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Progressive Estate Taxation
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ABSTRACT

For an economy with altruistic parents facing productivity shocks, the optimal estate taxation is progressive: fortunate parents should face lower net returns on their inheritances. This progressivity reflects optimal mean reversion in consumption, which ensures that a long-run steady state exists with bounded inequality - avoiding immiseration.

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and insurance of newborns. Our main result is that estate taxation should be progressive: fortunate parents should face a higher marginal tax rate on their bequests.

We begin with a two-period Mirrleesian economy with two generations linked by parental altruism; we then extend our analysis to an infinite horizon economy similar to [Atkeson and Lucas \(1995\)](#) and [Albanesi and Sleet \(2004\)](#). In our simplest economy, a continuum of parents live during the first period. In the second period each is replaced by a single descendent and parents are altruistic towards this child. Parents work, consume and bequeath; children simply consume.¹ Following [Mirrlees \(1971\)](#), parents first observe a random productivity draw and then exert work effort. Both productivity and work effort are private information; only output, the product of the two, is publicly observable. We study the entire set of constrained Pareto efficient allocations and derive their implications for marginal tax rates.

For this economy, if one assumes that the social welfare criterion coincides with the parent's expected utility, then [Atkinson and Stiglitz's \(1976\)](#) celebrated uniform-taxation result applies: the optimal estate tax is zero. That is, when no direct weight is placed on the welfare of children, income should be taxed nonlinearly (as in [Mirrlees, 1971](#)), but bequests should go untaxed. This arrangement ensures that the intertemporal consumption choice made by parents—trading off their own consumption against their child's consumption—is undistorted. As a result, the inheritability of welfare across generations is perfect: a child's consumption rises one-for-one with that of its parent. In effect, efficiency dictates that altruism be exploited to provide higher incentives for parents, by manipulating their children's consumption. Inequality for the children's generation is created as a byproduct, since their expected welfare is of no direct concern. Indeed, in this economy, if parent were not altruistic, the children's expected utility would be higher at any efficient allocation.

While this describes one efficient allocation, the picture is incomplete. In this economy the parent and child are distinct individuals, albeit linked through parental altruism, a form of externality. Thus, a complete welfare analysis requires examining the ex-ante utility of both parents and children. [Figure 1](#) depicts our economy's Pareto frontier graphically, which is peaked because altruistic parent's welfare decreases if the child is made relatively too miserable. The allocation discussed in the previous paragraph is a particular point lying on the Pareto frontier: the peak which maximizes the welfare of parents; marked as point A in the figure. In this paper we explore other efficient arrangements representing points lying on the downward sloping section of the the Pareto frontier, to the right of its peak.

Away from point A, a role for estate taxation emerges: efficient allocations which lie to the right of the peak can be implemented with a simple tax system that confronts parents with separate nonlinear schedules for income and estate taxes. Our main result is that the

¹ Although some readers have remarked that they find this assumption realistic, it will be relaxed when we extend the time horizon.

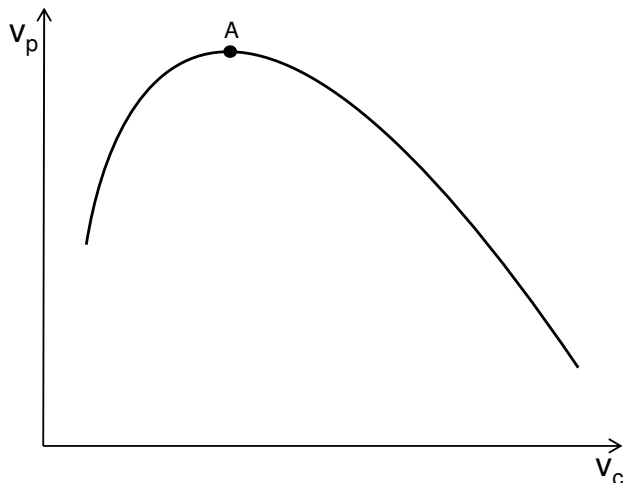


Figure 1: Pareto frontier between ex-ante utility for parent, v_p , and child, v_c .

optimal estate tax schedule is convex: fortunate high-skilled parents face a higher marginal tax rate on their bequests.

The intuition for this result is that progressive estate taxation arises to insure children against their parent’s luck. The progressive estate tax lowers consumption inequality within the children’s generation—which is desirable as long as some weight, however small, is placed on them in the social welfare criterion—while still providing incentives to parents. A child’s consumption still varies with their parent’s, providing some incentives, but now does so less than one-for-one, providing some insurance. In other words, consumption mean reverts across generations, making the inheritability of welfare is imperfect. The optimal progressivity in estate taxes reflects this mean reversion: fortunate dynasties must face a lower net return on bequests so that they choose a consumption path declining towards the mean.

Our stark conclusion on the progressivity of estate taxation strongly contrasts with the theoretical ambiguity in the shape of the optimal income tax schedule. [Mirrlees’s \(1971\)](#) seminal paper showed that for bounded distributions of skills the optimal marginal income tax rates are regressive at the top (see also [Seade, 1982](#); [Tuomala, 1990](#); [Ebert, 1992](#)). More recently, [Diamond \(1998\)](#) has shown that the opposite—progressivity at the top—is true with certain unbounded skill distributions (see also [Saez, 2001](#)). In contrast, our results on the progressivity of the estate tax do not depend on any assumptions regarding the distribution of skills.

We then extend our analysis for the two-period setup to an infinite-horizon economy with non-overlapping generations. This extension is important for at least two reasons.

First, it provides a motivation for studying efficient allocations which do not simply maximize the expected utility of the very first generation—the analogues of point A from [Figure 1](#)

for the infinite-horizon economy. Indeed, these allocations have everyone in distant generations converging to misery, with zero consumption. This is a version of the immiseration result shown by [Atkeson and Lucas \(1992\)](#) for a taste shock economy, which we extend here to the Mirrleesian economy. Loosely speaking, if we continue to plot the expected utility of the last generation on the horizontal axis, then as we extend the horizon point A moves further and further to the left, towards misery. This provides a rationale for focusing on efficient allocations that place positive weight on future generations, the analogues of the downward sloping section of the Pareto frontier, to the right of point A on the figure. However, as we show here by extending the analysis in [Farhi and Werning \(2005\)](#), this result is special, by placing weight on future generations, so that the social discount factor is greater than the private one, a steady state exists, misery is avoided and there is social mobility.

Second, the infinite horizon version of our model allows us to make contact with a growing literature on dynamic Mirrleesian models, such as [Golosov, Kocherlakota and Tsyvinski \(2003\)](#), [Albanesi and Sleet \(2004\)](#) and others (see further references in [Golosov, Tsyvinski and Werning, 2006](#)). In our model each individual lives for a single period, observes a productivity draw and works, and is then replaced by a single descendant in the next period. As usual, perfect altruism implies that each dynasty behaves as a single infinitely-lived individual, so our model environment is identical to [Albanesi and Sleet \(2004\)](#). However, our intergenerational interpretation of the infinite horizon leads us to study a different planning problem, one that puts direct weight on the expected utility of future generations, or equivalently, one that has a social discount factor that is higher than the private one. Indeed, to avoid the immiseration result mentioned above, [Albanesi and Sleet](#) impose an ad hoc lower bound on continuation utility along the equilibrium path; in contrast, our steady-state analysis requires no such lower imposition.

The progressivity of estate taxes extends to this infinite horizon setup: fortunate parents face a higher average marginal tax rate on their bequests. Indeed, the average marginal estate tax rate formula is the same as in the two-period economy. The main difference between the two-period and infinite horizon economies is that tax implementations are more involved in the latter. We adapt [Kocherlakota's \(2004\)](#) implementation, which yields a marginal tax estate rate that is zero, on average, for all parents when only the first generation is weighed in the welfare criterion, the analogue of point A.

Throughout this paper, we study an economy without capital, where aggregate consumption equals aggregate produced output plus an endowment. This no-aggregate-savings assumption allows us to focus on redistribution *within* generations and abstract from transfers *across* generations. Unfortunately, it does not allow us to pin down the level of estate taxation, only its shape. [Farhi, Kocherlakota and Werning \(2005\)](#) extend this model among several

dimensions—including capital accumulation, life-cycle elements and general skill processes—and show that our main result on progressive estate taxation is insensitive to this assumption.

Our work relates to a number of recent papers that have explored the implications of including future generations in the social welfare criterion. [Phelan \(2005\)](#) considered a social planning problem that weighted all generations equally, which is equivalent to not discounting the future at all. [Farhi and Werning \(2005\)](#) considered intermediate cases, where future generations receive a geometrically declining weight. This is equivalent to a social discount factor that is less than one and higher than the private one. [Sleet and Yeltekin \(2005\)](#) have studied how such a higher social discount factor may arise from a utilitarian planner without commitment. None of these papers consider implications for estate taxation.

We organized the rest of the paper in the following way. [Section 1](#) describes the two period model environment and [Section 2](#) introduces the associated planning problem. Our main results for this two-period economy are in [Section 3](#). In [Section 4](#) we describe the extension to an infinite horizon. The main results for that economy are contained in [Section 5](#). We use [Section 6](#) for concluding remarks.

1 Parent and Child: A Two Period Economy

There are two periods labelled $t = 0, 1$. The parent lives during $t = 0$ and is replaced by a single child at $t = 1$. The parent works and consumes, while the child only consumes. Thus, an allocation is a triplet of functions $(c_0(w_0), c_1(w_0), y_0(w_0))$, where c_0 and y_0 represents the parent's consumption and output, and c_1 represents the child's consumption.

The parent is altruistic towards the child

$$v_0 = \mathbb{E} \left[u(c_0) - h \left(\frac{y_0}{w_0} \right) + \beta v_1 \right], \quad (1)$$

where the expectations is over w_0 and $\beta < 1$. The child's utility is simply

$$v_1 = u(c_1) \quad (2)$$

The utility function $u(c)$ is increasing, concave and differentiable; the disutility function $h(n)$ is assumed increasing, convex and differentiable.

Substituting [equation \(2\)](#) into [equation \(1\)](#) yields the alternative expression for the parent's utility:

$$v_0 = \mathbb{E} \left[u(c_0) + \beta u(c_1) - h \left(\frac{y_0}{w_0} \right) \right] \quad (3)$$

As usual, the parent's expected utility can be reinterpreted as that of a fictitious dynasty that

lives for two periods and discounts at rate β .

Following (Atkeson and Lucas, 1992) and others, we abstract from capital accumulation to concentrate on the distributional assignment of goods across agents within a period, and not over time. An allocation is *resource feasible* if aggregate consumption in both periods is not greater than the sum of endowments and production:

$$\int_0^\infty c_0(w_0)dF(w_0) \leq e_0 + \int_0^\infty y_0(w_0)dF(w_0) \quad (4)$$

$$\int_0^\infty c_1(w_0)dF(w_0) \leq e_1 \quad (5)$$

Productivity is private information so incentives need to be provided for truthful revelation. We say that an allocation is *incentive compatible* if the parent finds it optimal to reveal her shock truthfully:

$$u(c_0(w_0)) + \beta u(c_1(w_0)) - h\left(\frac{y_0(w_0)}{w_0}\right) \geq u(c_0(w)) + \beta u(c_1(w)) - h\left(\frac{y_0(w)}{w_0}\right) \quad (6)$$

for all productivity realizations w_0 .

2 Social Welfare and Efficient Allocations

To study all constrained efficient allocations for the two-period economy it is useful to work with the general welfare criterion

$$W \equiv v_0 + \alpha \mathbb{E}v_1, \quad (7)$$

which places some weight $\alpha \geq 0$ on the expected utility of children. As we vary α we can trace out the entire Pareto frontier, since the latter is convex, as illustrated in [Figure 1](#).

Substituting [equations \(2\)](#) and [\(3\)](#) into [equation \(7\)](#) implies the alternative expression

$$W = \mathbb{E}[u(c_0) + (\beta + \alpha)u(c_1) - h(y_0/w_0)].$$

Thus, the social welfare function is equivalent to the parent's preferences but with a social discount factor $\hat{\beta} = \beta + \alpha$ that is higher than the private one as long as $\alpha > 0$.

The planning problem maximizes the welfare criterion W over allocations that are resource feasible and incentive compatible. Formally, the problem is

$$\max_{c_0, c_1, y_0} \int_0^\infty [u(c_0(w_0)) + \hat{\beta}u(c_1(w_1)) - h(y_0(w_0)/w_0)]dF(w_0)$$

subject to the resource constraints in [equations \(4\)-\(5\)](#) and the incentive compatibility con-

straints in [equation \(6\)](#).

It is useful to divide the planning problem into two stages. In the first stage the planner chooses the profile of output $y_0(w_0)$ and a schedule of incentives $\Delta(w_0)$, which is equal to utility from consumption $u(c_0(w_0)) + \beta u(c_1(w_0))$ up to a constant. In the second stage, the planner solves the subproblem of how best to provide the incentives $\Delta(w_0)$, using $c_0(w_0)$ and $c_1(w_0)$. The key feature is that the second stage involves no incentive constraints, these are imposed in the first stage. Formally, by introducing Δ and \underline{U} the full problem can be written as

$$\max_{c_0, c_1, y_0, \Delta, \underline{U}} \int_0^\infty [u(c_0(w_0)) + \hat{\beta}u(c_1(w_1)) - h(y_0(w_0)/w_0)]dF(w_0)$$

subject to $\Delta(w) + \underline{U} = u(c_0(w_0)) + \beta u(c_1(w_0))$, the resource constraints in [equations \(4\)-\(5\)](#) and the incentive compatibility constraints $\Delta(w_0) - h(y(w_0)/w_0) \geq \Delta(w) - h(y(w)/w_0)$ for all w_0 . Note that the incentive constraint does not involve c_0 , c_1 or \underline{U} ; only Δ and y_0 .

For our purposes, it suffices to focus on the second stage that takes Δ and y_0 as given, which allows us to drop the incentive constraint:

$$\max_{c_0, c_1, \underline{U}} \int_0^\infty [u(c_0(w_0)) + \hat{\beta}u(c_1(w_1))]dF(w_0)$$

subject to $\Delta(w_0) + \underline{U} = u(c_0(w_0)) + \beta u(c_1(w_1))$ and the resource constraints in [equations \(4\)-\(5\)](#).

It is convenient to rewrite this problem by changing variables, from consumption to utility assignments $U_0(w) = u(c_0(w))$ and $U_1(w) = u(c_1(w))$, since then the objective is then linear and the constraints strictly convex. After substituting $U_0(w_0) = \Delta(w_0) + \underline{U} - \beta U_1(w_1)$ out the problem becomes

$$\max_{U_1, \underline{U}} \int_0^\infty [\underline{U} + (\hat{\beta} - \beta)U_1(w_1)]dF(w_0)$$

subject to

$$\int_0^\infty C(\Delta(w_0) + \underline{U} - \beta U_1(w_1))dF(w_0) \leq e_0 + \int_0^\infty y_0(w_0)dF(w_0)$$

$$\int_0^\infty C(U_1(w_0))dF(w_0) \leq e_1$$

It is easy to see that both resource constraints must bind at an optimum.

3 The Main Result: Progressive Estate Taxation

In this section we derive two main results for the two-period economy laid out in the previous section. We first show that implicit marginal tax rates on bequests must be progressive. We then provide a simple tax implementation that relies on two separate schedules for labor income and estates.

3.1 Implicit Marginal Taxes

For any allocation and constant $R > 0$ we can define the associated marginal tax rates $\tau(w_0)$ solving the Euler equation

$$1 = \beta R(1 - \tau(w_0)) \frac{u'(c_1(w_0))}{u'(c_0(w_0))}. \quad (8)$$

Below, the constant R plays the role of the pre-tax gross interest rate. Since our economy has no savings technology, this value is not uniquely pinned down in equilibrium—it is completely unimportant for anything that follows. Different values of R are associated with different levels for the tax, but they do not affect its shape.

The first-order condition for $U_1(w_0)$, which is necessary and sufficient for optimality, is

$$\hat{\beta} - \beta + \beta \lambda_0 C'(U_0(w_0)) = \lambda_1 C'(U_1(w_0)).$$

where λ_t is strictly positive lagrange multiplier on the resource constraint for period t . From this equation it follows that $U_0(w_0)$ and $U_1(w_0)$ move in the same direction with w_0 . Since $U_0(w_0) + \beta U_1(w_0)$ must be increasing, in order to provide incentives, it follows that both $U_0(w_0)$ and $U_1(w_0)$ are increasing; hence, both consumptions $c_0(w_0)$ and $c_1(w_0)$ are increasing in w_0 .

Using the fact that $C(u)$ is the inverse of $u(c)$, so that $C'(U_t(w_0)) = 1/u'(c_t(w_0))$, and rearranging we obtain

$$1 = \beta \frac{\lambda_0}{\lambda_1} \left(1 + \left(\frac{\hat{\beta}}{\beta} - 1 \right) \frac{u'(c_0(w_0))}{\lambda_0} \right) \frac{u'(c_1(w_0))}{u'(c_0(w_0))}. \quad (9)$$

From the first order condition for \underline{U} it follows that $1/\lambda_0 = \int_0^\infty (1/u'(c_0(w))) dF(w)$. For what follows we normalize so that $R = \lambda_0/\lambda_1$.

Our first result, derived from [equation \(9\)](#) when $\hat{\beta} = \beta$, can be viewed as simply restating the celebrated Atkinson-Stiglitz uniform taxation result for our economy.

Proposition 1. *When $\hat{\beta} = \beta$ the optimal allocation implies a zero marginal estate tax rate: $\tau(w_0) = 0$ in [equation \(8\)](#) and the marginal rate of substitution $u'(c_1(w_0))/u'(c_0(w_0))$ is equated*

across all dynasties, *i.e.* for all w_0 .

Atkinson and Stiglitz (1976) showed that, provided preferences over a group of goods is separable from work effort, then consumption within this group should not be distorted. In other words, the implied marginal taxes for these goods should be equalized to avoid distorting their relative consumption—uniform taxation is optimal. In our context, this result applies to consumption at both dates, c_0 and c_1 , and implies that the ratio of marginal utilities is equalized across agents—the estate tax can be normalized to zero.²

In contrast, whenever $\hat{\beta} > \beta$ equation (9) implies that the ratio of marginal utilities is not equalized across agents: there must be some distortion, so the marginal estate tax cannot be zero. Indeed, since consumption increases with productivity estate taxation must be progressive.

Proposition 2. *When $\hat{\beta} > \beta$ the optimal allocation implies a nonzero and progressive marginal estate tax: $\tau(w_0) \neq 0$ for all w_0 and $\tau(w_0)$ is increasing in w_0 . For $R = \hat{\beta}$ the marginal tax rate is*

$$\tau(w_0) = -(\hat{\beta}/\beta - 1)u'(c_0(w_0)) \left(\int_0^\infty u'(c_0(w))^{-1} dF(w) \right) \quad (10)$$

and $c_0(w_0)$, $c_1(w_0)$ and $y_0(w_0)$ are increasing in w_0 .

We emphasize that the interesting implication for the tax rate here is that it increases with productivity: taxation is progressive. Without an aggregate savings technology the overall level of estate tax cannot be uniquely pinned down, it is completely irrelevant. Farhi, Kocherlakota and Werning (2005) extends the analysis to an economy with capital, which pins down the level of estate taxation.

3.2 A Simple Tax Implementation

We next show that we can implement efficient allocations, and the progressive implicit marginal tax rates that go with them, with a simple tax system. In our implementation, the government confronts parents with two separate schedules: an income tax and an estate tax. We say that an allocation is *implementable by non-linear income and estate taxation* $T_1^y(y_0)$, T_2^y and $T^b(b)$ if, for all w_0 , the allocation $(c_0(w_0), c_1(w_0), y_0(w_0))$ solves

$$\max_{c_0, c_1, y_0} \{u(c_0) + \beta u(c_1) - h(y_0/w_0)\}$$

² One difference is that Atkinson and Stiglitz (1976) assume a linear technological transformation between goods, whereas we assume no possible transformation. Their result on uniform taxation implies that marginal rates of substitution are equalized across agents *and* that they are all equal to the marginal rate of transformation. Our result only emphasizes the former.

subject to

$$\begin{aligned} c_0 + b_1 &= y_0 - T^b(b_1) - T_1^y(y_0), \\ c_1 &= Rb_1 + y_2 - T_2^y. \end{aligned}$$

It is trivial to change things so that it is the child that pays the estate tax at $t = 1$. Furthermore, without loss of generality we can assume that $y_2 - T_2^y = 0$. To see this, define $\hat{b}_1 \equiv b_1 + (y_2 - T_2)/R$ then

$$\begin{aligned} c_0 + \hat{b}_1 &= y_0 - T^b(\hat{b}_1 - (y_2 - T_2)/R) - T_1^y(y_0) - T_2^y(y_0) \\ &= y_0 - \hat{T}^b(\hat{b}_1) - \hat{T}^y(y_0) \end{aligned}$$

where $\hat{T}^y(y_0) \equiv T_1^y(y_0) + T_2^y(y_0)$ and $\hat{T}^b(\hat{b}_1) = T^b(\hat{b}_1 - (y_2 - T_2)/R)$.

Our next result establishes formally that efficient allocations can be implemented with separate nonlinear income and estate taxation. The idea is to define $T^b(b)$ so that

$$\frac{1}{1 + T^{b'}(c_1(w))} = 1 - \tau(w)$$

The proof then exploits the fact that marginal tax rates are progressive to ensure that the bequest problem faced by the parent is convex.

Proposition 3. *Suppose $c_0(w_0)$, $c_1(w_0)$, $y_0(w_0)$ and $\tau(w_0)$ are increasing functions. Then there exists tax functions $T^y(y)$ and $T^b(b)$ that implements this allocation, with $T^b(b)$ convex.*

Proof. Use the generalized inverse of $c_1(w)$, where possible flat portions of $c_1(w)$ define discontinuous jumps, to define

$$T^{b'}(c) = \frac{1}{1 - \tau((c_1)^{-1}(c))} - 1 \tag{11}$$

and normalize so that $T^b(0) = 0$. Note that by the monotonicity of $\tau(w)$ and $c_0(w)$, the function $T^b(b)$ is convex. Next define net income

$$I(w_0) \equiv c_0(w_0) + R^{-1}c_1(w_0) + T^b(c_1(w_0))$$

We can express this in terms of output y by using the inverse of $y_0(w_0)$: $I^y(y) \equiv I(y_0^{-1}(y))$. Then we let $T^y(y_0) \equiv y_0 - I^y(y_0)$. Finally, let the consumption allocation as a function of net income I be: $(\hat{c}_0(I), \hat{c}_1(I)) \equiv (c_0(I^{-1}(I)), c_1(I^{-1}(I)))$.

We now show that the constructed tax functions, $T^y(y)$ and $T^b(b)$, implement the alloca-

tion. For any given net income I the consumer solves the subproblem:

$$V(I) \equiv \max\{u(c_0) + \beta u(c_1)\}$$

subject to $c_0 + R^{-1}c_1 + T^b(c_1) \leq I$. This problem is convex, the objective is concave and the constraint set is convex, since T^b is convex. It follows that the first-order condition

$$1 = \frac{\beta R}{1 + T^{b'}(b)} \frac{u'(c_1)}{u'(c_0)}$$

sufficient for optimality. Combining [equation \(8\)](#) and [equation \(11\)](#) it follows that these conditions for optimality are satisfied by $\hat{c}_0(I), \hat{c}_1(I)$ for all I . Hence $V(I) = u(\hat{c}_0(I)) + \beta u(\hat{c}_1(I))$.

Next, consider the worker's maximization over y_0 given by

$$\max_y \{V(I(y)) - h(y/w_0)\}.$$

We need to show that $y_0(w_0)$ solves this problem, which implies that the allocation is implemented since consumption would be given by $\hat{c}_0(I(y_0(w_0))) = c_0(w_0)$ and $\hat{c}_1(I(y_0(w_0))) = c_1(w_0)$. Now, from the previous paragraph and our definitions it follows that

$$\begin{aligned} y_0(w_0) &\in \arg \max_y \{V(I(y)) - h(y; w_0)\} \\ &\Leftrightarrow y_0(w_0) \in \arg \max_y \{u(\hat{c}_0(I(y))) + \beta u(\hat{c}_1(I(y))) - h(y/w_0)\} \\ &\Leftrightarrow w_0 \in \arg \max_w \{u(c_0(w)) + \beta u(c_1(w)) - h(y_0(w)/w_0)\} \end{aligned}$$

Thus, the first line follows from the last, which is guaranteed by the assumed incentive compatibility of the allocation, [equation \(6\)](#). Hence, $y_0(w_0)$ is optimal and it follows that $(c_0(w_0), c_1(w_0), y_0(w_0))$ is implemented by the constructed tax functions. \square

3.3 Discussion

Without estate taxation there is perfect inheritability of welfare. In particular, consumption of parents and child move in tandem, one-for-one. This situation is only optimal when the children are not considered independently in the welfare criterion, so that insuring them against the risk of their parent's fortune is not valued.

In contrast, when insurance is provided to the children's generation their consumption still varies with their parent's, but less than one-for-one. The intergenerational transmission of welfare is imperfect: consumption mean reverts across generations. The progressivity of

the estate tax schedule reflects this mean reversion. Fortunate parents must face a lower net returns on bequests in order to give them incentives to tilt their consumption towards the present, that is, towards themselves. Likewise poorer parents need to face higher net returns so that their consumption slopes upward. This explains the progressivity of estate taxes.

Another intuition is based on the interpretation of altruism as a form of externality. In the presence of externalities, some form of corrective Pigouvian taxes are generally desirable. Think of a parental bequest as a consumption good with a positive externality to the child; then the Pigouvian logic implies that we should subsidize bequests. Since expected utility is our concern, and utility is concave, this externality is greatest for children with low consumption. Thus, the subsidy rate should be highest—or equivalently, the negative tax should be lowest—for poor parents. Optimal estate taxation is thus progressive. Since our economy has no capital, the Pigouvian level of taxation turns out to be irrelevant—we may tax or subsidize estates. However, the relative tax conclusion in this argument remains robust.

None of these arguments require the private-information structure. However, if productivity or effort were observable, then the first-best allocation would be achievable. Consumption and wealth would then be equated across parents. Although one can still think of a progressive estate tax in this situation for out-of-equilibrium levels of parental wealth, it becomes irrelevant given the lack of parental inequality. In this sense, our results rely on an interaction between redistributive and corrective motives for taxation (see also [Amador, Angeletos and Werning, 2005](#)).

4 A Mirrleesian Economy with Infinite Horizon

We now turn to a repeated version of this economy with an infinite horizon, as in [Albanesi and Sleet \(2004\)](#). All generations work and receive a random productivity draw. An individual born into generation t has ex-ante welfare v_t solving

$$v_t = \mathbb{E}_{t-1}[u(c_t) - h(n_t) + \beta v_{t+1}],$$

where $\beta < 1$ is the coefficient of altruism. We assume that the utility function over consumption satisfies the Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$. We adopt a power disutility function $h(n) = n^\gamma/\gamma$ with $\gamma > 1$ to ensure that the planning problem is convex.

An individual with productivity w , exerting work effort n , produces output $y = w \cdot n$. Utility can then be written as

$$V_t = \sum_{s=0}^{\infty} \beta^s \mathbb{E}_{t-1}[u(c_{t+s}) - \theta_{t+s} h(y_{t+s})] \tag{12}$$

where $\theta_t \equiv w_t^{-\gamma}$ can be interpreted as a taste shock to producing output. Productivity w_t , and hence θ_t , is independently and identically distributed across dynasties and generations $t = 0, 1, \dots$. With innate talents assumed nonheritable, intergenerational transmission of welfare is not mechanically linked through the environment but may arise to provide incentives for altruistic parents.

Since productivity shocks are assumed to be privately observed by individuals and their descendants each dynasty faces a sequence of consumption functions $\{c_t\}$, where $c_t(\theta^t)$ represents an individual's consumption after reporting the history $\theta^t \equiv (\theta_0, \theta_1, \dots, \theta_t)$. A dynasty's reporting strategy $\sigma \equiv \{\sigma_t\}$ is a sequence of functions $\sigma_t : \Theta^{t+1} \rightarrow \Theta$ that maps histories of shocks θ^t into a current report $\hat{\theta}_t$. Any strategy σ induces a history of reports $\sigma^t : \Theta^{t+1} \rightarrow \Theta^{t+1}$. We use σ^* to denote the truth-telling strategy with $\sigma_t^*(\theta^t) = \theta_t$ for all $\theta^t \in \Theta^{t+1}$.

Given an allocation $\{c_t\}$, the utility obtained from any reporting strategy σ is

$$U(\{c_t\}, \sigma; \beta) \equiv \sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \beta^t [u(c_t(\sigma^t(\theta^t))) - \theta_t h(y(\sigma^t(\theta^t)))] \Pr(\theta^t).$$

An allocation $\{c_t\}$ is *incentive compatible* if truth-telling is optimal, so that

$$U(\{c_t\}, \sigma^*; \beta) \geq U(\{c_t\}, \sigma; \beta) \quad (13)$$

for all strategies σ .

We identify dynasties by their initial utility entitlement v_0 with distribution ψ in the population. An allocation is a sequence of functions $\{c_t^v, y_t^v\}$ for each v , where $c_t^v(\theta^t)$ and $y_t^v(\theta^t)$ represents the consumption and income that a dynasty with initial entitlement v gets at date t after reporting the sequence of shocks θ^t . For any given initial distribution of entitlements ψ and resources e , we say that an allocation $\{c_t^v\}$ is *feasible* if: (i) it is incentive compatible for all dynasties; (ii) it delivers expected utility of v to all initial dynasties entitled to v ; and (iii) average consumption in the population does not exceed the fixed endowment e plus income generated in all periods:

$$\int \sum_{\theta^t} c_t^v(\theta^t) \Pr(\theta^t) d\psi(v) \leq e + \int \sum_{\theta^t} y_t^v(\theta^t) \Pr(\theta^t) d\psi(v) \quad t = 0, 1, \dots \quad (14)$$

Consider the sum of expected utilities weighted by geometric Pareto weights $\alpha_t = \hat{\beta}^t$

$$\sum_{t=0}^{\infty} \alpha_t \mathbb{E}_{-1} v_t = \left(1 - \frac{1}{\hat{\beta} - \beta}\right) v_0 + \frac{1}{\hat{\beta} - \beta} \sum_{t=0}^{\infty} \hat{\beta}^t \mathbb{E}_{-1} [u(c_t) - \theta_t h(y_t)]. \quad (15)$$

with $\hat{\beta} > \beta$. The first term is exogenously given, since we take as given a distribution for the

initial utility entitlements v_0 . Thus, the welfare criterion is given by

$$\sum_{t=0}^{\infty} \hat{\beta}^t \mathbb{E}_{-1} [u(c_t) - \theta_t h(y_t)] \quad (16)$$

Future generations are already indirectly valued through the altruism of the current generation. If, in addition, they are also *directly* included in the welfare function the social discount factor must be higher than the private one (see [Farhi and Werning, 2005](#), for more details).

When $\hat{\beta} = \beta$, the planning problem seeks the lowest constant resource level e to ensure that there exists a feasible allocation that delivers the distribution of utility entitlements ψ . This is precisely the efficiency problem studied in [Albanesi and Sleet \(2004\)](#). When $\hat{\beta} > \beta$ we define the social optimum as maximizing the average social welfare function (16), weighed by ψ , over all feasible allocations. That is, the *social planning problem* given an initial distribution of entitlements ψ and an endowment level e is to maximize

$$\int U(\{c_t^v\}, \sigma^*, \hat{\beta}) d\psi(v) \quad (17)$$

subject to the the resource constraints (14), as well as the promise keeping and incentive constraints: $v = U(\{c_t^v\}, \sigma^*; \beta)$ and $U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta)$ for all initial entitlements v and strategies σ .

We are interested in distributions of utility entitlements ψ such that the solution to the planning problem features, in each period, a cross-sectional distribution of continuation utilities v_t that is also distributed according to ψ . We also require the cross-sectional distribution of consumption and income to replicate itself over time. We term any initial distribution of entitlements with these properties a *steady state* and denote them by ψ^* . Following [Farhi and Werning \(2005\)](#), we approach the planning problem by studying a relaxed version of it. The solutions to both problems coincide for steady state distributions ψ^* , which is all we seek to characterize. The relaxed problem has continuation utility as a state variable that follows a Markov process. Steady states are then invariant distributions of this Markov process.

Define the *relaxed planning problem* to be equivalent to the *social planning problem* except that the sequence of resource constraints (14) is replaced by the single intertemporal condition

$$\int \sum_{t=0}^{\infty} \hat{\beta}^t \sum_{\theta^t} (c_t^v(\theta^t) - y_t^v(\theta^t) \Pr(\theta^t)) d\psi(v) \leq \frac{1}{1 - \hat{\beta}} e. \quad (18)$$

Letting $\hat{\lambda}$ be the multiplier for this intertemporal resource constraint we form the Lagrangian

$L \equiv \int L^v d\psi(v)$ where

$$L^v \equiv \sum_{t=0}^{\infty} \sum_{\theta^t} \hat{\beta}^t \left(u(c_t^v(\theta^t)) - \hat{\lambda}c_t^v(\theta^t) - \theta_t h(y_t^v(\theta^t)) + \hat{\lambda}y_t^v(\theta^t) \right) \Pr(\theta^t) \quad (19)$$

and study the maximization of L subject to $v = U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta)$ for all v and σ . For any endowment level e , there exists a unique positive multiplier $\hat{\lambda}(e)$ so that the maximizing this Lagrangian is equivalent to solving the relaxed problem. Maximizing L is equivalent to the pointwise optimization, for each v , of the subproblem:

$$k(v) \equiv \sup L^v \quad (20)$$

subject to $v = U(\{c(u_t^v)\}, \sigma^*; \beta) \geq U(\{c(u_t^v)\}, \sigma; \beta)$ for all σ .

The value function of the component planning problem $k(v)$ defined by [equation \(20\)](#) is continuous, concave, and satisfies the Bellman equation

$$k(v) = \max_{u, h, w} \mathbb{E} \left[u(\theta) - \hat{\lambda}c(u(\theta)) - \theta h(\theta) + \hat{\lambda}y(h(\theta)) + \hat{\beta}k(w(\theta)) \right] \quad (21)$$

subject to

$$v = \mathbb{E} \left[u(\theta) - \theta h(\theta) + \beta w(\theta) \right] \quad (22)$$

$$u(\theta) - \theta h(\theta) + \beta w(\theta) \geq u(\theta') - \theta h(\theta') + \beta w(\theta') \quad \text{for all } \theta, \theta' \in \Theta. \quad (23)$$

Denote by $g^w(v, \theta)$ and $g^u(v, \theta)$ the optimal policy function for w and u . The next lemma characterizes some key properties of the value function $k(v)$.

Lemma 1. *The value function $k(v)$ is strictly concave and continuously differentiable on (\underline{v}, \bar{v}) where $\underline{v} = -\infty$; it is unbounded below on both sides $\lim_{v \rightarrow \underline{v}} k(v) = \lim_{v \rightarrow \bar{v}} k(v) = -\infty$; and the derivative has $\lim_{v \rightarrow \underline{v}} k'(v) = 1$ and $\lim_{v \rightarrow \bar{v}} k'(v) = -\infty$.*

5 Steady States and Progressive Taxation

We are interested in steady state distributions ψ^* that have no mass at misery \underline{v} . Our first result is that this is not possible when future generations are not weighed directly, so that $\hat{\beta} = \beta$. We then show that, in contrast, whenever $\hat{\beta} > \beta$ a steady state distribution exists with no mass at misery. The efficient allocation displays a form of mean reversion across generations that keeps inequality bounded. The mean reversion is characterized by a modified inverse Euler equation which implies that estate taxation is progressive.

5.1 An Immiseration Result

For $\beta = \hat{\beta}$, we have to modify our definition for the Social Planning problem. For any distribution ψ of initial welfare entitlements, the planning problem is to minimize the net resources required to deliver the utility entitlements in an incentive compatible way:

$$\inf e \tag{24}$$

subject to,

$$\int \sum_{\theta^t} (c_t^v(\theta^t) - y_t^v(\theta^t)) d\psi(v) \leq e \tag{25}$$

$$U(\{c_t^v\}, \sigma; \beta) = v \text{ for all } v \tag{26}$$

$$U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta) \text{ for all } v \text{ and } \sigma \tag{27}$$

From this program, we can define an invariant distribution exactly as in Section 4 of the paper. We are interested in steady state distributions ψ^* without full mass at misery. Our first result is that this is basically not possible when $\beta = \hat{\beta}$.

Proposition 4. *Suppose that $\lim_{u \rightarrow \infty} \sup c''(u)/c'(u) < \infty$. Then if $\beta = \hat{\beta}$, there exists no invariant distribution ψ^* without full mass at misery.*

This result extends the immiseration result in [Atkeson and Lucas \(1992\)](#), who study an endowment economy with privately observed taste shocks, instead of the Mirrleesian production economy with privately observed productivity shocks studied here. They show that the cross-sectional distribution of consumption disperses steadily over time, with inequality growing without bound. As a result, almost everyone converges to the misery, consuming nothing, while a vanishing fraction tend towards bliss, consuming the entire aggregate endowment. Thus, no steady state distribution with positive consumption exists. To the best of our knowledge [Proposition 4](#) is the first formal statement of an analogous result in the context of a Mirrleesian economy, where private information is regarding productivity shocks. Researchers that assume $\hat{\beta} = \beta$ have been typically forced to impose an ad hoc lower bound on continuation utility to avoid misery and ensure that an steady-state distribution exists ([Atkeson and Lucas, 1995](#); [Albanesi and Sleet, 2004](#)).

5.2 Steady States and a Modified Inverse Euler Equation

We now return to efficient allocations where future generations are given positive weight. We first derive an important intertemporal condition that must be satisfied by the optimal

allocation. This condition has interesting implications for the optimal estate tax, computed later.

Let λ be the multiplier on the promise-keeping constraint and let $\mu(\theta, \theta')$ represent the multipliers on the incentive constraints. Then the first-order conditions for interior solutions for $u(\theta)$ and $w(\theta)$ are

$$p(\theta) - \hat{\lambda}c'(u(\theta))p(\theta) - \lambda p(\theta) - \sum_{\theta'} \mu(\theta, \theta') + \sum_{\theta'} \mu(\theta', \theta) = 0 \quad (28)$$

$$\hat{\beta}k'(w(\theta))p(\theta) - \beta\lambda p(\theta) - \beta \sum_{\theta'} \mu(\theta, \theta') + \beta \sum_{\theta'} \mu(\theta', \theta) = 0 \quad (29)$$

The envelope condition is $k'(v) = \lambda$. From the first-order condition for $w(\theta)$ we obtain the CLAR equation

$$\frac{\beta}{\hat{\beta}}k'(v) = \sum_{\theta} k'(g^w(v, \theta))p(\theta). \quad (30)$$

This equation encapsulates the mean-reversion force in the model. In sequential notation

$$\frac{\beta}{\hat{\beta}}k'(v_t) = \mathbb{E}_t[k'(v_{t+1})], \quad (31)$$

so that $\beta/\hat{\beta} < 1$ acts as an autoregressive coefficient ensuring that over time the derivative $k'(v_t)$ mean reverts back to zero, where the function $k(v)$ finds its interior maximum. The mean-reverting force provided by $\hat{\beta} > \beta$ is crucial for the existence of steady state distributions with bounded inequality, which we prove below. In contrast, when $\hat{\beta} = \beta$ no such central tendency exists, increasing inequality and immiseration ensues and no steady state exists ([Proposition 4](#)).

The optimal resolution of the tradeoff between incentives for altruistic parents and insurance for newborns gives rise to a less than one-for-one intergenerational transmission of welfare—in contrast to the case where $\hat{\beta} = \beta$. The descendants of a rich parent are more fortunate than those of a poor parent, but less and less so the more distant is the descendant: the impact of the initial fortune of dynasties dies out over generations.

The more weight is put on future generations, the higher is $\hat{\beta}$ compared to β , and the less intense is the link between the welfare of parents and child. But as we will now show, even the smallest amount of mean-reversion in the form of $\hat{\beta} > \beta$ puts enough limits on the transmission of shocks across generations to prevent the distribution of consumption and welfare from exploding.

The first-order conditions (28)–(29) imply that

$$\frac{\hat{\beta}}{\beta} k'(w(\theta)) = 1 - \hat{\lambda} c'(u(\theta)) \quad \text{and} \quad \frac{\hat{\beta}}{\beta} k'(v) = 1 - \hat{\lambda} c'(u_-), \quad (32)$$

where u_- should be interpreted as the previous period's assignment of utility from consumption. Substituting relations in equation (32) into the CLAR equation (30) we arrive at a *Modified Inverse Euler equation*

$$\frac{1}{u'(c_-)} = \frac{\hat{\beta}}{\beta} \sum_{\theta} \frac{1}{u'(c(\theta))} p(\theta) - \hat{\lambda}^{-1} \left(\frac{\hat{\beta}}{\beta} - 1 \right). \quad (33)$$

The left-hand side together with the first term on the right-hand side is the standard inverse Euler equation. The second term on the right-hand side is novel, since it is zero when $\beta = \hat{\beta}$ and is strictly negative when $\hat{\beta} > \beta$.³

We now show that a steady state exists whenever the welfare criterion places direct weight on children so that $\hat{\beta} > \beta$. The proof follows Farhi and Werning (2005) quite closely, which proves such a result for an economy with taste shocks.

Proposition 5. (a) *There exists an invariant distribution ψ^* for the Markov process $\{v_t\}$ implied by g^w . Moreover any invariant distribution ψ^* has a support bounded away from misery \underline{v} .* (b) *Suppose that $\lim_{u \rightarrow \infty} \sup c''(u)/c'(u) < \infty$, then any invariant distribution necessarily has a support bounded away from \bar{v} .*

An invariant distribution always exists, but when absolute risk aversion is bounded, so that $\lim_{u \rightarrow \infty} \sup c''(u)/c'(u) < \infty$,⁴ the invariant distribution has a compact support, that is bounded away from misery. It follows directly that the allocation implied by the invariant distribution has consumption and work effort that are bounded above. This ensures that the invariant ψ^* is also a steady state of the original planning problem, for some endowment level e .⁵

The result relies heavily on the force for mean reversion that is behind equation (31) and equation (33). To see this mean-reversion force most clearly consider, as an example, the logarithmic utility case, $u(c) = \log(c)$. Then $1/u'(c) = c$ and equation (33) can be written

³Farhi, Kocherlakota and Werning (2005) show that this equation, and its implications for estate taxation, generalize to an economy with capital and an arbitrary process for skills.

⁴This is the case for most common preference specifications, such as CARA or CRRA utility functions.

⁵Indeed, the proof of this result actually shows that promised continuation utility v_t is bounded for all realizations of the shocks, starting from any v_0 in the bounded support. It follows that promised utility v_t is bounded for all reporting strategies. This in turn implies that the proposed allocation is incentive compatible, that is, that the temporary incentive constraints in equation (23) imply equation (13) (see Theorem 2 in Farhi and Werning, 2005).

with sequential time notation as

$$\mathbb{E}_t[c_{t+1}] = \frac{\beta}{\hat{\beta}}c_t + \left(1 - \frac{\beta}{\hat{\beta}}\right)\bar{c},$$

or simply

$$c_{t+1} = \frac{\beta}{\hat{\beta}}c_t + \left(1 - \frac{\beta}{\hat{\beta}}\right)\bar{c} + \varepsilon_{t+1} \quad \text{with } \mathbb{E}_t[\varepsilon_{t+1}] = 0$$

where $\bar{c} \equiv \hat{\lambda}^{-1}$ is average consumption at the steady-state cross-sectional distribution. As the last expression indicates, with logarithmic utility, consumption itself is autoregressive with an autoregressive coefficient equal to $\beta/\hat{\beta} < 1$.

5.3 Tax Implementation

Any allocation that is incentive compatible and feasible, and has strictly positive consumption, can be implemented by a combination of taxes on labor income and estates. Here we first describe this implementation, and explore some features of the optimal estate tax in the next subsection.

For any incentive-compatible and feasible allocation $\{c_t^v(\theta^t), y_t^v(\theta^t)\}$ we propose an implementation along the lines of [Kocherlakota \(2004\)](#). In each period, conditional on the history of their dynasty's reports $\hat{\theta}^{t-1}$ and any inherited wealth, individuals report their current shock $\hat{\theta}_t$, produce, consume, pay taxes and bequeath wealth subject to the following set of budget constraints

$$c_t + b_t \leq y_t(\hat{\theta}^t) - T_t(\hat{\theta}^t) + (1 - \tau_t(\hat{\theta}^t))R_{t-1,t}b_{t-1} \quad t = 0, 1, \dots \quad (34)$$

where $R_{t-1,t}$ is the before-tax interest rate across generations, and initially $b_{-1} = 0$. Individuals are subject to two forms of taxation: a labor income tax $T_t(\hat{\theta}^t)$, and a proportional tax on inherited wealth $R_{t-1,t}b_{t-1}$ at rate $\tau_t(\hat{\theta}^t)$.⁶

Given a tax policy $\{T_t^v(\theta^t), \tau_t^v(\theta^t), y_t^v(\theta^t)\}$, an equilibrium consists of a sequence of interest rates $\{R_{t,t+1}\}$; an allocation for consumption, labor income and bequests $\{c_t^v(\theta^t), b_t^v(\theta^t)\}$; and a reporting strategy $\{\sigma_t^v(\theta^t)\}$ such that: (i) $\{c_t, b_t, \sigma_t\}$ maximize dynastic utility subject to (34), taking the sequence of interest rates $\{R_{t,t+1}\}$ and the tax policy $\{T_t, \tau_t, y_t\}$ as given; and (ii) the asset market clears so that $\int \mathbb{E}_{-1}[b_t^v(\theta^t)] d\phi(v) = 0$ for all $t = 0, 1, \dots$. We say that

⁶In this formulation, taxes are a function of the entire history of reports, and labor income y_t is mandated given this history. However, if the labor income histories $y^t: \Theta^t \rightarrow \mathbb{R}^t$ being implemented are invertible, then by the taxation principle we can rewrite T and τ as functions of this history of labor income and avoid having to mandate labor income. Under this arrangement, individuals do not make reports on their shocks, but instead simply choose a budget-feasible allocation of consumption and labor income, taking as given prices and the tax system.

a competitive equilibrium is incentive compatible if, in addition, it induces truth telling.

For any feasible, incentive-compatible allocation $\{c_t^v, y_t^v\}$, with strictly positive consumption we construct an incentive-compatible competitive equilibrium with no bequests by setting $T_t^v(\theta^t) = y_t(\theta^t) - c_t(\theta^t)$ and

$$\tau_t^v(\theta^t) = 1 - \frac{1}{\beta R_{t-1,t}} \frac{u'(c_{t-1}^v(\theta^{t-1}))}{u'(c_t^v(\theta^t))} \quad (35)$$

for any sequence of interest rates $\{R_{t-1,t}\}$. These choices work because the estate tax ensures that for any reporting strategy σ , the resulting consumption allocation $\{c_t^v(\sigma^t(\theta^t))\}$ with no bequests $b_t^v(\theta^t) = 0$ satisfies the consumption Euler equation

$$u'(c_t^v(\sigma^t(\theta^t))) = \beta R_{t,t+1} \sum_{\theta_{t+1}} u'(c_{t+1}^v(\sigma^{t+1}(\theta^t, \theta_{t+1}))) (1 - \tau_{t+1}^v(\sigma^{t+1}(\theta^t, \theta_{t+1}))) \Pr(\theta_{t+1}).$$

The labor income tax is such that the budget constraints are satisfied with this consumption allocation and no bequests. Thus, this no-bequest choice is optimal for the individual regardless of the reporting strategy followed. Since the resulting allocation is incentive compatible, by hypothesis, it follows that truth telling is optimal. The resource constraints together with the budget constraints then ensure that the asset market clears.⁷

As noted above, in our economy without capital only the after-tax interest rate matters so the implementation allows any equilibrium before-tax interest rate $\{R_{t-1,t}\}$. In the next subsection, we set the interest rate to the reciprocal of the social discount factor, $R_{t-1,t} = \hat{\beta}^{-1}$. This choice is natural because it represents the interest rate that would prevail at the steady state in a version of our economy with capital.

5.4 Optimal Progressive Estate Taxation

In our environment, the relevant past history is encoded in the continuation utility so the estate tax $\tau(\theta^{t-1}, \theta_t)$ can actually be reexpressed as a function of $v_t(\theta^{t-1})$ and θ_t . Abusing notation we then denote the estate tax by $\tau_t(v, \theta_t)$. Since we focus on the steady-state, invariant distribution, we also drop the time subscripts and write $\tau(v, \theta)$.

The average estate tax rate $\bar{\tau}(v)$ is then defined by

$$1 - \bar{\tau}(v) \equiv \sum_{\theta} (1 - \tau(v, \theta)) p(\theta) \quad (36)$$

⁷Since the consumption Euler equation holds with equality, the same estate tax can be used to implement allocations with any other bequest plan with income taxes that are consistent with the budget constraints.

Using the modified inverse Euler equation (33) we obtain

$$\bar{\tau}(v) = -\hat{\lambda}^{-1} u'(c_-(v)) \left(\frac{\hat{\beta}}{\beta} - 1 \right)$$

In particular, this implies that the average estate tax rate is negative, $\bar{\tau}(v) < 0$, so that bequests are subsidized. However, recall that before-tax interest rates are not uniquely determined in our implementation. As a consequence, neither are the estate taxes computed by (35). With our particular choice for the before-tax interest rate, however, the tax rates are pinned down and acquires a corrective, Pigouvian role. Differences in discounting can be interpreted as a form of externalities from future consumption, and the negative average tax can then be seen as a way of countering these externalities as prescribed by Pigou. In our setup without capital, this result depends on the choice of the before-tax interest rate. However, the negative tax on estates would be a robust steady-state outcome in a version of our economy with capital.

In our model it is more interesting to understand how the average tax varies with the history of past shocks encoded in the promised continuation utility v . The average tax is an increasing function of consumption, which, in turn, is an increasing function of v . Thus, estate taxation is progressive: the average tax on transfers for more fortunate parents is higher.

Proposition 6. *In the repeated Mirrlees economy, an optimal allocation with strictly positive consumption can be implemented by a combination of income and estate taxes. At a steady-state, invariant distribution ψ^* , the optimal average estate tax $\bar{\tau}(v)$ defined by (35) and (36) is increasing in promised continuation utility v .*

The progressivity of the estate tax reflects the mean-reversion in consumption. The fortunate must face lower net rates of return so that their consumption path decreases towards the mean.⁸

6 Concluding Remarks

When only the first generation's welfare is of concern, we obtain familiar results that echo those obtained in intragenerational settings. In particular, in our simple two-period economy we recover Atkinson-Stiglitz's uniform-taxation result. As a consequence, bequests should be undistorted and the transmission of welfare perfect: consumption of parent and child should move one-for-one. In our infinite-horizon model, we prove an immiseration result that parallels

⁸Farhi, Kocherlakota and Werning (2005) explore more general versions of this result and discuss other intuitions.

Atkeson-Lucas': a dynasty's consumption inherits a random walk property, inequality grows without bound and everyone converges to misery.

In contrast, when the expected welfare of future generations is taken into account, the planner values insuring children against the family they are born into. We characterize efficient allocations and study the role that estate taxation can play in implementing these allocations. We find that the estate tax should be progressive to ensure that consumption and welfare exhibit mean-reversion across generations. Inequality is then bounded: a steady-state cross-section for consumption and welfare exists.

Farhi, Kocherlakota and Werning (2005) explore some extensions—by including physical capital accumulation, modeling life-cycle elements and allowing skills to be correlated across generations—and show that the main result on progressive estate taxation holds. However, a number of issues are still unexplored. For example, the effects of parental investments in the child's human capital, of endogenous and variable fertility, and of *intervivo* transfers all remain open questions for future research.

Appendix

A Proof of Lemma 1

Strict concavity and differentiability follow from standard arguments. In order to derive the limits of k and k' at the bounds of the domain, we derive a lower bound k^{\min} and an upper bound k^{\max} , for which we can easily compute the corresponding limits.

Consider the solution $\{u^{v_0}(\theta^t), y^{v_0}(\theta^t)\}$ to the relaxed planning problem for a given v_0 . For all $v \leq v_0$, define $\{u_v^{v_0}(\theta^t), y_v^{v_0}(\theta^t)\}$ by

$$\begin{aligned} u_v^{v_0}(\theta^t) &= u^{v_0}(\theta^t) \text{ for all } t \geq 0 \\ h(y_v^{v_0}(\theta^0)) &= h(y^{v_0}(\theta^0)) + v_0 - v \\ y_v^{v_0}(\theta^t) &= y^{v_0}(\theta^t) \text{ for all } t \geq 1 \end{aligned}$$

Let

$$k^{\min}(v) = \sum_{t=0}^{\infty} \hat{\beta}^t E_{-1} [u_v^{v_0}(\theta^t) - \hat{\lambda}c(u_v^{v_0}(\theta^t)) + \hat{\lambda}y_v^{v_0}(\theta^t) - \theta_t h(y_v^{v_0}(\theta^t))]$$

Since $\{u_v^{v_0}(\theta^t), y_v^{v_0}(\theta^t)\}$ is incentive compatible and delivers welfare level v , we have $k(v) \geq k^{\min}(v)$, for all $v \leq v_0$. We have

$$k^{\min'}(v) = 1 - \hat{\lambda}E \left[\frac{1}{h'(h(y^{v_0}(\theta^0)) + v_0 - v)} \right]$$

Hence

$$\lim_{v \rightarrow -\infty} k^{\min'}(v) = 1$$

Since $k(v) \geq k^{\min}(v)$, for all $v \leq v_0$ and both k and k^{\min} are concave, this implies that

$$\lim_{v \rightarrow -\infty} k'(v) \leq 1$$

Next define

$$\bar{k}(v) = \sup \sum_{t=0}^{\infty} \hat{\beta}^t E_{-1} [u(\theta^t) - \hat{\lambda}c(u(\theta^t)) + \hat{\lambda}y(\theta^t) - \theta_t h(y(\theta^t))]$$

s.t.

$$v = \sum_{t=0}^{\infty} \beta^t E_{-1} [u(\theta^t) - \theta^t h(y(\theta^t))]$$

This corresponds to the relaxed planning problem, but without the incentive constraints.

Hence we have $k(v) \leq \bar{k}(v)$.

Let

$$m = \max_{u, y, \theta} u - \hat{\lambda}c(u) + \hat{\lambda}y - \theta h(y)$$

Then

$$\begin{aligned} \bar{k}(v) &\leq \sup \sum_{t=0}^{\infty} \beta^t E_{-1}[u(\theta^t) - \hat{\lambda}c(u(\theta^t)) + \hat{\lambda}y(\theta^t) - \theta_t h(y(\theta^t))] + m \left[\frac{1}{1 - \hat{\beta}} - \frac{1}{1 - \beta} \right] \\ &\leq v + \sup \left\{ \sum_{t=0}^{\infty} \beta^t E_{-1}[-\hat{\lambda}c(u(\theta^t)) + \hat{\lambda}y(\theta^t)] \right\} + m \left[\frac{1}{1 - \hat{\beta}} - \frac{1}{1 - \beta} \right] \end{aligned}$$

Hence if we define

$$C(v) = \inf \sum_{t=0}^{\infty} \beta^t E_{-1}[c(u(\theta^t)) - y(\theta^t)]$$

s.t

$$v = \sum_{t=0}^{\infty} \beta^t E_{-1}[u(\theta^t) - \theta_t h(y(\theta^t))]$$

and

$$k^{\max}(v) = v - C(v) + m \left[\frac{1}{1 - \hat{\beta}} - \frac{1}{1 - \beta} \right]$$

we have

$$\bar{k}(v) \leq k^{\max}(v)$$

Denote by $\{u^C(\theta^t, v), y^C(\theta^t, v)\}$ the solution of the program defining C . Combining the first order conditions for $u(\theta^t)$ and the envelope theorem, we get

$$\begin{aligned} c'(u^C(\theta^t, v)) &= C'(v) \text{ for all } t \geq 0 \\ \frac{1}{\theta_t h'(y^C(\theta^t, v))} &= C'(v) \text{ for all } t \geq 0 \end{aligned}$$

This implies that

$$\begin{aligned} \lim_{v \rightarrow -\infty} C'(v) &= 0 \\ \lim_{v \rightarrow -\infty} u^C(\theta^t, v) &= \underline{u} \\ \lim_{v \rightarrow -\infty} y^C(\theta^t, v) &= \infty \end{aligned}$$

Hence

$$\lim_{v \rightarrow -\infty} k^{\max'}(v) = 1$$

Since $k \leq k^{\max}$ and both k and k^{\max} are concave, this implies that

$$\lim_{v \rightarrow -\infty} k'(v) \geq 1$$

Since we already have

$$k'(v) \leq 1$$

this implies that

$$\lim_{v \rightarrow -\infty} k'(v) = 1$$

Note that we always have

$$\begin{aligned} \lim_{v \rightarrow \bar{v}} C'(v) &= +\infty \\ \lim_{v \rightarrow \bar{v}} k^{\max'}(v) &= -\infty \end{aligned}$$

Since $k(v) \leq k^{\max}(v)$, and both k and k^{\max} are concave, this implies that

$$\lim_{v \rightarrow \bar{v}} k'(v) = -\infty$$

Finally, note that

$$\lim_{v \rightarrow \bar{v}} k^{\max}(v) = \lim_{v \rightarrow \underline{v}} k^{\max}(v) = -\infty$$

Hence

$$\lim_{v \rightarrow \bar{v}} k(v) = \lim_{v \rightarrow \underline{v}} k(v) = -\infty$$

B Proof of Proposition 4

In order to characterize the optimal allocation it is convenient to study a relaxed problem. The Lagrangian theorem guarantees that there exists a unique sequence of multipliers $\{q_t\}$ with $q_0 = 1$ on (25) such that solving (24) is equivalent to solving the following program:

$$\inf \sum_{t \geq 0} q_t \int \sum_{\theta^t} (c_t^v(\theta^t) - y_t^v(\theta^t)) d\psi(v)$$

subject to (26) and (27). Note that this problem is equivalent to the minimization v by v of

$$C(v; \{q_t\}) = \sum_{t \geq 0} q_t \int \sum_{\theta^t} (c_t^v(\theta^t) - y_t^v(\theta^t))$$

subject to

$$U(\{c_t^v\}, \sigma; \beta) = v$$

$$U(\{c_t^v\}, \sigma^*; \beta) \geq U(\{c_t^v\}, \sigma; \beta) \text{ for all } \sigma$$

Hence $C(v; \{q_t\})$ is the least possible cost of an incentive compatible allocation delivering welfare v to the first generation. It is trivial to see that $C(v; \{q_t\})$ is the solution of the following Bellman equation

$$C(v; \{q_{t+s}\}_{s \geq 1}) = \inf E[c(u_\theta) - y(h_\theta) + q_{t+1}C(w_\theta, \{\frac{q_{t+s}}{q_{t+1}}\}_{s \geq 2})] \quad (37)$$

subject to

$$v = E[u_\theta + \beta w_\theta - \theta h_\theta]$$

$$u_\theta + \beta w_\theta - \theta h_\theta \geq u_{\theta'} + \beta w_{\theta'} - \theta h_{\theta'}$$

For future use, let us denote by $g^w(v, \theta^t)$ the continuation utility after a history of shock θ^t when the initial welfare entitlement is v .

Suppose there exists an invariant distribution ψ^* , and let $\{q_t\}$ be the associated sequence of multipliers. Since ψ is a state variable for (24), this shows that q_{t+1}/q_t is independent of t . Hence there exists $0 < q < 1$ such that $q_{t+1}/q_t = q$ for all t . We can therefore drop the time dependence on the sequence $\{q_t\}$ in $C_t(v; \{q_t\})$, and simply write $C(v)$ as a shortcut for $C(v, \{q^t\}_{t \geq 0})$.

Lemma 2. *Suppose there exists an invariant distribution ψ^* without full mass at misery. Then $q \geq \beta$.*

Proof. We will make use of two possible state variables. The first state variable is the natural one: v , promised future utility. The other one is utility attained by the previous generation u_- . Indeed, from the first order conditions, it is easy to see that these two state variables are related by

$$c'(u_-) = \frac{q}{\beta} C'(v)$$

The existence of an invariant distribution $\psi^*(v)$ with not mass at misery is equivalent to the existence of an invariant distribution $\hat{\psi}^*(u_-)$ with no mass at misery.

Let $x_\theta = u_\theta + \beta w_\theta$. Then we can rewrite the Bellman equation (37) as

$$C(v) = \inf E[c(u_\theta) - y(h_\theta) + qC(w_\theta)]$$

subject to

$$v = \mathbb{E}[x_\theta - \theta h_\theta]$$

$$x_\theta - \theta h_\theta \geq x_{\theta'} - \theta h_{\theta'}$$

$$u_\theta + \beta w_\theta = x_\theta$$

Hence, given a value x for x_θ , u_θ and w_θ are given by the sub-program

$$\min c(u) + qC(w)$$

subject to

$$u + \beta w = x$$

The solution is given by the first order condition

$$c'(x - \beta w) + \frac{q}{\beta} C'(w) = 0$$

Using the implicit function theorem, we can then compute

$$\frac{du}{dx} = \frac{\frac{q}{\beta} C''(w)}{\frac{q}{\beta} C''(w) + \beta c''(x - \beta w)}$$

Hence

$$0 \leq \frac{du}{dx} \leq 1$$

This in turn implies that there exists $M > 0$ such that

$$\max_{\theta, \theta'} |u_{\theta'} - u_\theta| < M \max_{\theta} h_\theta$$

The first order conditions for u_θ in in (37) imply that

$$\frac{\beta}{q} c'(u_-) = E[c'(u_\theta)]$$

Hence

$$\frac{\beta}{q} c'(u_-) = E[c'(u_\theta)] \leq c'(u_\theta)$$

Therefore,

$$\log\left(\frac{\beta}{q}\right) + \log(c'(u_-)) \leq \log(c'(u_\theta))$$

and hence

$$\log\left(\frac{\beta}{q}\right) + \log(c'(u_-)) \leq \log(c'(u_-)) + \left(\max_{u \in [u_-, u_\theta]} \frac{c''(u)}{c'(u)} \right) (u_\theta - u_-)$$

which we can rewrite as

$$\frac{\log(\frac{\beta}{q})}{\left(\max_{u \in [u_-, u_\theta]} \frac{c''(u)}{c'(u)}\right)} \leq u_\theta - u_-$$

Hence for all $\theta \in \Theta$,

$$\frac{\log(\frac{\beta}{q})}{\left(\max_{u \in [u_-, u_\theta]} \frac{c''(u)}{c'(u)}\right)} - M \max_{\theta' \in \Theta} h_{\theta'} \leq u_\theta - u_-$$

In order to allow for bunching in (37), it is convenient to consider the following program

$$\inf_{u, w} \sum_n \bar{p}_n \{c(u_n) - y(h_n) + qC(w_n)\}$$

$$v = \sum_n \bar{p}_n (u_n + \beta w_n - \bar{\theta}_n h_n)$$

$$-\theta_n h_n + u_n + \beta w_n \geq -\theta_n h_{n+1} + u_{n+1} + \beta w_{n+1} \text{ for } n = 1, 2, \dots, K-1,$$

This problem and its notation require some discussion. We do not incorporate the monotonicity constraint on h . But this notation allows us to consider bunching in the following way. If any set of neighboring agents is bunched, then we group these agents under a single index and let \bar{p}_n be the total probability of this group. Likewise let $\bar{\theta}_n$ represent the conditional average of θ within this group, which is what is relevant for the promise-keeping constraint and the objective function. Let θ_n be the shock of the highest agent in the group. The incentive constraint must rule the highest agent in each group from deviating and taking the allocation of the group above him.

Of course, every combination of bunched agents leads to a different program. The optimal allocation of our problem must solve one of these programs with a strictly monotone allocation—since bunching can be characterized by regrouping agents. Thus, below we characterize solutions to these programs with strict monotonicity of the solution.

The first order conditions for h_n is

$$y'(h_n) = C'(v)\bar{\theta}_n + \theta_n \mu_{n,n+1} - \theta_{n-1} \mu_{n-1,n}.$$

This implies in particular that at the optimum, for any of these programs (and hence for the program solved by the true optimal allocation),

$$y'(h_\theta) \geq C'(v)\underline{\theta}.$$

It is easy to verify that $C \geq \hat{C}$, where \hat{C} is the solution to (37) without the incentive

compatibility constraints. Let \bar{v} be the upper bound of the domain for v . Since both C and \hat{C} are increasing and convex, and since

$$\lim_{v \rightarrow \bar{v}} \hat{C}(v) = \infty \quad \text{and} \quad \lim_{\bar{v} \rightarrow \infty} \hat{C}'(v) = \infty$$

we have

$$\lim_{v \rightarrow \bar{v}} C(v) = \infty \quad \text{and} \quad \lim_{v \rightarrow \bar{v}} C'(v) = \infty$$

Therefore,

$$\lim_{v \rightarrow \bar{v}} y'(h_\theta(v)) = \infty \quad \Rightarrow \quad \lim_{v \rightarrow \bar{v}} h_\theta(v) = 0$$

and since h_θ has is decreasing in θ ,

$$\lim_{v \rightarrow \bar{v}} h_\theta(v) = 0 \quad \text{for all } \theta \in \Theta$$

But this in turn implies that

$$\frac{\log(\frac{\beta}{q})}{\left(\max_{u \in [u_-, u_\theta]} \frac{c''(u)}{c'(u)}\right)} \leq \liminf_{v \rightarrow \bar{v}} (u_\theta - u_-)$$

Suppose that $q < \beta$. This implies that for v or equivalently u_- high enough, the policy functions u_θ are all such that $u_\theta > u_-$. This in turn implies that $\hat{\psi}^*$ necessarily has a support bounded away from \bar{u} . This in turn implies that

$$\int C'(v) d\psi^*(v) = \int c'(u_-) d\hat{\psi}^*(u_-) < \infty$$

Integrating

$$\frac{\beta}{q} C'(v) = E[C'(w_\theta)]$$

over v , we get

$$\int C'(v) d\psi^*(v) = \frac{\beta}{q} \int C'(v) d\psi^*(v)$$

Since ψ^* doesn't have full mass at misery, we have $\int C'(v) d\psi^*(v) > 0$. This in turn implies that $\beta = q$, a contradiction. \square

We have therefore proved that $q \geq \beta$ at ψ^* . But then from the equation

$$\frac{\beta}{q}C'(v) = \mathbb{E}[C'(w_\theta)]$$

we see that $C'(v_t)$ is a positive supermartingale. By the martingale convergence theorem, for any initial value v_0 for v , the sequence of random variables $\{v_t\}$ converges almost surely to a random variable $C'_v{}^\infty$ with

$$\mathbb{E}[C'_v{}^\infty] \leq C'(v).$$

Suppose that there must exist a v^* such that $\Pr(C'_{v^*}{}^\infty > 0) > 0$. We will show that this is not possible.

For any realization θ^∞ define the set of periods where θ_t takes on some particular value $\theta \in \Theta$ as

$$O_\theta(\theta^\infty) \equiv \{t, \theta_t(\theta^\infty) = \theta\}.$$

Then since Θ is finite, we have that with probability one all values of θ occur infinitely often

$$\Pr(\#O_\theta(\theta^\infty) = \infty \text{ for all } \theta \in \Theta) = 1.$$

Hence there exists an event θ^∞ such that $C'(g^w(v^*, \theta^t(\theta^\infty)))$ converges to a positive and finite value, and $\#O_\theta(\theta^\infty) = \infty$ for all $\theta \in \Theta$. Hence $g^w(v^*, \theta^t(\theta^\infty))$ converges to a finite value w^* . Since $g^w(v, \theta)$ is continuous in v , and $\#O_\theta(\theta^\infty) = \infty$ this implies that $g^w(w^*, \theta) = w^*$ for all $\theta \in \Theta$. This implies that the incentive constraints are not binding at w^* , a contradiction.

Hence $\Pr(C'_v{}^\infty > 0) = 0$ for all v . Therefore for all v , $C'(g^w(v, \theta^t))$ converges almost surely to 0. This in turn implies that the stochastic process $C'(v_t)$ converges almost surely to 0. This implies that $C'(v_t)$ converges in distribution to 0. Since ψ^* is an invariant distribution, $C'(v_t)$ is distributed as $C'(v_0)$. This implies that the distribution of $C'(v_0)$ has full mass at zero, i.e. that ψ^* has full mass at misery.

C Proof of Proposition 5

We start with two lemmas, and then proceed to prove the proposition.

Lemma 3. *The following inequalities hold*

$$\underline{\gamma}(1 - k'(v)) + \left(1 - \frac{\beta}{\hat{\beta}}\right) \leq 1 - k'(g^w(\theta, v)) \leq \bar{\gamma}(1 - k'(v)) + \left(1 - \frac{\beta}{\hat{\beta}}\right)$$

for all $\theta \in \Theta$, where the constants are given by $\bar{\gamma} \equiv (\beta/\hat{\beta}) \max_{1 \leq n \leq N} \{(1 + \theta_n - \mathbb{E}[\theta \mid \theta \leq \theta_n])/\theta_n\}$ and $\underline{\gamma} \equiv (\beta/\hat{\beta}) \min_{2 \leq n \leq N} \{1 + \theta_{n-1} - \mathbb{E}[\theta \mid \theta \geq \theta_n]/\theta_{n-1}\}$.

Proof. Consider the program

$$\max_{u,w} \sum_n \bar{p}_n \{u_n - \hat{\lambda}c(u_n) + \hat{\lambda}y(h_n) - \bar{\theta}_n h_n + \hat{\beta}k(w_n)\}$$

$$v = \sum_n \bar{p}_n (u_n + \beta w_n - \bar{\theta}_n h_n)$$

$$-\theta_n h_n + u_n + \beta w_n \geq -\theta_n h_{n+1} + u_{n+1} + \beta w_{n+1} \text{ for } n = 1, 2, \dots, K-1,$$

This problem and its notation require some discussion. We do not incorporate the monotonicity constraint on h . But this notation allows us to consider bunching in the following way. If any set of neighboring agents is bunched, then we group these agents under a single index and let \bar{p}_n be the total probability of this group. Likewise let $\bar{\theta}_n$ represent the conditional average of θ within this group, which is what is relevant for the promise-keeping constraint and the objective function. Let θ_n be the shock of the highest agent in the group. The incentive constraint must rule the highest agent in each group from deviating and taking the allocation of the group above him.

Of course, every combination of bunched agents leads to a different program. The optimal allocation of our problem must solve one of these programs with a strictly monotone allocation—since bunching can be characterized by regrouping agents. Thus, below we characterize solutions to these programs with strict monotonicity of the solution.

The first-order conditions are

$$\bar{p}_n \{\hat{\lambda}y'(h_n) - \bar{\theta}_n + \lambda\bar{\theta}_n\} - \theta_n \mu_n + \theta_{n-1} \mu_{n-1} \leq 0$$

$$\bar{p}_n \{\hat{\beta}k'(w_n) - \beta\lambda\} + \beta(\mu_n - \mu_{n-1}) = 0$$

where, by the envelope condition $\lambda = k'(v)$.

Summing the first-order conditions for h_n , we get

$$\hat{\lambda} \mathbb{E}[y'(h(\theta))] = 1 - k'(v)$$

Summing up the first-order conditions for w_n , we get

$$\mathbb{E}[k'(g^w(v, \theta))] = \frac{\beta}{\hat{\beta}} k'(v)$$

The first-order conditions for $n = 1$ imply

$$(1 - \lambda) + \frac{\theta_1}{\bar{\theta}_1} \frac{\mu_1}{\bar{p}_1} = \frac{\hat{\lambda}y'(h_1)}{\bar{\theta}_1} \leq \frac{\hat{\lambda} \mathbb{E}[y'(h_\theta)]}{\bar{\theta}_1} = \frac{1 - \lambda}{\bar{\theta}_1}.$$

This implies

$$\frac{\mu_1}{\bar{p}_1} \leq \frac{1-\lambda}{\theta_1} - (1-\lambda) \frac{\bar{\theta}_1}{\theta_1}.$$

Using

$$k'(w_1) = \frac{\beta}{\hat{\beta}} \lambda - \frac{\beta \mu_1}{\hat{\beta} \bar{p}_1},$$

we get

$$k'(w_1) \geq \frac{\beta}{\hat{\beta}} \left[\lambda - \frac{1-\lambda}{\theta_1} + (1-\lambda) \frac{\bar{\theta}_1}{\theta_1} \right] = \frac{\beta}{\hat{\beta}} \left[1 + \frac{1}{\theta_1} - \frac{\bar{\theta}_1}{\theta_1} \right] k'(v) + \frac{\beta}{\hat{\beta}} \left[\frac{\bar{\theta}_1}{\theta_1} - \frac{1}{\theta_1} \right].$$

Similarly, writing the first-order conditions for $n = K$, we get

$$(1-\lambda) - \frac{\theta_{K-1} \mu_{K-1}}{\bar{\theta}_K \bar{p}_K} = \frac{\hat{\lambda} y'(h_K)}{\bar{\theta}_K} \geq \frac{\hat{\lambda} \mathbb{E}[c'(h_\theta)]}{\bar{\theta}_K} = \frac{1-\lambda}{\bar{\theta}_K}.$$

This implies

$$-\frac{\mu_{K-1}}{\bar{p}_K} \geq \frac{1-\lambda}{\theta_{K-1}} - (1-\lambda) \frac{\bar{\theta}_K}{\theta_{K-1}}.$$

Using

$$k'(w_K) = \frac{\beta}{\hat{\beta}} \lambda + \frac{\beta \mu_{K-1}}{\hat{\beta} \bar{p}_K},$$

we get

$$k'(w_K) \leq \frac{\beta}{\hat{\beta}} \left[\lambda - \frac{1-\lambda}{\theta_{K-1}} + (1-\lambda) \frac{\bar{\theta}_K}{\theta_{K-1}} \right] = \frac{\beta}{\hat{\beta}} \left[1 + \frac{1}{\theta_{K-1}} - \frac{\bar{\theta}_K}{\theta_{K-1}} \right] k'(v) + \frac{\beta}{\hat{\beta}} \left[\frac{\bar{\theta}_K}{\theta_{K-1}} - \frac{1}{\theta_{K-1}} \right].$$

For any n , $w_K \leq w_n \leq w_1$,

$$\begin{aligned} \frac{\beta}{\hat{\beta}} \left[1 + \frac{1}{\theta_1} - \frac{\bar{\theta}_1}{\theta_1} \right] k'(v) + \frac{\beta}{\hat{\beta}} \left[\frac{\bar{\theta}_1}{\theta_1} - \frac{1}{\theta_1} \right] &\leq k'(w_n) \\ &\leq \frac{\beta}{\hat{\beta}} \left[1 + \frac{1}{\theta_{K-1}} - \frac{\bar{\theta}_K}{\theta_{K-1}} \right] k'(v) + \frac{\beta}{\hat{\beta}} \left[\frac{\bar{\theta}_K}{\theta_{K-1}} - \frac{1}{\theta_{K-1}} \right]. \end{aligned}$$

After rearranging, we obtain

$$\begin{aligned} \frac{\beta}{\hat{\beta}} \left[1 + \frac{1}{\theta_1} - \frac{\bar{\theta}_1}{\theta_1} \right] (1 - k'(v)) + 1 - \frac{\beta}{\hat{\beta}} &\geq 1 - k'(g^w(\theta, v)) \\ &\geq \frac{\beta}{\hat{\beta}} \left[1 + \frac{1}{\theta_{K-1}} - \frac{\bar{\theta}_K}{\theta_{K-1}} \right] (1 - k'(v)) + 1 - \frac{\beta}{\hat{\beta}}. \end{aligned}$$

□

By Lemma 1 we have that $\lim_{v \rightarrow \underline{v}} k'(v) = 1$, then using the bounds we obtain that

$$\lim_{v \rightarrow \underline{v}} k'(g^w(v, \theta)) = \frac{\beta}{\hat{\beta}} < 1 = \lim_{v \rightarrow \underline{v}} k'(v),$$

for all $\theta \in \Theta$.

The following lemma describes the behavior of the optimal allocation when v goes to \underline{v} .

Lemma 4. *We have $g^u(v, \theta) > \underline{u}$ and $\lim_{v \rightarrow \underline{v}} g^u(v, \theta) = \underline{u}$, $\lim_{v \rightarrow \underline{v}} g^h(v, \theta) = \infty$*

Proof. Consider the program

$$\max_{u, w} \sum_n \bar{p}_n \{u_n - \hat{\lambda}c(u_n) + \hat{\lambda}y(h_n) - \bar{\theta}_n h_n + \hat{\beta}k(w_n)\}$$

$$v = \sum_n \bar{p}_n (u_n + \beta w_n - \bar{\theta}_n h_n)$$

$$-\theta_n h_n + u_n + \beta w_n \geq -\theta_n h_{n+1} + u_{n+1} + \beta w_{n+1} \text{ for } n = 1, 2, \dots, K-1,$$

The first order condition for u_n is $1 - \hat{\lambda}c'(u_n) = \frac{\hat{\beta}}{\beta}k'(w_n)$. Hence $1 - \hat{\lambda}c'(g^u(v, \theta)) = \frac{\hat{\beta}}{\beta}k'(g^w(v, \theta))$. Since $k'(g^w(v, \theta)) < \frac{\beta}{\hat{\beta}}$, we have $u_n > \underline{u}$. Moreover, since $\lim_{v \rightarrow \underline{v}} k'(v, \theta) = \frac{\beta}{\hat{\beta}}$, we have

$$\lim_{v \rightarrow \underline{v}} c'(g^u(v, \theta)) = 0 \quad \text{or equivalently} \quad \lim_{v \rightarrow \underline{v}} g^u(v, \theta) = \underline{u}$$

That $\lim_{v \rightarrow \underline{v}} g^h(v, \theta) = \infty$ follows from

$$\hat{\lambda}\mathbb{E}[y'(g^h(v, \theta))] = 1 - k'(v)$$

and $\lim_{v \rightarrow \underline{v}} k'(v) = 1$. □

Since the derivative $k'(v)$ is continuous and strictly decreasing, we can define the transition function

$$Q(x, \theta) = k'(g^w((k')^{-1}(x), \theta))$$

for all $x < l$ if utility is unbounded below. For any probability distribution μ , let $T_Q(\mu)$ be the probability distribution defined by

$$T_Q(\mu)(A) = \int \mathbf{1}_{\{Q(x, \theta) \in A\}} d\mu(x) dp(\theta)$$

for any Borel set A . Define

$$T_{Q,n} \equiv \frac{T_Q + T_Q^2 + \dots + T_Q^n}{n}$$

For example, $T_{Q,n}(\delta_x)$ is the empirical average of $\{k'(v_t)\}_{t=1}^n$ over all histories of length n starting with $k'(v_0) = x$. The following lemma establishes the existence of an invariant distribution by considering the limits of $\{T_{Q,n}\}$.

We are now able to prove a proposition that implies the first part [Proposition 5](#), and describes an algorithm to construct an invariant distribution.

Proposition 7. *For each $x < l$ there exists a subsequence $\{T_{Q,\phi(n)}(\delta_x)\}$ that converges weakly, i.e. in distribution, to an invariant distribution on $(-\infty, 1)$ under Q .*

Proof. For all $\theta \in \Theta$

$$\lim_{x \uparrow 1} Q(x, \theta) = \lim_{v \rightarrow -\infty} k'(g^w(\theta, v)) = \frac{\beta}{\hat{\beta}} < 1.$$

Note that we have a continuous transition function $Q(x, \theta): (-\infty, 1) \times \Theta \rightarrow (-\infty, 1)$.

We next show that the sequence $\{T_Q^n(\delta_x)\}$ is tight, in that for any $\varepsilon > 0$ there exists a compact set K_ε such that $T_Q^n(\delta_x)(K_\varepsilon) \geq 1 - \varepsilon$, for all n . The expected value of the distribution $T_Q^n(\delta_x)$ is simply $\mathbb{E}_{-1}[k'(v_t(\theta^{t-1}))]$ with $x = k'(v_0) < 1$. Recall that $\mathbb{E}_{-1}[k'(v_t(\theta^{t-1}))] = (\beta/\hat{\beta})^t k'(v_0) \rightarrow 0$. This implies that

$$\begin{aligned} \min\{0, k'(v_0)\} &\leq \mathbb{E}_{-1}[k'(v_t(\theta^{t-1}))] \\ &\leq T_Q^n(\delta_x)(-\infty, -A)(-A) + (1 - T_Q^n(\delta_x)(-\infty, -A))1 \end{aligned}$$

for all $A > 0$. Rearranging,

$$T_Q^n(\delta_x)(-\infty, -A) \leq \frac{1 - \min\{0, x\}}{A + 1}$$

Hence we can find $A_\varepsilon > 0$ such that

$$T_Q^n(\delta_x)(-\infty, -A_\varepsilon) \leq \frac{\varepsilon}{2}$$

Define a_ε by

$$1 - a_\varepsilon = \sup_{\substack{x \in [A_\varepsilon, 1] \\ \theta \in \Theta}} Q(x, \theta)$$

Since for all $\theta \in \Theta$, $\lim_{v \rightarrow -\infty} k'(v, \theta) < \frac{\beta}{\hat{\beta}} < 1$, we have $a_\varepsilon > 0$. In addition, for all $n \geq 1$, $T_Q^n(\delta_x) = T_Q(T_Q^{n-1}(\delta_x))$, so that

$$T_Q^n(\delta_x)(1 - a_\varepsilon, 1) \leq T_Q^{n-1}(\delta_x)(-\infty, A_\varepsilon) \leq \frac{\varepsilon}{2}$$

Since we also have

$$T_Q^n(\delta_x)(-\infty, -A_\varepsilon) \leq \frac{\varepsilon}{2}$$

this implies

$$T_Q^n(\delta_x)[A_\varepsilon, 1 - a_\varepsilon] \geq \varepsilon$$

Taking $K_\varepsilon = [A_\varepsilon, 1 - a_\varepsilon]$, this implies that $\{T_Q^n(\delta_x)\}_{n \geq 1}$ is tight, and therefore $\{T_Q^n(\delta_x)\}_{n \geq 0}$, is tight.

Tightness implies that there exists a subsequence $T_Q^{\phi(n)}(\delta_x)$ that converges weakly, i.e. in distribution, to some probability distribution π on $(-\infty, 1)$. Since $Q(x, \theta)$ is continuous in x , then $T_Q(T_Q^{\phi(n)}(\delta_x))$ converges weakly to $T_Q(\pi)$. But the linearity of T_Q implies that

$$T_Q(T_Q^{\phi(n)}(\delta_x)) = \frac{T_Q^{\phi(n)+1}(\delta_x) - T_Q(\delta_x)}{\phi(n)} + T_Q^{\phi(n)}(\delta_x)$$

and since $\phi(n) \rightarrow \infty$ we must have $T_Q(\pi) = \pi$. \square

Note that for any invariant distribution π , $T_Q(\pi) = \pi$ implies that the support of π is contained in $(-\infty, \frac{\beta}{\beta}]$. This proves the second part of [Proposition 5](#). We finally prove a lemma that implies the last part [Proposition 5](#).

Lemma 5. *Suppose that $\lim_{u \rightarrow \infty} \sup c''(u)/c'(u) < \infty$. Then any invariant distribution $\hat{\psi}$ necessarily has a support bounded away from \bar{v} .*

Proof. We will make use of two possible state variables. The first state variable is the natural one: v , promised future utility. The other one is utility attained by the previous generation u_- . Indeed, from the first order conditions, it is easy to see that these two state variables are related by

$$1 - \hat{\lambda}c'(u_-) = \frac{\hat{\beta}}{\beta}k'(v)$$

The existence of an invariant distribution $\psi^*(v)$ with not mass at misery is equivalent to the existence of an invariant distribution $\hat{\psi}^*(u_-)$ with no mass at misery.

Let $x_\theta = u_\theta + \beta w_\theta$. Then we can rewrite the Bellman equation [\(21\)](#) as

$$k(v) = \sup E[u_\theta - \hat{\lambda}c(u_\theta) - \theta h_\theta + \hat{\lambda}y(h_\theta) + \hat{\beta}k(w_\theta)]$$

subject to

$$v = E[x_\theta - \theta h_\theta]$$

$$x_\theta - \theta h_\theta \geq x_{\theta'} - \theta h_{\theta'}$$

$$u_\theta + \beta w_\theta = x_\theta$$

Hence, given a value x for x_θ , u_θ and w_θ are given by the sub-program

$$\max u - \hat{\lambda}c(u) + \hat{\beta}k(w)$$

subject to

$$u + \beta w = x$$

The solution is given by the first order condition

$$1 - \hat{\lambda}c'(u) = \frac{\hat{\beta}}{\beta}k'\left(\frac{x-u}{\beta}\right) = 0$$

Using the implicit function theorem, we can then compute

$$\frac{du}{dx} = \frac{-\frac{\hat{\beta}}{\beta}k''\left(\frac{x-u}{\beta}\right)}{-\frac{\hat{\beta}}{\beta}k''\left(\frac{x-u}{\beta}\right) + \hat{\lambda}c''(u)}$$

Hence

$$0 \leq \frac{du}{dx} \leq 1$$

This in turn implies that there exists $M > 0$ such that

$$\max_{\theta, \theta'} |u_{\theta'} - u_\theta| < M \max_{\theta} h_\theta$$

Consider the program (C). The first order condition for h_n is

$$\bar{p}_n \{ \hat{\lambda}y'(h_n) - \bar{\theta}_n + \lambda\bar{\theta}_n \} - \theta_n \mu_n + \theta_{n-1} \mu_{n-1} \leq 0$$

where $\lambda = k'(v)$. This implies that

$$y'(h_\theta) \geq \frac{\theta}{\lambda}(1 - k'(v))$$

This shows that

$$\lim_{v \rightarrow \bar{v}} y'(h_\theta(v)) = \infty \quad \Rightarrow \quad \lim_{v \rightarrow \bar{v}} h_\theta(v) = 0$$

and since h_θ has is decreasing in θ ,

$$\lim_{v \rightarrow \bar{v}} h_\theta(v) = 0 \text{ for all } \theta \in \Theta$$

The first order condition(33) implies that

$$c'(u_-) \geq \frac{\hat{\beta}}{\beta} c'(u_\theta) - \hat{\lambda}^{-1} \left(\frac{\hat{\beta}}{\beta} - 1 \right)$$

which can be rewritten as

$$\frac{\beta}{\hat{\beta}} c'(u_-) + \hat{\lambda}^{-1} \left(1 - \frac{\beta}{\hat{\beta}} \right) \geq c'(u_\theta)$$

This in turn implies that for all $\theta \in \Theta$

$$\exp \left(M \max_{\theta} h_{\theta} \max_{u \in [u_{\bar{v}}, u_{\theta}]} \frac{c''(u)}{c'(u)} \right) \left(\frac{\beta}{\hat{\beta}} c'(u_-) + \hat{\lambda}^{-1} \left(1 - \frac{\beta}{\hat{\beta}} \right) \right) \geq c'(u_\theta)$$

Since

$$\lim_{v \rightarrow \bar{v}} h_{\theta}(v) = 0 \text{ for all } \theta \in \Theta$$

we have

$$\lim_{u_- \rightarrow \bar{u}} \exp \left(M \max_{\theta} h_{\theta} \max_{u \in [u_{\bar{v}}, u_{\theta}]} \frac{c''(u)}{c'(u)} \right) = 1 \text{ for all } \theta \in \Theta$$

This in turn proves that for u_- high enough, all the policy functions u_θ are such that $u_\theta < u_-$. Hence any invariant distribution $\hat{\psi}^*$ necessarily has a support bounded away from \bar{u} . This is equivalent to saying that any invariant distribution $\hat{\psi}$ necessarily has a support bounded away from \bar{v} . \square

References

- Albanesi, Stefania and Christopher Sleet**, “Dynamic Optimal Taxation with Private Information,” 2004. mimeo. [2](#), [4](#), [12](#), [14](#), [16](#)
- Amador, Manuel, George-Marios Angeletos, and Iván Werning**, “Corrective vs. Redistributive Taxation,” 2005. work in progress. [12](#)
- Atkeson, Andrew and Robert E. Lucas Jr.**, “On Efficient Distribution with Private Information,” *Review of Economic Studies*, 1992, *v59* (3), 427–453. [4](#), [6](#), [16](#)
- and **Robert Lucas**, “Efficiency and Equality in a Simple Model of Efficiency Unemployment Insurance,” *Journal of Economic Theory*, 1995, *66*, 64–88. [2](#), [16](#)
- Atkinson, A.B. and J.E. Stiglitz**, “The Design of Tax Structure: Direct vs. Indirect Taxation,” *Journal of Public Economics*, 1976, *6*, 55–75. [2](#), [9](#)
- Diamond, Peter A.**, “Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates,” *American Economic Review*, 1998, *88* (1), 83–95. [3](#)
- Ebert, Udo**, “A reexamination of the optimal nonlinear income tax,” *Journal of Public Economics*, 1992, *49* (1), 47–73. [3](#)
- Farhi, Emmanuel and Iván Werning**, “Inequality, Social Discounting and Progressive Estate Taxation,” NBER Working Papers 11408 2005. [4](#), [5](#), [14](#), [18](#)
- , **Narayana Kocherlakota, and Iván Werning**, 2005. work in progress. [4](#), [9](#), [18](#), [21](#), [22](#)
- Golosov, Mikhail, Aleh Tsyvinski, and Iván Werning**, “New Dynamic Public Finance: A User’s Guide,” *forthcoming in NBER Macroeconomics Annual 2006*, 2006. [4](#)
- , **Narayana Kocherlakota, and Aleh Tsyvinski**, “Optimal Indirect and Capital Taxation,” *Review of Economic Studies*, 2003, *70* (3), 569–587. [4](#)
- Kocherlakota, Narayana**, “Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation,” 2004. mimeo. [4](#), [19](#)
- Mirrlees, James A.**, “An Exploration in the Theory of Optimum Income Taxation,” *Review of Economic Studies*, 1971, *38* (2), 175–208. [2](#), [3](#)
- Phelan, Christopher**, “Opportunity and Social Mobility,” *Federal Reserve Bank of Minneapolis, Research Department Staff Report*, 2005, *323*. [5](#)
- Saez, Emmanuel**, “Using Elasticities to Derive Optimal Income Tax Rates,” *Review of Economic Studies*, 2001, *68* (1), 205–29. [3](#)
- Seade, Jesus**, “On the Sign of the Optimum Marginal Income Tax,” *Review of Economic Studies*, 1982, *49* (4), 637–43. [3](#)
- Sleet, Christopher and Sevin Yeltekin**, “Credible Social Insurance,” 2005. mimeo. [5](#)
- Tuomala, Matti**, *Optimal Income Taxation and Redistribution*, Oxford University Press, Clarendon Press, 1990. [3](#)