount of claims he can issue for each continuation s^{t+1} of s^t , or equivalently, a lower bound

$$a^j(s^{t+1}) \geq \phi^j(s^{t+1}) \quad (3)$$

on his net financial position at s^{t+1} . These debt limits are endogenously determined to make the debt self-enforcing, i.e., to give the agents the right repayment incentives. Formally, the sequence of debt limits $\Phi^j \equiv \{\phi^j(s^t)\}_{s^t \in \mathcal{S}}$ is set in such a way that, if an agent's net financial position reaches $a^j(s^t) = \phi^j(s^t)$, then at s^t he is exactly indifferent between repayment and default. If an agent's net financial position were to fall below $\phi^j(s^t)$ at s^t , the agent would have an incentive to default on his promises, whereas if his net asset position remains above $\phi^j(s^t)$, he prefers to repay rather than default. In equilibrium, all other market participants anticipate this, and are hence willing to

⁴The assumption that any positive holdings of other agents' claims are confiscated in case of default is made only for analytic and expositional purposes; it implies that agents can default only on their *net* financial position. We will discuss later how it can be relaxed; it turns out to have no impact on our results. Therefore, the only disciplining element that may prevent agents from defaulting is losing the privilege to borrow in future periods.

extend credit up to the point where the agent's net financial position reaches $\phi^j(s^t)$, thus allowing the agent to borrow any amount that he can credibly commit to repay.

Equilibrium definition: For each type j , and history s^t , let $V^j(a, \Phi^j(s^t); s^t)$ denote the life-time utility of a consumer who starts from s^t with net assets a , faces debt limits $\Phi^j(s^t) = \{\phi^j(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$, and never defaults. $V^j(a, \Phi^j(s^t); s^t)$ is defined by the following optimization problem (P1):

$$V^j(a, \Phi^j(s^t); s^t) = \max_{\{a^j(s^\tau)\}_{s^{t+\tau} \in \mathcal{S}(s^t)}} \sum_{s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}} \beta^\tau \pi(s^{t+\tau} | s^t) u(c^j(s^{t+\tau})) \quad (4)$$

such that for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$, $c^j(s^{t+\tau})$ and $\{a^j(s^{t+\tau+1})\}_{s^{t+\tau+1} \succ_{s^{t+\tau}}}$ satisfy the budget constraint (2), the borrowing constraint (3) and $c^j(s^{t+\tau}) \geq 0$, with given $a^j(s^t) = a$.

Likewise, let $D^j(a; s^t)$ denote the life-time utility of a consumer who has defaulted in the past and hence has to hold a non-negative financial position at all future periods. For any $a \geq 0$, $D^j(a; s^t)$ is defined by the optimization problem of maximizing (4) such that for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$, $c^j(s^{t+\tau})$ and $\{a^j(s^{t+\tau+1})\}_{s^{t+\tau+1} \succ_{s^{t+\tau}}}$ satisfy the budget constraint (2), the borrowing constraint $a^j(s^{t+\tau+1}) \geq 0$, and $c^j(s^{t+\tau}) \geq 0$, with given $a^j(s^t) = a$; therefore, $D^j(a; s^t) = V^j(a, O(s^t); s^t)$, where $O(s^t)$ stands for the sequence of borrowing constraints equal to zero at every continuation history of s^t .

$V^j(a, \Phi^j(s^t); s^t)$ and $D^j(a; s^t)$ are both strictly increasing in a . Since, by assumption, an agent who defaults starts from a net asset position of 0, an agent with net asset position a will find it optimal not to default whenever $V^j(a, \Phi^j(s^t); s^t) \geq D^j(0; s^t)$. We thus have the following definition of self-enforcement:

Definition 1 *The debt limits $\Phi^j : \mathcal{S} \rightarrow \mathbb{R}$ are self-enforcing (SE), if and only if:*

$$V^j(\phi^j(s^t), \Phi^j(s^t); s^t) = D^j(0; s^t) \text{ for all } s^t \in \mathcal{S}. \quad (5)$$

This leads to the following definition of a competitive equilibrium with self-enforcing private debt:

Definition 2 *A competitive equilibrium with self-enforcing private debt $\{C^j, a^j, \Phi^j; p\}_{j=1, \dots, J}$ is defined by a sequence of consumption allocations $C^j : \mathcal{S} \rightarrow \mathbb{R}^+$ and net financial positions $a^j : \mathcal{S} \rightarrow \mathbb{R}$ for each consumer type j , a sequence of debt limits $\Phi^j : \mathcal{S} \rightarrow \mathbb{R}^-$ for each consumer type, and a price sequence $p : \mathcal{S} \rightarrow \mathbb{R}^+$, for which:*

- (i) *Optimality: for each j , C^j and a^j solve (P1) at s^0 , given initial asset holdings $a^j(s^0) = 0$.*

- (ii) *Self-enforcement: the debt limits Φ^j are self-enforcing.*
 (iii) *Market clearing: $\sum_j c^j(s^t) = \sum_j y^j(s^t)$ and $\sum_j a^j(s^t) = 0$ for all $s^t \in \mathcal{S}$.*

This equilibrium definition follows Alvarez and Jermann (2000) in that debt limits must be self-enforcing, i.e., not give agents any incentive to default, but departs from them by assuming denial of future credit instead of complete autarky as the consequence of a default. From the perspective of the individual, the debt limits are treated much like prices in Walrasian markets in that individuals optimize taking prices and debt limits as given, and these adjust to satisfy market-clearing and self-enforcement.

Furthermore, our definition of self-enforcement implies that debt limits adjust in equilibrium to allow for the maximum amount of credit. This is akin to Alvarez and Jermann's (2000) notion of debt limits being 'not too tight.' In principle, any set of debt limits for which $V^j(\phi^j(s^t), \Phi^j(s^t); s^t) \geq D^j(0; s^t)$ for all $s^t \in \mathcal{S}$ gives agents no incentive to default. However, if it were the case that $V^j(\phi^j(s^t), \Phi^j(s^t); s^t) > D^j(0; s^t)$, an agent facing a binding debt limit at $\phi^j(s^t)$ would be willing to borrow at a rate slightly higher than the market interest rate and market participants would not be willing to refuse him credit. Our debt limits are thus set so that (i) no borrower has an incentive to default, (ii) no lender has an incentive to extend credit beyond a borrower's debt limit, and (iii) no lender has an incentive to refuse credit to a borrower below the borrower's debt limit.

Unbacked public debt: Finally, for our equivalence result, we consider an alternative economy, in which agents are not allowed to borrow, but can smooth consumption using government-issued securities that are not backed by taxation, i.e., the government must issue new securities to honor current outstanding claims.

As before, we suppose that at each event s^t , agents may purchase contingent claims $a(s^{t+1})$, which are traded in complete sequential Walrasian markets. However, unlike before, the agents can no longer issue these claims themselves; instead they are provided by a government, which rolls over a fixed initial stock of claims $d(s^0)$ period by period by issuing new securities. The government's roll-over condition thus requires that $d(s^t) \leq \sum_{s^{t+1} \succ s^t} q(s^{t+1}|s^t) d(s^{t+1})$, i.e., that the amount of resources raised by issuing new claims for s^{t+1} at history s^t is sufficient to honor the previous period's commitments. We focus on environments where in each period, the government's roll-over condition is satisfied with equality, or

$$d(s^t) = \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} d(s^{t+1}) \text{ for all } s^t \in \mathcal{S}. \quad (6)$$

Let $d^j(s^0) \geq 0$ denote the initial allocation of claims to type j agents, with $\sum_j d^j(s^0) = d(s^0)$.

For a given sequence of prices, the consumers' problem is then defined as above by (P1), with debt limits characterized by $O(s^0)$. Thus, a competitive equilibrium with unbacked public debt is defined as follows:

Definition 3 *A competitive equilibrium $\{C^j, \hat{a}^j, d^j(s^0); D, p\}_{j=1, \dots, J}$ with unbacked public debt is defined by a sequence of consumption allocations $C^j : \mathcal{S} \rightarrow \mathbb{R}^+$ and net financial positions $a^j : \mathcal{S} \rightarrow \mathbb{R}^+$ for each consumer type j , an initial debt allocation $d^j(s^0)$ for each consumer type, a price sequence $p : \mathcal{S} \rightarrow \mathbb{R}^+$, and a sequence of debt circulations $D : \mathcal{S} \rightarrow \mathbb{R}^+$, for which:*

(i) *Optimality: for each j , C^j and a^j solve (P1) at $t = 0$, given initial asset holdings $a^j(s^0) = d^j(s^0) \geq 0$ and debt limits $O(s^0)$.*

(ii) *Market clearing: $\sum_j c^j(s^t) = \sum_j y^j(s^t)$ and $\sum_j a^j(s^t) = d(s^t)$ for all $s^t \in \mathcal{S}$.*

(iii) *Government Budget constraint: (6) is satisfied for all $s^t \in \mathcal{S}$.*

3 A Simple Example

In this section, we illustrate the main results of our paper by means of a simple deterministic example with alternating endowments.⁵ There are two types of consumers, odd and even, which are characterized by different endowment profiles: Odd consumers receive the endowment $\{y_t^o\} = \{\theta_0 \bar{e}, \theta_1 \underline{e}, \theta_2 \bar{e}, \dots\}$ and even consumers the endowment $\{y_t^e\} = \{\theta_0 \underline{e}, \theta_1 \bar{e}, \theta_2 \underline{e}, \dots\}$. The consumers have identical preferences over consumption sequences $\{c_t\}$, represented by the utility function $\sum_{t=0}^{\infty} \beta^t \log c_t$. The productivity parameter $\theta_t = g^t$ grows exogenously at a rate $g > 0$, and we assume that $\beta \bar{e} > \underline{e}$, which implies that endowment fluctuations are sufficiently large.

In the absence of enforcement frictions, the market equilibrium of this economy achieves a Pareto-optimal allocation in which the consumption of all consumers grows at the rate g and the one-period bond price is given by β/g . This setup goes back at least to Townsend (1980) and is analyzed in detail by Woodford (1990) and Ljungqvist and Sargent (2000, Chap. 18) as an example of an economy where private or public debt instruments are traded to smooth idiosyncratic income fluctuations.

Steady states with self-enforcing private debt: Suppose now that agents trade self-enforcing private debt to smooth endowment fluctuations. Since the environment is deterministic, they simply trade non-contingent debt. To derive the steady-state equilibria, we conjecture (and verify) that individual debt limits for each type grow at a constant rate g , i.e., they take the form

⁵To simplify notation, we replace the dependence on histories s^t by the time subscript t .

$\phi_t^j = \phi_j g^t$ for $j \in \{o, e\}$ with $\phi_j \leq 0$. Furthermore, we conjecture that the steady-state bond price is q_o in periods when odd types have high endowments, and q_e in periods when even types have high endowments. Finally, we conjecture that the equilibrium bond prices are greater or equal than their frictionless level, i.e., $q_o, q_e \geq \beta/g$.

Consider first the behavior of a non-defaulting agent. Given our conjecture on prices and debt limits, Lemma 1 derives optimal consumption and asset holdings that solve (P1) at date t , with initial asset holdings $a_t = \phi_j$.

Lemma 1 For $j \in \{o, e\}$, define

$$\bar{c}_j = \frac{1}{1+\beta} [\bar{e} + (q_j g) \underline{e} - \phi_j ((q_o g) (q_e g) - 1)], \quad (7)$$

$$\underline{c}_j = \frac{\beta/(q_j g)}{1+\beta} [\bar{e} + (q_j g) \underline{e} - \phi_j ((q_o g) (q_e g) - 1)], \quad (8)$$

$$a_j = \frac{1/(q_j g)}{1+\beta} [\beta \bar{e} - (q_j g) \underline{e} + \phi_j (\beta + (q_o g) (q_e g))]. \quad (9)$$

Consider a consumer of type j , who has a high endowment in period t and has net assets $a_t = \phi_j$. Suppose $a_j \geq \phi_j$. If the consumer never defaults on or after date t , his optimal consumption is $\bar{c}_j g^{t+\tau}$ in high endowment periods and $\underline{c}_j g^{t+\tau}$ in low endowment periods. Optimal asset holdings are $a_j g^{t+\tau}$ in low endowment periods and $-\phi_j g^{t+\tau}$ in high endowment periods.

To find a steady-state equilibrium, we impose market-clearing in the asset market:

$$-\phi_{-j} = a_j, \quad (10)$$

for $j \in \{o, e\}$. For given values of ϕ_o and ϕ_e , we can substitute (9) in (10) and obtain two equations which give us the equilibrium prices q_o and q_e . Market clearing in the goods markets follows by Walras' Law. Using (10), we also have $a_j \geq 0$, which guarantees that the debt limit condition $a_j \geq \phi_j$ in Lemma 1 is satisfied, and that it is never optimal to default in low-endowment periods.

It remains to be determined under what conditions the debt limits ϕ_o and ϕ_e are self-enforcing. Lemma 2 determines optimal consumption allocations and asset holdings for an agent who defaults in the high endowment period.

Lemma 2 For $j \in \{o, e\}$, define

$$\bar{c}_j^d = \frac{1}{1+\beta} [\bar{e} + (q_j g) \underline{e}], \quad (11)$$

$$\underline{c}_j^d = \frac{\beta/(q_j g)}{1+\beta} [\bar{e} + (q_j g) \underline{e}], \quad (12)$$

$$a_j^d = \frac{1/(q_j g)}{1+\beta} [\beta \bar{e} - (q_j g) \underline{e}].$$

Consider a consumer of type j , who has a high endowment in period t . If the consumer defaults his optimal consumption after default is $\bar{c}_j^d g^{t+\tau}$ in high endowment periods and $\underline{c}_j^d g^{t+\tau}$ in low endowment periods. Optimal asset holdings are $a_j^d g^t$ in low endowment periods, and 0 in high endowment periods.

If an agent defaults, his initial asset position after default is 0. From then on, the agent's optimal consumption path alternates between \bar{c}_j^d and \underline{c}_j^d . If he does not default, his consumption alternates between \bar{c}_j and \underline{c}_j . By comparing (7)-(8) with (11)-(12), it immediately follows that agents prefer no-default as long as either $(q_o g)(q_e g) \geq 1$ or $\phi_j = 0$. Since we are interested in equilibria with positive amounts of borrowing, i.e., non autarkic equilibria, let us focus on the case $\phi_j < 0$. Moreover, agents are exactly indifferent between default and no-default if and only if

$$(q_o g)(q_e g) = 1. \quad (13)$$

Since we seek to determine ϕ_o and ϕ_e in such a way that, whenever the agents are debt constrained, they are also indifferent between repayment and default, we impose (13) as an equilibrium condition. Going back to the asset market clearing condition, we find

$$-\phi_j - \phi_{-j} = \frac{1/(q_j g)}{1 + \beta} [\beta \bar{e} - (q_j g) \underline{e}], \quad (14)$$

for $j \in \{o, e\}$. Therefore, in an equilibrium with self-enforcing private debt, the bond price q_j has to be the same in odd and even periods, and, by condition (13), has to be equal to $q_o = q_e = 1/g$. Substituting this into equation (14), we then find the values for the debt limits ϕ_o and ϕ_e compatible with a self-enforcing equilibrium. The assumption that $\beta \bar{e} > \underline{e}$ guarantees that both limits can be set at values smaller or equal than zero. However, apart from this restriction, any pair of ϕ_o, ϕ_e which satisfies (14) is compatible with self-enforcement. These results are summarized in the following proposition.

Proposition 1 *Whenever $\beta \bar{e} > \underline{e}$, there exists a non-autarkic steady-state equilibrium with self-enforcing private debt, in which $q_o g = q_e g = 1$. Borrowing constraints adjust so that*

$$\phi_o + \phi_e = -\frac{1}{1 + \beta} [\beta \bar{e} - \underline{e}], \quad (15)$$

and are otherwise indeterminate. Consumption allocations are determinate and are given by $\bar{c} = \frac{1}{1+\beta} [\bar{e} + \underline{e}]$ in high endowment periods and $\underline{c} = \frac{\beta}{1+\beta} [\bar{e} + \underline{e}]$ in low endowment periods. In addition, there always exists an autarkic equilibrium, in which $\phi_o = \phi_e = 0$, $\bar{c} = \bar{e}$, $\underline{c} = \underline{e}$, and $q_e, q_o \geq \frac{\beta \bar{e}}{g \underline{e}}$.

This characterization illustrates the main point of our paper. Contrary to the zero-borrowing result of BR, who consider a small open economy borrowing at given world interest rates, the general equilibrium environment considered here leads to positive levels of borrowing and lending. This is sustained in equilibrium by an interest rate equal to the growth rate of the economy's endowment. As debt limits grow at the same rate as the economy, this interest rate ensures that the incentives for repayment are satisfied and debt is sustainable.⁶

This general equilibrium result derives from the observation that an agent's repayment incentives depend not only on whether an agent is allowed to borrow or lend in the future, but also on the interest rate at which borrowing and lending will take place. The higher the interest rate, the less appealing is the opportunity to borrow in the future, and the more appealing the option to be a net lender after default.

Self-enforcement imposes an upper bound on the interest rate, so as to reduce the returns to savings in case of a default, and to reduce the cost of borrowing. In steady-state, this upper bound exactly pins down the interest rate as equal to the steady-state growth rate. Debt limits then adjust to make sure that this interest rate clears the market. This contrasts with a partial equilibrium approach which takes interest rates as given and seeks to find sustainable debt limits. This would require the debt limits to grow at the given real rate of interest. With positive interest rates, the agents' debt limits would then eventually exceed the present value of their life-time future endowments, which would be inconsistent with market clearing in general equilibrium. This is why, under the assumptions of BR, our form of self-enforcing debt could not arise.

It is useful to stress one property of the self-enforcing debt limits $\{\phi_t^j\}$. Consider a borrower who is constrained in periods t and $t + 2$. In a steady state with self-enforcing borrowing, this borrower could just roll-over his current debt between periods t and $t + 2$. This roll-over does not require any real resources from consumer j , since he can repay $\phi^j g^t$ by issuing $\phi^j g^{t+1}$ at the price $q_t = g^{-1}$, and then repay $\phi^j g^{t+1}$ by issuing $\phi^j g^{t+2}$ in the same manner. This exact roll-over is not necessarily optimal for the consumer. In fact, in the example considered it is not optimal along the equilibrium path. However, the fact that this roll-over is feasible turns out to be an essential property of self-enforcing borrowing limits, as we will show in the next section.⁷

⁶In our environment, the consumers' transversality condition $\lim_{t \rightarrow \infty} \beta^t u'(c_t) a_t = 0$ does not imply the "no bubble" condition $\lim_{t \rightarrow \infty} p_t a_t = 0$. As discussed in Kocherlakota (1992) and in Santos and Woodford (1997), in the presence of borrowing constraints the former can hold independently of the latter because the Euler equation does not hold at all points in time as an equality.

⁷Although our discussion above has not explicitly considered default incentives for periods in which the debt limit is not binding, our characterization of debt limits is valid in those periods as well. This can be shown either by

Finally, note that the equilibrium determines the aggregate amount of debt $\phi_o + \phi_e$, but not how these borrowing privileges are split between the two types. Steady-state consumption allocations do not depend on this distribution of borrowing privileges. In our model, the ability to borrow is a form of private seignorage, which acts like a wealth transfer to the borrower and raises his consumption in all periods. The sign and magnitude of this rent are directly proportional to $(q_o g)(q_e g) - 1$. In equilibrium, competition amongst borrowers eliminates this rent, and drives interest rates up to the point, where $(q_o g)(q_e g) = 1$. At this point the wealth transfer associated with the access to credit is zero, so that steady-state consumption allocations are independent of an agent's allowed debt limit. As we will see below, this indeterminacy in the allocation of borrowing limits also holds more generally.⁸

Equivalence with fiat money economy: The equilibrium allocations characterized above have the property that steady-state interest rates must equal the steady state growth rate, $qg = 1$. This is not a coincidence. As is well known, the same steady-state property arises in models in which agents only trade a fixed supply of fiat money. To formalize this, consider the alternative environment described above (p. 8), with no borrowing and with unbacked public debt. With deterministic endowments, unbacked public debt is identical to a fixed supply of fiat money.⁹ Let the fixed money stock be denoted by M . Let Q_t denote the period- t price of money in terms of consumption goods, and $1/q_t = Q_{t+1}/Q_t$ the real return on holding one unit of money from t to $t + 1$. Proposition 2 characterizes steady-state equilibria of the economy with fiat money.

Proposition 2 *Whenever $\beta\bar{e} > \underline{e}$, there exists a non-autarkic steady-state, in which optimal consumption equals $\bar{c}g^t$ in periods of high endowment, $\underline{c}g^t$ in periods of low endowment, and optimal money holdings are 0 in high endowment periods and M in low endowment periods. Q_t in turn equals Qg^t , where $QM = \frac{1}{1+\beta}[\beta\bar{e} - \underline{e}]$. \bar{c} and \underline{c} are defined as in Proposition 1. In addition, there always exists an autarkic steady-state equilibrium, in which $Q_t = 0$ and $\bar{c} = \bar{e}$, $\underline{c} = \underline{e}$ for all t .*

directly comparing the optimal default or non-default consumption plans starting in a low-endowment period, or by relying on an arbitrage argument of the sort that we will use in the next section to establish more general results.

⁸However, when one takes into account transitional dynamics, the effects of this allocation on the consumption allocation is no longer neutral. We discuss such transitional dynamics in section 5 and provide a formal analysis in Appendix C.

⁹Just think of the money stock as one-period government bonds with a zero nominal interest rate. The government rolls over this stock of debt each period, thus keeping its supply constant. Allowing for a positive nominal interest rate and positive money growth would not affect our results, since the real interest rate would still be pinned down in steady state.

The steady-state consumption allocations and prices with fiat money are thus identical to the ones characterized in Proposition 1. This illustrates our second result, that the set of equilibrium allocations with self-enforcing private debt and limited market exclusion is equivalent to the set of equilibrium allocations in the fiat money economy. We can therefore use equilibrium characterizations that are well known for the fiat money environment to establish the existence of an equilibrium with positive levels of self-enforcing private debt, as well as characterize the resulting level of consumption-smoothing that takes place in equilibrium.

In the general stochastic environment, this equivalence result requires that the same set of state-contingent securities is available to agents in the economy with private debt and in the economy with unbacked public debt. This condition is met by generalizing the fiat money economy to an economy with unbacked state-contingent public debt as described in Section 2.¹⁰

4 Main Results

In this section, we show that the results derived for our deterministic example extend to a general environment with stochastic endowments. First, we provide a necessary and sufficient condition for borrowing constraints to be self-enforcing, which requires that at each history agents must be able to exactly finance their previous debt obligation by issuing new claims. We call this condition “exact roll-over.” Then, we show that if debt limits satisfy the exact roll-over condition, equilibrium allocations are equivalent to the ones in the economy with unbacked public debt. This result implies that if the economy primitives are such that unbacked public debt is valued in equilibrium, then there also exist equilibria with positive levels of self-enforcing private debt.

4.1 Self-enforcement and exact roll-over

Here, we show that debt limits are self-enforcing if and only if they allow for exact roll-over. Debt limits are said to allow for exact roll-over, if, at each history s^t , an agent can finance the maximum amount of outstanding promises, $-\phi^j(s^t)$, by issuing the maximum amount of new promises, $-\phi^j(s^{t+1})$, for each s^{t+1} following s^t .

Definition 4 *The debt limits $\Phi^j : \mathcal{S} \rightarrow \mathbb{R}$ allow for exact roll-over (ER), if and only if:*

$$\phi^j(s^t) = \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} \phi^j(s^{t+1}) \text{ for all } s^t \in \mathcal{S}. \quad (16)$$

¹⁰Moreover, if unbacked public debt and self-enforcing private debt are allowed to co-exist, the set of steady-state allocations remains the same. We will return to this point in Section 5.

Our first main result is then stated as follows:

Theorem 1 *The debt limits $\Phi^j : \mathcal{S} \rightarrow \mathbb{R}$ are self-enforcing (SE), if and only if they also allow for exact roll-over (ER).*

Theorem 1 shows that exact roll-over is a necessary and sufficient condition for debt limits to be self-enforcing. It generalizes the condition from the deterministic case, in which debt limits must grow at the real interest rate $1/q_t$, that is, the condition $\phi_t^j = q_t \phi_{t+1}^j$. In the remainder of this subsection, we discuss the formal steps used to establish Theorem 1, and provide a simple heuristic explanation of the relation between self-enforcement and exact roll-over for the deterministic case.

Exact roll-over implies self-enforcement: As a first step, Proposition 3 shows that if debt limits satisfy exact roll-over, they are also self-enforcing.

Proposition 3 *Suppose that the debt limits $\Phi^j : \mathcal{S} \rightarrow \mathbb{R}$ allow for exact roll-over. Then*

$$V^j(a, \Phi^j(s^t); s^t) = D^j(a - \phi^j(s^t); s^t) \text{ for all } s^t \in \mathcal{S}, \text{ for all } a \geq \phi^j(s^t). \quad (17)$$

Proof. We establish this result by comparing the set of feasible consumption plans without default and initial asset holdings $a \geq \phi^j(s^t)$ and the set of feasible consumption plans with default and initial asset holdings $a - \phi^j(s^t) \geq 0$, and showing that these sets are identical, for all $s^t \in \mathcal{S}$, and for all $a \geq \phi^j(s^t)$. This then immediately implies $V^j(a, \Phi^j(s^t); s^t) = D^j(a - \phi^j(s^t); s^t)$.

Consider therefore an arbitrary event s^t , $a \geq \phi^j(s^t)$, and $\{a(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ and $\{d(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$, s.t. $d(s^{t+\tau}) = a(s^{t+\tau}) - \phi^j(s^{t+\tau})$ for all $s^{t+\tau} \succ s^t$. Clearly, $d(s^{t+\tau}) \geq 0$ if and only if $a(s^{t+\tau}) \geq \phi^j(s^{t+\tau})$ and therefore $\{a(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ is feasible without default, if and only if $\{d(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ is feasible with a default.

To complete the proof, we therefore show that $\{a(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ and $\{d(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ also lead to the same consumption allocations. Using (16), we have that for all $s^{t+\tau} \succ s^t$,

$$y^j(s^{t+\tau}) + a(s^{t+\tau}) - \sum_{s^{t+\tau+1} \succ s^{t+\tau}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} a(s^{t+\tau+1}) = y^j(s^{t+\tau}) + d(s^{t+\tau}) - \sum_{s^{t+\tau+1} \succ s^{t+\tau}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} d(s^{t+\tau+1})$$

Likewise, for s^t , we have

$$y^j(s^t) + a - \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} a(s^{t+1}) = y^j(s^t) + a - \phi^j(s^t) - \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} d(s^{t+1})$$

Therefore, using (2), a consumption allocation consisting $\{c(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}}$ is feasible under asset plan $\{a(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ given $a(s^t) = a$, if and only if it is feasible under asset plan

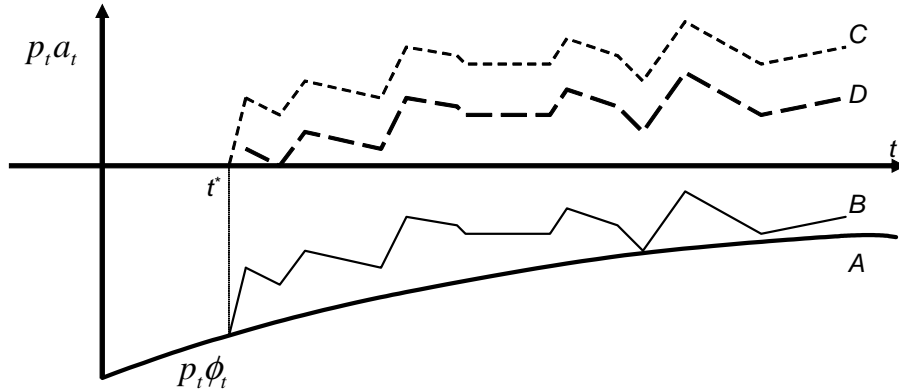


Figure 1: Bulow and Rogoff (1989a)

$\{d(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$, given $d(s^t) = a - \phi^j(s^t)$, and hence the resulting consumption allocations are identical. ■

Condition (17) implies the self-enforcement condition (5) by setting $a = \phi^j(s^t)$. It has the additional implication that the assumption that agents default on net asset positions and start with a net financial position of zero after default is not necessary for providing repayment incentives. Condition (17) states that if $a \geq \phi^j(s^t)$, an agent who defaults on his maximum gross amount of debt $|\phi^j(s^t)|$, but keeps his own asset holdings $a - \phi^j(s^t)$ after a default is always exactly indifferent between defaulting and not defaulting. With exact roll-over, the assumption that agents default on net asset positions can therefore be relaxed without weakening repayment incentives.

A graphical illustration: We can illustrate this characterization of self-enforcing debt limits and the relation to the roll-over condition with a series of figures. For this, we assume that endowment fluctuations are deterministic, as in the example of Section 3, and bonds are uncontingent.¹¹ The agents' budget constraint can then be rewritten as

$$c_t = y_t^j + \frac{1}{p_t} (p_t a_t - p_{t+1} a_{t+1}).$$

For a given sequence of prices $\{p_t\}$, we can thus compare the consumption profiles resulting from different asset plans simply by comparing the period-by-period changes in the present value of asset holdings, $p_t a_t - p_{t+1} a_{t+1}$. We use this observation to give a graphical illustration of an agent's incentives to default.

First, let us revisit the no lending result of BR. In BR, it is assumed that $Y_t^j = \sum_{\tau=0}^{\infty} p_{t+\tau} y_{t+\tau}^j / p_t$ is finite, and that $\phi_t^j \geq -Y_t^j$, i.e., that a borrower is never allowed to borrow more than the present

¹¹As in section 3, we replace the dependence on s^t by a time subscript to simplify notation.

value of all his future endowments. Any sequence of debt limits that satisfies these assumptions must also satisfy $\lim_{t \rightarrow \infty} p_t \phi_t^j = 0$.

In Figure 1 we present a graphic argument that shows that if the sequence of debt limits Φ^j satisfies $\lim_{t \rightarrow \infty} p_t \phi_t^j = 0$, and $\phi_s^j < 0$ for some date s , then the consumer will find it optimal to default at some date $t^* \geq s$. Clearly, along any such sequence of debt limits $p_t \phi_t^j$ must reach a minimum at some date t , and there exists t^* , such that $p_{t^*} \phi_{t^*}^j < \inf_{t > t^*} p_t \phi_t^j$, i.e., the borrower's debt limit is less tight in present value terms at t^* than at any subsequent date $t > t^*$. This is illustrated in Figure 1, where the thick line labelled A plots a possible path for the present value of the borrowing limit, $p_t \phi_t^j$, which satisfies this requirement. The thin line labelled B then plots an arbitrary asset profile that starts from $a_{t^*} = \phi_{t^*}^j$ and is consistent with this debt limit. Now, the short-dashed line labelled C , starting from an asset position of 0 at time t^* , plots an asset profile that represents a parallel upwards shift of B . From t^* on, this profile implements exactly the same consumption sequence as B , and since it only requires positive asset positions for all $t > t^*$, it is implementable after a default. Finally, the asset profile represented by the long-dashed line labelled D is a parallel downward shift of C starting from $t^* + 1$, which still maintains non-negative asset positions in all periods. Profile D thus remains feasible after a default and implements exactly the same consumption profile from date $t + 1$ on as B and C , but it delivers strictly higher consumption in period t . Hence D must be preferred to both B and C . Since B was chosen arbitrarily, we conclude that any non-default asset profile can be strictly improved upon by default at time t^* .

This argument shows that a sequence of debt limits Φ^j that satisfies the two assumptions made by BR can be self-enforcing if and only if $\phi_t^j = 0$ for all t , i.e., if no lending is allowed. In fact, if $\phi_t^j < 0$ for some t , self-enforcement requires that $\lim_{t \rightarrow \infty} p_t \phi_t^j < 0$, i.e., a consumer's debt limit must asymptotically grow at least at the rate of interest, contradicting either BR's assumption that $Y_t^j < \infty$, or that $-Y_t^j$ bounds ϕ_t^j from below.

A similar argument can be used to illustrate Proposition 3. Figure 2 considers debt limits Φ^j , such that $p_t \phi_t^j \leq p_{t+1} \phi_{t+1}^j$ for all t . As an equality, this condition represents the deterministic version of the exact roll-over condition. It implies that at each date t , a consumer must be able to exactly roll-over his maximum outstanding debt ϕ_t^j by issuing new claims which are valued at $(p_{t+1}/p_t) \phi_{t+1}^j$. In Figure 2, line A plots such debt limits, which are weakly expanding (and strictly expanding at date t'). Line B then plots an asset profile that is feasible for an agent who defaults at date t^* . Line C represents a parallel downward shift of B , thus implementing the same consumption profile, starting from an initial asset position of $\phi_{t^*}^j$ at date t^* . Since C never violates the debt limit

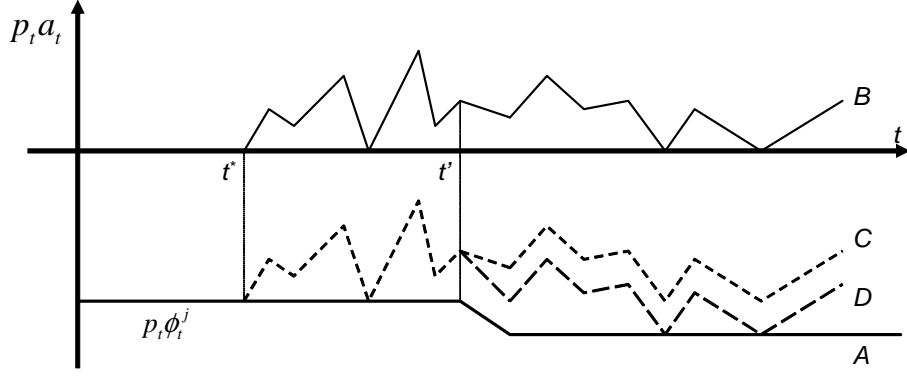


Figure 2: Expanding debt limits

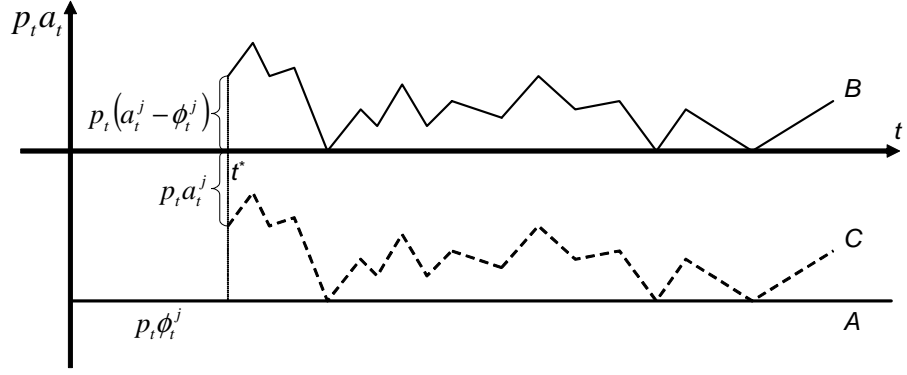


Figure 3: Exact roll-over and the strong self-enforcement condition

A , it is feasible for an agent who does not default. Finally, profile D uses the fact that the debt limit is strictly expanding in present value terms at t' to increase consumption in that period. D remains feasible without a default and is strictly preferred to B and C ; since B was picked arbitrarily, we conclude that no default is always at least as good as default, and it is strictly better than default, if Φ^j is strictly expanding at some subsequent date.

Finally, Figure 3 illustrates the knife-edge case where Φ^j satisfies the deterministic exact roll-over condition with equality in every period. As before, the thick line A plots the debt limit, $p_t \phi_t^j$. B and C are two asset profiles which start, respectively, from asset positions of $a - \phi_{t^*}^j \geq 0$ and $a \geq \phi_{t^*}^j$ at date t^* , and are parallel to each other; hence they lead to the same consumption sequences. When the exact roll-over condition holds with equality, C satisfies the debt limit, if and only if B is feasible after a default. Since this applies to any such pair of asset profiles B and C , we therefore conclude that the same consumption sequences can be implemented starting with assets

$a - \phi_{t^*}^j$ after a default, or starting with assets $a \geq \phi_{t^*}^j$, with no default. Therefore, the agent must be just indifferent between the two, or $V_t^j(a, \Phi_t^j) = D_t^j(a - \phi_t^j)$, for $a \geq \phi_t^j$ and all t .

Figures 2 and 3 thus provide a graphical illustration of the strong self-enforcement condition in Proposition 3. The roll-over condition, expressed as an inequality, is sufficient to deter default. If it holds as a strict inequality for some $t' > t$, then the agent is strictly better off not defaulting. If, instead, it holds as an equality in all periods then the agents is always indifferent between defaulting and not defaulting.

Self-enforcement implies exact roll-over: To complete the proof of Theorem 1, we show that the converse of Proposition 3 also holds.

Proposition 4 *Suppose that the debt limits $\Phi^j : \mathcal{S} \rightarrow \mathbb{R}$ are self-enforcing (SE). Then they allow for exact roll-over (ER).*

The proof of this proposition combines arbitrage arguments with specific properties of the agents' optimization problem. We present the complete proof in Appendix B; here we sketch the main steps.

Suppose that the sequence of borrowing limits $\Phi^j : \mathcal{S} \rightarrow \mathbb{R}$ is self-enforcing. Consider the problem of a type j agent who starts at an arbitrary event s^t with asset holdings $a(s^t) = \phi^j(s^t)$, and let $\{a^*(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ denote the resulting optimal asset profile. Now, construct the sequence of "shadow debt limits" $\tilde{\Phi}^j(s^t) : \mathcal{S}(s^t) \cup \{s^t\} \rightarrow \mathbb{R}$, which satisfies the following recursive condition:

$$\tilde{\phi}^j(s^{t+\tau}) = \begin{cases} \phi^j(s^{t+\tau}) & \text{if } a^*(s^{t+\tau}) = \phi^j(s^{t+\tau}) \\ \sum_{s^{t+\tau+1} \succ s^{t+\tau}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \min \left\{ \phi^j(s^{t+\tau+1}), \tilde{\phi}^j(s^{t+\tau+1}) \right\} & \text{otherwise} \end{cases} \quad (18)$$

In the appendix, we show that this sequence is well-defined and finite, for each $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$. The sequence of shadow debt limits $\tilde{\Phi}^j(s^t)$ is defined recursively as the sum over the continuation histories' shadow or actual debt limits, picking for each $s^{t+\tau+1} \succ s^{t+\tau}$ the larger of the two, in absolute value.

Next, we show that if $\phi^j(s^{t+\tau}) < \tilde{\phi}^j(s^{t+\tau})$ for some $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$, then an arbitrage argument shows that default is strictly better than no default at $s^{t+\tau}$. Therefore, for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$, $\phi^j(s^{t+\tau}) \geq \tilde{\phi}^j(s^{t+\tau})$, i.e., the sequence $\tilde{\Phi}^j(s^t)$ provides a lower bound for $\Phi^j(s^t)$. Moreover, by construction, $\tilde{\Phi}^j(s^t)$ can be strictly lower only if $a^*(s^{t+\tau}) > \phi^j(s^{t+\tau})$, i.e., at events where the borrowing constraint is non-binding.

This implies that the value of the no-default problem P1 starting from asset holdings of $a(s^t) = \phi^j(s^t)$ at s^t is the same under the original debt limits, Φ^j , as under the shadow debt limits, $\tilde{\Phi}^j(s^t)$.

Since the original debt limits are self-enforcing we obtain:

$$D(0; s^t) = V(\phi^j(s^t); \Phi^j(s^t), s^t) = V(\phi^j(s^t); \tilde{\Phi}^j(s^t), s^t) \geq V(\tilde{\phi}^j(s^t); \tilde{\Phi}^j(s^t), s^t),$$

where the last inequality follows from the monotonicity of V and the fact that $\phi^j(s^t) \geq \tilde{\phi}^j(s^t)$. On the other hand, we can show that for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$, the shadow debt limits $\tilde{\Phi}^j(s^t)$ satisfy the exact roll-over condition as a weak inequality:

$$\begin{aligned} \tilde{\phi}^j(s^{t+\tau}) &\geq \sum_{s^{t+\tau+1} \succ s^{t+\tau}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \tilde{\phi}^j(s^{t+\tau+1}) \quad \text{if } a^*(s^{t+\tau}) = \phi^j(s^{t+\tau}), \\ \tilde{\phi}^j(s^{t+\tau}) &= \sum_{s^{t+\tau+1} \succ s^{t+\tau}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \tilde{\phi}^j(s^{t+\tau+1}) \quad \text{otherwise.} \end{aligned}$$

The first line is established by arbitrage, and the second line follows from the characterization of $\tilde{\phi}^j(s^{t+\tau})$, using the fact that $\phi^j(s^{t+\tau+1}) \geq \tilde{\phi}^j(s^{t+\tau+1})$ for all $s^{t+\tau+1} \succ s^{t+\tau}$. Along the same lines as Proposition 3, an arbitrage argument then implies that starting from s^t with asset position $\tilde{\phi}^j(s^t)$, no-default must be weakly preferred to default at s^t , and the preference is strict, if $\tilde{\phi}^j(s^{t+\tau}) > \sum_{s^{t+\tau+1} \succ s^{t+\tau}} p(s^{t+\tau+1})/p(s^{t+\tau}) \tilde{\phi}^j(s^{t+\tau+1})$ for some $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$. In other words, we also have

$$V(\tilde{\phi}^j(s^t); \tilde{\Phi}^j, s^t) \geq D(0; s^t).$$

Therefore, both inequalities must hold with equality implying $\phi^j(s^t) = \tilde{\phi}^j(s^t)$, i.e., that the current actual debt limit equals the shadow debt limit, and that $\tilde{\Phi}^j$ satisfies the exact roll-over condition as an equality, for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$.

Using the definition of $\tilde{\Phi}^j$, this implies that for all $s^t \in \mathcal{S}$,

$$p(s^t) \phi^j(s^t) = \sum_{s^{t+\tau} \in \mathcal{B}(s^t)} p(s^{t+\tau}) \phi^j(s^{t+\tau}),$$

where $\mathcal{B}(s^t)$ is the set of histories $s^{t+\tau} \succ s^t$, at which the debt limit is binding for the first time after s^t , for an agent who starts at s^t with asset position $\phi^j(s^t)$. This can further be decomposed as

$$\begin{aligned} p(s^t) \phi^j(s^t) &= \sum_{s^{t+1} \succ s^t; s^{t+1} \in \mathcal{B}(s^t)} p(s^{t+1}) \phi^j(s^{t+1}) \\ &\quad + \sum_{s^{t+1} \succ s^t; s^{t+1} \notin \mathcal{B}(s^t)} \sum_{s^{t+\tau} \succ s^{t+1}; s^{t+\tau} \in \mathcal{B}(s^t)} p(s^{t+\tau}) \phi^j(s^{t+\tau}) \end{aligned}$$

i.e., we divide the set $\mathcal{B}(s^t)$ at which the debt limit is binding, into immediate successors $s^{t+1} \succ s^t$, and into histories $s^{t+\tau} \succ s^t$, with $\tau > 1$, which are not immediate successors of s^t . The latter set is then divided into subsets of histories $s^{t+\tau}$ that are each successors of the same immediate successor

$s^{t+1} \succ s^t$. As a final step of our proof, we exploit the monotonicity properties of the optimal asset plan to show that

$$\sum_{s^{t+\tau} \succ s^{t+1}; s^{t+\tau} \in \mathcal{B}(s^t)} p(s^{t+\tau}) \phi^j(s^{t+\tau}) = \sum_{s^{t+\tau} \in \mathcal{B}(s^{t+1})} p(s^{t+\tau}) \phi^j(s^{t+\tau})$$

Combining this with $p(s^{t+1}) \phi^j(s^{t+1}) = \sum_{s^{t+\tau} \in \mathcal{B}(s^{t+1})} p(s^{t+\tau}) \phi^j(s^{t+\tau})$, it follows that Φ^j satisfies the exact roll-over condition.

4.2 Allocational equivalence

We now turn to the question whether there exist equilibria with positive levels of self-enforcing debt, and how they can be characterized. We answer this question in Theorem 2, which states that a given consumption allocation and price vector constitute a competitive equilibrium with self-enforcing private debt, if and only if the same allocation and prices are an equilibrium of the corresponding economy with unbacked public debt. For the latter economy, there are known existence and characterization results (e.g. Santos and Woodford 1997), which then extend immediately to the economy with self-enforcing private debt.

Theorem 2 (i) *If $\{C^j, a^j, \Phi^j; p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with self-enforcing private debt, then $\{C^j, \hat{a}^j, d^j(s^0); D, p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with unbacked public debt, where $\hat{a}^j(s^t) = a^j(s^t) - \phi^j(s^t)$ for all j , $s^t \in \mathcal{S}$, $d^j(s^0) = -\phi^j(s^0)$ for all j , and $d(s^t) = \sum_{j=1}^J |\phi^j(s^t)|$ for all $s^t \in \mathcal{S}$.*

(ii) *If $\{C^j, \hat{a}^j, d^j(s^0); D, p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with unbacked public debt, then $\{C^j, \tilde{a}^j, \tilde{\Phi}^j; p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with self-enforcing private debt, where $\tilde{a}^j(s^t) = \hat{a}^j(s^t) + \tilde{\phi}^j(s^t)$, and $\tilde{\phi}^j(s^t) = -\frac{d^j(s^0)}{d(s^0)} d(s^t)$ for all j , $s^t \in \mathcal{S}$.*

Proof. (i) Fix a price sequence p , and suppose that Φ^j satisfies exact roll-over. Then, for $j = 1, \dots, J$, Proposition 3 implies $V^j(a, \Phi^j; s^0) = D^j(a - \phi^j(s^0); s^0)$, for all $a \geq \phi^j(s^0)$. Therefore, since $\{C^j, a^j\}$ solves (P1) for given debt limits Φ^j , prices p and zero initial asset holdings, $\{C^j, \hat{a}^j\}$ solves (P1), given zero debt limits, prices p , and initial asset holdings of $d^j(s^0) = a - \phi^j(s^0)$. But then, for given prices p , $\{C^j, \hat{a}^j\}$ are optimal allocations in the economy with unbacked public debt given initial asset holdings $d^j(s^0)$. Next, notice that if Φ^j allows for exact roll-over for all j , then $d(s^t) = \sum_{j=1}^J |\phi^j(s^t)|$ satisfies the government budget constraint (6), for all $s^t \in \mathcal{S}$. Finally, to show that $\{C^j, \hat{a}^j\}$ clears markets in the public debt economy, notice that asset market clearing in the private debt economy requires $\sum_{j=1}^J a^j(s^t) = 0$, for all $s^t \in \mathcal{S}$, which implies $\sum_{j=1}^J [\hat{a}^j(s^t) + \phi^j(s^t)] = 0$, or $\sum_{j=1}^J \hat{a}^j(s^t) = \sum_{j=1}^J |\phi^j(s^t)| = d(s^t)$, so $\{\hat{a}^j\}$ clears asset markets

in the public debt economy. Good markets clearing is immediate, since market clearing condition for goods markets equires that $\sum_{j=1}^J c^j (s^t) = \sum_{j=1}^J y^j (s^t)$ for all $s^t \in \mathcal{S}$, in both environments.

(ii) First, notice that if $d(s^t)$ satisfies (6), for all $s^t \in \mathcal{S}$, then the sequence of debt limits $\tilde{\Phi}^j$ as constructed above allows for exact roll-over, for all j . Now, reversing the above argument, if $\{C^j, \hat{a}^j\}$ is optimal in the public debt economy, given initial debt holdings of $d^j(s^0)$, then $\{C^j, \tilde{a}^j\}$ is optimal in the private debt economy, given borrowing limits $\tilde{\Phi}^j$ and zero initial asset holdings. Finally, asset market clearing implies that for all $s^t \in \mathcal{S}$, $\sum_{j=1}^J \hat{a}^j (s^t) = d(s^t) = -\sum_{j=1}^J \tilde{\phi}^j (s^t)$ or $\sum_{j=1}^J [\hat{a}^j (s^t) + \tilde{\phi}^j (s^t)] = \sum_{j=1}^J \tilde{a}^j (s^t) = 0$, which implies that $\{C^j, \tilde{a}^j\}$ also clears asset markets in the private debt economy. That $\{C^j, \tilde{a}^j\}$ also clears goods markets is then immediate. ■

Theorem 2 determines a mapping between the initial borrowing limits in the private debt economy and the initial holdings of public debt in the public debt economy. It also determines a mapping, for each s^t , between the aggregate private debt in circulation in the first economy and the aggregate public debt in circulation in the second. Once the initial borrowing limits and asset positions are aligned and the dynamics of aggregate debt are the same, then the same consumption allocation and the same real rates of return emerge in the two economies.

The argument of this theorem is established in three steps. First, for a given set of bond returns, the strong self-enforcement condition (17) in Proposition 3 implies that once each agent's initial debt holdings are equated to his initial private debt limit, the set of feasible consumption allocations coincide in the public and private debt economies. A given equilibrium consumption allocation is then optimal in both economies.

Second, starting from the debt limits $\tilde{\Phi}^j$, construct debt levels for the public debt economy as $d(s^t) = \sum_{j=1}^J |\phi^j (s^t)|$. If $\tilde{\Phi}^j$ allows for exact roll-over, it must be the case that the sequence $\{d(s^t)\}$ satisfies the government budget constraint (6). The reverse is also true: for any sequence of public debt levels that satisfy (6), we can construct debt limits for the private debt economy, such that initial debt limits are equated to initial public debt holdings for each type, debt limits allow for exact roll-over, and $d(s^t) = \sum_{j=1}^J |\phi^j (s^t)|$ for all histories. As in the example of Section 3, these debt limits are not uniquely determined for each individual, but the aggregate private debt in circulation is.

Finally, we check that the resulting allocations and asset holdings clear the markets in the private debt economy, if and only if they clear markets in the economy with unbacked public debt. For goods markets, this is immediate since market-clearing requires in both cases that $\sum_{j=1}^J c^j (s^t) = \sum_{j=1}^J y^j (s^t)$ for all $s^t \in \mathcal{S}$.

The equivalence result is not restricted to steady-states, but applies to any competitive equilibrium. To interpret this result consider the following. If we aggregate the total debt in circulation in the private debt economy, and use the exact roll-over condition, we see that aggregate debt satisfies the same aggregate law of motion as the unbacked public debt issued by the government. The public debt economy is an economy where the government cannot use taxation to finance the repayment of its claims, and must instead roll them over indefinitely. On the other hand, in the private debt economy agents have extremely limited power to enforce private debt, to the extent that agents can only issue claims that can be repaid by issuing new claims, whereas any contract that required a net transfer of resources from some date forward would not be sustainable. The lack of taxation power on the government side matches exactly the lack of commitment on the agents' side. The equivalence thus arises when both the public sector and the private sector have very limited power to collect payments from market participants.

We conclude this section with a brief comment about the preference assumptions. Time-additive separability, strict concavity and boundedness of the life-time utility function enter only in Proposition 4. Since Proposition 3 and Theorem 2 relied purely on arbitrage arguments, they only require strict monotonicity of U w.r.t. C , and therefore hold much more generally, if (for Theorem 2) one restricts attention to debt limits that allow for exact roll-over as one particular class of equilibria with self-enforcing private debt.

5 Extensions

In this section, we discuss some extensions of the results derived and some related applications in the context of monetary models and in international finance.

Out-of-steady-state dynamics: As noticed above, Theorem 2 applies to all equilibria, not only to steady-states. To elaborate on that observation, one can go back to the example in Section 3 and look at (i) transitional dynamics and (ii) non-stationary equilibria.

In an economy with fiat money, it is well known that the initial allocation of fiat money across agents will determine the transition to steady-state consumption allocations. Theorem 2 implies that, in the same way, the initial borrowing limits will determine the transition to steady-state in the private debt economy. The analysis in Appendix C derives explicitly the transition path for the model of Section 3, and shows the mapping between the private debt and the fiat money economy.

Likewise, it is well known that in models with fiat money there are non-stationary equilibria with hyperinflation, where the real value of money collapses over time. By Theorem 2, there must

also exist equilibria of the private debt economy with similar features, i.e., non-stationary equilibria in which the real value of private debt is collapsing over time. In Appendix C we analyze these equilibria formally. The logic of these equilibria is the following: if agents anticipate that there will be a tightening of borrowing constraints in the future, this reduces repayment incentives today. This means that there are equilibrium sequences of debt limits that go to zero over time, with the equilibrium allocation converging to autarky.

Co-existence of public and private debt and implications for monetary theory: It is straightforward to extend our model to allow for the co-existence of unbacked public debt and self-enforcing private debt. This does not change the set of equilibrium consumption allocations but introduces a source of indeterminacy regarding the real value of public and private debt in circulation. If we go back to the example of Section 3, it is possible to show that the same steady state allocation described in Propositions 1 and 2 can be supported in an equilibrium where money and private debt co-exist. In this equilibrium the price of money Q and the debt limits ϕ_o, ϕ_e must satisfy:

$$QM + [-\phi_o - \phi_e] = \frac{1}{1 + \beta} [\beta \bar{e} - \underline{e}].$$

If both are positively valued, they must be perfect substitutes, and their rates of return must be identical. This is reminiscent of the indeterminacy result of Kareken and Wallace (1981) regarding the co-existence of multiple fiat currencies.

In the absence of aggregate shocks, unbacked public debt may be interpreted as fiat money, while self-enforcing private debt may be interpreted as a form of inside money, such as bank deposits. Within our environment, one is sustainable if and only if the other is, and they lead to identical real allocations. When both are available, this merely leads to a further indeterminacy in how much each is used in transactions. We view this as a useful benchmark result. The existing monetary literature discusses the circulation of fiat and inside money largely in separation from each other. The circulation of fiat money requires that an intrinsically useless asset is traded at a positive price, which connects the analysis to the possibility of rational bubbles. The circulation of inside money (demandable debt) instead relies on having the proper reputational mechanisms in place to guarantee that outstanding claims are honored; for this, the Bulow-Rogoff puzzle is directly relevant. Although on the surface these seem distinct conceptual problems, our analysis shows that the sustainability of fiat and inside money are actually closely related.¹²

¹²See Cavalcanti, Erosa and Temzelides (1999) and Berentsen, Camera and Waller (2005) for matching models of private debt circulation under limited commitment. Cavalcanti et al. study a monetary matching model, in which a fixed subset of agents is allowed to issue notes, which are sustained by the loss of a non-competitive note-issuing rent

The case where both public and private debt co-exist can also be used to clarify the informational and enforcement frictions under which the BR assumption that agents after a default are only denied credit, but cannot be prevented from savings emerges as the equilibrium punishment. This assumption requires the public monitoring of *negative* asset holdings, to enforce the borrowing limits, but, at the same time, this assumption rules out the monitoring and confiscation of *positive* asset holdings after the default episode. This assumption arises if some debt contracts take the form of bearer bonds, i.e., bonds whose ownership is not monitored. This is a particularly natural assumption for government-issued fiat money. In an environment with government-issued fiat money, the BR assumption then emerges naturally as the strongest possible punishment, since after a default, agents are able to use fiat money anonymously for savings, even if they are completely excluded from private debt transactions.

Sovereign debt and international capital flows: The original motivation of Bulow and Rogoff’s was to study the sustainability of sovereign debt by reputation. Our analysis suggests that such debt may indeed be sustainable, provided that prices and debt ceilings adjust accordingly. Whereas much of the existing literature following BR has treated the no-lending result as a theoretical puzzle, our results suggest that debt sustainability should first be addressed as a quantitative question. In particular, the relevant issue is to examine whether or not debt limits and international rates of return are consistent with repayment incentives. A quantitative evaluation of this issue would require us to enrich our model to allow for a number of features of actual international capital flows, in particular the presence of gross positions on different types of public and private instruments. This evaluation is outside the scope of this paper. However, several observers have recently noticed that the largest world debtor, the US, does indeed pay a low rate of return on its external liabilities.¹³ Under our approach, this low rate of return provides the US with a broad form of “seignorage,” reflected in the fact that a *positive* financial flow is associated to the net debtor position. The threat of losing this seignorage provides a simple discipline device that gives the proper incentives to the sovereign debt issuer.¹⁴

if outstanding notes are not redeemed on demand. Related to our results, Berentsen et al. study a matching model with money and competitive supply of bank credit, and show among other things that with lack of commitment, such credit is sustainable only if the inflation rate is non-negative.

¹³See e.g. Gourinchas and Rey (2005).

¹⁴This net flow of resources is due to (1) the fact that the debt of the US grows over time, and (2) the fact that the US receives a higher return on its gross asset positions than what it pays on his gross liabilities. To capture the second element would require a model with an explicit treatment of gross financial positions.

Our theoretical analysis is related also to the model of private international capital flows analyzed by Jeske (2006) and Wright (2006). In their model, individuals can borrow either with full commitment in a domestic capital market, or with limited commitment in an international market; after a default on a borrower's outstanding international debt, the borrower is excluded from the international, but not the domestic capital market. With perfect domestic commitment, all agents in a country are either simultaneously constrained or unconstrained in international markets (otherwise the unconstrained agents could profitably intermediate the constrained agents' access to international markets), and if the country is unconstrained, domestic rates of return are the same as international rates of return. Therefore, with perfect access to domestic credit markets, agents are able to save at international rates of return, with these savings being intermediated by the agents' compatriots, but they lose access to credit at international rates and they actually prefer not to borrow at the domestic rates. The allocations that can be supported by private international capital flows then replicate exactly the general equilibrium allocations of our Bulow-Rogoff economy, and our equivalence with rational bubbles can also be used to show existence and characterize equilibria in the Jeske-Wright model.¹⁵

Other extensions: Our analysis has focused on an endowment economy. Extending our results to the case of production economies presents a number of challenges, which are well known from the literature on rational bubbles. Namely, allowing for capital accumulation restricts the sustainability of rational bubbles, or of positively priced fiat money, if capital is freely tradeable. Generalizing our equivalence result to production economies with capital accumulation then implies that these same factors also restrict the sustainability of private debt. One way around this problem is to assume that capital accumulation and capital transfers are subject to real or financial frictions. This avenue is pursued, for example, in Woodford (1990) and Ventura (2004), who show that if the financial friction is sufficiently strong, bubbles can still arise in equilibrium.

It may also be interesting to explore the implications of our model for income and consumption inequality and risk-sharing. Krueger and Perri (2006), for example, ask whether a limited commitment model can account for the trends of increasing income and consumption inequality over the last 25 years. A key insight from their analysis is that an increase in income volatility may generate only a much smaller increase, or possibly a decrease, in consumption inequality, because the increasing income volatility makes a default less appealing, which improves the opportunities for risk-sharing. The magnitude of this effect depends primarily on the extent to which agents can smooth consumption after a default, and it may therefore be worthwhile to study the risk-sharing

¹⁵See Wright (2006), Section 5, for a formal discussion.

implications of alternative consequences of default.

To account for consumption and income volatility and risk sharing, it may also be useful to extend our model to allow for unobserved idiosyncratic shocks and incomplete markets. This extension is important also from a theoretical perspective: early versions of the Bewley-style fiat money model assume explicitly that agents face unobserved idiosyncratic endowment shocks. A generalization of our results to incomplete market economies would then further reinforce the equivalence between self-enforcing private and unbacked public securities.¹⁶

6 Conclusion

In this paper, we have studied a general equilibrium economy with self-enforcing private debt, in which, after a default, borrowers are excluded from future credit, but retain the ability to save in the market. For a partial equilibrium version of this model, in which a small open economy borrows internationally at fixed, positive interest rates, Bulow and Rogoff (1989a) have shown that debt cannot be sustainable by reputational mechanisms only: eventually, the country always has an incentive to default. In contrast, we show that positive levels of debt can be sustained in general equilibrium. The key to our result is that interest rates adjust downwards to provide the right repayment incentives.

More generally, we have established two results. First, we show that with future exclusion from credit as the only consequence of default, debt limits are self-enforcing if and only if they allow agents to exactly honor their outstanding payment obligations by issuing new debt (exact roll-over). Second, if debt limits satisfy exact roll-over, the resulting equilibrium allocations are equilibrium allocations of an economy, in which a government issues unbacked public securities and rolls them over period by period. For the latter environment, there exist well known existence results for non-autarkic equilibria with positive levels of debt.

We believe that our characterization results may be useful for a variety of applications. In the context of sovereign debt, to which this model was originally applied, our analysis suggests that the sustainability of debt should be viewed not so much as a theoretical puzzle, but as a quantitative issue. We leave an exploration of these and other applications to future work.

¹⁶One complication of such a generalization is the possibility that agents may actually use default explicitly to obtain better insurance against idiosyncratic risk; i.e to “complete the market”, in a sense.

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7 Appendix A: Proofs

Proof of Lemma 1. For $j \in \{o, e\}$, a consumption and asset profile $\left\{c_t^j, a_{t+1}^j\right\}_{t=0}^{\infty}$ is a solution to (P1), if and only if it satisfies the sequence of flow budget constraints, the sequence of Euler equations $q_t \frac{1}{c_t^j} \geq \frac{\beta}{c_{t+1}^j}$ for all t , which must hold with equality if $a_{t+1}^j > \phi_{t+1}^j$, and the transversality condition $\lim_{t \rightarrow \infty} \beta^t \frac{1}{c_t^j} a_{t+1}^j = 0$. We check that our proposed solution satisfies these conditions for an agent starting with an asset profile $a_t^j = \phi_t^j$ in a high endowment period.

Therefore need to check that the solution satisfies

$$\begin{aligned}
c_{t+\tau}^j &= y_{t+\tau}^j + a_{t+\tau}^j - q_{t+\tau} a_{t+\tau+1}^j, \\
c_{t+\tau+1}^j &= y_{t+\tau+1}^j + a_{t+\tau+1}^j - q_{t+\tau+1} a_{t+\tau+2}^j, \\
c_{t+\tau}^j &= c_{t+\tau+1}^j \cdot q_{t+\tau} / \beta, \quad a_{t+\tau+1}^j \geq \phi_{t+\tau+1}^j, \\
c_{t+\tau+1}^j &\leq c_{t+\tau+2}^j \cdot q_{t+\tau+1} / \beta, \quad a_{t+\tau+1}^j = \phi_{t+\tau+2}^j, \\
\lim_{t \rightarrow \infty} \beta^t \frac{1}{c_t^j} a_{t+1}^j &= 0.
\end{aligned}$$

for each $\tau = 2 \cdot k$.

Substituting $y_t^j = \bar{e}g^t$ and $y_{t+1}^j = \underline{e}g^{t+1}$, our conjectures for bond prices $q_t = q_j$, $q_{t+1} = q_{-j}$, debt limits $\phi_t^j = \phi_j g^t$ and $\phi_{t+2}^j = \phi_j g^{t+2}$, consumption profiles $c_t^j = \bar{c}_j g^t$ and $c_{t+1}^j = \underline{c}_j g^{t+1}$, and asset holdings $a_{t+1}^j = a_j g^{t+1}$, we can solve the first three conditions for \bar{c}_j , \underline{c}_j and a_j and find (7)-(9) as a solution. Since $\frac{\bar{c}_j}{\underline{c}_j} = gq_{-j}/\beta$ and $q_{-j}, q_j \geq \beta/g$, we obtain the following chain of inequalities

$$\frac{\bar{c}_j}{\underline{c}_j} = \beta / (gq_{-j}) \leq 1 \leq gq_j / \beta,$$

so the Euler inequality for $t+1$ is also satisfied. Moreover, $|a_{t+1}^j|/c_t^j \leq \frac{1}{\underline{c}_j} g \max\{a, \phi\}$, so the transversality condition is satisfied as well. Finally, the inequality $a_{t+1}^j \geq \phi_{t+1}^j$ is satisfied by hypothesis. ■

Proof of Lemma 2. We again check that the proposed allocation satisfies the consumer's flow budget constraints, the Euler equations and the transversality condition, starting from a period t with high endowments, in which $a_t^j = 0$. This is immediate, since the problem is the same as that in Lemma 1, after setting $\phi_j = 0$. In this case, we can check that the condition $a_j^d \geq 0$ always holds, because $\beta\bar{e} \geq q_j g \underline{e}$ follows from the fact that $\bar{e} \geq \underline{e}$ and $q_j \leq \beta/g$. ■

Proof of Proposition 1. Apart from the autarkic case, the proof is in the text. Suppose that $\phi_o = \phi_e = 0$. Then, market clearing requires that $a_t^j = 0$ for all j, t . Given our solution to (P2), it is easy to check that $a_t^j = 0$ for all t is optimal if and only if $\beta\bar{e} \leq (q_j g) \underline{e}$. Thus, whenever $(q_j g) \geq \frac{\beta\bar{e}}{\underline{e}}$ for $j \in \{o, e\}$, the autarkic allocation is indeed optimal, and the bond market clears without any trade actually occurring. ■

Proof of Proposition 2. For a given sequence of prices $Q_t > 0$, the household's problem is

$$\begin{aligned}
V(M_0) &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t. } c_t &\leq y_t^j + Q_t(M_t - M_{t+1}) \\
M_{t+1} &\geq 0
\end{aligned}$$

Defining $a_t = Q_t M_t$ and $q_t = Q_t / Q_{t+1}$, this problem is identical to (P1), with zero debt limits and initial asset holdings of $a_0 = Q_0 M_0$. Therefore, conjecturing that $q_t = q$ is constant in steady-state, the optimal steady-state allocations are characterized from Lemma 1, as $\bar{c}g^t$ in high endowment periods, and $\underline{c}g^t$ in low endowment periods, where

$$\bar{c} = \frac{1}{1+\beta} [\bar{e} + (qg) \underline{e}] \quad \text{and} \quad \underline{c} = \frac{\beta/(qg)}{1+\beta} [\bar{e} + (qg) \underline{e}]$$

These allocations clear the market, if and only if $\bar{e} + \underline{e} = \bar{c} + \underline{c}$, or

$$\begin{aligned} \bar{e} - \frac{1}{1+\beta} [\bar{e} + (qg) \underline{e}] &= \frac{\beta/(qg)}{1+\beta} [\bar{e} + (qg) \underline{e}] - \underline{e} \\ \text{or } \frac{1}{1+\beta} [\beta\bar{e} - (qg) \underline{e}] &= \frac{1/(qg)}{1+\beta} [\beta\bar{e} - (qg) \underline{e}] \end{aligned}$$

Thus, markets clear if and only if $qg = 1$, which corresponds to the non-autarkic equilibrium. In addition, there exists an autarkic equilibrium, in which $Q_t = 0$ and $c_t^j = y_t^j$ for all t, j . ■

8 Appendix B: Proof of Proposition 4

Here, we discuss the proof of Proposition 4. Since this result is significantly more involved than the others, we discuss its proof separately from the other results, and have divided it into seven lemmas. Lemma 3 establishes useful properties of the solution to the household problem (P1). Lemmas 4-7 then establish the existence and characterization of the shadow debt limits $\tilde{\Phi}$. Lemma 8 establishes the exact roll-over condition for $\tilde{\Phi}$, and Lemma 9 uses this to show that ER also holds for Φ . To simplify notation, we omit the superscript j throughout the proof.

Suppose that $V(\phi(s^t), \Phi(s^t); s^t) = D(0; s^t)$ for all $s^t \in \mathcal{S}$. If $\phi(s^t) = 0$ for all $s^t \in \mathcal{S}$, the proposition holds trivially. Suppose therefore that $\phi(s^t) < 0$ for some s^t . Let $\{a^*(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ be the optimal asset profile and $\{c^*(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}}$ the associated optimal consumption profile, starting from asset holdings of $\phi(s^t)$ at history s^t . Lemma 3 establishes some useful properties for the solution to problem (P1), the value functions $V(a, \Phi(s^t); s^t)$ and the optimal asset plan $\{a^*(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$.

Lemma 3 (i) $V(a, \Phi(s^t); s^t)$ is strictly increasing, differentiable and strictly concave in a , for $a \geq \underline{a}(s^t) \equiv -y(s^t) + \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} \phi(s^{t+1})$.

(ii) $\phi(s^t) > \underline{a}(s^t)$.

(iii) if $\{\tilde{a}(s^{t+k})\}_{s^{t+k} \in \mathcal{S}(s^{t+\tau})}$ is optimal starting from initial asset holdings $\phi(s^{t+\tau})$ at history $s^{t+\tau}$, and $a^*(s^{t+\tau}) > \phi(s^{t+\tau})$, then $a^*(s^{t+k}) \geq \tilde{a}(s^{t+k})$ for all $s^{t+k} \in \mathcal{S}(s^{t+\tau})$.

Proof. Part (i) follows immediately from the properties of problem (P1). To characterize \underline{a} notice that the budget constraint at s^t can be rewritten as

$$c(s^t) \leq y(s^t) + a(s^t) - \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} a(s^{t+1})$$

Since $c(s^t)$ must be non-negative, and $a(s^{t+1}) \geq \phi(s^{t+1})$, the budget set:

$$\left\{ c(s^t), \{a(s^{t+1})\} : c(s^t) \in \left[0, y(s^t) + a(s^t) - \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} a(s^{t+1}) \right]; a(s^{t+1}) \geq \phi(s^{t+1}) \right\}$$

is non-empty only if $y(s^t) + a(s^t) - \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} \phi(s^{t+1}) \geq 0$, or equivalently $a(s^t) \geq \underline{a}(s^t)$.

Part (ii): if $a(s^t) = \underline{a}(s^t)$, the only feasible allocation without default yields $c(s^t) = 0$, and $a(s^{t+1}) = \phi(s^{t+1})$ for all $s^{t+1} \succ s^t$. This yields a life-time expected utility of $u(0) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V(\phi(s^{t+1}), \Phi(s^{t+1}), s^{t+1})$. If instead the agent defaults and sets $d(s^{t+1}) = 0$ for all $s^{t+1} \succ s^t$, his life-time expected utility is $u(y(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) D(0, s^{t+1})$. Since $D(0, s^{t+1}) = V(\phi(s^{t+1}), \Phi(s^{t+1}), s^{t+1})$ for all s^{t+1} , it follows immediately that at $a(s^t) = \underline{a}(s^t)$ default is strictly preferred to no default, and hence $\phi(s^t) > \underline{a}(s^t)$.

Part (iii): For given a , the first-order conditions for $c(s^t)$ and $\{a(s^{t+1})\}$ are

$$\begin{aligned} u'(c(s^t)) &= \lambda(s^t) \\ \beta \pi(s^{t+1}|s^t) V_a(a(s^{t+1}), \Phi(s^{t+1}), s^{t+1}) &= \lambda(s^t) \frac{p(s^{t+1})}{p(s^t)} + \mu(s^{t+1}) \end{aligned}$$

where $\lambda(s^t)$ and $\mu(s^{t+1})$ are, respectively, the Lagrange multipliers on the budget constraint at s^t and the debt limit for s^{t+1} . We can rewrite these conditions as

$$c(s^t) = \gamma(\lambda(s^t)) \text{ and } a(s^{t+1}) = \max\{\psi(\lambda(s^t); s^{t+1}); \phi(s^{t+1})\}$$

where $\gamma(\cdot) = (u')^{-1}(\cdot)$, and $\psi(\cdot; s^{t+1}) = (V_a)^{-1}\left(\lambda \frac{p(s^{t+1})}{p(s^t)} \frac{1}{\beta \pi(s^{t+1}|s^t)}\right)$. The budget constraint can then be rewritten as

$$\gamma(\lambda(s^t)) + \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} \max\{\psi(\lambda(s^t); s^{t+1}); \phi(s^{t+1})\} = y(s^t) + a$$

Since u is concave in c and V is concave in a , $\gamma(\cdot)$ and $\psi(\cdot; s^{t+1})$ are decreasing in λ , and therefore there exists a unique value of $\lambda(a; s^t)$ which solves the budget constraint; moreover $\lambda(\cdot; s^t)$ is strictly decreasing in a . Therefore, at the optimum, $c(s^t)$ and $\{a(s^{t+1})\}$ are all non-decreasing in $a(s^t)$. But then, the proposition follows immediately. ■

Starting from s^t and $\mathcal{N}_0(s^t) \equiv \{s^t\}$, we now define the following sets of events:

$$\begin{aligned}\mathcal{N}_\tau(s^t) &= \{s^{t+\tau} \succ s^t : a^*(s^{t+\tau}) > \phi(s^{t+\tau}) \text{ and } \sigma(s^{t+\tau}) \in \mathcal{N}_{\tau-1}(s^t)\} \\ \mathcal{B}_\tau(s^t) &= \{s^{t+\tau} \succ s^t : a^*(s^{t+\tau}) = \phi(s^{t+\tau}) \text{ and } \sigma(s^{t+\tau}) \in \mathcal{N}_{\tau-1}(s^t)\} \\ \mathcal{N}(s^t) &= \bigcup_{\tau=1}^{\infty} \mathcal{N}_\tau(s^t), \mathcal{B}(s^t) = \bigcup_{\tau=1}^{\infty} \mathcal{B}_\tau(s^t).\end{aligned}$$

$\mathcal{N}_\tau(s^t)$ denotes the set of histories $s^{t+\tau}$ along which the debt limit was never binding between event s^t and $s^{t+\tau}$, and $\mathcal{N}(s^t)$ the union of all such sets. $\mathcal{B}_\tau(s^t)$ denotes the set of histories $s^{t+\tau}$ at which the debt limit is binding for the first time after s^t , and $\mathcal{B}(s^t)$ the union of all such sets. If the debt limit never binds, then $\mathcal{B}(s^t)$ is empty.

Next, we recursively define the following ‘auxiliary’ debt limits $\tilde{\Phi}(s^t) : \mathcal{S}(s^t) \cup \{s^t\} \rightarrow \mathbb{R}$:

$$\tilde{\phi}(s^{t+\tau}) = \begin{cases} \sum_{s^{t+\tau+1} \succ s^{t+\tau}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \min \left\{ \phi(s^{t+\tau+1}), \tilde{\phi}(s^{t+\tau+1}) \right\} & \text{if } a^*(s^{t+\tau}) > \phi(s^{t+\tau}) \\ \phi(s^{t+\tau}) & \text{if } a^*(s^{t+\tau}) = \phi(s^{t+\tau}) \end{cases} \quad (19)$$

It is immediate to check that a solution $\tilde{\Phi}(s^t)$ to (19) exists, if $\tilde{\Phi}(s^t)$ is allowed to take on the value of $-\infty$. Our next task is to establish that a finite-valued solution $\tilde{\Phi}(s^t)$ exists, and to characterize this solution; this is immediate for all $s^{t+\tau}$ s.t. $a^*(s^{t+\tau}) = \phi(s^{t+\tau})$, but not otherwise. We complete this in four steps, that are formulated by the next four lemmas. The first three characterize the solution for all $s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}$. The fourth lemma extends this characterization to all of $\mathcal{S}(s^t)$.

Lemma 4 For all $s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}$, define $\hat{\phi}(s^{t+\tau})$ and $Y(s^{t+\tau})$ by

$$\begin{aligned}\hat{\phi}(s^{t+\tau}) &= \sum_{s^{t+\tau+k} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+k})}{p(s^{t+\tau})} \phi(s^{t+\tau+k}) \\ Y(s^{t+\tau}) &\equiv y(s^{t+\tau}) + \sum_{s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+k})}{p(s^{t+\tau})} y(s^{t+\tau+k})\end{aligned}$$

Then, $Y(s^{t+\tau}) < \infty$ and $\phi(s^{t+\tau}) + Y(s^{t+\tau}) \geq \hat{\phi}(s^{t+\tau}) > -\infty$.

Proof. Summing the agent’s budget constraint over $s^{t+\tau}$ and all $s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})$, we

get

$$\begin{aligned}
& \sum_{s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+k}) c^*(s^{t+\tau+k}) + p(s^{t+\tau}) c^*(s^{t+\tau}) \\
= & p(s^{t+\tau}) y(s^{t+\tau}) + \sum_{s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+k}) y(s^{t+\tau+k}) + p(s^{t+\tau}) a^*(s^{t+\tau}) \\
& - \sum_{s^{t+\tau+k} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+k}) \phi(s^{t+\tau+k}) - \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) a^*(s^{t+\tau+K}) \\
= & p(s^{t+\tau}) \left[a^*(s^{t+\tau}) + Y(s^{t+\tau}) - \hat{\phi}(s^{t+\tau}) \right] \\
& - \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) a^*(s^{t+\tau+K})
\end{aligned}$$

For all $s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})$, the first-order condition holds with equality, implying

$$p(s^{t+\tau+k}) c^*(s^{t+\tau+k}) = \frac{p(s^{t+\tau})}{u'(c^*(s^{t+\tau}))} \beta^k \frac{\pi(s^{t+\tau+k})}{\pi(s^{t+\tau})} u'(c^*(s^{t+\tau+k})) c^*(s^{t+\tau+k})$$

Substituting this into the LHS of the budget constraint and using $u'(c^*(s^{t+\tau+k})) c^*(s^{t+\tau+k}) \leq (u(c^*(s^{t+\tau+k})) - u(0)) \leq \bar{U}$, we find

$$\begin{aligned}
& \sum_{s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+k}) c^*(s^{t+\tau+k}) \\
= & \frac{p(s^{t+\tau})}{u'(c^*(s^{t+\tau}))} \sum_{s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \beta^k \frac{\pi(s^{t+\tau+k})}{\pi(s^{t+\tau})} u'(c^*(s^{t+\tau+k})) c^*(s^{t+\tau+k}) \\
\leq & \frac{p(s^{t+\tau})}{u'(c^*(s^{t+\tau}))} \sum_{s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \beta^k \frac{\pi(s^{t+\tau+k})}{\pi(s^{t+\tau})} \bar{U} \leq \frac{p(s^{t+\tau})}{u'(c^*(s^{t+\tau}))} \frac{1}{1-\beta} \bar{U}
\end{aligned}$$

For the RHS, the agents' transversality condition implies that

$$\begin{aligned}
0 &= \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{S}(s^{t+\tau})} \beta^K \frac{\pi(s^{t+\tau+K})}{\pi(s^{t+\tau})} u'(c^*(s^{t+\tau+K})) a^*(s^{t+\tau+K}) \\
&= \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \beta^K \frac{\pi(s^{t+\tau+K})}{\pi(s^{t+\tau})} u'(c^*(s^{t+\tau+K})) a^*(s^{t+\tau+K}) \\
&= \frac{u'(c^*(s^{t+\tau}))}{p(s^{t+\tau})} \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) a^*(s^{t+\tau+K})
\end{aligned}$$

where the first equality makes use of the transversality conditions for all $s^{t+\tau+K} \in \mathcal{S}_K(s^{t+\tau}) \setminus \mathcal{N}_{\tau+K}(s^t)$, and the second equality again uses the agents' first-order condition. Hence, we have

$$\lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) a^*(s^{t+\tau+K}) = 0.$$

But then, it follows immediately that $Y(s^{t+\tau}) < \infty$ and $\hat{\phi}(s^{t+\tau}) > -\infty$. Moreover, by Lemma 3 (ii), we have $y(s^t) + \phi(s^t) - \sum_{s^{t+1} \succ s^t} \frac{p(s^{t+1})}{p(s^t)} \phi(s^{t+1}) \geq 0$ for all $s^t \in \mathcal{S}$. Summing this inequality over $s^{t+\tau}$ and all $s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})$, we find

$$p(s^{t+\tau}) Y(s^{t+\tau}) + p(s^{t+\tau}) \phi(s^{t+\tau}) - \sum_{s^{t+\tau+k} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+k}) \phi(s^{t+\tau+k}) \geq 0$$

or $Y(s^{t+\tau}) + \phi(s^{t+\tau}) - \tilde{\phi}(s^{t+\tau}) \geq 0$. ■

The second lemma establishes the existence of a solution to (19), for all $s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}$:

Lemma 5 *There exists a solution $\tilde{\Phi}(s^t) : \mathcal{N}(s^t) \cup \{s^t\} \rightarrow \mathbb{R}$ to (19). Moreover, for all $s^{t+\tau} \in \mathcal{N}_\tau(s^t) \cup \{s^t\}$, $\tilde{\Phi}$ satisfies $\tilde{\phi}(s^{t+\tau}) \in [\hat{\phi}(s^{t+\tau}) - Y(s^{t+\tau}), 0]$, as well as the limit property*

$$\lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) \tilde{\phi}(s^{t+\tau+K}) = 0.$$

Proof. Let $\{\phi^{(0)}(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}}$ be defined by $\phi^{(0)}(s^{t+\tau}) = \hat{\phi}(s^{t+\tau}) - Y(s^{t+\tau})$, and define $\{\phi^{(K)}(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}}$ recursively by

$$\phi^{(K)}(s^{t+\tau}) = \sum_{s^{t+\tau+1} \succ s^{t+\tau}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \min \left\{ \phi(s^{t+\tau+1}), \phi^{(K-1)}(s^{t+\tau+1}) \right\}.$$

Using the preceding lemma, we have

$$\begin{aligned} \phi^{(1)}(s^{t+\tau}) &= \sum_{s^{t+\tau+1} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \phi(s^{t+\tau+1}) \\ &\quad + \sum_{s^{t+\tau+1} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \left[\hat{\phi}(s^{t+\tau+1}) - Y(s^{t+\tau+1}) \right] \\ &= \hat{\phi}(s^{t+\tau}) - \sum_{s^{t+\tau+1} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} Y(s^{t+\tau+1}) \\ &= \hat{\phi}(s^{t+\tau}) - Y(s^{t+\tau}) + y(s^{t+\tau}) \geq \phi^{(0)}(s^{t+\tau}) \end{aligned}$$

for all $s^{t+\tau} \in \mathcal{N}_\tau(s^t) \cup \{s^t\}$. But then, since $\{\phi^{(K)}(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}}$ is non-decreasing in $\{\phi^{(K-1)}(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}}$, the sequence of sequences $\{\phi^{(K)}(s^{t+\tau})\}_{K=0,1,\dots}$ is non-decreasing. Moreover, $\phi^{(K)}(s^{t+\tau}) \leq 0$, for all K and all $s^{t+\tau}$, $\{\phi^{(K)}(s^{t+\tau})\}_{K=0,1,\dots}$ must converge to a finite limit $\tilde{\phi}(s^{t+\tau}) = \lim_{K \rightarrow \infty} \phi^{(K)}(s^{t+\tau}) \in [\phi^{(0)}(s^{t+\tau}), 0]$, which satisfies equation (19).

For the limit property, we have

$$\begin{aligned}
0 &\geq \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) \tilde{\phi}(s^{t+\tau+K}) \geq \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) \phi^{(0)}(s^{t+\tau+K}) \\
&= \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) [\phi(s^{t+\tau+K}) - Y(s^{t+\tau+K})] \\
&= 0
\end{aligned}$$

where $\lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) Y(s^{t+\tau+K}) = 0$ follows from $Y(s^{t+\tau}) < \infty$, and $\lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} p(s^{t+\tau+K}) \phi(s^{t+\tau+K}) = 0$ is established from the household's transversality condition, after redefining (P1) terms of $\delta(s^{t+\tau}) = a(s^{t+\tau}) - \phi(s^{t+\tau})$, with debt limits $\delta(s^{t+\tau}) \geq 0$, and using the same argument as in the preceding lemma. ■

Our next lemma shows that the auxiliary debt limits $\tilde{\Phi}(s^t)$ provide a lower bound for the actual sequence of debt limits $\Phi(s^t)$. From this, it follows immediately that $\tilde{\phi}(s^{t+\tau}) = \hat{\phi}(s^{t+\tau})$ for all $s^{t+\tau}$, and that $\tilde{\phi}(s^{t+\tau})$ (or equivalently $\hat{\phi}(s^{t+\tau})$) satisfies the exact roll-over property for all $s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}$.

Lemma 6 (i) For all $s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}$, $\phi(s^{t+\tau}) \geq \tilde{\phi}(s^{t+\tau}) = \hat{\phi}(s^{t+\tau})$.

(ii) For all $s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}$, $p(s^{t+\tau}) \tilde{\phi}(s^{t+\tau}) = \sum_{s^{t+\tau+1} \succ_{s^{t+\tau}} s^{t+\tau}} p(s^{t+\tau+1}) \tilde{\phi}(s^{t+\tau+1})$.

Proof. Part (i): Suppose that $\phi(s^{t+\tau}) < \tilde{\phi}(s^{t+\tau})$ for some $s^{t+\tau} \in \mathcal{N}(s^t) \cup \{s^t\}$, and let $\{\tilde{a}(s^{t+\tau+k})\}_{s^{t+\tau+k} \in \mathcal{S}(s^t)}$ denote the optimal asset profile without default starting from a position of $\phi(s^{t+\tau})$ at event $s^{t+\tau}$, and $\{\tilde{c}(s^{t+\tau+k})\}_{s^{t+\tau+k} \in \mathcal{S}(s^t) \cup \{s^t\}}$ the corresponding consumption profile. From Lemma 3(iii), we have that $\tilde{a}(s^{t+\tau+k}) = \phi(s^{t+\tau+k})$, whenever $s^{t+\tau+k} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})$. Suppose now that the agent defaults and sets

$$d(s^{t+\tau+k}) = \tilde{a}(s^{t+\tau+k}) - \min \left\{ \phi(s^{t+\tau+k}), \tilde{\phi}(s^{t+\tau+k}) \right\} \text{ for all } s^{t+\tau+k} \in [\mathcal{N}(s^t) \cup \mathcal{B}(s^t)] \cap \mathcal{S}(s^{t+\tau}),$$

and let $\{c^d(s^{t+\tau+k})\}_{s^{t+\tau+k} \in \mathcal{S}(s^t) \cup \{s^t\}}$ be the corresponding consumption profile. Clearly, this asset profile is feasible; we show that this profile also leads to strictly higher utility, and hence default must be optimal - a contradiction to the hypothesis that the debt limit of $\phi(s^{t+\tau})$ is self-enforcing. For $s^{t+\tau+k} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})$,

$$\begin{aligned}
c^d(s^{t+\tau+k}) &= \tilde{c}(s^{t+\tau+k}) - \min \left\{ \phi(s^{t+\tau+k}), \tilde{\phi}(s^{t+\tau+k}) \right\} \\
&\quad + \sum_{s^{t+\tau+k+1} \succ_{s^{t+\tau+k}} s^{t+\tau+k}} \frac{p(s^{t+\tau+k+1})}{p(s^{t+\tau+k})} \min \left\{ \phi(s^{t+\tau+k+1}), \tilde{\phi}(s^{t+\tau+k+1}) \right\} \\
&= \tilde{c}(s^{t+\tau+k}) - \min \left\{ \phi(s^{t+\tau+k}), \tilde{\phi}(s^{t+\tau+k}) \right\} + \tilde{\phi}(s^{t+\tau+k}) \\
&= \tilde{c}(s^{t+\tau+k}) - \min \left\{ \phi(s^{t+\tau+k}) - \tilde{\phi}(s^{t+\tau+k}), 0 \right\} \geq \tilde{c}(s^{t+\tau+k})
\end{aligned}$$

For any $s^{t+\tau+k} \in \mathcal{B}_{\tau+k}(s^t) \cap \mathcal{S}^{(k)}(s^{t+\tau})$, $d(s^{t+\tau+k}) = 0$ after a default, which yields a life-time expected discounted utility of $D(0; s^{t+\tau+k})$. Without default, the asset position at $s^{t+\tau+k}$ is $\phi(s^{t+\tau+k})$, which yields a life-time expected discounted utility of $V(\phi(s^{t+\tau+k}), \Phi(s^{t+\tau+k}), s^{t+\tau+k}) = D(0; s^{t+\tau+k})$.

Finally, for $s^{t+\tau}$,

$$\begin{aligned} c^d(s^{t+\tau}) &= \tilde{c}(s^{t+\tau}) - \phi(s^{t+\tau}) + \sum_{s^{t+\tau+1} \succ_{s^{t+\tau}}} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \min \left\{ \phi(s^{t+\tau+1}), \tilde{\phi}(s^{t+\tau+1}) \right\} \\ &= \tilde{c}(s^{t+\tau}) - \phi(s^{t+\tau}) + \tilde{\phi}(s^{t+\tau}) > \tilde{c}(s^{t+\tau}) \end{aligned}$$

We conclude that the expected life-time utility for an agent is strictly larger if he defaults at $s^{t+\tau}$.

Therefore, using $\phi(s^{t+\tau}) \geq \tilde{\phi}(s^{t+\tau})$, we can solve forward for $\tilde{\phi}(s^{t+\tau})$:

$$\begin{aligned} \tilde{\phi}(s^{t+\tau}) &= \sum_{s^{t+\tau+1} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \phi(s^{t+\tau+1}) + \sum_{s^{t+\tau+1} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+1})}{p(s^{t+\tau})} \tilde{\phi}(s^{t+\tau+1}) \\ &= \sum_{k=1}^K \sum_{s^{t+\tau+k} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+k})}{p(s^{t+\tau})} \phi(s^{t+\tau+k}) + \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+K})}{p(s^{t+\tau})} \tilde{\phi}(s^{t+\tau+K}) \\ &= \hat{\phi}(s^{t+\tau}) + \lim_{K \rightarrow \infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})} \frac{p(s^{t+\tau+K})}{p(s^{t+\tau})} \tilde{\phi}(s^{t+\tau+K}) = \hat{\phi}(s^{t+\tau}). \end{aligned}$$

Part (ii) then follows immediately from the definition of $\tilde{\phi}(s^{t+\tau})$ and noting that $\phi(s^{t+\tau+1}) = \tilde{\phi}(s^{t+\tau+1})$ for all $s^{t+\tau+1} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+\tau})$ and $\phi(s^{t+\tau+1}) \geq \tilde{\phi}(s^{t+\tau+1})$ for all $s^{t+\tau+1} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+\tau})$. ■

Our next lemma extends the existence and characterization of $\tilde{\Phi}(s^t)$ to all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$. Moreover, it shows that this solution must establish a version of the roll-over property as a weak inequality, i.e., if the borrowing constraints were given by $\tilde{\Phi}(s^t)$, an agent would always be able to roll over existing claims by issuing new claims, without necessarily always exhausting his debt limits.

Lemma 7 (i) *A finite solution to (19) exists for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$.*

(ii) *For all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$, $p(s^{t+\tau}) \tilde{\phi}(s^{t+\tau}) \geq \sum_{s^{t+\tau+1} \succ_{s^{t+\tau}}} p(s^{t+\tau+1}) \tilde{\phi}(s^{t+\tau+1})$.*

Proof. Part (i): Define $\mathcal{B}^{(1)}(s^t) = \mathcal{B}(s^t)$ and

$$\mathcal{B}^{(k)}(s^t) = \bigcup_{\tau=k}^{\infty} \left\{ s^{t+\tau} \succ_{s^t} : a^*(s^{t+\tau}) = \phi(s^{t+\tau}) \text{ and } \sigma(s^{t+\tau}) \in \mathcal{N}(s^{t+\tau'}) \text{ for some } s^{t+\tau'} \in \mathcal{B}^{(k-1)}(s^t) \right\}$$

$\mathcal{B}^{(k)}(s^t)$ is the subset of histories in $\mathcal{S}(s^t)$, at which the debt limit is binding for the k th time after s^t . Recall that $\{a^*(s^{t+\tau})\}_{s^{t+\tau} \succ_{s^t}}$ defines the solution to (P1), starting from s^t with asset position

$\phi(s^t)$. Since $a^*(s^{t+\tau}) = \phi(s^{t+\tau})$ for all $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}(s^t)$, $\{a^*(s^{t+\tau+k})\}_{s^{t+\tau+k} \in \mathcal{S}(s^{t+\tau})}$ also solves (P1), starting from any $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}(s^t)$ with asset position $\phi(s^{t+\tau})$. Since our analysis holds for arbitrary s^t , we can replicate the same arguments as above for all $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}(s^t), \dots$ to construct a solution $\tilde{\Phi}(s^t)$ to (19) for all $s^{t+\tau} \in \mathcal{S}(s^t)$.

Part (ii): From Lemma 6(ii), it follows that the exact roll-over condition holds with equality, whenever $a^*(s^{t+\tau}) > \phi(s^{t+\tau})$, or $s^{t+\tau} \notin \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}(s^t)$. If instead $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}(s^t)$, the same arbitrage argument as in the proof of Lemma 6(i) establishes that whenever $p(s^{t+\tau})\phi(s^{t+\tau}) < \sum_{s^{t+\tau+1} \succ s^{t+\tau}} p(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1})$, a default is strictly better than no default. ■

In Lemma 8, we show that $\phi(s^t) = \tilde{\phi}(s^t) = \hat{\phi}(s^t)$. For any $s^t \in \mathcal{S}$, $\phi(s^t)$ must be equal to the sum of the present discounted values of the debt limits at all events $s^{t+\tau} \in \mathcal{B}(s^t)$, where the debt limit is binding for the first time after s^t . This also implies that $\tilde{\Phi}(s^t)$ must satisfy (ER) with equality, for all $s^{t+\tau} \in \mathcal{S}(s^t)$.

Lemma 8 For all $s^t \in \mathcal{S}$, $\phi(s^t) = \sum_{s^{t+\tau} \in \mathcal{B}(s^t)} p(s^{t+\tau})/p(s^t)\phi(s^{t+\tau})$.

Proof. Starting from any $s^t \in \mathcal{S}$, construct the sequence of debt limits $\tilde{\Phi}(s^t) : \mathcal{S}(s^t) \cup \{s^t\} \rightarrow \mathbb{R}$ as characterized by Lemmas (5)-(7). Consider the problem (P1), but with the borrowing constraints equal to $\tilde{\Phi}(s^t)$. Since $\phi(s^{t+\tau}) \geq \tilde{\phi}(s^{t+\tau})$ for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$, $V(\phi(s^t), \tilde{\Phi}(s^t), s^t) \geq V(\phi(s^t), \Phi(s^t), s^t)$. However, since the objective is strictly concave, and $\phi(s^{t+\tau}) > \tilde{\phi}(s^{t+\tau})$ only if $a^*(s^{t+\tau}) > \phi(s^{t+\tau})$, it follows that $\{a^*(s^{t+\tau})\}_{s^{t+\tau} \succ s^t}$ also satisfies the optimality conditions for the relaxed problem with borrowing constraints $\tilde{\Phi}(s^t)$, and therefore, $V(\phi(s^t), \tilde{\Phi}(s^t), s^t) = V(\phi(s^t), \Phi(s^t), s^t)$. Combining with the self-enforcement hypothesis, and using the fact that $V(a, \tilde{\Phi}(s^t), s^t)$ is monotone in a and $\phi(s^t) \geq \tilde{\phi}(s^t)$, we find

$$D(0, s^t) = V(\phi(s^t), \Phi(s^t), s^t) = V(\phi(s^t), \tilde{\Phi}(s^t), s^t) \geq V(\tilde{\phi}(s^t), \tilde{\Phi}(s^t), s^t).$$

On the other hand, since $\tilde{\phi}(s^{t+\tau})p(s^{t+\tau}) = \sum_{s^{t+\tau+1} \succ s^{t+\tau}} p(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1})$, whenever $a^*(s^{t+\tau}) > \phi(s^{t+\tau})$, and $\tilde{\phi}(s^{t+\tau})p(s^{t+\tau}) \geq \sum_{s^{t+\tau+1} \succ s^{t+\tau}} p(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1})$, whenever $a^*(s^{t+\tau}) = \phi(s^{t+\tau})$, it follows from a direct extension of the proof in Proposition 3 that $V(\tilde{\phi}(s^t), \tilde{\Phi}(s^t), s^t) \geq D(0, s^t)$, and $V(\tilde{\phi}(s^t), \tilde{\Phi}(s^t), s^t) > D(0, s^t)$, whenever $\tilde{\phi}(s^{t+\tau})p(s^{t+\tau}) > \sum_{s^{t+\tau+1} \succ s^{t+\tau}} p(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1})$ for some $s^{t+\tau} \in \mathcal{S}(s^t)$. Combining we have the following equalities and inequalities:

$$V(\tilde{\phi}(s^t), \tilde{\Phi}(s^t), s^t) \geq D(0, s^t)$$

Together these inequalities can hold only as equalities - which requires that $\phi(s^t) = \tilde{\phi}(s^t)$, and $\tilde{\phi}(s^{t+\tau})p(s^{t+\tau}) = \sum_{s^{t+\tau+1} \succ s^{t+\tau}} p(s^{t+\tau+1})\tilde{\phi}(s^{t+\tau+1})$ for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$. The latter in

turn requires that

$$\phi(s^{t+\tau}) = \tilde{\phi}(s^{t+\tau}) = \sum_{s^{t+\tau+k} \in \mathcal{B}(s^{t+\tau})} \frac{p(s^{t+\tau+k})}{p(s^{t+\tau})} \phi(s^{t+\tau+k}) \text{ for all } s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}(s^t).$$

■

Lemma 9 completes the proof by establishing the exact roll-over condition for $\Phi(s^t)$.

Lemma 9 For all $s^t \in \mathcal{S}$, $\phi(s^t) p(s^t) = \sum_{s^{t+1} \succ_{s^t} s^t} p(s^{t+1}) \phi(s^{t+1})$.

Proof. >From the previous lemma,

$$\begin{aligned} \phi(s^t) p(s^t) &= \sum_{s^{t+\tau} \in \mathcal{B}(s^t)} p(s^{t+\tau}) \phi(s^{t+\tau}) \\ &= \sum_{s^{t+1} \in \mathcal{B}_1(s^t)} p(s^{t+1}) \phi(s^{t+1}) + \sum_{s^{t+1} \in \mathcal{N}_1(s^t)} p(s^{t+1}) \hat{\phi}(s^{t+1}). \end{aligned}$$

It therefore suffices to show that for all $s^{t+1} \in \mathcal{N}_1(s^t)$, $\phi(s^{t+1}) = \hat{\phi}(s^{t+1})$, or equivalently

$$\sum_{s^{t+\tau} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+1})} p(s^{t+\tau}) \phi(s^{t+\tau}) = \sum_{s^{t+\tau} \in \mathcal{B}(s^{t+1})} p(s^{t+\tau}) \phi(s^{t+\tau}).$$

$\mathcal{B}(s^{t+1})$ denotes the set of histories $s^{t+\tau}$, at which the borrowing constraint is binding for the first time for an agent who starts from an asset position of $\phi(s^{t+1})$ at history s^{t+1} . $\mathcal{B}(s^t) \cap \mathcal{S}(s^{t+1})$ is the set of histories $s^{t+\tau}$, at which the borrowing constraint is binding for the first time for an agent who starts at s^{t+1} with an asset position of $a^*(s^{t+1}) > \phi(s^{t+1})$. Lemma 3(iii) then implies that $\mathcal{N}(s^{t+1}) \subseteq \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+1})$, $\mathcal{B}(s^{t+1}) \subseteq [\mathcal{B}(s^t) \cup \mathcal{N}(s^t)] \cap \mathcal{S}(s^{t+1})$ and $\mathcal{B}(s^t) \cap \mathcal{S}(s^{t+1}) \subseteq \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}(s^{t+1})$. Moreover, the set $\mathcal{N}(s^t) \cap \mathcal{S}(s^{t+1})$ is countable, and we can order the histories in $\mathcal{N}(s^t) \cap \mathcal{S}(s^{t+1})$ into a sequence $\{s(k)\}_{k=1}^{\infty}$, so that any $s^{t+\tau} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+1})$ is lower-ranked than all of its successor nodes $s^{t+\tau'} \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+1})$.

We then construct the following sequence of sets $\{\mathcal{M}_k(s^{t+1})\}$: Let $\mathcal{M}_0(s^{t+1}) = \mathcal{B}(s^{t+1})$, and define $\mathcal{M}_k(s^{t+1})$ from $\mathcal{M}_{k-1}(s^{t+1})$ as follows: Let $s(k)$ denote the k -th history in our ordering of events in $\mathcal{N}(s^t) \cap \mathcal{S}(s^{t+1})$. If $s(k) \notin \mathcal{M}_{k-1}(s^{t+1})$, let $\mathcal{M}_k(s^{t+1}) = \mathcal{M}_{k-1}(s^{t+1})$. If $s(k) \in \mathcal{B}_{\tau-1}(s^{t+1})$, let

$$\mathcal{M}_k(s^{t+1}) = \mathcal{M}_{k-1}(s^{t+1}) \cup \mathcal{B}(s(k)) \setminus \{s(k)\}$$

That is, for each event $s(k) \in \mathcal{N}(s^t) \cap \mathcal{S}(s^{t+1})$, we check whether it is in $\mathcal{M}_{k-1}(s^{t+1})$, and if it is, we eliminate it, and replace it with the set of events $\mathcal{B}(s(k))$ at which the debt limit is binding for the first time after event $s(k)$. By construction, $\lim_{k \rightarrow \infty} \mathcal{M}_k(s^{t+1}) = \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+1})$. Moreover,

since from the previous lemma, $p(s(k))\phi(s(k)) = \sum_{s \in \mathcal{B}(s(k))} p(s)\phi(s)$, it follows immediately that for all k ,

$$\begin{aligned} \sum_{s^{t+\tau} \in \mathcal{B}(s^{t+1})} p(s^{t+\tau})\phi(s^{t+\tau}) &= \sum_{s \in \mathcal{M}_k(s^{t+1})} p(s)\phi(s) \\ &= \lim_{k \rightarrow \infty} \sum_{s \in \mathcal{M}_k(s^{t+1})} p(s)\phi(s) = \sum_{s^{t+\tau} \in \mathcal{B}(s^t) \cap \mathcal{S}(s^{t+1})} p(s)\phi(s) \end{aligned}$$

which completes the proof. ■

Remark: None of the preceding arguments rely on $\mathcal{B}(s^t)$ being non-empty. If instead $\mathcal{B}(s^t) = \emptyset$, then $\mathcal{N}(s^t) = \mathcal{S}(s^t)$, and all the same steps above imply that $\hat{\phi}(s^{t+\tau}) = 0$ and hence $\phi(s^{t+\tau}) = \tilde{\phi}(s^{t+\tau}) = 0$ for all $s^{t+\tau} \in \mathcal{S}(s^t) \cup \{s^t\}$. Therefore, if $\phi(s^t) < 0$, it must be the case that $\mathcal{B}(s^t)$ is non-empty, and that there exists a history $s^{t+\tau} \in \mathcal{B}(s^t)$, s.t. $\phi(s^{t+\tau}) < 0$.

9 Appendix C: Extensions

Transitional dynamics in the model of Section 3. Let the initial borrowing constraints satisfy $\phi_1^o + \phi_1^e = \frac{1}{1+\beta} [\beta\bar{e} - \underline{e}]$, so that there exists a steady-state with positive levels of debt. The transition to steady-state is complete after one period and $q_t = 1$ for $t \geq 1$, optimal consumption allocations are given by the steady-state allocations $\bar{c} = \frac{1}{1+\beta} [\bar{e} + \underline{e}]$ and $\underline{c} = \frac{\beta}{1+\beta} [\bar{e} + \underline{e}]$ starting from period 1. In period 0, the even agent's optimal consumption is determined by the initial borrowing constraint, and is equal to $c_0^e = \underline{e} + q_0\phi_1^e$. Odd types are unconstrained, and hence c_0^o and a_1^o are determined from the Euler equation in period 0,

$$c_0^o = \bar{e} - q_0 a_1^o = \frac{q_0}{\beta} [\underline{e} + a_1^o + \phi_1^o],$$

which yields $a_1^o = \frac{1}{q_0(1+\beta)} [\beta\bar{e} - q_0\underline{e}] - \frac{1}{1+\beta}\phi_1^o$ and $c_0^o = \frac{1}{1+\beta} [\bar{e} + q_0\underline{e}] + \frac{q_0}{1+\beta}\phi_1^o$. The market-clearing condition in period 0 then requires $c_0^o + c_0^e = \bar{e} + \underline{e}$. Substituting, we find $q_0 [\underline{e} + \phi_1^o + (1+\beta)\phi_1^e] = \beta\bar{e}$, or, after using the fact that $\phi_1^o + \phi_1^e = \frac{1}{1+\beta} [\beta\bar{e} - \underline{e}]$,

$$q_0 = \frac{\bar{e}}{\phi_1^e + \frac{1}{1+\beta} (\bar{e} + \underline{e})}.$$

Moreover, since $\phi_1^e \leq \frac{1}{1+\beta} [\beta\bar{e} - \underline{e}]$, it is easy to check that

$$\begin{aligned} c_0^e &= \underline{e} + q_0\phi_1^e = \underline{e} + \frac{\bar{e}\phi_1^e}{\phi_1^e + \frac{1}{1+\beta} (\bar{e} + \underline{e})} \leq \underline{e} + \frac{\bar{e}\frac{1}{1+\beta} [\beta\bar{e} - \underline{e}]}{\frac{1}{1+\beta} [\beta\bar{e} - \underline{e}] + \frac{1}{1+\beta} (\bar{e} + \underline{e})} \\ &= \underline{e} + \frac{\beta\bar{e} - \underline{e}}{1+\beta} = c_1^e \leq \frac{1}{\beta} c_1^e \end{aligned}$$

so that the even agent's Euler equation in the initial period is also satisfied.

Therefore, the lower is the borrowing capacity of the even agent who initially has low endowments, the higher is the first-period bond price, the lower is the period 0 consumption by even agents, and the higher is the period 0 consumption by odd agents. At one extreme, $\phi_1^e = 0$ and $\phi_1^o = \frac{1}{1+\beta} [\beta\bar{e} - \underline{e}]$, i.e., even agents are unable to undertake any borrowing, in which case the period zero consumption is equal to the endowment. At the other extreme, $\phi_1^e = \frac{1}{1+\beta} [\beta\bar{e} - \underline{e}]$ and $\phi_1^o = 0$. In that case, all the borrowing is undertaken by the even agents, $q_0 = 1$, and first period consumption allocations jump immediately to the steady-state.

Private Debt Collapses: We return to the example of Section 3, and set $g = 1$, for simplicity. We consider non-stationary equilibria, in which borrowing constraints are binding every other period, i.e., $a_t^j = -\phi_t^j$, whenever $y_t^j = \bar{e}$, and $a_t^j > -\phi_t^j$ whenever $y_t^j = \underline{e}$. By self-enforcement, we have $\phi_t^j = q_t q_{t+1} \phi_{t+2}^j$. Given a sequence of bond prices $\{q_t\}_{t=0}^\infty$, optimal consumption allocations are then characterized by $(\bar{c}_t, \underline{c}_{t+1}, a_{t+1}^j)$, where \bar{c}_t and \underline{c}_{t+1} denote consumption in period t and $t+1$, respectively, for an agent who has high endowment in period t and low endowment in period $t+1$, and a_{t+1}^j denotes this agent's asset holdings in period $t+1$. These must satisfy the first-order condition at date t , as well as the budget constraints at t and $t+1$ with equality:

$$\frac{1}{\bar{c}_t} = \frac{\beta}{q_t} \frac{1}{\underline{c}_{t+1}}, \quad \bar{c}_t = \bar{e} - \phi_t^j - q_t a_{t+1}^j \quad \text{and} \quad \underline{c}_{t+1} = \underline{e} + a_{t+1}^j + q_{t+1} \phi_{t+2}^j$$

Solving for $(\bar{c}_t, \underline{c}_{t+1}, a_{t+1}^j)$, the consumption allocations thus characterized by

$$\begin{aligned} \bar{c}_t &= \frac{1}{1+\beta} [\bar{e} + q_t \underline{e}] \quad \text{and} \quad \underline{c}_{t+1} = \frac{\beta}{q_t (1+\beta)} [\bar{e} + q_t \underline{e}] \\ a_{t+1}^j &= \frac{1}{q_t (1+\beta)} [\bar{e} - (q_t/\beta) \underline{e}] - \phi_{t+1}^j \end{aligned}$$

where we have used the fact that $\phi_t^j = q_t \phi_{t+1}^j = q_t q_{t+1} \phi_{t+2}^j$. Now, the market-clearing condition requires that $a_{t+1}^j = \phi_{t+1}^{-j}$ and $a_{t+2}^{-j} = \phi_{t+2}^j$, or

$$\frac{\beta}{q_t (1+\beta)} \left\{ \bar{e} - \frac{q_t}{\beta} \underline{e} \right\} = \phi_{t+1}^o + \phi_{t+1}^e \quad \text{and for all } t.$$

Combining with $\phi_t^o + \phi_t^e = q_t (\phi_{t+1}^o + \phi_{t+1}^e)$, we then have

$$\frac{\beta}{q_t (1+\beta)} \left\{ \bar{e} - \frac{q_t}{\beta} \underline{e} \right\} = \frac{\beta}{1+\beta} \left\{ \bar{e} - \frac{q_{t+1}}{\beta} \underline{e} \right\}$$

which immediately yields

$$q_{t+1} = \frac{\beta \bar{e}}{\underline{e}} \left[1 - \frac{1}{q_t} \right] + 1 \tag{20}$$

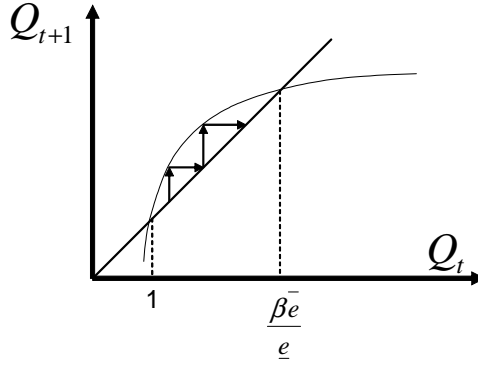


Figure 4: Private Debt Collapses

Finally, notice that for all $q_t \in [1, \beta \bar{e}/\underline{e}]$, $\beta \bar{e}/\underline{e} > q_{t+2} > q_{t+1} > q_t$, and hence $1/\bar{c}_{t+2} \leq 1/\bar{c}_t = (\beta/q_t)/\bar{c}_{t+1} \leq 1/\bar{c}_{t+1}$. Thus, for $q_t \in [1, \beta \bar{e}/\underline{e}]$, the Euler Equations are also satisfied as an inequality when agents are borrowing constrained.

Condition (20) characterizes equilibrium dynamics and is plotted in figure 4. There exist two stationary equilibria. In the first one, $q_t = 1$, for all t , which corresponds to the steady-state with positive levels of debt as characterized in Proposition 2. In the other one, $q_t = \frac{\bar{e}\beta}{\underline{e}} > 1$, borrowing constraints are equal to zero, and agents are in autarky. Finally, for intermediate initial levels of borrowing, there exist transition paths, in which there is a self-fulfilling collapse of the real amount of debt in circulation: q_t initially is larger than 1, and keeps increasing over time, while the borrowing constraints ϕ_t^j collapse to zero. In the limit, consumption allocations approach the endowments, i.e., there is convergence to autarky. Moreover, these transition paths to autarky may start arbitrarily closely to the steady-state with positive levels of debt.

Coexistence of Private and Public Debt: It is possible also to allow for the coexistence of self-enforcing private debt and unbacked public debt. If both types of securities are in circulation, one can immediately establish the existence of equilibria in which only one form of debt is positively valued, and either private debt limits are all equal to zero (replicating the equilibrium with only public debt), or the real value of the public debt in circulation is zero (replicating the equilibrium with only private debt).

The question then arises whether there can be equilibria in which both types of securities are positively valued, and what allocations can be sustained this way. For this, we notice first that in such an equilibrium, both types of securities must be perfect substitutes, and hence offer the

same real returns. We can therefore redefine an agent's net financial position $a^j(s^t)$ as the sum of the net holdings of private debt claims, and the holdings of unbacked public debt. As before, the net financial position is bounded below by the debt limit, $\phi^j(s^t)$, which must be self-enforcing. In addition, the total amount of public debt in circulation still has to satisfy the government roll-over condition, and markets must clear, i.e., the agents' net financial positions add up to the government debt in circulation. This leads to the following equilibrium definition:

Definition 5 *A competitive equilibrium with self-enforcing private and unbacked public debt is defined as $\{C^j; a^j; \Phi^j; d^j(s^0); p; D\}$ s.t.*

- (i) *given Φ^j, p and initial asset holdings $d^j(s^0)$, $\{C^j; a^j\}$ solve the consumer's problem (P1),*
- (ii) *the borrowing constraints Φ^j are self-enforcing,*
- (iii) *the public debt sequence D satisfies (6), and*
- (iv) *all markets clear: $\sum_j c^j(s^t) = \sum_j y^j(s^t)$ and $\sum_j a^j(s^{t+1}) = d(s^{t+1})$ for $s^{t+1} \succ s^t$, for all j and all $s^t \in \mathcal{S}$.*

We then get the following generalization of Theorem 2:

Proposition 5 (i) *If $\{C^j, a^j, \Phi^j, d^j(s^0); D, p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with self-enforcing private and unbacked public debt, then $\{C^j, \hat{a}^j, \hat{d}^j(s^0); \hat{D}, p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with unbacked public debt, where $\hat{a}^j(s^t) = a^j(s^t) - \phi^j(s^t)$ for all j , $s^t \in \mathcal{S}$, $\hat{d}^j(s^0) = d^j(s^0) - \phi^j(s^0)$ for all j , and $\hat{d}(s^t) = d(s^t) + \sum_{j=1}^J |\phi^j(s^t)|$ for all $s^t \in \mathcal{S}$.*

(ii) *If $\{C^j, \hat{a}^j, d^j(s^0); D, p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with unbacked public debt, then $\{C^j, \tilde{a}^j, \tilde{\Phi}^j, \tilde{d}^j(s^0); \tilde{D}, p\}_{j=1, \dots, J}$ is a non-autarkic competitive equilibrium with self-enforcing private debt, where $\tilde{a}^j(s^t) = \hat{a}^j(s^t) + \tilde{\phi}^j(s^t)$ for all j , $s^t \in \mathcal{S}$, (ii) $\tilde{\phi}^j(s^t) = -k \frac{d^j(s^0)}{d(s^0)} d(s^t)$ for all j , $s^t \in \mathcal{S}$, and (iii) $\tilde{d}^j(s^0) = (1 - k) d(s^0)$ for all j and $\tilde{d}(s^t) = (1 - k) d(s^t)$ for all $s^t \in \mathcal{S}$, for any $k \in [0, 1]$.*

Proof. (i) Self-enforcement still requires that Φ^j satisfies exact roll-over. Proposition 3 then implies $V^j(a, \Phi^j; s^0) = D^j(a - \phi^j(s^0); s^0)$, for all $a \geq \phi^j(s^0)$. Therefore, since $\{C^j, a^j\}$ solves (P1) for given debt limits Φ^j , prices p and initial asset holdings of $d^j(s^0)$, $\{C^j, \hat{a}^j\}$ solves (P1), given zero debt limits, prices p , and initial asset holdings of $\hat{d}^j(s^0) = d^j(s^0) - \phi^j(s^0)$. Next, notice that if Φ^j allows for exact roll-over for all j , then $\hat{d}(s^t) = d(s^t) + \sum_{j=1}^J |\phi^j(s^t)|$ satisfies the government budget constraint (6), for all $s^t \in \mathcal{S}$. Finally, to show that $\{C^j, \hat{a}^j\}$ clears markets in the public debt economy, notice that asset market clearing in the economy with public and private

debt requires $\sum_{j=1}^J a^j (s^t) = d (s^t)$, for all $s^t \in \mathcal{S}$, which implies $\sum_{j=1}^J [\hat{a}^j (s^t) + \phi^j (s^t)] = d (s^t)$, or $\sum_{j=1}^J \hat{a}^j (s^t) = \hat{d} (s^t)$. Again, goods market-clearing is immediate.

(ii) If $d (s^t)$ satisfies (6), for all $s^t \in \mathcal{S}$, then the sequence of debt limits $\tilde{\Phi}^j$ allows for exact roll-over, for all j , and any $k \in [0, 1]$ Now, if $\{C^j, \hat{a}^j\}$ is optimal in the public debt economy, given initial debt holdings of $d^j (s^0)$, then $\{C^j, \tilde{a}^j\}$ is optimal in the economy with public and private debt, given borrowing limits $\tilde{\Phi}^j$ and initial asset holdings of $\tilde{d}^j (s^0)$. Finally, asset market clearing implies that for all $s^t \in \mathcal{S}$, $\sum_{j=1}^J \hat{a}^j (s^t) = d (s^t) = \sum_{j=1}^J \tilde{\phi}^j (s^t) + \tilde{d} (s^t)$ or $\sum_{j=1}^J [\hat{a}^j (s^t) + \tilde{\phi}^j (s^t)] = \sum_{j=1}^J \tilde{a}^j (s^t) = \tilde{d} (s^t)$, which implies that $\{C^j, \tilde{a}^j\}$ also clears asset markets in the private debt economy. ■