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THE EFFICIENCY OF REFERENDA

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ABSTRACT

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A Simple Scheme to Improve the Efficiency of Referenda[*]

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Abstract

This paper proposes a simple scheme designed to elicit and reward intensity of preferences in referenda: voters faced with a number of binary proposals are given one regular vote for each proposal plus an additional number of bonus votes to cast as desired. Decisions are taken according to the majority of votes cast. In our base case, where there is no systematic difference between proposals' supporters and opponents, there is always a positive number of bonus votes such that ex ante utility is increased by the scheme, relative to simple majority voting. When the distributions of valuations of supporters and opponents differ, the improvement in efficiency is guaranteed only if the distributions can be ranked according to first order stochastic dominance. If they are, however, the existence of welfare gains is independent of the exact number of bonus votes.

1   Introduction

In binary decisions—when a proposal can either pass or fail—majority voting performs rather well. In fact, it has only one obvious drawback: it fails to account for the intensity of preferences. It is this failing that leads to several related problems: the blocking of proposals that would increase conventional measures of social welfare, and the related need to pretect minorities from the worst abuses; the temptation to recur to log-rolling in committees, and the resulting lack of transparency. It is natural to ask then whether a voting system

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as simple as majority voting but rewarding intense preferences could be designed for binary decisions.

The functioning of prices in a market offers some inspiration: prices elicit consumers' intensity of preferences by differentiating across goods and functioning in tandem with a budget constraint. The budget constraint plays a central role and suggests an immediate idea: suppose voters were given a stock of votes, and asked to allocate them as they see fit over a series of binary proposals, each of which would then be decided on the basis of the majority of votes cast. Would voters be led to cast more votes over those issues to which they attach more importance? And would the final result then be an expected welfare gain, relative to simple majority voting, as the probability of winning a vote shifts for each voter from issues of relatively less importance towards issues of relatively more importance? We have proposed a voting system of this type—*storable votes*—in two recent papers (Casella, 2002, and Casella, Gelman and Palfrey 2003).[1] The simple intuition proves correct: both in theory and in experiments subjects cast more votes when the intensity of their preferences is higher. The efficiency gains are also borne out: both in theory and in the experiments, ex ante utility is typically higher with storable votes (although some counterexamples exist).

A particularly clear example is the application of these ideas to referenda[2]. Referenda are important in practice because they are commonly and increasingly used,[3] and are attractive in theory because the large population of voters eliminates most of the strategic considerations that complicate the analysis of voting choices in committees. In addition, referenda are often submitted to voters in bundles—think of a set of propositions on a California ballot, or indeed in many US states and European countries. Consider then a voting mechanism where voters are faced with a number of contemporaneous, unrelated referenda, and are asked to cast one vote on each referendum but in addition are given a number of "bonus votes" to cast as desired over the different referenda. Each referendum is then decided according to the majority of the votes. Does the addition of the bonus votes allow voters to express the intensity of their preferences and increase their ex ante welfare, relative to simple majority voting? This is the question studied in this paper.

We find that the answer is positive if the number of bonus votes is chosen correctly. Intuitively, the bonus votes give voters the possibility to target the single issue that is most important to them (in equilibrium the bonus votes are

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[1] Storable votes apply to series of binary decisions taken over time. Hortala-Vallve (2004) has proposed a similar system in a simultaneous voting game with two voters and two binary decisions.

[2] We use the term "referenda" to indicate any proposition decided by popular majority voting, whether initiated by the government (referenda in the proper sense) or by the people (initiatives).

[3] Gerber (1999), Matsusaka (2004), the Inititative and Referendum Institute at www.iandrinstitute.org, and the Direct Democracy Institute at www.c2d.unige.ch provide a wealth of information on the history and practice of direct democracy around the world. Referenda are now used in many democracies (in Switzerland, of course, but also in the US, the European Union, Australia, and other countries), and their number is rising (in US states, for example, the number of referenda has increased in every decade since 1970, at an average rate of seventy per cent per decade).

never split), but at the cost of more uncertainty over the other proposals. In our base case, where for any proposal there is no systematic difference between the distribution of valuations of proponents and opponents, the result is clean: if the edge between the representative voter's highest expected valuation and his mean valuation over all proposals is large, the first effect dominates, and the number of bonus votes should be large; if on the other hand such an edge is small, the number of bonus votes should be correspondingly small. But the number should not be zero: for all distributions of valuations there is a positive number of bonus votes such that ex ante welfare rises, relative to simple majority voting.

After having presented our analysis in the simplest setting, we discuss two natural questions on the design of the mechanism. Should the bonus votes be perfectly divisible, as opposed to being a discrete number, allowing voters to fine-tune the ranking of the different referenda? Should the bonus votes be negative, as opposed to positive, granting voters the power to cast "partial votes," weaker votes relative to the regular ones? We show that generalizing the scheme to admit these possibilities is not difficult, but the resulting design has less intuitive appeal. The increase in complexity is not matched by a corresponding increase in efficiency.

The theoretical result of our base case is clean and robust. The difficulty lies in quantifying the importance of the welfare gain. Symmetrical distributions of valuations between supporters and opponents of any given proposal make the analysis manageable and are commonly assumed in the literature, but constrain the problem severely: a direct implication of the results in Ledyard and Palfrey (2002) is that with symmetrical distributions our bonus votes mechanism, simple majority voting, and random decision making all are asymptotically efficient. The positive welfare gain associated with bonus votes becomes vanishingly small in the limit, as the population approaches infinity. (Of course the same can be said of majority voting over random decision making.) Introducing asymmetries is then important, but their choice can be arbitrary. In a thorough empirical analysis of more than 800 ballot propositions in California from 1912 through 1989, Matsusaka (1992) identifies an equally split electorate as characteristic of propositions submitted to popular vote (as opposed to being decided by the legislature). Anchoring our model with this observation, we assume that the population is equally split on all proposals, but mean intensity is higher on one side. In this case, bonus votes are guaranteed to increase ex ante utility if the distribution of valuations on the side with higher mean first-order stochastically dominates the distribution on the opposite side; that is, if the mass of voters with more intense preferences is larger on the side with higher mean. When this sufficient condition is satisfied, the superiority of bonus votes over majority voting holds independently of the exact number of bonus votes and remains true asymptotically (whereas majority voting again converges to random decision-making). First order stochastic dominance is a sufficient condition for robust welfare gains, but our numerical exercises suggest that the result is more general: especially when the number of referenda is not large, counterexamples where simple majority voting is superior are not easy to construct.

It is this more general case of asymmetric distributions that better captures the basic intuition for bonus votes discussed earlier.[4] If voters are equally split on a proposal, efficiency demands that the side with the higher intensity of preferences prevails; and if the voters are not equally split, a strongly affected minority should at time prevail over a less affected majority. This is the outcome that bonus votes can deliver. Notice that the conclusion need not involve inter-personal comparisons of utility: in the ex ante evaluation, at a constitutional stage taking place before specific ballots are realized, all voters are identical and the representative voter weighs the probabilities of his yet unrevealed valuations.

But is the need for stronger minority representation a real need in practice? Anecdotal reports abound on the distorting effects of money in direct democracy, and more precisely on the disproportionate power of narrow business interests.[5] Is there room for a voting scheme that is designed to increase further the power of minorities? Perhaps surprisingly, the informed answer seems to be yes. Gerber (1999) and Matsusaka (2004) provide exhaustive empirical analyses of direct democracy in US states, where money spent in referenda campaigns is largest and unlimited. Although their emphasis differs, they both conclude that there is no evidence that business interests are succeeding at manipulating the process in their favor any more than grass-root citizens' groups (or, according to Matsusaka, away from the wishes of the majority). In fact both books isolate the need to protect minorities, stripped of the checks and balances of representative democracy and of the pragmatic recourse to log-rolling, as the most urgent task in improving the process.[6]

The protection of minorities is the heart of the existing voting system that most closely resembles the mechanism described here. "Cumulative voting" applies to a single multi-candidate election and grants each voter a number of votes equal to the number of positions to be filled, with the proviso that the votes can either be spread or cumulated on as few of the candidates as desired. The system has been advocated as an effective protection of minority rights (Guinier, 1994) and has been recommended by the courts as redress to violations of fair representation in local elections (Issacharoff, Karlan and Pildes, 2001). There is some evidence, theoretical (Cox, 1990), empirical (Bowler, Donovan and Brockington, 2003), and experimental (Gerber, Morton and Rietz, 1998) that cumulative voting does indeed work in the direction intended. The bonus votes scheme discussed in this paper differs because it applies to a series of

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[4] The asymmetry of the distribution seems natural when talking informally, but is difficult to justify in analyses based on a single referendum. The approach posits cardinal valuations, but on what basis can one side claim a larger mean valuation than the other? A normalization, a reference criterion, is required. Studying multiple proposals contemporaneously provides such a reference.

[5] See, for example, Broder (2000), with the expressive title *Democracy Derailed*. Opposite views on the promise of direct democracy, held with equal strength, are also common: see for example *The Economist*, Dec 21, 1996 ("The idea that people should govern themselves can at last mean just that") or *The Economist*, Jan 23, 2003.

[6] Even if their conclusion were incorrect, a priori it is unclear how granting bonus votes would interact with the existence of special interests and their differential access to campaign money.

independent decisions, each of which can either pass or fail, but the intuition inspiring it is similar.

The idea of eliciting preferences by linking independent decisions through a common budget constraint can be exploited quite generally, as shown by Jackson and Sonnenschein (2003). Their paper proposes a specific mechanism that allows individuals to assign different priority to different actions while constraining their choices in a tightly specified manner. The mechanism converges to the first best allocation as the number of decisions grows large, but the design of the correct menu of choices offered to the agents is complex, and the informational demands on the planner severe—the first best result comes at the cost of the mechanism's complexity. The recourse to bonus votes in referenda that we discuss in the present paper builds on the same principle but with a somewhat different goal: a mechanism with desirable if not fully optimal properties that is simple enough to be put in practice. It is this simplicity that we particularly value: we hope that the reader will keep in mind that we propose and study a minor, plausible modification to existing voting practices.

The paper proceeds as follows. Section 2 describes the model; sections 3 and 4 establish the first result and discuss its intuition in the simplest setting, when the distributions of valuations are identical across individuals and proposals, and are symmetrical between opponents and supporters of each proposal. Section 5 extends the analysis to the case where distributions differ across proposals. Sections 6 and 7 study possible extensions of the mechanism—should the bonus votes be a continuous variable, as opposed to a discrete number? Should the extra votes be positive or should they instead be negative, mimicking "limited voting" in multi-candidate elections where voters have fewer votes than the number of open positions? Section 8 addresses the case of asymmetric distributions, and section 9 concludes. The Appendix contains some of the proofs.

2   The Basic model

A large number n of voters are asked to vote, contemporaneously, on a set of k unrelated proposals (with $k > 1$). Each proposal can either pass or fail, and we will refer to each vote as an unrelated referendum. Each voter is asked to cast one vote in each referendum, but in addition is given a set of m bonus votes. It is natural to think of each bonus vote as equivalent to one regular vote, but we can suppose, more generally, that each bonus vote is worth ϑ regular votes, with $\vartheta > 0$. For example, imagine regular votes as green, and bonus votes as blue; if $\vartheta = 1/2$, it takes 2 blue votes to counter 1 green vote, and viceversa if $\vartheta = 2$. The parameter ϑ can take any value between $1/C$ and C, where C is an integer, small relative to n but otherwise arbitrary. We denote by θ the aggregate value of all bonus votes: $\theta \equiv m\vartheta$.[7]

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[7] The set-up generalizes trivially to the case where different bonus votes are allowed to have different values, relative to regular votes. As we shall see, the only important parameter is the aggregate value θ.

The valuation that voter i attaches to proposal r is summarized by v_{ir}. A negative valuation indicates that an individual is against the proposal, while a positive valuation indicates that he or she is in favor, and the valuation's absolute value, which we denote by v_{ir}, summarizes the intensity of i's preferences: voter i's payoff from proposal r is v_{ir} if the referendum is resolved in his preferred direction, and 0 otherwise. Individual valuations are drawn, indipendently across individuals and across proposals from probability distributions $F_r(\mathrm{v})$ that can vary across proposals but are common knowledge and have full support normalized to $[-1, 1]$. In our basic model, we maintain the traditional assumption that the distributions $F_r(\mathrm{v})$ are symmetrical around 0: there is no systematic difference between voters who oppose and voters who favor any proposal. We will come back to this assumption later. Each individual knows his own valuation over each proposal, but only the probability distribution of the others' valuations. There is no cost of voting.[8]

We concentrate on symmetric equilibria where, conditional on their set of valuations, all voters select the same optimal strategy—a restriction that will allow us to make use of standard limit theorems even in our case of discrete votes. In addition, we select equilibria where voters do not use weakly dominated strategies. Since there can be no gain from voting against one's preferences, in these equilibria voters vote sincerely. The only decision is the number of votes x_r to cast in each referendum, where we use the convention that negative votes are votes cast against a proposal and positive votes are votes cast in favor. Voter i's strategy is then indicated by $x_{ir}(\mathrm{v}_{ir}, m, \vartheta, F_r)$.

We will characterize the Bayesian equilibrium of the game. But before doing so, it is helpful to establish some preliminary results.

3   Three preliminary results

We show in this section that the number of candidate equilibria can be drastically reduced by three observations, summarized here as lemmas. All three lemmas are proved in the Appendix, but the intuitions behind them can be described simply.

Lemma 1. *In equilibrium, $x_{ir}(\mathrm{v}_{ir}, m, \vartheta, F_r) = -x_{ir}(-\mathrm{v}_{ir}, m, \vartheta, F_r)\ \forall i,\ \forall r$: for all voters and in all referenda the number of votes cast is independent of the sign of the voter's valuation.*

Voters spend their bonus votes on referenda where they expect the rest of the electorate to be equally split, and in this case the number of votes cast is independent of whether the voter favors the proposal or opposes it. The only complication could occur if all referenda were expected to be won by one side with probability close to 1. But then undominated strategies imply that voters

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[8] Introducing voting costs would be easier in a probabilistic model of voting (as in Campbell (1999)), but should not have qualitative effects on our results as long as the distribution of costs and the distributions of valuations are independent (see the discussion of Ledyard (1984) in the conclusions of Campbell (1999)).

would act *as if* they had a positive probability of being pivotal, i.e. as if the rest of the electorate were equally split, replicating the conclusion above (and in fact ensuring, with $F_r(\mathrm{v})$ symmetrical around 0, that the outcome of all referenda must be uncertain).

One implication of Lemma 1 is that we can simplify our notation. As long as the distributions $F_r(\mathrm{v})$ are symmetric, we can work with distributions $G_r(v)$ defined over absolute valuations and support $[0, 1]$, and we will understand the strategies $x_{ir}(v_{ir}, m, \vartheta, F_r)$ to refer to the absolute number of votes cast.

Lemma 2. *In all equilibria with weakly undominated strategies, voters cumulate their bonus votes on one referendum. All equilibria are equilibria of the simpler game where a single bonus vote of value θ is granted to all voters.*

In all referenda, the asymptotic distribution of votes is such that the probability of being pivotal is approximately proportional to the number of votes cast (see the Appendix). The implication is that the problem faced by a voter when choosing how to allocate bonus votes is linear in the number of votes and has a corner solution: all bonus votes should be cumulated on one proposal.

Lemma 2 allows us to simplify the problem drastically. We can model the menu of bonus votes as a single bonus vote of value θ and reduce the voters' problem to the choice of the single referendum in which the bonus vote will be cast. From a practical point of view, granting a single bonus vote seems a preferable design: it simplifies the voters' problem and has no effect in equilibrium. For the remainder of the paper, we will then refer to a single bonus vote.

Call ϕ_r the share of voters expected to cast their bonus vote in referendum r, where $\sum_{r=1}^{k} \phi_r = 1$. If all voters were expected to cast their bonus vote in the same referendum r', all referenda would be decided by a simple majority of voters, as if no bonus votes were granted. We show in the Appendix that with probability arbitrarily close to 1 a fraction of the population would then prefer to withdraw its bonus vote from r'. It follows that in equilibrium $\phi_r < 1$ for all r. We can then establish:

Lemma 3. *Call p_r the probability that voter i obtains his or her desired outcome in referendum r when casting a regular vote only, and $p_{\theta r}$ the corresponding probability when adding the bonus vote. Then:*

$$p_r \simeq \frac{1}{2} + \frac{1}{\sqrt{2\pi n \left[1 + \phi_r(\theta^2 + 2\theta)\right]}}$$

$$\tag{1}$$

$$p_{\theta r} \simeq \frac{1}{2} + \frac{1 + \theta}{\sqrt{2\pi n \left[1 + \phi_r(\theta^2 + 2\theta)\right]}}$$

Voter i's optimal strategy is to cast a bonus vote in referendum s if and only if :

$$\frac{v_{is}}{v_{ir}} > \sqrt{\frac{1 + \phi_s(\theta^2 + 2\theta)}{1 + \phi_r(\theta^2 + 2\theta)}} \qquad \forall r \neq s \tag{2}$$

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The probabilities result from the application of the local limit theorem and are valid up to an approximation of order $O(n^{-3/2})$ (see the Appendix). Voter i will choose to cast his or her bonus vote in referendum s over referendum r if and only if $v_{is}p_{\theta s} + v_{ir}p_r > v_{ir}p_{\theta r} + v_{is}p_s$. Substituting (1), we obtain (2). If (2) holds for all r's different from s, then the voter will cast the extra vote on referendum s.

We are now ready to characterize the equilibria, and we begin by the simpler case where $G_r(v) = G(v)$ for all r.

4   Identical distributions

When the distributions of valuations are identical across proposals, intuition suggests a simple strategy: let each voter cast the bonus vote in the referendum to which he or she attaches the highest valuation. Indeed we can show:

Proposition 1. *If $G_r(v) = G(v)$ $\forall r$, then there exists a unique equilibrium where each voter casts the bonus vote in referendum s if and only if $v_{is} \geq v_{ir}$ $\forall r$.*

Proof of Proposition 1. Given Lemma 3, the proof of proposition 1 is straightforward. (i) To see that the candidate strategy is indeed an equilibrium strategy, suppose all voters but i cast their bonus vote in the referendum with highest absolute valuation. Since all valuations are drawn from the same probability distribution, with k draws each has probability $1/k$ of being the highest, implying $\phi_s = \phi_r = 1/k$ $\forall r$. Thus the square root in (2) equals 1 and by Lemma 3 voter i should follow the same strategy, establishing that it is indeed an equilibrium. (ii) To see that the equilibrium is unique, suppose to the contrary that there is an equilibrium where not all ϕ_r's are equal, and call s the referendum such that $\phi_s = \max\{\phi_r\}$. Then $\sqrt{[1 + \phi_s(\theta^2 + 2\theta)]/[1 + \phi_r(\theta^2 + 2\theta)]} \equiv \alpha(s,r) \geq 1$ $\forall r$, with at least one strict inequality. Call r' one of the referenda for which the strict inequality holds (at least one r' must exist). Then, by (1), in the bilateral comparison between s and r' the expected share of voters casting their bonus vote on s is lower than on r'. We can have $\phi_s = \max\{\phi_r\}$ only if there exists at least one r'' such that $\alpha(s,r'') < \alpha(r',r'')$. But this requires $\phi_{r'} > \phi_s$, contradicting $\phi_s = \max\{\phi_r\}$. $\square$

In our opinion, both the uniqueness of the equilibrium strategy and its simplicity are strong assets of the mechanism. The immediate response to being allowed to cast a bonus vote is to cast it over the issue that matters most. It seems to us important in practice that the best strategy associated with a mechanism be both sincere and simple, and that voters' concerns with strategic calculations be limited to a minimum.

To evaluate the potential for welfare gains, we use as criterion ex ante efficiency: the expected utility of a voter before having drawn his or her valuations

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(or equivalently before being informed of the exact slate of proposals on the ballot). By Proposition 1, the expected share of voters casting their bonus vote is equal in all referenda ($\phi_r = 1/k \ \forall r$), impliying, by (1), that the probability of obtaining the desired outcome depends on whether the bonus vote is cast, but not on the specific referendum: $p_r = p$, $p_{\theta r} = p_\theta \ \forall r$. Denote by Ev the expected absolute valuation over any proposal, and by $Ev_{(j)}$ the expected jth order statistics among each individual's k absolute valuations (where therefore $Ev_{(k)}$ is the expected value of each voter's highest absolute valuation). Since voters cast their bonus vote in the referendum associated with the highest valuation, expected ex ante utility EU is given by:

$$EU = Ev_{(k)}p_\theta + \sum_{j=1}^{k-1} Ev_{(j)}p = k(Ev)p + Ev_{(k)}(p_\theta - p). \qquad (3)$$

Substituting (1) and $\phi_r = 1/k \ \forall r$, we can write:

$$EU \simeq kEv \left(\frac{1}{2} + \frac{\sqrt{k}}{\sqrt{2\pi n(k + \theta^2 + 2\theta)}} \right) + Ev_{(k)} \left(\frac{\theta\sqrt{k}}{\sqrt{2\pi n(k + \theta^2 + 2\theta)}} \right). \qquad (4)$$

Our reference is expected ex ante utility with a series of simple majority referenda, which we denote EW, where, as established in Lemma A.1 in the Appendix:

$$EW \simeq kEv \left(\frac{1}{2} + \frac{1}{\sqrt{2\pi n}} \right), \qquad (5)$$

again ignoring terms of order $O(n^{-3/2})$. Comparing (4) and (5), we see that both mechanisms dominate randomness (where each proposal is resolved in either direction with probability $1/2$), although, in line with the arguments in Ledyard and Palfrey (2002) both converge to randomness, and to each other, as the population approaches infinity (a point we will discuss in more detail later). Thus a plausible scaling of efficiency is the relative improvement of the two mechanisms over randomness. Calling ER expected utility with random decision-making, we define our measure of welfare improvement as ω, where

$$\omega \equiv \left(\frac{EU - ER}{EW - ER} \right). \qquad (6)$$

We will state that the voting mechanism improves efficiency over a series of simple majority referenda if $\omega > 1$.

Substituting (4) and (5) and $ER = kEv(1/2)$, we derive immediately[9]

$$\omega = \frac{kEv + \theta Ev_{(k)}}{Ev\sqrt{k^2 + k\theta^2 + 2k\theta)}}, \qquad (7)$$

which then implies:

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Proposition 2. *For any distribution $G(v)$ and any number of referenda $k > 1$, there exists a $\bar{\theta} > 0$ such that $\omega > 1$ for all $\theta < \bar{\theta}$.*

The proposition follows immediately from (7). Indeed, a simple manipulation shows,

$$\omega > 1 \quad \begin{cases} \forall \theta > 0 & \text{if } (Ev_{(k)})^2 \geq k(Ev)^2 \\ \forall \theta < \frac{2kEv(Ev_{(k)} - Ev)}{k(Ev)^2 - (Ev_{(k)})^2} & \text{if } (Ev_{(k)})^2 < k(Ev)^2. \end{cases} \tag{8}$$

Given a specific distribution, the admissible range of θ values is easily pinned down. Suppose for example that $G(v)$ is the uniform distribution; then $Ev = 1/2$ and $Ev_{(k)} = k/(k+1)$, implying that efficiency improves for all $\theta < 2(k+1)/(k-1)$. If $k = 2$, the constraint is $\theta < 6$—the bonus vote cannot count more than 6 regular votes; if $k = 5$, the constraint is $\theta < 3$, and so forth. Because the ceiling on θ is declining in k, its limit as k approaches infinity provides a sufficient condition for efficiency gains: for any number of referenda, $\theta < 2$ guarantees $\omega > 1$.

In fact we can do more: from (7) we can derive the optimal θ, the value of the bonus vote that maximizes the efficiency gains, which we denote by θ^*. For arbitrary $G(v)$,[10]

$$\theta^* = \frac{k(Ev_{(k)} - Ev)}{kEv - Ev_{(k)}}. \tag{9}$$

If $G(v)$ is a uniform distribution, then $\theta^* = 1$ for any value of k: regardless of the number of referenda, the optimal value of the bonus vote is 1—that is, the bonus vote should be equivalent to a regular vote. At $\theta = 1$ and for a uniform $G(v)$, $\omega = \sqrt{k(3+k)}/(1+k)$, always larger than 1, but maximal at $k^* = 3$: given the optimal choice of θ, the number of contemporaneous referenda that maximizes efficiency gains is 3. At these parameter values, the welfare gain relative to simple majority, as defined by ω, is 6 percent.

There results, so surprisingly clean, extend easily to a general power distribution, and we summarize them in the following example:

Example 1. *Suppose that $G(v)$ can be parameterized as a power distribution: $G(v) = v^b$, $b > 0$. Then, ignoring integer constraints: (i) For all k, $\omega > 1$ if $\theta < 2/b$. (ii) For all k, $\theta^* = 1/b$. (iii) If $\theta = \theta^*$, $k^* = 2 + 1/b$.*

The parameter b determines the shape of the distribution, reducing to the uniform if $b = 1$. If $b < 1$, $G(v)$ is unimodal at 0, and the mass of voters declines monotonically as the valuations become more extreme; with $b > 1$, on the contrary, the distribution is unimodal at 1, the upper boundary of the support, and the mass of voters increases with the intensity of the valuations. The mean of the distribution is $b/(1+b)$, and the mean of the kth order statistic

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[10] For arbitrary $G(v)$,

$$Ev_{(k)} = k \int_0^1 v[G(v)]^{k-1} dG(v),$$

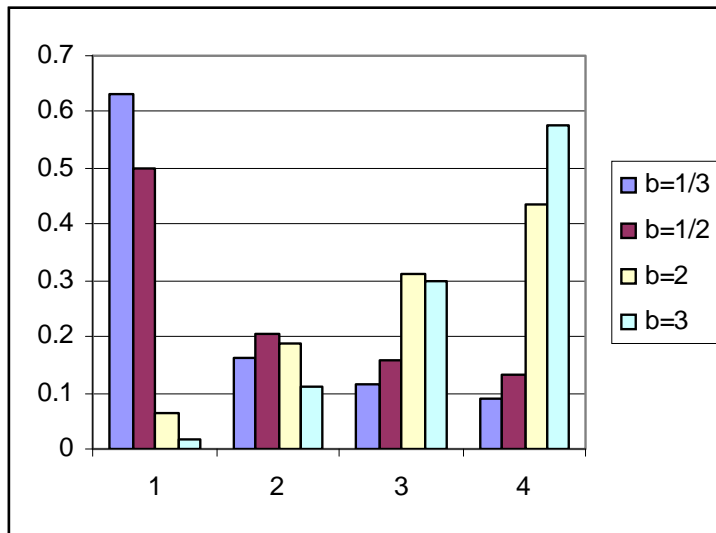ensuring that the denominator in (9) is always positive.

Figure 1: Power distributions: histograms. Shares of the electorate with valuations in the first, second, third and fourth quarter of the support for different values of the b parameter.

is $bk/(bk + 1)$. For a more intuitive understanding of what the distribution implies, suppose for example that voters were asked to rank an issue as "not important," "somewhat important," "important," or "very important," and that these labels corresponded to a partition of the range of possible intensities into 4 intervals of equal size, from $[0, 0.25]$ to $[0.75, 1]$. For a uniform distribution of valuations, a quarter of the voters would choose each interval; with $b = 1/2$, half of the voters would classify the issue as "not important" and about 13 percent as "very important"; with $b = 2$, the percentages become 6 percent for "not important" and close to 45 percent for "very important" (see Figure 1). With a power distribution, the parameter b is a measure of the saliency of the issue: the higher is b, the higher the share of voters who feel strongly about the proposal. The more salient the set of issues, the smaller is the optimal value of the bonus vote: with $b = 1/2$, the bonus vote should count as 2 regular votes; with $b = 2$ as half, and with $b = 3$ as a third.

The sufficient condition (i) above is important. Without precise knowledge of the distribution, a policy-maker cannot set the optimal value of the bonus vote, but if the more modest goal of *some* improvement over simple majority is acceptable, this can be achieved by choosing a conservatively small θ. Consider for example setting $\theta = 1/2$—then, for all k, efficiency gains are achieved as long as $b < 4$. With $b = 4$, almost 70 percent of the voters consider the issue "very important," more than 90 percent either "important" or "very important" and less than 1 percent "not important." As long as saliency is not higher, welfare

is improved by the bonus vote.

Why is there a ceiling on the acceptable values of the bonus vote? And why does this ceiling depend on the shape of the distribution? Taking θ as given, we can rewrite the necessary condition for efficiency gains as:

$$\omega > 1 \Leftrightarrow \frac{Ev_{(k)}}{Ev} > \frac{k}{\theta} \left(\sqrt{1 + \frac{\theta^2 + 2\theta}{k}} - 1 \right) > 1 \quad \forall \theta > 0. \qquad (10)$$

Condition (10) makes clear that an improvement in efficiency requires a sufficient wedge between the mean valuation and the highest expected valuation draw. The problem is that the introduction of the bonus vote creates noise and redistributes the probability of winning towards the referendum where the bonus vote is utilized but away from the others. Efficiency can increase only if the higher probability of being on the winning side is enjoyed over a decision that really matters to the voter, a decision that matters enough to compensate for the decline in influence in the other referenda. Predictably, the required wedge is increasing in θ: the higher the value of the bonus vote, the larger the noise in the votes distribution and the larger the shift in the probability of winning towards the referendum judged most important. Equations (1) show this effect very clearly. Similarly, the wedge is increasing in k: the larger is k, the more issues over which the probability of winning declines $(k-1)$, and thus again the larger must be the valuation attached to the referendum over which the bonus vote is spent.[11]

For our purposes, the ratio $Ev_{(k)}/Ev$ summarizes all that matters about the distribution of valuations. With a power distribution the ratio equals $(k + bk)/(1+bk)$, an expression that is declining in b: the more salient the issues—the higher b—the smaller the expected difference between the highest draw and the mean valuation, and the smaller must then θ be if (10) is to be satisfied. Hence the result described above. More generally, given $Ev_{(k)}/Ev$ and k, condition (8) specifies the constraint on θ and (9) θ's optimal value.[12]

Summarizing, the voting scheme exploits the variation in valuations to ensure that the added noise created by the bonus vote is compensated by a higher probability of winning a decision that really matters. The more intense the average valuations—the more polarized the society—the higher the variance must be for a given value of the bonus vote, or equivalently, the smaller must be the value of the bonus vote; the less intense the average valuations, the lower the required variance or equivalently the higher the optimal value of the bonus vote.[13]

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[11]But $Ev_{(k)}$ is also increasing in k. Whether fulfilling (10) becomes more or less difficult as k increases depends on the distribution.

[12]It was tempting to conjecture a link between the ordering of distributions in terms of the ratio $Ev_{(k)}/Ev$ and first-order stochastic dominance—until Russell Davidson provided a counterexample.

[13]The ratio $Ev_{(k)}/Ev$ depends both on the variance of $G(v)$ and on the mean. A power distribution conflates the two, since both depend on b. (The variance equals $b/[(1+b)^2(2+b)]$ with a maximum at $b = 0.62$). A beta distribution is more flexible and isolates the two effects,

5   Heterogeneous distributions

The assumption that valuations are identically distributed over all proposals is, in general, unrealistic: many issues put to referendum are typically of interest only to a small minority—the calendar of the hunting season, the decision to grant landmark status to a building, the details of government procedures—while some on the contrary evoke strong feelings from most voters—divorce in Italy, affirmative action and taxation in California, equal rights for women in Switzerland.[14] Allowing for different distributions makes the problem less transparent, but does not change its logic and in fact increases the expected dispersion in valuations that makes the voting scheme valuable.

The first step is the characterization of the equilibrium—the choice of the referendum on which to cast the bonus vote. Lemmas 1 to 3 continue to apply, but now voters' bonus votes will not be spread equally over all referenda—the more salient issues will receive a larger share of bonus votes. In equilibrium, ϕ_r, the expected share of voters casting their bonus vote in referendum r, must satisfy

$$\phi_r = \int_0^1 \prod_{s \neq r} G_s(\min(\alpha_{sr}v, 1))g_r(v)dv, \tag{11}$$

where

$$\alpha_{sr} \equiv \sqrt{\frac{1 + \phi_s(\theta^2 + 2\theta)}{1 + \phi_r(\theta^2 + 2\theta)}}. \tag{12}$$

When $G_r(v) = G_s(v) \; \forall r, s$, as in the previous section, (11) and (12) simplify to $\phi_r = 1/k$ and $\alpha_{sr} = 1$. This is not the case now.

The equilibrium remains unique[15] but is less intuitive than in the case of identical distributions: if a referendum evokes more intense preferences and more voters are expected to cast their bonus vote on that issue, then the impact of the bonus vote will be higher elsewhere. It may be referable to cast one's bonus vote in a different referendum, even if the valuation is slightly lower. For example, in the case of 2 referenda and power distributions, suppose $b_1 = 1$ and $b_2 = 2$. Then $\alpha_{12} = 0.89$—a voter casts the bonus vote on issue 1 as long as

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but does not provide a closed form solution for the kth order statistics. We can nevertheless check conditions (8) or (10) numerically. Suppose for example $\theta = 1/2$. Then if $E(v) = 1/2$, (10) is satisfied for all k as long as the variance is larger than 0.008 (or equivalently as long as not more than 3/4 of the population are concentrated in the two deciles around the mean). But if the mean is 3/4, the minimum variance rises to 0.02 (or not more than 50 percent of the population in the two deciles around the mean); if instead the mean is 1/4, the minimum variance falls to 0.002 (or not more than 98 percent of the population in the two deciles around the mean). The necessary floor on the variance rises as the mean increases.

[14] The distinction is equivalent to Matsusaka's (1992) empirical classification of initiatives into "efficiency" (low salience) and "distributional" (high salience).

[15] Consider an equilibrium $\{\phi_r'\}$. Posit a second equilibrium where $\phi_s = \phi_s'' > \phi_s'$. Then, given (11) and (12) there must exist at least one issue r such that $\phi_r''/\phi_r' > \phi_s''/\phi_s'$. Call z the issue such that ϕ_z''/ϕ_z' is maximum. Then $\alpha_{rz}(\phi_z'', \phi_r'') < \alpha_{rz}(\phi_z', \phi_r') \; \forall r \neq z$, and by (11) $\phi_z'' < \phi_z'$ establishing a contradiction. Reversing the signs, the identical argument can be used to show that there cannot be an equilibrium with $\phi_s'' < \phi_s'$.

$v_1 \geq 0.89\ v_2$—and the expected shares of bonus votes cast on the two referenda are $\phi_1 = 0.41$ and $\phi_2 = 0.59$. If $b_2 = 4$, the numbers become $\alpha_{12} = 0.82$, $\phi_1 = 0.34$ and $\phi_2 = 0.66$.

The condition for efficiency gains over simple majority again follows the logic described earlier, but is made less transparent by the need to account for the different distributions and for the factors of proportionality α_{rs}:

$$\omega > 1 \Leftrightarrow \sum_{r=1}^{k} \left[\left(\int_0^1 \prod_{s \neq r} G_s(\min(\alpha_{sr}v, 1)) v g_r(v) dv \right) \theta \beta_r \right] > \sum_{r=1}^{k} E_r(v) (1 - \beta_r), \tag{13}$$

where

$$\beta_r \equiv \frac{1}{\sqrt{1 + \phi_r(\theta^2 + 2\theta)}}.$$

Condition (13) is analogous to (10), but because the parameters β_r and α_{sr} differ across distributions and α_{sr} in general differs from 1, it does not reduce to a simple condition on the ratio of the expected highest valuation draw to the mean valuation. Nevertheless, it remains possible to state:

Proposition 3. *For any set of distributions $\{G_r(v)\}$ symmetric around zero and with full support and for any number of referenda $k > 1$, there exists a $\widetilde{\theta} > 0$ such that $\omega > 1$ for all $\theta < \widetilde{\theta}$.*

The proposition is proved in the Appendix. It states that the result we had previously established in the case of identical distributions is in fact more general, and continues to apply with heterogeneous distributions.

In practical applications, two concerns remain. The first is that calculating the correct equilibrium factors of proportionality α_{rs} is not easy. How well would voters fare if they followed the plausible rule of thumb of casting the bonus vote on the highest valuation proposal? It seems wise to make sure that the desirable properties of the mechanism are robust to the most likely off-equilibrium behavior. In fact, Proposition 3 extends immediately to this case:

Proposition 3b. *Suppose voters set $\alpha_{sr} = 1\ \forall s, r$. Proposition 3 continues to hold.* (See the Appendix).

The second concern was voiced earlier. If the planner is not fully informed on the shape of the distributions, or if the value of θ is to be chosen once and for all, for example in a constitutional setting, can we identify sufficient conditions on θ that ensure efficiency gains for a large range of distributions? The answer is complicated by the factors α_{rs} and thus by the lack of a simple closed-form solution even when we specialize the distributions to simple functional forms. However, in our reference example of power distributions and in the "rule-of-thumb" case where voters cast the bonus vote on the highest valuation proposal, we obtain an interesting result:

Example 2. *Suppose $G_r(v) = v^{b_r}$, $b_r > 0\ \forall r$, and set $\alpha_{sr} = 1\ \forall s, r$. Call $b_k \equiv \max\{b_r\}$. Then for all $k > 1$, $\omega > 1$ if $\theta \leq 1/b_k$.*

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The example is proved in the Appendix. As in the case of identical distributions, we can derive a simple sufficient condition ensuring welfare gains: the value of the bonus vote can be safely set on the basis of the distribution of valuations in the most strongly felt of the issues under consideration. If we return to our previous discussion and partition the support of the valuations into four equal size intervals, setting $\theta = 1/4$ or $1/5$ would seem a reasonably prudent policy.[16] Intuitively, we expect the condition to be stronger than needed: the heterogeneity of the distributions should help in providing the spread in expected valuations that underlies the voting scheme's efficiency gains. Indeed, in all our numerical exercises with power distributions we achieved welfare gains by setting $\theta \leq k / \sum_{r=1}^{k} b_r$, the inverse of the mean b parameter, a looser constraint than $1/b_k$.[17]

This section allows us to conclude that the properties of the voting scheme, so transparent in the simple case of identical distributions, extend to the more plausible scenario of heterogeneous distributions. Having established our result in the most intuitive settings, we can now address two questions that seems particularly natural.

6   Why discrete bonus votes?

The equivalence between granting voters a single bonus vote or multiple extra votes is driven by the discreteness we have attributed to the votes. Why not consider instead a continuous bonus vote that voters can split as they see fit between the different issues? The scheme is intuitive and more general than the one we have considered so far; indeed, from a theoretical point of view, it is a more natural starting point. There is also reason to expect that the generalization could help. We saw that in the discrete scheme the value of the bonus vote has to be tuned correctly: when the dispersion in valuations is small, the bonus vote runs the risk of being too blunt an instrument to finely differentiate between them. Why not let voters do the fine-tuning themselves, choosing the extent to which they want to divide their extra vote over the different issues? We show in this section that although some theoretical improvement over the discrete scheme is possible, some complications arise. In our opinion, the balance of the arguments comes down in favor of the discrete bonus vote we have described so far.

The main points will be clearer in the simplest setting, and in what follows we assume that there are only two proposals ($k = 2$), the distributions of valuations are identical over both proposals ($G_r(v) = G(v)$, $r = 1, 2$), and the value of the continuous bonus vote θ is set to 1.[18] Call $x_i \in [0, 1]$ the fraction of the

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[16] With $b = 5$, more than $3/4$ of all voters consider the issue "very important," 97 percent consider it either "important" or "very important," and less than 1 in a thousand "not important."

[17] This was true whether we looked at the equilibrium or at the $\alpha_{sr} = 1$ case. With $k = 2$, efficiency gains in the "rule-of-thumb" scenario are sufficient for efficiency gains in equilibrium, but not with $k > 2$.

[18] The analysis extends immediately to $\theta < 1$. If $\theta > 1$, the logic is unchanged but the

bonus vote cast by voter i on the issue with highest valuation. Thus i will cast $(1 + x_i)$ votes in one referendum and $(2 - x_i)$ in the other. We continue to focus on equilibria in undominated strategies where each voter conditions his or her voting strategy on a positive probability of being pivotal in both issues. Hence all the previous arguments continue to apply and we can restrict candidate equilibria to symmetrical scenarios where strategies are contingent on absolute valuations and the distribution of the vote differential[19] faced by voter i is identical in both referenda. Denoting by $-i$ the choices of the other voters, such a distribution[20] must be normal with mean 0 and variance

$$\sigma^2 = n \left[5/2 + Ex^2_{-i} - Ex_{-i} \right].$$

Labeling v_{1i} the higher valuation, the expected utility of voter i after valuations are drawn is given by

$$Eu_i = v_{1i} \Phi \left(\frac{1 + x_i}{\sqrt{\sigma^2}} \right) + v_{2i} \Phi \left(\frac{2 - x_i}{\sqrt{\sigma^2}} \right), \tag{14}$$

where $\Phi(\cdot)$ is the normal cumulative distribution function—hence $\Phi \left(\frac{1+x_i}{\sqrt{\sigma^2}} \right)$, for example, is the probability that i's preferred outcome in referendum 1 is winning or tied (with probability $1/2$) or is losing by not more than $1 + x_i$ (with probability $\Phi \left(\frac{1+x_i}{\sqrt{\sigma^2}} \right) - 1/2$). Before proceeding further, notice that there is always an equilibrium where no-one splits the bonus vote. If none of the other voters splits a bonus vote, the distribution of the vote differential faced by voter i will have steps at all discrete number of votes, $\Phi \left(\frac{1+x_i}{\sqrt{\sigma^2}} \right) = \Phi \left(\frac{2-x_i}{\sqrt{\sigma^2}} \right)$ $\forall x_i \in (0, 1)$, and the only relevant choices are $x_i = 0$ or $x_i = 1$. The analysis in the first part of this paper remains the correct analysis here, and the equilibrium with discrete voting identified there remains an equilibrium here.[21] Thus the first observation is that allowing the bonus vote to be divisible must always increase the number of equilibria.

Consider now a candidate equilibrium where x_{-i} is continuous over the whole interval $[0, 1]$. The distribution of the vote differential is then continuous and equation (14) can be differentiated to with respect to x_i, yielding,

$$v_{1i} \varphi \left(\frac{1 + x_i^*}{\sqrt{\sigma^2}} \right) = v_{2i} \varphi \left(\frac{2 - x_i^*}{\sqrt{\sigma^2}} \right) \quad \text{if } x_i^* \in [0, 1], \tag{15}$$

equations need to be amended.

[19] The vote differential faced by i is the net sum of all votes cast by the other voters, where votes against the proposal are counted as negative votes.

[20] In equilibrium, in either referendum half of the voters vote $\pm(1 + x_{-i})$ and the remainder $\pm(2 - x_{-i})$:
$$\sigma^2 = n[(1/2)E(1 + x_{-i})^2 + (1/2)E(2 - x_{-i})^2]$$

[21] The logic extends immediately to all other possible discrete jumps in the proportion of the bonus vote cast in the two referenda. But then we revert to the case of discrete votes, and to the result reached earlier: the bonus vote should be cumulated on the one most important issue. In equilibrium the only relevant case is then the $0 - 1$ split.

where the star indicates x_i's optimal value and $\varphi(\cdot)$ is the normal density function. We can rewrite (15) as:

$$\log v_{1i} - \frac{(1+x_i^*)^2}{2\sigma^2} = \log v_{2i} - \frac{(2-x_i^*)^2}{2\sigma^2} \quad \text{if } x_i^* \in [0,1].$$

Taking into account $x_i^* \in [0,1]$ and substituting σ^2 from above:

$$x_i^*(v_{1i}, v_{2i}) = \min\left[1, \frac{1}{2} + n\log\left(\frac{v_{1i}}{v_{2i}}\right)\left(\frac{5}{6} + \frac{1}{3}(Ex_{-i}^2 - Ex_{-i})\right)\right],$$

or, in equilibrium,

$$x_i^* = \begin{cases} \frac{1}{2} + \frac{1}{2t(n,x_{-i}^*)}\log(v_{1i}/v_{2i}) & \text{if } \log(v_{1i}/v_{2i}) \in [0, t(n, x_{-i}^*)] \\ \\ 1 & \text{if } \log(v_{1i}/v_{2i}) > t(n, x_{-i}^*), \end{cases} \tag{16}$$

where

$$t(n, x_{-i}^*) \equiv \frac{3}{n[5 + 2E(x_{-i}^*)^2 - 2Ex_{-i}^*]}.$$

The main observation can be made without an explicit solution for x^* and is in fact immediate: for large n the option of splitting the bonus vote becomes irrelevant. The reason is that $t(n, x_{-i}^*)$, the upper boundary on the logarithm of relative valuations consistent with splitting the bonus vote, approaches zero at rate n. Since $v_{1i} > v_{2i}$ by definition, the probability of splitting the bonus vote approaches zero at rate n.

To gain a more precise sense of what this means, suppose that $G(v)$ is a uniform distribution. In the interval where the bonus vote is split, $\log(v_1/v_2)$ is of order $O(n^{-1})$ and thus, ignoring terms of order $O(n^{-2})$, can be approximated by $(v_1/v_2) - 1$. Call η the share of the population that splits the bonus vote, or equivalently the probability of splitting one's vote (where, with a uniform distribution, $\eta = t$). In a symmetrical equilibrium with $n = 100$, $\eta = 0.006$; with $n = 1,000$, $\eta = 0.0006$—as we increase the order of magnitude of the population, the number of voters expected to split their vote remains less than a single one.[22] The same result can be stated in terms of welfare: with $n = 100$, the equilibrium with continuous voting slightly improves our measure of welfare; but for all $n \geq 1,000$ the precision of our numerical simulations is not sufficient to detect any difference.[23]

Summarizing, we have reached two conclusions. First, the equilibrium where the bonus vote is not split continues to exist when the bonus vote is perfectly divisible—moving from discrete bonus votes to a continuum increases the number of equilibria. Second, the distinction becomes irrelevant in large populations,

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[22] The product ηn is approximately constant because t approaches $3/(5n)$; since η equals t, ηn must be approximately $3/5 = 0.6$.

[23] With $G(v)$ uniform, $k = 2$, $\theta = 1$ and one indivisible bonus vote, $\omega = 1.054 \; \forall n$. In the symmetrical equilibrium with continuous splitting, $\omega^C = 1.055$ if $n = 100$, but $\omega^C = 1.054$ $\forall n \geq 1,000$.

both in terms of the proportion of voters who exploit it in equilibrium and in terms of its welfare consequences.[24]

7   Bonus votes or partial votes?

Nothing in the logic of our scheme requires θ to be positive, an assumption we have maintained so far. A negative θ corresponds to granting voters a "partial vote," a vote worth less than the others, and voters can select the issue on which they would then "partially abstain." The idea is not new: if granting bonus votes is similar to cumulative voting in multi-candidate elections, a partial vote recalls *limited voting*, a voting system advocated and adopted with some frequency as a simpler alternative to cumulative voting. Limited voting amounts to granting voters fewer votes than the number of seats to be filled, without allowing voters to cumulate the votes.[25] In this section, we study the properties of the voting mechanism if $\theta < 0$.

To keep the comparison to bonus votes straightforward, we limit ourselves to the case where $\theta \in (-1,0)$: as in the rest of the paper, voters will select a single referendum on which to cast their modified (here, partial) vote. As θ approaches -1, the partial vote approaches full abstention, and voters are endowed with one fewer vote than the number of referenda.[26] Because the only relevant variable is the value of the partial vote relative to the regular votes, the first observation is that a partial vote or a bonus vote mechanism are identical when $k = 2$. If the number of referenda on the ballot is larger than 2, on the other hand, in general the two schemes will differ.

Suppose, for simplicity that all distributions of valuations are identical $(G_r(v) = G(v), r = 1, 2, \ldots, k)$. The logic followed in the case of a bonus vote continues to apply, establishing that there is a unique equilibrium where all voters cast their partial vote in the referendum with lowest valuation.[27] The welfare criterion ω, the percentage increase in ex ante utility relative to simple majority voting,

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[24] Notice a corollary to the last observation: in large populations, a continuous bonus vote cannot be used to guarantee welfare gains, relative to simple majority referenda, if the value of the bonus vote is not chosen correctly.

[25] The argument is that limited voting is strategically more straightforward than cumulative voting. See the discussion in Issacharoff, Karlan and Pildes (2001), chapter 13.

[26] Given that every voter has one regular vote in each referendum, by imposing the constraint $\theta \geq -1$, we are excluding negative (total) votes.

[27] In equilibrium, the probabilities of winning casting a regular vote or the partial vote are given by:

$$p \simeq \frac{1}{2} + \frac{\sqrt{k}}{\sqrt{2\pi n(k + \theta^2 + 2\theta)}}$$

$$p_\theta \simeq \frac{1}{2} + \frac{\sqrt{k}(1+\theta)}{\sqrt{2\pi n(k + \theta^2 + 2\theta)}}.$$

Because $\theta < 0$, the probability of winning is smaller in the referendum where the partial vote is cast, and the variance of the vote differential is smaller than with bonus votes or with simple majority voting.

expressing both as improvement over randomness, now becomes:

$$\omega^P = \frac{kEv + \theta Ev_{(1)}}{Ev\sqrt{k^2 + k\theta^2 + 2k\theta}}, \tag{17}$$

where the superscript P stands for partial vote, $Ev_{(1)}$ is the expected value of the lowest valuation and θ is negative.

It remains true that there always exists a value of $\theta \in (-1,0)$ for which ω^P is larger than 1. From (17), $\omega^P > 1$ requires

$$|\theta| < \frac{2kEv(Ev - Ev_{(1)})}{k(Ev)^2 - (Ev_{(1)})^2}.$$

The optimal value of θ is then

$$\theta^{P*} = -\frac{k(Ev - Ev_{(1)})}{kEv - Ev_{(1)}}.$$

How do the partial vote and the bonus vote schemes compare with one another? If θ is chosen optimally in both cases:

$$\omega^{P*} = \sqrt{1 + \frac{(Ev - Ev_{(1)})^2}{(k-1)(Ev)^2}} \qquad\qquad \omega^* = \sqrt{1 + \frac{(Ev_{(k)} - Ev)^2}{(k-1)(Ev)^2}}.$$

Hence

$$\omega^* > \omega^{P*} \Leftrightarrow \frac{Ev_{(k)} + Ev_{(1)}}{2} > Ev.$$

Not surprisingly, the answer depends on the distribution: if the midrange—the arithmetic mean of the expected highest and lowest draws—is higher than the mean, the bonus vote scheme is preferable; if the opposite holds, the partial vote yields higher gains. Thus the two schemes are identical if $k = 2$, as mentioned above, or if the distribution of valuations is symmetric around a 0.5 mean (for example in the case of a uniform). If the distribution is not symmetric, then the comparison depends on the shape of the distribution. For example, if $G(v) = v^b$, $\omega^* > \omega^{P*} \Leftrightarrow b < 1.$[28] If preferences are polarized and the mean intensity is high, the wedge between the mean and the highest valuation is expected to be relatively small—if there is an outlier it is more likely to be on the low side, the one referendum over which preferences are weak, and a partial vote scheme is preferable. But if preferences are not very polarized, then the possible outlier is expected to be on the high side, and a bonus vote scheme is preferable.

It is not difficult to generalize the results of this paper to include the option of a negative θ. In our exposition, we have chosen to concentrate on $\theta > 0$ for

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[28] For arbitrary $G(v)$,

$$Ev_{(1)} = k \int_0^1 v[1 - G(v)]^{k-1} dG(v).$$

practical reasons: the bonus vote scheme is simply easier to design. In both cases, the only variable that matters is the relative value of the "special" vote, but while that is seen immediately when $\theta > 0$, it is less transparent with $\theta < 0$. For example, granting voters one fewer vote than the number of issues may seem somewhat equivalent, in practice, to granting one more vote. In fact, the first option corresponds to an extreme choice while the second does not, and the first will thus be appropriate in a much more restricted set of circumstances.

We turn now to the robustness of our conclusions if we relax one important assumption maintained so far: the symmetry of the distributions of valuations.

8 Asymmetrical distributions of valuations

The assumption that the distributions of valuations, though possibly different for different issues, are all symmetrical around 0 implies that in any referendum there is no systematic difference in the distribution of valuations between voters who oppose the proposal and voters who favor it. The assumption is standard in the literature, but how important is it for our results? Intuitively we expect that asymmetries should make the bonus vote mechanism more valuable: its role is to give weight to differences in intensities that majority voting cannot capture, and thus it is precisely when intensities on the two sides of an issue differ that bonus votes should matter most.

When the distributions of valuations are symmetric and the value of the bonus vote is chosen correctly, the bonus vote mechanism yields efficiency gains over majority voting, always improving over randomness more than majority voting does. However, we have also remarked that the absolute difference in ex ante efficiency between the two voting mechanisms converges to zero asymptotically, as the population grows without bound. As noted earlier, the convergence of the two mechanisms in the symmetric case is implied by the result in Ledyard and Palfrey (2002) establishing the asymptotic optimality of referenda when the threshold for approval is chosen correctly.[29] Exploiting the dominant nature of sincere voting in referenda, Ledyard and Palfrey define the optimal threshold by the requirement that the mean sample valuation when the threshold is exactly reached should equal zero (in our setting). Because the sample mean converges to the distribution's mean as the size of the electorate grows without bound, this condition ensures that the proposal passes only if the mean valuation is positive, as efficiency demands. When the distribution of valuations is symmetric around zero, the optimal threshold is 50 percent, the simple majority requirement (hence also the asymptotic optimality of randomness in this case). But if the distribution of valuations is not symmetric, and in particular if the mean and the median valuations differ in sign, the optimal threshold is not 50 percent, and a simple majority referendum does not yield the optimal decision (and neither of course does randomness) even in the limit. Bonus votes, on the other hand, are designed to give weight to intensity of preferences—i.e. to

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[29]Ledyard and Palfrey establish the convergence of the optimal referendum to the interim efficient mechanism in a model of public good provision that includes our set-up.

reflect the mean valuation more than the median. If they do so successfully, the absolute difference in ex ante utility from the two voting systems will remain bounded away from zero as the size of the electorate grows without bound.[30]

To study the problem in the simplest setting, suppose that the distributions of valuations are identical over all proposals, but now for each proposal call $P(v)$ the distribution of valuations of voters in favor, and $C(v)$ the distribution of valuations of voters against the proposal (where both distributions can be stated in terms of absolute valuations). In line with the reading of the empirical evidence in the literature[31] and with the model so far, suppose that the median of the distribution remains at 0: assume that both $P(v)$ and $C(v)$ have full support $[0, 1]$ and that $P(1) = C(1)$. But the two distributions have different means: for concreteness, suppose $E_P(v) > E_C(v)$, implying that in each referendum the mean valuation over the whole electorate is positive. We assign the higher mean valuation to the "pro" side with no loss of generality—which side has higher mean is irrelevant and we could trivially generalize the model to allow the side with higher mean to change across proposals. The important point is that in each referendum the mean valuation over the whole electorate differs from 0 while the median equals 0. All valuations are independent, across voters and proposals.

We begin by characterizing equilibrium strategies. The following Lemma is proved in the Appendix:

Lemma 5. *If $P_r(v) = P(v)$ $\forall r$ and $C_r(v) = C(v)$ $\forall r$, then there exists a unique equilibrium where each voter casts a bonus vote in referendum s if and only if $v_{is} \geq v_{ir}$ $\forall r$.*

The equilibrium is pinned down by the requirement that the impact of the bonus vote is equalized across referenda. In practice, if the asymmetry in the distributions is not trivial the equilibrium probability of being pivotal is negligible in all referenda, and it is the simplicity of the strategy, more than the infinitesimal loss that a deviating voter would incur, that recommends focussing on it. Once again, casting the bonus vote on the highest valuation is both the equilibrium strategy and the most plausible heuristic rule for the voters. We can then establish:

Proposition 5. *If $P(v)$ first-order stochastically dominates $C(v)$, then the bonus votes mechanism improves efficiency over majority voting for any $\theta > 0$.* (The proof is in the Appendix).

The intuition behind the proposition is that, with first-order stochastic dominance, the probability mass of favorable valuations is concentrated towards higher values than is the case for negative valuations. By Lemma 5, bonus

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[30] The principle discussed by Ledyard and Palfrey identifies the optimal threshold for referenda with any distribution. Because in practice most referenda are decided according to simple majority, it is this case that we consider for comparison in this paper.

[31] Studying all ballot propositions in California in the period 1912-89, Matsusaka (1992) identifies equally split electorates as the defining empirical regularity of proposals subject to popular vote (as opposed to being decided by the legislature).

votes are then correspondingly concentrated on favorable votes. The probability that a referendum passes given that the mean valuation is positive converges asymptotically to 1 for any positive value of the bonus vote, as opposed to approaching $1/2$ in the case of simple majority. First-order stochastic dominance guarantees that, as conjectured above, bonus votes shift the outcome in the direction of the mean, and hence increase efficiency.

Consider then a sequence of bonus votes referenda indexed by the size of the population n, and similarly index our welfare criteria. As shown in the Appendix, the following result also holds:

Corollary. *If $P(v)$ first-order stochastically dominates $C(v)$, then as $n \rightarrow \infty$, $EU_n/EW_n \rightarrow 1 + [E_P(v) - E_C(v)]/[E_P(v) + E_C(v)] > 1$, and $\omega_n \rightarrow \infty$ for any $\theta > 0$.*

In words, as the size of the population appraoches infinity, majority voting approaches randomness, but bonus votes do not, and the difference in ex ante utility between the two voting mechanisms does not disappear, in fact, relative to randomenss the welfare gain associated with bonus votes grows arbitrarily large.

First-order stochastic dominance is satisfied by the power distribution we have used as recurring example:

Example 3. *Suppose that both $P(v)$ and $C(v)$ are power distributions with parameters b_p and b_c, where $b_p > b_c$. Then for any $\theta > 0$ and n large, $\omega > 1$ and as $n \rightarrow \infty$, $\omega_n \rightarrow \infty$.*

To see what first-order stochastic dominance implies in practice, suppose once again that the public's intensity of preferences at best can be identified through a partition of the support of (absolute) valuations into four equally sized intervals. Consider a referendum where proponents on average have more intense preferences than opponents. First-order stochastic dominance requires some monotonicity in the manner in which voters on the two sides are distributed in the four intervals. Among those judging the proposal "very important" most should be proponents, and similarly among those considering it either "very important" or "important"; among those judging the proposal "not important" most should be opponents, and similarly among those considering it either "not important" or "somewhat important" (see for example Figure 1). In practice, the requirement may be not too restrictive.

But first-order stochastic dominance is stronger than needed—it is a sufficient condition for welfare gains from bonus votes, but not a necessary one. Consider for example two beta distributions: $P(v) = \int_0^v x^{a_P-1}(1-x)^{b_P-1}/\text{Beta}[a_P, b_P]$, $C(v) = \int_0^v x^{a_C-1}(1-x)^{b_C-1}/\text{Beta}[a_C, b_C]$, and suppose $E_P(v) > E_C(v)$ in each referendum. Call ϕ_P (ϕ_C) the expected fraction of voters casting their bonus vote in favor of (against) any given proposal. Figure 2 shows the ratio ϕ_P/ϕ_C when $a_P/b_P = 2$, $a_C/b_C = 1$, $b_P \in [1/2, 5]$, and $b_C \in [1/2, 5]$. Since $E(v) = a/(a+b)$, the constraints $a_P/b_P = 2$ and $a_C/b_C = 1$ imply $E_P(v) = 2/3$ and $E_C(v) = 1/2$ (or $E_P(v)/E_C(v) = 4/3$), while the range in the parameters b_P and b_C allows us to change the shape of the distributions from bimodal at

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0 and 1 (when $b < 1$) to uniform ($b = 1$) to unimodal (when $b > 1$). If ϕ_P/ϕ_C is larger than 1, then the probability that the given referendum will pass converges to 1, as demanded by efficiency. In Figure 2.a $k = 2$, and ϕ_P/ϕ_C is always larger than 1, although $P(v)$ and $C(v)$ cannot in general be ranked in terms of first-order stochastic dominance.

It is also true however that in the absence of stochastic dominance bonus votes need not improve welfare.[32] The problematic cases are those where the side with higher mean is concentrated in its valuation, while the opposite side is dispersed (particularly so if it is bimodal at 0 and 1). With probability increasing in k, the number of referenda, it is then possible for the bonus votes to be used predominantly by the side with lower mean valuation (the larger the number of draws, the higher the probability that the highest draw will come from the more dispersed distribution). In our beta-distribution example, for some of our parameter values ϕ_P/ϕ_C falls below 1 if we increase k to 4 (see the upper left corner of Figure 2.b). Even though $P(v)$ has higher mean, a referendum may fail to pass if $P(v)$ is very concentrated around its mode (b_P is high), while $C(v)$ is not (b_C is low). But as the figure shows, the range of parameter values for which this occurs is small.[33] Intuition suggests that it should be smaller still if the distributions differ across referenda.

ϕ_P/ϕ_C.
Beta distributions: $E_P(v) = 2/3$; $E_C(v) = 1/2$;
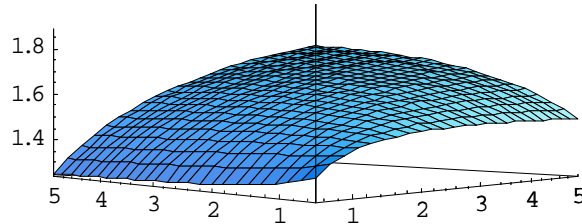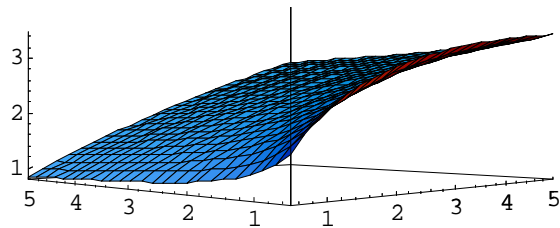$b_P \in [0.5, 5]$, $b_C \in [0.5, 5]$;

Figure 2.a: $k = 2$

Figure 2.b: $k = 4$

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[32] Bonus votes are not a fully efficient mechanism, even asymptotically.

[33] With $k = 4$ the upper bound on b_C for which problems can occur is 0.7, and this requires $b_P \geq 5$. There is a trade-off involved in the choice of k: the higher is k the larger is $(EU - EW)$ if $\phi_P/\phi_C > 1$; but if $P(v)$ does not first-order stochastically dominate $C(v)$, the lower is k, the smaller the range of distributions for which $\phi_P/\phi_C < 1$. Thus the optimal k depends on the precision of the information on the shape of the distributions.

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9 Conclusions

This paper has discussed an easy scheme to improve the efficiency of referenda: when several referenda are held simultaneously, grant voters, in addition to their regular votes, a stock of special votes—or even more simply a single special vote—that can be allocated freely among the different referenda. By concentrating these bonus votes on the one issue to which each voter attaches most importance, voters can shift the probability of obtaining the outcome they prefer towards the issue they care most about. If the value of the bonus votes is chosen correctly, and in particular if it is positive but not too large, the result is an increase in ex ante utility, relative to simple majority voting. If we use random decision-making as a natural lower bound against which the voting mechanisms are evaluated, then when the distributions of valuations are symmetric between voters who favor and voters who oppose each proposal, bonus votes (of the correct value) always improve over randomness by more than majority voting does. The analysis can be generalized easily to bonus votes of negative value (allowing voters to cast "partial" or weaker votes) or to a continuous, perfectly divisible bonus vote that voters can divide freely over the multiple proposals, but we argue in the paper that the more general, more complex schemes need not be preferable.

The symmetry assumption is strong and, although standard in the literature, has special implications: in particular, as the population approaches infinity, random decision-making, majority voting and bonus votes are all asymptotically efficient. In the final section of the paper we have extended the analysis to asymmetrical distributions, and in particular to situations where the population is equally split between voters favoring or opposing any proposal, but the mean intensity of preferences is higher on one side. In this case, bonus votes are guaranteed to increase ex ante utility if the distribution of valuations on the side with higher mean first-order stochastically dominates the distribution on the opposite side; that is, if the mass of voters with more intense preferences is larger on the side with higher mean. If this condition is satisfied, bonus votes induce a welfare gain over simple majority regardless of their exact value, and the improvement does not disappear asymptotically (whereas majority voting continues to approach random decision-making). If this sufficient condition is not satisfied, counterexamples exist, but expecially when the number of referenda is small, constructing them requires some ingenuity.

Bonus votes are a simple mechanism allowing some expression of voters' intensity of preferences. They recall cumulative voting—an existing voting scheme employed in multi-candidate elections with the expressed goal of protecting minority interests. The protection of minorities built into these mechanisms is a particularly important objective as recourse to direct democracy increases. In fact the need to safeguard minorities, and in particular minorities with little access to financial resources, is the single point of agreement in the often heated debate on initiatives and referenda (for example Matsusaka (2004) and

Gerber (1999)).[34] The important objective is designing voting mechanisms that increase minority representation without aggregate efficiency losses, and this is why in this paper we have insisted on the pure efficiency properties of bonus votes.[35]

There is one open question that the paper has not addressed: the composition of the agenda. In our analysis, the slate of referenda is exogeneous, and in particular is not affected by the voting system. We believe this is the correct starting point, and one that allows us to begin by assuming the independence of individual valuations. Modeling any agenda-formation process is famously controversial, and in our case in addition requires identifying specific (large) groups with common interests, taking a stance on the forces holding the groups together and on the correlations of the group members valuations across different issues. From a technical point of view, it implies renouncing the assumption of independence and thus the power of the limit theorems we have exploited repeatedly. Intuitively, the final outcome seems difficult to predict: bonus votes may increase the incentive to manipulate the agenda, but also the ability to nullify the manipulation. We leave serious work on this question for the future.

10   References

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[34] According to Gerber (1999), narrow business interests have a comparative advantage in influencing popular votes through financial resources, while grass-root movements have a comparative advantage in gathering votes. If this logic is correct, bonus votes would both help to protect resource-less minorities against the power of the majority and reduce the power of money in direct democracy.

[35] In the legal literature, Cooter (2002) compares "median democracy" (direct democracy) to "bargain democracy" (legislatures) and argues for the practical superiority of the former, while admitting that the latter is "ideally" more efficient. Increasing the representation of intense preferences in a direct democracy is a step towards higher efficiency.

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11 Appendix

Before proving the lemmas in the text, we begin by a preliminary result that will be used repeatedly. Define votes in favor as positive votes and votes against as negative votes, and the vote differential, the sum of all votes cast in referendum r, as V_r.

Lemma A.1. *Consider the voting problem in the absence of bonus votes when everybody votes sincerely. Call p_r a voter's probability of obtaining the desired outcome in referendum r. Then if $F_r(v)$ is symmetric around 0, $V_r \sim N(0, n)$ and $p_r \simeq 1/2 + 1/\sqrt{2\pi n}$.*

Proof of Lemma A.1. The derivation of the asymptotic distribution of the vote differential is standard (see for example Feller, 1968, pp. 179 -182). The distribution is normal with mean given by the sample mean $(1/2(-1) + 1/2(1) = 0)$ and variance given by the sum of the variances of the summands: $n[(1/2)(-1)^2 + (1/2)(1)^2] = n$. Because the distribution does not depend on $F_r(v)$, we can ignore the subscript r. Taking into account possible ties, the probability of obtaining one's desired outcome is:

$$p = prob(V_i \leq 0) + (1/2)prob(V_i = 1)$$

where V_i is the voting differential excluding voter i. Given the discreteness of the votes:

$$prob(V_i \leq 0) \simeq \frac{1}{2}\left(1 + \frac{1}{\sqrt{2\pi(n-1)}}e^{-0}\right) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2\pi(n-1)}}\right)$$

$$(1/2)prob(V_i = 1) \simeq \frac{1}{2}\left(\frac{1}{\sqrt{2\pi(n-1)}}e^{-\frac{(1)^2}{2(n-1)}}\right)$$

$$\simeq \frac{1}{2}\left[\frac{1}{\sqrt{2\pi(n-1)}}\left(1 - \frac{1}{2(n-1)}\right)\right] = \frac{1}{2}\left(\frac{1}{\sqrt{2\pi(n-1)}} - O(n^{-3/2})\right)$$

Given n large and ignoring terms of order $O(n^{-3/2})$, $p \sim 1/2 + 1/\sqrt{2\pi n}$. $\square$

We can now begin to study our model, where voters cast both regular and bonus votes.

Proof of Lemma 1. When we add to the problem bonus votes of arbitrary value, we need to be more careful about the discreteness of the asymptotic distribution of the vote differential. The subtlety is in sizing correctly the steps of the distribution. The proof proceeds in three stages. First, we present the local limit theorem relevant to our problem. Then we show that casting bonus votes in a referendum where the expected value of the vote differential is non-zero cannot affect the probability of being pivotal. Finally we show that this argument implies Lemma 1.

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(i). Consider the problem facing voter i in referendum r when bonus votes are available. The voter has to evaluate the probability of obtaining his or her desired outcome when casting x_i votes, where x_i equals ± 1 if i casts no bonus votes and $\pm(1 + z\vartheta)$ if he or she casts z bonus votes. All voters have the same set of feasible options $X = \{x_j\} = \{\pm(1 + z\vartheta), \ z = 0, 1, \ldots, m\}$ (where j indexes any point in the support that has positive probability). If voters use symmetric strategies, x_i is iid for all i's, we can drop the subscript i and use local limit theorems to characterize the asymptotic distribution of the votes differential in any referendum. The random variable x is distributed according to a lattice distribution—a discrete distribution such that all possible $x_j \in X$ can be expressed as $a + s_j h$ with $h > 0$ and s_j integer $\forall j$. We need to impose the correct normalization. Following Gnedenko local limit theorem (Gnedenko and Kolmogorov, 1968, ch.9), select the representation $x_j = a + s_j h^0$ such that h^0 is the largest common divisor of all possible pairwise differences $x_j - x_{j'}$, and consider the normalized random variable $x' \equiv (x - a)/h^0$ and the normalized sum $V' = \sum_{i=1}^{n} x_i'$. If x' has finite mean $E(x')$ and non-zero variance $\sigma_{x'}^2$, then:

$$prob\{V' = y\} \to \frac{1}{\sigma_{x'}\sqrt{2\pi n}} Exp[\frac{-(y - nE(x'))^2}{2n\sigma_{x'}^2}] \text{ as } n \to \infty$$

For our purposes, we need to consider two cases. If $\vartheta \geq 1$, no normalization is required; if $\vartheta = 1/C$ with C discrete and larger than 1, normalize the problem so that all x are set in terms of the smallest possible integers: set $h^0 = 1/C$ and $a = 0$, and thus $X' = \{-(C+m), \ldots, -(C+1), -C, C, (C+1), \ldots, (C+m)\}$. In both cases, call $\pm\rho$ the normalized value of the regular vote, and $\pm\xi$ the normalized value of one bonus vote. In addition, define as $\phi_{x'r}$ the probability that any voter casts x' votes in referendum r, again distinguishing between positive and negative votes, $E_r(x') \equiv \mu_r = \sum_{x' \in X'} \phi_{x'r} x'$ and $\sigma_r^2 = \sum_{x' \in X'} \phi_{x'r}(x' - \mu_r)^2$.

(ii) Suppose i opposes the proposal. The probability that i obtains the desired outcome when voting $-\rho$, $p_{-\rho r}$, equals:

$$p_{-\rho r} = prob(V_{ir}' < \rho) + (1/2)prob(V_{ir}' = \rho)$$

The corresponding probability when adding $z\xi$ bonus votes equals

$$p_{-z\xi r} = prob(V_{ir}' < \rho + z\xi) + (1/2)prob(V_{ir}' = \rho + z\xi)$$

Thus

$$p_{-z\xi r} - p_{-\rho r} = (1/2)[prob(V_{ir}' = \rho) + prob(V_{ir}' = \rho + z\xi)] + prob(V_{ir}' \in (\rho, \rho + z\xi))$$

or, by Gnedenko's theorem,

$$p_{-z\xi r} - p_{-\rho r} \simeq \frac{1}{2\sqrt{2\pi n\sigma_r^2}} e^{-\frac{(\rho - E(V_r'))^2}{2n\sigma_r^2}} + \frac{1}{2\sqrt{2\pi n\sigma_r^2}} e^{-\frac{(\rho + z\xi - E(V_r'))^2}{2n\sigma_r^2}} + \sum_{V' \in (\rho, \rho + z\xi)} \frac{e^{-\frac{(V' - E(V_r'))^2}{2n\sigma_r^2}}}{\sqrt{2\pi n\sigma_r^2}}$$

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But recall $E(V_r') = n\mu_r$. As long as $(\rho + z\xi)/\sqrt{n}$ (or $C/\sqrt{n}$) approaches 0, we

conclude:
$$p_{-z\xi r} - p_{-\rho r} < O(e^{-n}) \text{ if } \mu_r \neq 0$$

The conclusion is identical if i favors the proposal, and thus casting bonus votes cannot increase the probability of being pivotal (up to an approximation smaller than $O(e^{-n})$) in referenda where the votes differential has non-zero mean.

(iii) Consider a candidate equilibrium where there exist some r, r' such that $E(V_r') = 0$, $E(V_{r'}') \neq 0$. By Lemma A.2 no voter casts any bonus vote in referenda r'. With $F(\mathrm{v})$ symmetric around 0 and undominated strategies - hence sincere voters—the scenario cannot occur: either $E(V_r') = 0 \,\forall r$, or $E(V_r') \neq 0 \,\forall r$. Suppose then $E(V_r') \neq 0 \,\forall r$. By Lemma A.2, $p_{-z\xi r} - p_{-\rho r} < O(e^{-n}) \,\forall r, \,\forall z$, regardless of the sign of $E(V_r')$. By selecting weakly undominated strategies, voters vote *as if* $p_{-z\xi r} - p_{-\rho r} > 0 \,\forall r$, or as if $E(V_r') = 0 \,\forall r$, which implies $p_{-z\xi r} - p_{-\rho r} = p_{z\xi r} - p_{\rho r}$. But then the sign of v cannot matter: with payoff $\mathrm{v} \equiv |\mathrm{v}|$ and $p_{-z\xi r} - p_{-\rho r} = p_{z\xi r} - p_{\rho r}$ all weakly undominated strategies must depend only on v. But then with $F(\mathrm{v})$ is symmetric around 0, $E(V_r') \neq 0$ $\forall r$ cannot occur. We are left with the case $E(V_r') = 0 \,\forall r$. By Gnedenko's theorem, and exploiting the approximation $Exp[y/n] \simeq 1 - y/n$, the probability of obtaining one's desired outcome when casting $|x'|$ votes in referendum r, can be approximated by:
$$p_{|x'|r} \sim \frac{1}{2} + \frac{|x'|}{\sqrt{2\pi n \sigma_r^2}} \tag{A1}$$

up to terms of order $O(n^{-3/2})$, whether the votes are for or against. (The normalization discussed earlier is important. (A.1) is derived separately for the two cases that are relevant for the model: $\rho = \xi = 1$ and $\rho = C > 1$, $\xi = 1$). Once again, with payoff $\mathrm{v} \equiv |\mathrm{v}|$ and $p_{|x'|r}$ independent of the sign of the vote, the optimal $|x_r'|$ must depend on v only. Since $|x_r'|$ is just a normalization of $|x_r|$, the conclusion applies to $|x_r|$ too and Lemma 1 is established. $\square$ The

following Corollary follows immediately from the proof:

Corollary to Lemma 1. *If $F(v)$ is symmetric around 0, in all undominated strategies equilibria $E(V_r') = 0 \,\forall r$.*

Proof of Lemma 2. Recall that we now understand the strategies $x_{ir}'(v_{ir}, m, \vartheta, F_r)$ to refer to the absolute number of votes cast. Consider the problem faced by voter i, deciding how to allocate bonus votes so as to maximize expected utility:

$$\max_{\{x'_r\}} \sum_{r=1}^{k} \left(\frac{1}{2} + \frac{x_r'}{\sqrt{2\pi n \sigma_r^2}} \right) v_{ir} \tag{A2}$$

$$\text{s.t. } x_r' = 1 + z_r\xi \text{ where } z_r \in \{0, 1, \dots, m\} \text{ and } \sum_{r=1}^{k} z_r = m$$

Problem (A2) is linear in z_r, and the solution must be at a corner: voter i will cast all bonus votes on referendum s such that $v_{is}/\sqrt{2\pi n \sigma_s^2} > v_{ir}/\sqrt{2\pi n \sigma_r^2}$

$\forall r \neq s$. (The variance of the vote differential facing voter i in referendum r is $n\sigma_r^2 = n\sum_{x'\in X'}\phi_{x'r}(x'-\mu_r)^2$ and is taken as given by i.) $\square$

Proof of Lemma 3. We begin by establishing one preliminary result.

Lemma A.2. In equilibrium, $\phi_r < 1$ $\forall r$.

Proof of Lemma A.2. Suppose that there exists a r' such that $\phi_{r'} = 1$, and consider voter i's problem. If i casts his bonus vote on r', then $p_{\rho r} = p_{m\xi r'} \simeq 1/2 + 1/\sqrt{2\pi n}$ $\forall r$. If i shifts the bonus vote to s, $p_{\rho r'} \simeq 1/2 + 1/(2\sqrt{2\pi n})$ (where his vote now makes a difference only in case of a tie) but $p_{m\xi s} \simeq 1/2 + 1/\sqrt{2\pi n} + 1/(2\sqrt{2\pi n})(1+d)$ where d is the largest positive integer smaller or equal to than $m\xi/\rho$, if one exists, or 0. Thus voter i's expected utility would increase by: $1/(2\sqrt{2\pi n})[(1+d)v_{is} - v_{ir'}] > 0$ $\forall v_{is} > v_{ir'}/(1+d)$. The probability that i finds it profitable to deviate is the probability that there exists at least one s such that $v_{is} > v_{ir'}/(1+d)$, or $1 - \int_0^1 \prod_{s\neq r'} G_s(\frac{v}{1+d})g_{r'}(v)dv$. With valuations that are iid across voters, the probability that any one voter deviates is then

$$1 - \left[\int_0^1 \prod_{s\neq r'} G_s(\tfrac{v}{1+d})g_r(v)dv\right]^n > 0 \text{ (in fact arbitrarily close to 1 for } n \text{ large}$$

enough). $\square$

We can now return to Lemma 3. By Lemma 1, Lemma 2 and Lemma A.2, $E(V_r') = 0$ $\forall r$ and $n\sigma_r^2 = n[\phi_r(\rho+m\xi)^2 + (1-\phi_r)\rho^2] = n[\rho^2 + \phi_r(2\rho m\xi + (m\xi)^2)]$, with $\phi_r \in [0,1)$ $\forall r$. Thus:

$$p_{\rho r} \sim \frac{1}{2} + \frac{\rho}{\sqrt{2\pi n[\rho^2 + \phi_r(2\rho m\xi + (m\xi)^2)]}}$$

$$p_{m\xi r} \sim \frac{1}{2} + \frac{\rho + m\xi}{\sqrt{2\pi n[\rho^2 + \phi_r(2\rho m\xi + (m\xi)^2)]}}$$

up to terms of order $O(n^{-3/2})$. If we renormalize $\rho \equiv 1$ and $\theta \equiv m\xi/\rho$, we can write:

$$p_r \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi n[1 + \phi_r(\theta^2 + 2\theta)]}}$$

$$p_{\theta r} \sim \frac{1}{2} + \frac{1 + \theta}{\sqrt{2\pi n[1 + \phi_r(\theta^2 + 2\theta)]}}$$

The second part of the Lemma is proved in the text. $\square$.

Proof of Proposition 3. At $\theta = 0$, $\omega = 1$—as must be the case by the definition of ω and as can be verified by setting $\theta = 0$ in (13). Proposition 3 must hold if ω is increasing in θ at $\theta = 0$: although θ must be a rational number larger than $1/\sqrt{n}$, there is always a value of n such that θ can take values arbitrarily close to 0. Because in addition ω is continuous in θ in the neighborhood of

$\theta = 0$, in this neighborhood we can treat θ as a continuous variable. Showing $d\omega/d\theta > 0$ at $\theta = 0$ amounts to differentiating (13), taking into account (11) and (12). The derivative is greatly simplified by being evaluated at $\theta = 0$. In fact it is not difficult to show that it reduces to:

$$\frac{d\omega}{d\theta}\Big|_{\theta=0} = \sum_{r=1}^{k} \left[\left(\int_0^1 \prod_{s \neq r} G_s(v) v g_r(v) dv \right) \right] - \sum_{r=1}^{k} \left[Ev_r(v) \left(\int_0^1 \prod_{s \neq r} G_s(v) g_r(v) dv \right) \right]$$
(A3)

Integrating by part the first summation, we obtain:

$$\sum_{r=1}^{k} \left[\left(\int_0^1 \prod_{s \neq r} G_s(v) v g_r(v) dv \right) \right] = \int_0^1 \left(1 - \prod_{r=1}^{k} G_r(v) \right) dv = Ev_{(k)}$$

where $Ev_{(k)}$ now stands for the expected highest draw over all distributions. Thus (A3) can be rewritten more simply as:

$$\frac{d\omega}{d\theta}\Big|_{\theta=0} = Ev_{(k)} - \sum_{r=1}^{k} Ev_r(v) \ \phi_r|_{\theta=0}$$

But since $\phi_r|_{\theta=0} \in (0,1) \forall r$, the expression must be strictly positive, and the proposition is established.□

Proof of Proposition 3b. The proof proceed identically to the proof of Proposition 3. Indeed, $\frac{d\omega}{d\theta}\big|_{\theta=0} = \frac{d\omega^R}{d\theta}\big|_{\theta=0}$, where $\omega^R \equiv \omega(\alpha_{sr} = 1 \forall s, r)$, (A3) continues to hold, and the argument is unchanged.□

Proof of Example 2. Here it turns out to be easier to work with $\tau \equiv 1/\theta$, the value of the regular votes relative to the bonus vote. With power distributions, the condition $\omega > 1$ then corresponds to:

$$\omega > 1 \Leftrightarrow \sum_{r=1}^{k} \left(\frac{b_r c_r}{\sum_{r=1}^{k} b_r + 1} \right) + \tau \sum_{r=1}^{k} \left(\frac{b_r c_r}{1 + b_r} \right) > \sum_{r=1}^{k} \left(\frac{b_r}{1 + b_r} \right)$$
(A4)

where:

$$c_r \equiv \sqrt{\frac{\sum_{r=1}^{k} b_r}{b_r(1 + 2\tau) + \tau^2}}$$

The proof proceeds in three steps. First, we know from the proof of Proposition 3 that as τ approaches infinity, or equivalently θ approaches 0, ω approaches 1 from above. This immediately establishes that either $\omega > 1 \ \forall \tau$, or there exists at least one internal maximum at a finite value of τ. Second, we can derive the first-order condition that an internal maximum, if it exists, must satisfy. Differentiating the left-hand side of (A4) with respect to τ, we find that the first derivative equals zero at τ^* defined by the implicit equation:

$$\tau^* = \sum_{r=1}^{k} \left(\frac{\gamma_r(\tau^*)}{\sum_{r=1}^{k} \gamma_r(\tau^*)} \right) b_r,$$
(A5)

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where

$$\gamma_r(\tau^*) \equiv \frac{b_r}{1+b_r}\left[\sum_{s\neq r} b_s \left(\frac{1}{\sum_{j=1}^{k} b_j \tau^{*2} + b_r(1+2\tau^*)}\right)^{3/2}\right].$$

For our purposes, the important point is that any and all τ^* must be a weighted average of the distribution parameters $\{b_1, \ldots, b_k\}$, with weights $w_r(\tau^*) \equiv \left(\gamma_r(\tau^*)/\sum_{r=1}^{k}\gamma_r(\tau^*)\right) \forall\, r = 1, \ldots, k$ and such that $\sum_{r=1}^{k} w_r(\tau^*) = 1$. In particular, notice that each weight is strictly between 0 and 1 for all positive finite b_r and τ^* (including the limit as τ^* approaches 0). Thus any and all τ^* must satisfy $\tau^* < b_k$ where $b_k \equiv \max\{b_r\}$. Third, consider the limit of ω as τ approaches 0:

$$\lim_{\tau->0} \omega = \left(\frac{\sqrt{\sum_{r=1}^{k} b_r}(\sum_{r=1}^{k}\sqrt{b_r})}{1+\sum_{r=1}^{k} b_r}\right) \Big/ \left(\sum_{r=1}^{k}\left(\frac{b_r}{1+b_r}\right)\right) \tag{A6}$$

The limit is positive and finite. There are two possibilities. If (A6) is smaller than 1, then by step 1 above an internal maximum must exist. Call $\tau^{*\prime}$ the largest value of τ that satisfies (A5), and notice that $\omega(\tau^{*\prime})$ must be a maximum: then $\omega > 1 \forall \tau > \tau^{*\prime}$. And since $\tau^{*\prime} < b_k$, $\omega > 1 \,\forall \tau \geq b_k$. If (A6) is larger than 1, either no internal maximum exists and ω is larger than 1 for all τ—in which case, $\omega > 1 \,\forall \tau \geq b_k$ is trivially satisfied. Or an internal maximum exists, and the argument above continues to hold: $\omega > 1 \forall \tau > \tau^{*\prime}$, and since $\tau^{*\prime} < b_k$, $\omega > 1$ $\forall \tau \geq b_k$. Thus in all cases, $\omega > 1 \,\forall \tau \geq b_k$ or, equivalently, $\omega > 1 \,\forall \theta \leq 1/b_k$.□

Proof of Lemma 5. Call ϕ_{Pr} (ϕ_{Cr}) the share of all voters in favor of (against) proposal r who choose to cast their bonus vote in referendum r. The expected vote differential in referendum r is: $EV_r = (n/2)\theta(\phi_{Pr} - \phi_{Cr})$, and its variance: $\sigma_r^2 = (n/2)\left[(\theta^2 + 2\theta)(\phi_{Pr} + \phi_{Cr}) + 2\right]$. Consider two referenda, s and r. Voter i casts his or her bonus vote on referendum s over r if $v_{is}/v_{ir} > (p_{\theta r} - p_r)/(p_{\theta s} - p_s)$, or, ignoring terms of order $O(n^{-3/2})$ or smaller:

$$\frac{v_{is}}{v_{ir}} > \left(\frac{(\theta^2 + 2\theta)(\phi_{Ps} + \phi_{Cs}) + 2}{(\theta^2 + 2\theta)(\phi_{Pr} + \phi_{Cr}) + 2}\right) e^{-\frac{n}{4}(A_r - A_s)} \left(\frac{e^{I_r B_r} + e^{I_r(1+\theta)B_r}}{e^{I_s B_s} + e^{I_s(1+\theta)B_s}}\right) \tag{A7}$$

where

$$A_r \equiv \frac{\theta^2(\phi_{Pr} - \phi_{Cr})^2}{\left[(\theta^2 + 2\theta)(\phi_{Pr} + \phi_{Cr}) + 2\right]}$$

$$B_r \equiv \frac{\theta(\phi_{Pr} - \phi_{Cr})}{\left[(\theta^2 + 2\theta)(\phi_{Pr} + \phi_{Cr}) + 2\right]}$$

$$I_r \equiv \begin{cases} -1 \text{ if } v_r > 0 \\ 1 \text{ if } v_r < 0 \end{cases}$$

Regardless of the sign of the valuations, the terms in the large parentheses are always positive. Suppose first $A_r > A_s$. Then for all v_{is}, v_{ir} no bonus

votes should be cast in referendum r, and thus $\phi_{Pr} = \phi_{Cr} = 0$. But then $A_r = 0 \leq A_s$, contradicting $A_r > A_s$. Thus all equilibria must be such that $A_r = A_s$. One equilibrium is immediate and holds for all distributions $P(v)$ and $C(v)$: voter i casts his or her bonus vote in referendum s if and only if $v_s > v_r$ for all $r \neq s$, and thus $\phi_{Ps} = \phi_{Pr}$ and $\phi_{Cs} = \phi_{Cr}$ for all r, s. Other candidate equilibria are possible, but none exists for arbitrary $P(v)$ and $C(v)$ distributions: for any given pair of distributions, the best response strategies above yield values for ϕ_{Ps}, ϕ_{Pr}, ϕ_{Cs}, and ϕ_{Cr} that are derived independently of the equilibrium constraint $A_r = A_s$ and generically violate it. $\square$

Proof of Proposition 5. The proof proceeds by showing that if $P(v)$ first-order stochastically dominates $C(v)$ for all r, each referendum passes with probability approaching 1. With a large population this outcome is ex ante efficient and dominates the outcome of simple majority voting. Given Lemma 5:

$$\phi_P = \left(\frac{2}{k}\right)\left(\frac{1}{2}\right)^k \left[\sum_{s=0}^{k-1}(s+1)\binom{k}{s+1}\int_0^1 C(v)^{k-1-s}P(v)^s p(v)dv\right]$$

$$\phi_C = \left(\frac{2}{k}\right)\left(\frac{1}{2}\right)^k \left[\sum_{s=0}^{k-1}(s+1)\binom{k}{s+1}\int_0^1 P(v)^{k-1-s}C(v)^s c(v)dv\right]$$

or, identically, using the index $j \equiv k - 1 - s$:

$$\phi_C = \left(\frac{2}{k}\right)\left(\frac{1}{2}\right)^k \left[\sum_{j=0}^{k-1}(j+1)\binom{k}{j+1}\int_0^1 C(v)^{k-1-j}P(v)^j c(v)dv\right]$$

Because both $P(v)$ and $C(v)$ are strictly increasing in v, and $P(v)$ first-order stochastically dominates $C(v)$, each term summed in ϕ_P is larger than its corresponding term in ϕ_C, and thus $\phi_P > \phi_C$. The vote differential in each referendum is normally distributed with mean $EV = (n/2)\theta(\phi_P - \phi_C) > 0$ and variance $\sigma_V^2 = (n/2)[(\phi_P + \phi_C)(2\theta + \theta^2) + 2]$. Recall that $\Phi(x) \simeq 1 - x^{-1}\frac{1}{\sqrt{2\pi}}Exp[\frac{-x^2}{2}]$ when x is large and $\Phi(\cdot)$ is the standard normal distribution function (see for example Feller, 1968, chapter 7). Hence:

$$prob(V > 0) = prob\left[(\frac{V - EV}{\sigma_V}) > -\frac{EV}{\sigma_V}\right] =$$

$$= \Phi\left(\frac{EV}{\sigma_V}\right) \simeq 1 - \frac{1}{\sqrt{2\pi}}\frac{\sigma_V}{EV}e^{-EV^2/(2\sigma_V^2)} = 1 - \frac{1}{\sqrt{2\pi n}}O(e^{-n})$$

and the probability that proposal r passes equals

$$prob(V > 0) + \frac{1}{2}prob(V = 0) \simeq 1 - \frac{1}{2}\frac{1}{\sqrt{2\pi n}}O(e^{-n}) \simeq 1$$

Thus a proposal passes with probability approaching 1.

We can then write ex ante utility as:

$$EU \simeq \sum_r \frac{1}{2} E_{Pr}(v)$$

where $1/2$ is the ex ante probability of being in favor of any proposal (given the 0 median). With simple majority voting, on the other hand:

$$EW \simeq \sum_r \left(\frac{1}{2} + \frac{1}{\sqrt{2\pi n}} \right) \left(\frac{E_{Pr}(v) + E_{Cr}(v)}{2} \right)$$

We can conclude that there always exists a large but finite $\widetilde{n}$ such that for all $n > \widetilde{n}$, $EU > EW$. Notice that the result holds for any positive θ, independenlty of its precise value. In addition, if we consider a sequence of referenda with increasing n, as $n \to \infty$, $EW_n \to ER = 1/2 \sum_r \left(\frac{E_{Pr}(v) + E_{Cr}(v)}{2} \right)$, while $EU_n \to \sum_r \frac{1}{2} E_{Pr}(v)$ yielding the Corollary to Proposition 5 in the text. $\square$