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THE STRUCTURE OF FACTOR CONTENT PREDICTIONS

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ABSTRACT

The last decade witnessed an explosion of research into the impact of international technology differences on the factor content of trade. Yet the literature has failed to confront two pivotal issues. First, with international technology differences and traded intermediate inputs there does not exist a Vanek-consistent definition of the factor content of trade. Restated, we do not know what we are trying to explain! We fill this gap by providing the correct definition. Second, as Helpman and Krugman (1985) showed, many models beyond Heckscher-Ohlin imply the Vanek prediction. So what model is being tested? We completely characterize the class of models being tested by providing a familiar `consumption similarity' condition that is necessary and sufficient for the Vanek prediction. We illustrate with a unique dataset containing input-output tables for 41 rich and poor countries. We find modest support for the strong version of the Vanek prediction and impressive support for weaker versions of the prediction.

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1. Introduction

There was a time when the factor content of trade prediction was the exclusive domain of the Heckscher-Ohlin model. However, the prediction is now known to be consistent with a larger class of models. For example, in 1979 Deardorff opened up the possibility of a factor content prediction without factor price equalization, in 1982 Ethier implicitly derived a factor content prediction with international returns to scale, and by 1985 Helpman and Krugman were able to derive the Vanek (1968) factor content prediction under a variety of assumptions about increasing returns and imperfect competition. Having opened Pandora's box, just how general is the Vanek factor content prediction? We know that many models imply the Vanek prediction and that probably many more so imply it than have yet been explored. But how many? This paper completely characterizes the relevant class of models by providing a familiar 'consumption similarity' condition that is necessary and sufficient for a 'robust' Vanek prediction.

To understand robustness and why it is needed, consider the example of a standard monopolistic competition model (e.g., Helpman and Krugman, 1985, chapter 6) that is augmented by asymmetric trade barriers and international technology differences. Such a model will not in general yield a Vanek prediction, but it is possible that the Vanek prediction just happens to obtain for very particular values of the international technology difference parameters. Robustness is a weak condition that identifies such values and treats them for what they are, namely, uninteresting special cases.[1]

We have discussed the first goal of this paper, namely, a complete characterization of the class of models that imply a robust Vanek prediction. As should be clear from the definition of robustness, we are particularly interested in the role of international

[1]On a somewhat more technical level, we say that a model has a robust Vanek prediction if the prediction survives an almost irrelevant local perturbation of the underlying technology i.e., a perturbation that infinitesimally alters industry-level demands for primary factors without affecting (i) economy-wide factor prices or (ii) any equilibrium outcome in the markets for final goods and intermediate inputs.

technology differences for empirical studies of the Vanek prediction. We argue that this large literature is fundamentally flawed because it has failed to correctly provide a Vanek-consistent definition of the factor content of trade. That is, the empirical literature is using the wrong dependent variable. Our second goal is to provide the correct Vanek-consistent definition of the factor content of trade. Before being more specific, we first review the relevant literature.

International differences in technology and choice of techniques have finally emerged as the central issue in assessing the validity of the Vanek prediction. Trefler (1993, 1995) showed that international productivity differences explain at least some of the observed departures from the Vanek prediction. Using novel methodology, Davis et al. (1997) demonstrated that the failure of the Vanek prediction is in part due to international choice-of-technique differences. In a crucial contribution, Davis and Weinstein (2001) carefully estimated choice-of-technique matrices using data from ten OECD countries and provided strong evidence that allowing for Hicks-neutral technology differences and factor price differences greatly improves the fit of the model. Hakura (2001) echoed this result. Antweiler and Trefler (2002) incorporated increasing returns to scale, one source of international productivity differences, into the Vanek prediction and found scale returns to be very important. Debaere (2003) showed that the successes and failures of the Vanek prediction are intimately related to issues of economic development. Other papers that allow for international technology differences include Davis and Weinstein (2000, 2003), Trefler and Zhu (2000), Conway (2002), Trefler (2002) and Reimer (2003). International technology and choice-of-technique differences have thus emerged as the central issue in empirical studies of the Vanek prediction.

Yet factor content *theory* with international differences in technology and choice of techniques lags far behind *empirical* research. Thus Harrigan (1997, page 492) laments the problems created by the fact that "the effective factor content of trade is not well defined when there are nonneutral technology differences across sectors." Feenstra (2004, page 55) argues that current definitions of the factor content of trade are so problematic that great caution must be exercised in using them to test the Vanek prediction. And Davis and Weinstein (2003, page 129) complain that "understanding how to incorporate traded intermediates into factor content studies remains an important area for future research." The problem is that with intermediate inputs and international technology differences, no one knows how to define the factor content of trade in a way that is consistent with the Vanek prediction.

In light of this black hole it is not surprising to see healthy mud-slinging between Trefler and Zhu (2000) and Davis and Weinstein (2003). Each correctly finds error in what the other had done − a case of the kettle calling the stove black. Unfortunately, neither side was able to offer a correct Vanek-consistent definition of the factor content of trade. Our paper is the first to get the definition right. Remarkably, the correct definition bears no resemblance to the definitions used by either Trefler and Zhu or Davis and Weinstein.

The correct definition requires data that are not typically collected. Implementation of the definition thus requires data imputations that are closely related to the imputations of intermediate trade made by Feenstra and Hanson (1996, 1999), Hummels et al. (2001) and Yi (2003). Thus, our paper is unexpectedly related to the growing literature on outsourcing and vertical production networks.

We round out the paper with an empirical assessment of the Vanek prediction. We use a new data set that has input-output tables for 41 developed and developing countries. Previous research has been confined to at most 10 developed countries, thus missing North-South endowments-based trade. As compared to using just the U.S. input-output table, using 41 input-output tables significantly improves the fit of the Vanek prediction for labour and human capital, but not for physical capital. We also consider a Debaereinspired (2003) 'ratio' version of the Vanek prediction and find that it does very well for all three factors. Finally, we find dramatic support for an informal hypothesis relating endowments to the factors embodied in world trade. This relationship appears in figure 1. Each point is a country. The horizontal axis is the ratio of a country's endowments (the ratio of human capital to labour in the top panel and physical capital to labour in the bottom panel). The vertical axis is the ratio of a country's factor content of world exports (correctly defined). Figure 1 reveals that the factor content of world exports is strongly correlated with endowments across countries. We will have more to say about this in section 8.3.

The paper is organized as follows. Sections 2-3 provide the correct definition of the factor content of trade. Sections 4-5 completely characterize the class of models implying a robust Vanek prediction. Sections 6-7 review previous empirical work in light of our findings and section 8 presents new empirical work.

2. Setup

Let $g = 1, ..., G$ index goods, let i and $j = 1, ..., N$ index countries, and let $f = 1, ..., K$ index factors. Let V_i be the $K \times 1$ vector of country i endowments, let $V_w \equiv \Sigma_i V_i$ be the world endowment vector, and let F_i be the $K \times 1$ vector giving the factor content of trade for country i. Let s_i be the consumption share of country i, where $s_i > 0$ for all i and $\Sigma_i s_i = 1$. The object of analysis is the Vanek factor content of trade prediction, $F_i = V_i - s_i V_w$. By implication, if country i is abundant in factor f (element f of $V_i - s_i V_w$ is positive) then the country is a net exporter of the services of factor f (element f of F_i is positive).

Every good is consumed as a final product and/or used as an intermediate input. Let C_{ij} be a $G \times 1$ vector denoting country i consumption of goods produced in country j. Let Y_{ij} be a $G \times 1$ vector denoting i's usage of intermediate inputs produced in country

j. Country j's output Q_j is split between consumption and intermediate inputs:

$$Q_j \equiv \Sigma_i \left(C_{ij} + Y_{ij}\right). \tag{1}$$

World consumption of goods produced in country j is

$$C_{wj} \equiv \Sigma_i C_{ij}. \tag{2}$$

Let $B_{ij}\left(g,h\right)$ be the amount of intermediate input g used to produce one unit of good h, where g is made in country i and h is made in country j. Let $Q_j\left(h\right)$ be a typical element of Q_j. Then $B_{ij}\left(g,h\right)Q_j\left(h\right)$ is the amount of input g used to produce $Q_j\left(h\right)$ and $\Sigma_h B_{ij}\left(g,h\right)Q_j\left(h\right)$ is the amount of input g used by country j. Restated, $\Sigma_h B_{ij}\left(g,h\right)Q_j\left(h\right)$ is the gth element of Y_{ji}. In matrix notation,

$$Y_{ji} = B_{ij}Q_j \tag{3}$$

where B_{ij} is the $G \times G$ matrix with typical element $B_{ij}\left(g,h\right)$.

Let D_i be the matrix whose (f,g) element gives the average amount of factor f used directly to produce one unit of good g in country i. To ensure that factors are fully employed, we assume that D_i satisfies

$$D_i Q_i = V_i. \tag{4}$$

Equations (3) and (4) are best viewed as data identities that (partly) define B_{ij} and D_i, respectively.

Country i's vector of imports from country j is $M_{ij} \equiv Y_{ij} + C_{ij}$ for $j \neq i$. From equation (3), M_{ij} may alternatively be defined as

$$M_{ij} \equiv B_{ji}Q_i + C_{ij} \qquad j \neq i. \tag{5}$$

Country i's vector of exports to the world is $X_i \equiv \Sigma_{j\neq i}M_{ji} = \Sigma_{j\neq i}(Y_{ji}+C_{ji}) = \Sigma_j\,(Y_{ji}+C_{ji}) - Y_{ii} - C_{ii}$. Hence, from equations (1) and (3), X_i may alternatively be defined as

$$X_i \equiv Q_i - B_{ii}Q_i - C_{ii}. \tag{6}$$

This completes the definition of the variables that we will use.

3. The Factor Content of Trade

To pave the way for a Vanek-consistent definition of the factor content of trade, define

$$Q \equiv \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_N \end{bmatrix}, \qquad C \equiv \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{N1} \\ C_{12} & C_{22} & \cdots & C_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1N} & C_{2N} & \cdots & C_{NN} \end{bmatrix},$$

$$B \equiv \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1N} \\ B_{21} & B_{22} & \cdots & B_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1} & B_{N2} & \cdots & B_{NN} \end{bmatrix}, \quad T \equiv \begin{bmatrix} X_1 & -M_{21} & \cdots & -M_{N1} \\ -M_{12} & X_2 & \cdots & -M_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ -M_{1N} & -M_{2N} & \cdots & X_N \end{bmatrix},$$

and

$$D \equiv \begin{bmatrix} D_1 & D_2 & \cdots & D_N \end{bmatrix}.$$

Let T_i be the ith column of T so that $T = [T_1\ T_2\ \cdots\ T_N]$. Let I be the $NG \times NG$ identity matrix. These definitions are motivated by the following non-trivial theorem.

Theorem 1. *Assume that* $(I - B)$ *is invertible and define* $A \equiv D(I - B)^{-1}$. *Then*

$$F_i \equiv AT_i \tag{7}$$

is the factor content of country i*'s trade. Specifically,* F_i *is the amount of factors employed worldwide to produce* T_i.

No researcher, empirical or theoretical, has ever defined the factor content of trade as in theorem 1. We will show in the next section that F_i is the Vanek-consistent definition of the factor content of trade i.e., $F_i = V_i - s_i V_w$. It follows that no empirical researcher who is interested in the Vanek prediction with unrestricted international technology differences has ever used the right definition of the factor content of trade. This includes empirical work by Trefler (1993, 1995), Davis and Weinstein (2001), Hakura (2001), Conway (2002), Trefler (2002), Debaere (2003) and others. We will develop this point in sections 6-7 below.

There is a simple and elegant proof of theorem 1 that appears in appendix A.1. However, we start with a lengthier, but constructive proof.

Proof of Theorem 1:

Let Z be an arbitrary $G \times 1$ output vector. By the definition of D_i, production of Z in country i directly requires (in an input-output sense) $D_i Z$ units of primary factors. We will use this fact repeatedly.

Stacking equations (5) and (6) yields

$$T = (I - B)\, Q - C. \tag{8}$$

To fix ideas, consider momentarily the case of only 2 countries. The direct requirements of primary factors needed to produce country 1's exports to country 2 (i.e., to produce X_1) are $D_1 X_1$. The direct requirements of primary factors needed to produce country 1's imports from country 2 (i.e., to produce M_{12}) are $D_2 M_{12}$. Recalling that T_i is the ith column of T, the direct requirements of primary factors needed to produce T_1 are thus $D_1 X_1 - D_2 M_{12} = [D_1\ D_2] T_1 = DT_1$. Generalizing to many countries, DT_i is the direct factor requirements needed to produce T_i.

Production of T_i also requires intermediate inputs. These inputs themselves require primary factors. Returning to the 2-country case, production of X_1 uses domestic intermediate inputs $B_{11} X_1$ and imported intermediate inputs $B_{21} X_1$.[2] Production of M_{12} requires $B_{12} M_{12}$ units of intermediate inputs produced in country 1 and $B_{22} M_{12}$ units of intermediate inputs produced in country 2. These intermediate input requirements may be summarized as

$$\begin{bmatrix} B_{11} X_1 - B_{12} M_{12} \\ B_{21} X_1 - B_{22} M_{12} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ -M_{12} \end{bmatrix} = BT_1.$$

Generalizing to many countries, BT_i is the intermediate inputs needed to produce T_i. But these BT_i intermediate inputs must themselves be produced. Repeating the same logic with BT_i replacing T_i, the intermediate inputs needed to produce BT_i are $B(BT_i) = B^2 T_i$. Either by repeating the argument *ad infinitum* or by applying induction, the total amount of intermediate inputs needed to produce T_i must be $(\Sigma_{n=1}^{\infty} B^n) T_i$. Further, the primary factors needed to produce these intermediates are $D(\Sigma_{n=1}^{\infty} B^n) T_i$.

The sum of these indirect factor requirements plus the direct requirements DT_i is $D(\Sigma_{n=0}^{\infty} B^n) T_i$. Since $(\Sigma_{n=0}^{\infty} B^n) = (I - B)^{-1}$, $D(I - B)^{-1} T_i$ is the total (direct plus indirect) factor requirements needed to produce T_i. ■

[2] These expressions are just the right-hand side of equation (3) with $j = 1$, $i = 1, 2$ and Q_j replaced by X_1.

4. Sufficiency

We next turn to the question of which models imply the Vanek prediction $F_i = V_i - s_i V_w$. In this section we show that models which imply a familiar 'consumption similarity' condition imply the Vanek prediction. This is a generalization of a key result in Helpman and Krugman (1985).[3]

Lemma 1 establishes the relationship between the Vanek prediction and consumption patterns. It is useful to partition A as $A = [A_1 \; A_2 \; \cdots \; A_N]$.

Lemma 1. $F_i = (V_i - s_i V_w) - \Sigma_j A_j (C_{ij} - s_i C_{wj}) \; \forall i$.

All remaining proofs appear in appendices. Lemma 1 states that the definitions in equations (1)-(6) are all that is needed to show that the Vanek prediction is always wrong by an amount $\Sigma_j A_j (C_{ij} - s_i C_{wj})$. This is an assumption-free result. An immediate consequence of lemma 1 is the following.

Theorem 2. *(Sufficiency):* $C_{ij} = s_i C_{wj} \; \forall \; i$ *and* $j \Longrightarrow F_i = V_i - s_i V_w \; \forall \; i$.

In the next section we will show the converse, but this is much harder to show.

$C_{ij} = s_i C_{wj}$ for all i and j defines 'consumption similarity' in a way that makes the Vanek prediction hold even though choice of techniques vary across countries. Introducing g subscripts to denote elements of C_{ij} and C_{wj}, consumption similarity states that $C_{gij}/C_{gwj} = s_i$ for all g, i, and j. This means that country i consumes a fixed proportion s_i of the final goods produced by all other countries. This appears in models with taste for variety or ideal varieties (e.g., Helpman and Krugman, 1985).

If there is production specialization so that only one country produces the good then consumption similarity reduces to the usual Heckscher-Ohlin consumption similarity con-

[3] In the Trefler (1999) interview of Helpman, Helpman states that a key finding of Helpman and Krugman (1985) is that the Vanek prediction appears in many of the models considered in that book.

dition, namely, $\Sigma_j C_{ij} = s_i \Sigma_j C_{wj}$.[4] Production specialization is associated with scale returns (Helpman and Krugman, 1985), failure of factor price equalization (Deardorff, 1979) or both (Markusen and Venables, 1998).

In North-South models, choice of techniques differ across regions N and S, but are the same within regions. In this case, $A_i = A_S$ for all Southern countries and $A_i = A_N$ for all Northern countries. Then $\Sigma_j A_j (C_{ij} - s_i C_{wj}) = A_S \Sigma_{j\in S} (C_{ij} - s_i C_{wj}) + A_N \Sigma_{j\in N} (C_{ij} - s_i C_{wj})$. Thus, theorem 2 (and its converse below) hold with $C_{ij} = s_i C_{wj}$ for all j replaced by $\Sigma_{j\in R} C_{ij} = s_i \Sigma_{j\in R} C_{wj}$ for $R = N, S$. In the extreme where all countries share a common choice of technique, theorem 2 and its converse hold with $C_{ij} = s_i C_{wj}$ for all j replaced by the usual Heckscher-Ohlin condition $\Sigma_j C_{ij} = s_i \Sigma_j C_{wj}\ \forall\ i$. That is, location of production plays no role.

Note that $C_{ij} = s_i C_{wj}$ looks like the gravity equation. In the absence of intermediate inputs, $C_{ij} = s_i C_{wj}$ becomes $M_{ij} = s_i Q_j$. This is exactly the equation estimated by Harrigan (1996). Specifically, he estimated $\ln M_{ij} = \alpha + \beta \ln s_i Q_j$ for 28 ISIC industries in 22 OECD countries and found $\widehat{\beta} = 1.20$ and $\overline{R}^2 = 0.66$. The equation has since been estimated by many other researchers.

Finally, what models do not imply $C_{ij} = s_i C_{wj}$? There are three possibilities. The first is models with international differences in preferences. The second is models with income effects associated with non-homotheticities e.g., Hunter and Markusen (1988). This occurs when richer countries spend disproportionately more on certain types of goods such as health or better-quality goods. The third possibility is that consumers in different countries face different product prices. If consumers face different prices, they will not make choices consistent with $C_{ij} = s_i C_{wj}$. Tariffs and transportation costs are one source of international differences in product prices. Product price differences also appear in Balassa-Samuelson models where non-traded consumption goods such as

[4] If j^* is the only country that produces g, then $\Sigma_j C_{gij} = C_{gij^*}$ and $\Sigma_j C_{gwj} = C_{gwj^*}$ so that $C_{gij^*} = s_i C_{gwj^*}$ becomes $\Sigma_j C_{gij} = s_i \Sigma_j C_{gwj}$.

haircuts are cheaper in poor countries. Thus, non-tradeable final goods pose a serious challenge to the Vanek prediction. Summarizing, preference differences, income effects and price differences all lead to models with $C_{ij} \neq s_i C_{wj}$.

5. Necessity

We have shown that consumption similarity implies the Vanek prediction. Does the Vanek prediction imply consumption similarity? The answer is 'almost' in the following sense: if a model does not imply consumption similarity, then it does not imply the Vanek prediction except for very special and empirically uninteresting forms of international technology differences. Proving this without any assumptions about the form of product market competition and with few assumptions on technology is difficult so we break the problem down into three pieces. The reader who is not interested in the details should jump straight to section 5.3 or even to theorem 3.

5.1. Technology Primitives π and Factor Market Equilibrium

We assume the following.

Assumption 1. *(i) Factor markets are perfectly competitive: factors are mobile across firms within a country and firms are price takers in factor markets. (ii) There is no joint production. (iii) Cost functions are differentiable. (iv) All factor prices are strictly positive.*

Part (iv) is for notational convenience.

Let q_k be the amount of good g that firm k produces in country i. The cost of producing q_k is $c_k(\omega_i, q_k)$ where ω_i is a vector of factor prices. Let π be the underlying technology that generates the cost functions $\{c_k\}_{\forall k}$. We will write $c_k(\omega_i, q_k|\pi)$ as a function of π

in order to indicate that c_k is generated by π.[5] Under assumption 1, a firm's vector of cost-minimizing average factor inputs is given by

$$d_k \equiv \left(\frac{1}{q_k}\right) \frac{\partial c_k\left(\omega_i, q_k|\pi\right)}{\partial \omega_i} \tag{9}$$

for $q_k > 0$ and $d_k \equiv 0$ for $q_k = 0$.

The d_k are the firm-level factor demands that aggregate up to the national-level factor demands D_i. (For a formal statement of this, see appendix equation 25.) Assumption 1 together with equations (4) and (9) describe competitive factor markets with exogenous factor supplies V_i.[6]

5.2. Product Market Equilibrium Outcomes

We next turn to the problem of characterizing product market equilibrium *outcomes* without fully specifying the equilibrium concept. To this end, consider an economy with the following features. (*i*) Consumers maximize utilities. (*ii*) Producers maximize profits in a way that is consistent with equation (9). (*iii*) Factor markets clear according to equation (4).

The *exogenous* parameters of the economy are technology π, preferences, and endowments V_i. The endogenous variables include $d \equiv \{d_k\}_{\forall k}$, $D \equiv (D_1, \ldots, D_N)$, $\omega \equiv (\omega_1, \ldots, \omega_N)$, and $E \equiv \{p_k, q_k, s_i, C_{ij}, C_{wj}, Y_{ij}, Q_i, B\}_{\forall i,j,k}$ (where p_k is the price of firm k's product). E collects all the endogenous variables explicitly referred to below that relate to the markets for final goods and intermediate inputs. These endogenous variables are all functions of π.

[5]If this is too abstract, think about a world with Cobb-Douglas production functions in which α_{fgi} is the exponent on factor input f in the production of good g in country i. Then π collects all the α_{fgi}.

[6]It turns out that the competitive factor markets assumption is not necessary. All we need is an equation like equation (9) that makes d_k a function of π. It does not actually matter much what the function looks like. For expositional clarity we stick with the competitive factor markets assumption.

5.3. Economically Insignificant Perturbations of Technology

We turn last to defining what a robust Vanek prediction means. Pick an arbitrary technology primitive π and let $\Pi(\pi, \varepsilon)$ be a set of technology primitives that are 'close' to π in a sense yet to be described. Suppose that the Vanek prediction holds at π i.e., $F_i(\pi) = V_i - s_i(\pi) V_w$. If $F_i(\pi') = V_i - s_i(\pi') V_w$ for all $\pi' \in \Pi(\pi, \varepsilon)$, we say that the Vanek prediction is robust. Our aim is to show that if the Vanek prediction is robust then consumption similarity holds. We are only interested in robust Vanek predictions. We are not interested in a Vanek prediction that pops up only for very special values of π.

The smaller is the set $\Pi(\pi, \varepsilon)$, the weaker is the requirement of robustness and hence the stronger is our theorem. We thus define $\Pi(\pi, \varepsilon)$ narrowly.[7]

Definition 1. $\Pi(\pi, \varepsilon)$ *is the set of perturbations* π' *satisfying the following: (1)* $||D(\pi') - D(\pi)|| < \varepsilon$ *where* $||\cdot||$ *is the Euclidean norm. That is, the perturbation alters industry-level factor demands by an infinitesimal amount. (2) The perturbation does not alter any equilibrium outcomes in the markets for final goods and intermediate inputs. (3) The perturbation does not alter the economy-wide demand for factors. (4) The perturbation does not alter factor prices. (5) The perturbation does not alter industry-level factor payments.*

Clearly, perturbations of technology that are confined to $\Pi(\pi, \varepsilon)$ affect almost nothing in the economy. In this sense Π is small and robustness is a weak requirement.[8]

Theorem 3 is roughly the converse of theorem 2 and is a key result of this paper.

Theorem 3. *(Necessity): Under Assumption 1,*

[7] For those readers in need of a more formal statement of definition 1, $\Pi(\pi, \varepsilon)$ is the set of π' satisfying the following: *(1)* $||D(\pi') - D(\pi)|| < \varepsilon$. *(2)* $E(\pi') = E(\pi)$. *(3)* $D_i(\pi') Q_i(\pi') = V_i \; \forall i$. *(4)* $\omega_i(\pi') = \omega_i(\pi) \; \forall i$. *(5)* $\omega_i(\pi') D_i(\pi') = \omega_i(\pi) D_i(\pi) \; \forall i$.

[8] Of course, we do not want $\Pi(\pi, \varepsilon)$ to be so small that it contains only a single point i.e., π. Appendix lemma 3 shows that this is not a concern. The lemma also characterizes $\Pi(\pi, \varepsilon)$ in terms of the set of $D(\pi')$ it generates.

$\{F_i(\pi') = V_i - s_i(\pi') V_w\}_{i=1}^N$ *for all* π' *in* $\Pi(\pi, \varepsilon) \Longrightarrow C_{ij}(\pi) = s_i(\pi) \; C_{wj}(\pi) \;\; \forall \; i$ and j.

Theorem 3 states that if the Vanek prediction is robust to inconsequential perturbations of the underlying technology, then consumption must be similar across countries.

$\Pi(\pi, \varepsilon)$ plays a central role so that the reader should ask whether it captures the 'right' set of perturbations. $\Pi(\pi, \varepsilon)$ only includes perturbations that have no impact on (i) economy-wide factor prices or (ii) any equilibrium outcome in the markets for final goods or intermediate inputs. It is thus safe to say that $\Pi(\pi, \varepsilon)$ is not too large. Further, it is logically impossible for $\Pi(\pi, \varepsilon)$ to be too small: For suppose we expand Π by replacing it with a set Π^* that contains Π. If theorem 3 holds on Π, then it must also hold on Π^*.[9] That is, $\Pi(\pi, \varepsilon)$ cannot be too small.

This concludes our discussion of the necessary and sufficient conditions for a robust Vanek prediction. Deardorff (1979), Ethier (1982) and Helpman and Krugman (1985) showed us that many models imply the Vanek prediction. Our paper shows that the consumption similarity condition completely characterizes the set of models implying a robust Vanek prediction.

6. Empirical Counterpart of F_i

The factor content of trade F_i is a function of B. Unfortunately, data for B do not exist. To see this, recall that $B_{ji}(g,h)\,Q_i(h)$ is the amount of input g used to produce h where g is made in country j and h is made in country i. For $j \neq i$, $B_{ji}(g,h)\,Q_i(h)$ is an import of intermediate inputs. A firm that produces h will know how much g it needs. However, it will often not know which country produced g because g was bought from a

[9] Since $\Pi^* \supset \Pi$, $\{F_i(\pi') = V_i - s_i(\pi') V_w\}_{i=1}^N$ on Π^* implies $\{F_i(\pi') = V_i - s_i(\pi') V_w\}_{i=1}^N$ on Π which, by theorem 3, implies $C_i(\pi) = s_i(\pi)\, C_w(\pi) \; \forall i$. Hence, theorem 3 with Π implies a modified theorem 3 with Π^* replacing Π.

local distributor (Hummels and Hillberry, 2003). For example, an Atlanta construction firm will not know if the pine 2×4s it bought were produced in the state of Washington or the province of British Columbia. Of course, some firms like General Motors will know exactly where each part is sourced. However, statistical agencies do not ask sourcing questions even of these firms because the reporting requirements are too onerous. There are exceptions − for example, Brazil reports data on imported machinery (Muendler, 2004) − but these exceptions prove the rule. To summarize using matrix notation (and working with inputs per unit of output), national statistical agencies report

$$\overline{B}_i \equiv \Sigma_j B_{ji}.$$

They do not report the B_{ji}.[10]

What may puzzle the reader is that secondary disseminators of input-output tables, such as the OECD and GTAP, claim to know whether intermediate inputs are imported or produced locally. Correspondingly, they report input-output tables separately for locally produced intermediates (B_{ii}) and for imported intermediates ($\Sigma_{j\neq i} B_{ji}$). How can this be? The answer is that they impute the data using the 'proportionality' assumption. To quote from the OECD:

> "This technique assumes that an industry uses an import of a particular product in proportion to its total use of that product. For example if an industry such as motor vehicles uses steel in its production processes and 10 per cent of all steel is imported, it is assumed that 10 per cent of the steel used by the motor vehicle industry is imported." (Organisation for Economic Co-operation and Development, 2002, page 12)

To formalize this, let $Q_i(g)$, $X_i(g)$, $M_{ij}(g)$, and $M_i(g)$ be the gth elements of the vectors Q_i, X_i, M_{ij}, and $\Sigma_{j\neq i} M_{ij}$, respectively. For good g, $Q_i(g) + M_i(g) - X_i(g)$ is domestic

[10] Note that the B_{ji} *cannot* be recovered from import data on intermediate inputs. B_{ji} identifies not only which input is imported, but also which domestic industry purchases the input. Restated, the $G \times G$ matrix B_{ji} cannot be backed out of the $G \times 1$ vector M_{ij} of country i's imports from country j.

absorption i.e., the amount of g used by country i for both intermediate use and final consumption. Define

$$\theta_{ij}(g) \equiv \frac{M_{ij}(g)}{Q_i(g) + M_i(g) - X_i(g)} \qquad \text{for } j \neq i. \tag{10}$$

$\theta_{ij}(g)$ is the share of domestic absorption that is sourced from country j. Also define

$$\theta_{ii}(g) \equiv 1 - \Sigma_{j \neq i} \theta_{ij}(g) \tag{11}$$

which is the share of domestic absorption that is sourced locally. Finally, let $B_{ji}(g,h)$ and $\overline{B}_i(g,h)$ be elements of B_{ji} and $\overline{B}_i \equiv \Sigma_j B_{ji}$, respectively. Then the proportionality assumption is

$$\begin{aligned} \Sigma_{j \neq i} B_{ji}(g,h) &= \overline{B}_i(g,h) \Sigma_{j \neq i} \theta_{ij}(g) && \text{(imported intermediates)} \\ B_{ii}(g,h) &= \overline{B}_i(g,h) \theta_{ii}(g) && \text{(local intermediates)} \end{aligned} \tag{12}$$

This is how the OECD and GTAP break out domestic and foreign purchases. It is one of the assumptions that allows Hummels et al. (2001) and Yi (2003) to estimate the growth in world trade in intermediate inputs and in inputs used in vertical production networks. (See equations 2-3 in Hummels et al..) It is also the assumption used by Feenstra and Hanson (1996, 1999) to develop their broad measure of outsourcing.[11]

An obvious and simple extension of the proportionality assumption in equation (12) is

$$B_{ji}(g,h) = \overline{B}_i(g,h) \theta_{ij}(g) \qquad \text{for all } i \text{ and } j. \tag{13}$$

Equation (13) allows one to recover the B matrix from available data in a way that is

[11] Feenstra and Hanson care about outsourcing, but not about which intermediates g are outsourced. They thus sum equation (12) over intermediates g to obtain $\Sigma_g \overline{B}_i(g,h) \Sigma_{j \neq i} \theta_{ij}(g)$. This multiplied by $Q_i(h)$ is their measure of outsourcing.

consistent with the efforts of Feenstra and Hanson (1996, 1999), Hummels et al. (2001) and Yi (2003). In the empirical section below, we will use equation (13) to calculate B.

7. Previous Definitions of the Factor Content of Trade

In the literature on the Vanek prediction with international technology differences and traded intermediates we count at least five different and mutually incompatible definitions of the factor content of trade. Here we reconsider this literature in light of our new definition of the factor content of trade. In reviewing the literature, it is best to have a narrative or story line. In our view, this narrative has been the on-going challenge to come up with a definition of the factor content of trade that satisfies three criteria: (*1*) the definition holds without undue restrictions on the form of international choice-of-technique differences, (*2*) the definition makes sense independently of whether the Vanek prediction holds, and (*3*) the definition is correct.

We begin by defining

$$\overline{A}_i \equiv D_i(I - \overline{B}_i)^{-1}$$

where, as before, $\overline{B}_i \equiv \Sigma_j B_{ji}$ is the standard national input-output table i.e., the input requirements summed over both national and international sources of supply. All previous work on the Vanek prediction has used $\overline{A}_i$ rather than our A.

Trefler (1993) assumes that choice-of-technique differences take the form $\overline{B}_i = \overline{B}_{US}$ and $D_i = \Lambda_i^{-1} D_{US}$ where Λ_i is a diagonal matrix whose typical diagonal element gives the productivity of factor f in country i relative to the United States. Under Trefler's assumption, the full employment condition $D_i Q_i = V_i$ can be re-written as $D_{US} Q_i = V_i^*$ where $V_i^* \equiv \Lambda_i V_i$ is country i's endowments measured in productivity-equivalent units. This transforms the model into the standard Heckscher-Ohlin-Vanek model with internationally identical choice of techniques, but with factors measured in productivity-equivalent units. In particular, the Vanek prediction becomes $\overline{A}_{US}(X_i - M_i) = V_i^* - s_i \Sigma_j V_j^*$ where

$\overline{A}_{US}(X_i - M_i)$ is the factor content of trade measured in productivity-equivalent units. Variants of this approach are used by Trefler (1995, hypothesis T1), Davis and Weinstein (2001, hypothesis T3), Conway (2002), and Debaere (2003). This approach satisfies our second and third criteria above, but not our first.

When choice of techniques are allowed to differ internationally in more general ways, coming up with a sensible definition of the factor content of trade has proved far more difficult. For example, Davis et al. (1997) is a major contribution that improves on Trefler (1993) by relaxing all restrictions on the form of the international choice-of-technique differences. Absent such restrictions however, it is clear that their dependent variable $\overline{A}_{JAPAN}(X_i - M_i)$ is *not* the factor content of trade. After all, it evaluates goods produced in country i using Japan's choice of techniques.[12] Likewise for Hakura (2001) who moves from using a single country's input-output table to using the input-output tables of 4 OECD countries. Contrary to what Hakura claims, her dependent variable $\overline{A}_i(X_i - M_i)$ is *not* the factor content of trade: $\overline{A}_i(X_i - M_i)$ evaluates the factor content of i's imports using i's choice of techniques rather than using the producing country's choice of techniques.

For the case of general international choice-of-technique differences, only two serious definitions of the factor content of trade have been proposed. The Davis and Weinstein (2000, 2001, hypotheses T4-T7) definition makes sense independently of whether the Vanek prediction holds (criterion 2). Unfortunately, the definition is wrong (criterion 3). In contrast, Antweiler and Trefler (2002), Trefler and Zhu (2000) and a much earlier version of this paper proposed a definition that is correct. Unfortunately, the definition has no meaning when the Vanek prediction fails. This will require some explanation.

Davis and Weinstein (2001), in their core hypothesis T4, define the factor content of

[12] This statement should not be misunderstood to mean that the equations estimated by Davis et al. (1997) contain mathematical errors. The equations are correct. It is the interpretation of the dependent variable that we are questioning. This caveat applies to all the papers reviewed below.

trade as[13]

$$F_i^{DW} \equiv \overline{A}_i X_i - \Sigma_{j \neq i} \overline{A}_j M_{ij}.$$

This definition first appeared in Helpman and Krugman (1985, equation 1.11) and is very intuitive in the sense that it appears to evaluate the output of country j using country j's choice of techniques. That is, it evaluates M_{ij} using $\overline{A}_j$. Further, the definition looks a lot like our F_i. To see this, partition our A as $[A_1 \ A_2 \ \cdots \ A_N]$. Then F_i can be written as

$$F_i = A_i X_i - \Sigma_{j \neq i} A_j M_{ij}.$$

It follows that $F_i^{DW} = F_i$ when $\overline{A}_i = A_i$. Restated, F_i^{DW} is the factor content of trade when $\overline{A}_i = A_i$. When is $\overline{A}_i = A_i$? Without additional restrictions on B, a necessary and sufficient condition for $\overline{A}_i = A_i$ is $B_{ji} = 0$ for all $j \neq i$.[14]

$B_{ji} = 0$ means that country i does not import any intermediate inputs from country j. Thus, without additional restrictions on B, F_i^{DW} is the factor content of trade only when there is no trade in intermediate inputs. Clearly, this is an uncomfortable assumption in

[13] Their definition is actually more complicated, but these complications only obscure our main point without altering it. In particular, see Davis and Weinstein (2001, page 1425–26) and their hypotheses T5, T6, and T7.

[14] To see this, first consider the case of 2 countries. To keep the expression for F_1 manageable we assume that intermediate inputs flow only in one direction, from country 2 to country 1, so that $B_{21} = 0$. Then it is straightforward to show that our equation (7) definition of the factor content of trade reduces to

$$F_1 = D_1(I - B_{11})^{-1} X_1 - D_2(I - B_{22})^{-1} M_{12} - D_1(I - B_{11})^{-1} B_{12} (I - B_{22})^{-1} M_{12}$$

while Davis and Weinstein's definition reduces to

$$F_i^{DW} = \overline{A}_1 X_1 - \overline{A}_2 M_{12} = D_1(I - B_{11})^{-1} X_1 - D_2(I - B_{22} - B_{12})^{-1} M_{12}.$$

Clearly, these definitions are equivalent only in the special case where there is no intermediate trade i.e., where $B_{12} = 0$.

More generally, consider the definitions of $\overline{A}_i$ and A as well as the definition of B at the start of section 3. Then $\overline{A}_i = A_i$ when $(I - B)^{-1}$ is a block diagonal matrix with typical diagonal matrix $(I - \overline{B}_i)^{-1}$. Without further restrictions on B , a necessary and sufficient condition for this block-diagonality is that the off-diagonal elements of B equal 0 i.e., $B_{ji} = 0$ for all $j \neq i$. To see this, note that $B_{ji} = 0$ for all $j \neq i$ implies two things. First, $(I - B)^{-1}$ is block diagonal with typical diagonal element $(I - B_{ii})^{-1}$. Second, $\overline{B}_i \equiv \Sigma_j B_{ji} = B_{ii}$. Hence, $(I - B)^{-1}$ is block diagonal with typical diagonal element $(I - \overline{B}_i)^{-1}$, as required.

light of the enormous interest in global vertical production networks e.g., Feenstra and Hanson (1996, 1999), Hummels et al. (2001) and Yi (2003).

What is wrong with the Davis and Weinstein definition? The problem is that $\overline{A}_i$ shares with $\overline{B}_i$ a failure to distinguish intermediate inputs that are produced domestically from intermediate inputs that are produced abroad. $\overline{A}_i$ can therefore not be used in any simple way to evaluate the factor content of trade.

Antweiler and Trefler (2002), Trefler and Zhu (2000) and a much earlier version of this paper get around this problem, but at a cost. They define the factor content of trade as

$$F_i^T \equiv \overline{A}_i X_i^c - \Sigma_{j \neq i} \overline{A}_j M_{ij}^c + \overline{A}_i (X_i^y - M_i^y) - s_i \Sigma_j \overline{A}_j (X_j^y - M_j^y)$$

where X_i^c is i's exports of consumption goods, M_{ij}^c is i's imports of consumption goods produced in country j, X_i^y is i's exports of intermediate inputs, and M_i^y is i's imports of intermediate inputs. These authors show that under consumption similarity, $F_i^T = V_i - s_i V_w$. But under consumption similarity, $F_i = V_i - s_i V_w$. Hence, $F_i^T = F_i$. That is, under consumption similarity, F_i^T is the factor content of trade.

This places the literature at an impasse. F_i^{DW} is a factor content definition that makes sense independently of whether the Vanek prediction holds (criterion 2), but it is wrong (criterion 3). In contrast, F_i^T is a definition that is correct, but only when the Vanek prediction holds. One contribution of this paper is that it provides a factor content definition F_i that moves the discipline beyond this impasse. F_i is both correct and makes sense independently of whether the Vanek prediction holds. F_i thus satisfies all 3 criteria.

We are heavily indebted to Davis and Weinstein for discussing (and arguing!) these points with us. Indeed, we are doubly grateful to them. We had implicitly adopted an approach that placed our criterion 3 above their criterion 2. We now understand that both criteria are important. Without their input we would have continued to self-righteously use F_i^T rather than F_i and this paper would only have contributed further to the confusion in this literature.[15] [16]

This completes our review of what has proven to be a very confused literature. This should both clarify past research and point the way to improved future research.

8. New Empirical Work

8.1. Testing the Vanek Prediction

In this section we assess the Vanek prediction $F_i = V_i - s_i V_w$. Let F_{fi}, V_{fi}, and V_{fw} be elements of the vectors F_i, V_i, and V_w, respectively. The Vanek prediction is then $F_{fi} = V_{fi} - s_i V_{fw}$. We exploit the GTAP (version 5) dataset that contains 1997 input-output tables for 41 developed and developing countries.[17] The data set is documented in Dimaranan and McDougall (2002). We use these data together with equation (13) to compute the world B matrix. We construct D ourselves as described in appendix B. The dataset includes 3 factors: physical capital, labour, and human capital (measured as the number of grade-12 equivalent workers). Data are for 1997 whenever possible. Appendix B provides more details about the data.

In order to express factors in comparable units, we follow Antweiler and Trefler (2002) in scaling observation (f, i) of $F_{fi} = V_{fi} - s_i V_{fw}$ by a scalar σ_{fi} in order to ensure that

[15] In addition, we owe an apology to Feenstra (2004, page 55) who takes the Davis and Weinstein logic an extra step by arguing that one cannot test the model using F_i^T and therefore one must follow Davis and Weinstein in estimating the $\overline{A}_i$ before plugging them into F_i^T. This argument is not relevant when one uses F_i because it is the factor content of trade both under the Vanek null and under the alternative that the Vanek prediction is wrong.

[16] Davis and Weinstein (2003) claim that F_i^T is wrong ('tautological') and make a number of other misrepresentations about F_i^T. We take these licks as just desserts for having misrepresented their work in Trefler and Zhu (2000). However, it is important for the reader to understand that the Antweiler and Trefler (2002) results based on F_i^T are correct. The fact that F_i^T is not the factor content of trade when the maintained assumption of consumption similarity is relaxed is irrelevant to Antweiler and Trefler: they never relax the assumption. Their null hypothesis is consumption similarity plus constant returns to scale and their alternative hypothesis is consumption similarity plus increasing returns to scale.

[17] The 41 countries (ranked by per capita GDP in 1996) are the United States, Hong Kong, Singapore, Switzerland, Denmark, Japan, Canada, Austria, the Netherlands, Australia, Germany, Belgium, Sweden, Italy, France, the United Kingdom, Finland, Ireland, New Zealand, Taiwan, Spain, South Korea, Portugal, Greece, Argentina, Uruguay, Malaysia, Chile, Hungary, Poland, Mexico, Thailand, Venezuela, Brazil, Turkey, Colombia, Peru, Indonesia, Sri Lanka, the Philippines, and China.

	Full Sample			Trimmed Sample
	$F_i \equiv AT_i$	$\overline{A}_{US}(X_i - M_i)$	$\overline{A}_{CHINA}(X_i - M_i)$	$F_i \equiv AT_i$
	(1)	(2)	(3)	(4)
1. Spearman Correlation	.63 (.00)	.15 (.09)	.19 (.03)	.62 (.00)
2. Sign Statistic	.80 (.00)	.50 (.50)	.54 (.24)	.80 (.00)
3. Missing Trade Statistic	.13	.01	.42	.19
4. Slope (β)	.21 (.00)	.02 (.00)	.21 (.00)	.23 (.00)
5. R^2	.34	.07	.11	.28
Observations	123	123	123	109

Notes: Row 1 is the Spearman correlation between F_{fi} and $V_{fi} - s_i V_{fw}$. Row 2 is the percentage of observations for which F_{fi} and $V_{fi} - s_i V_{fw}$ have the same sign. Row 3 is the variance of F_{fi} divided by the variance of $V_{fi} - s_i V_{fw}$. Rows 4-5 are the slope and R^2 from the regression $F_{fi} = \alpha + \beta(V_{fi} - s_i V_{fw}) + \varepsilon_{fi}$. In columns 1 and 4, the correct (equation 7) definition of the factor content of trade is used. In column 2 (3), factor contents are calculated assuming that all countries use U.S. (Chinese) choice of techniques. The full sample contains 41 countries and 3 factors. The trimmed sample excludes the 14 observations with $|V_{fi} - s_i V_{fw}| > 0.25$. p-values are in parentheses. Low p-values indicate statistical significance. In row 4, the p-value is for the hypothesis $\beta = 1$.

Table 1: The Vanek Prediction, All Factors

the residual $(F_{fi} - V_{fi} + s_i V_{fw})/\sigma_{fi}$ has a unit variance.[18]

Table 1 reports some standard statistics about the performance of the Vanek prediction. Columns 1, 2 and 3 each uses a different definition of the factor content of trade. Column 1 uses the correct definition of equation (7) i.e., $F_i = AT_i$. Columns 2-3 assume that choice of techniques are internationally identical. In column 2, all countries use U.S. techniques and the factor content of trade for country i is defined as in the older literature

[18]We use $\sigma_{fi} \equiv s_i^{\mu} \sigma_f$ where σ_f^2 is the cross-country variance of $(F_{fi} - V_{fi} + s_i V_{fw})/s_i^{\mu}$ and $\mu = 0.9$ is the Antweiler and Trefler maximum likelihood estimate of μ. Almost identical results obtain with the more usual $\mu = 0.5$. To the extent that most of our results are reported by factor, σ_f is a constant that plays no role.

as $\overline{A}_{US}(X_i - M_i)$. In column 3, $\overline{A}_{CHINA}(X_i - M_i)$ is used.

Row 1 shows the Spearman (or rank) correlation between F_{fi} and $V_{fi} - s_i V_{fw}$. There are 123 observations (41 countries × 3 factors). The 0.63 correlation that holds when the factor content of trade is defined correctly is a dramatic improvement over the correlations of 0.15 and 0.19 that obtain using the incorrect U.S.- and China-based definitions of the factor content of trade.

Row 2 is the percentage of observations for which F_{fi} has the same sign as $V_{fi} - s_i V_{fw}$. Using incorrect definitions of the factor content of trade (columns 2-3), the sign statistics are about 0.50, just as in Trefler (1995). Trefler concluded from this that the model performs about as well as a coin toss. Using the correct definition of the factor content of trade, 80% of the observations have the correct sign. In addition, the p-value of the sign test is less than 0.01 which means that the probability of F_{fi} and $V_{fi} - s_i V_{fw}$ randomly having the same sign more than 80% of the time is less than 1%.

Row 3 reports Trefler's (1995) 'missing trade' statistic i.e., the variance of F_{fi} divided by the variance of $V_{fi} - s_i V_{fw}$. Previous research has always calculated the missing trade statistic using the $\overline{A}_{US}(X_i - M_i)$ definition of the factor content of trade that appears in column 2. The result is a 0.01 missing trade statistic i.e., a huge amount of trade is missing relative to its Vanek prediction. Using our definition of the factor content of trade, the missing trade statistic rises more than tenfold to 0.13. This is still low, but represents an order of magnitude improvement.[19] The fact that the missing trade problem is alleviated by using the correct factor content of trade definition is exactly what Helpman (1998, 1999) establishes theoretically.

One way to partly resolve the 'missing trade' problem is to use Trefler's (1993) productivity-equivalent transformation so that $V_i - s_i V_w$ is replaced by $V_i^* - s_i \Sigma_j V_j^*$. This

[19]There is even less missing trade when using $\overline{A}_{CHINA}(X_i - M_i)$ (column 3). This is because China is so unproductive that using $\overline{A}_{CHINA}$ dramatically inflates the amount of factors needed to produce $X_i - M_i$.

is partly why Trefler (1995) and Davis and Weinstein (2001) do not have as pronounced problems with missing trade. However, their use of the productivity-equivalent transformation disguises the impressive amount by which our missing-trade statistic improves upon those reported in Trefler (1995) and Davis and Weinstein (2001).

Another way of thinking about missing trade and the fit of the Vanek prediction is to report the slope and R^2 from the regression $F_{fi} = \alpha + \beta(V_{fi} - s_i V_{fw}) + \varepsilon_{fi}$. See Davis and Weinstein (2001). This is reported in rows 4 and 5 and gives the same impression as rows 1 and 3.

As we shall see shortly, there are a few outliers. In order to investigate whether the good fit of the Vanek prediction is driven by outliers, we trimmed the sample by excluding the 14 observations with $|V_{fi} - s_i V_{fw}| > 0.25$. Column 4 shows that trimming does not alter the conclusions.

We next examine the performance of the Vanek prediction by factor. Figure 2 plots F_{fi} against $V_{fi} - s_i V_{fw}$ by factor. The top panels are labour, the middle panels are human capital and the bottom panels are physical capital. Further, the left-hand panels are the full sample while the right-hand panels are the trimmed sample. *Figure 2 clearly shows that the Vanek prediction fits very well for labour and human capital, but fits very poorly for physical capital.*

Table 2 provides additional results by factor. From row 1, the Spearman correlation is a statistically significant 0.89 for labour and 0.85 for human capital, but a statistically insignificant 0.18 for physical capital. From row 2, F_{fi} and $V_{fi} - s_i V_{fw}$ have the same sign a statistically significant 98% of the time for labour and 85% for human capital, but a statistically insignificant 59% of the time for physical capital. Similar results obtain for the trimmed sample.

When we aggregate across factors, as we did in table 1, our conclusions echo those of Davis and Weinstein (2001). However, our results differ from theirs in three important

	Labour	Human Capital	Physical Capital	Human Capital − Labour	Physical Capital − Labour
	(1)	(2)	(3)	(4)	(5)
1. Spearman Correlation	.89	.85	.18	.84	.80
	(.00)	(.00)	(.27)	(.00)	(.00)
2. Sign Statistic	.98	.85	.59	.90	.80
	(.00)	(.00)	(.17)	(.00)	(.00)
3. Missing Trade Statistic	.07	.09	.30	N/A	N/A
4. Slope (β)	.24	.27	.08	.15	.15
	(.00)	(.00)	(.00)	(.00)	(.00)
5. R^2	.82	.79	.02	.56	.66
Observations	41	41	41	41	41

Notes: See the notes to table 1. The correct (equation 7) definition of the factor content of trade is used in this table. Columns 1, 2, and 3 deal with $F_{fi} = V_{fi} - s_i V_{fw}$. Column 4 and 5 deal with equations (14) and (15), respectively. p-values are in parentheses. Low p-values indicate statistical significance. In row 4, the p-value is for the null of $\beta = 1$.

Table 2: By Factors and the Differenced Vanek Prediction

ways. First, we are using data on 41 developed *and developing* countries whereas they used 10 OECD countries. When it comes to examining endowments-based theories of trade, the contrast between developed and developing countries provides a crucial source of sample variation. After all, it is precisely this developed-developing country contrast that these theories are intended to exploit. Second, when we examine the Vanek prediction by factor we obtain very different results than Davis and Weinstein. Their results for labour and physical capital are similar while we obtain horrible results for physical capital.[20] Third, they use the wrong definition of the factor content of trade for their core hypotheses T4-T7.

[20]Note that our results barely change when we follow Davis and Weinstein in estimating choice of techniques rather than using the actual choice of techniques. This is not an explanation of why our conclusions differ.

8.2. A Different View of the Vanek Prediction

So far we have worked with a strong version of the Vanek prediction, that is, a version that examines each factor separately. Following Debaere (2003), one can also look at the difference across factors. Dividing $F_{fi} = V_{fi} - s_i V_{fw}$ by $s_i V_{fw}$ to obtain $F_{fi}/(s_i V_{fw}) = V_{fi}/(s_i V_{fw}) - 1$ and differencing across factors yields

$$\frac{F_{Hi}}{s_i V_{Hw}} - \frac{F_{Li}}{s_i V_{Lw}} = \frac{V_{Hi}}{s_i V_{Hw}} - \frac{V_{Li}}{s_i V_{Lw}} \tag{14}$$

and

$$\frac{F_{Ki}}{s_i V_{Kw}} - \frac{F_{Li}}{s_i V_{Lw}} = \frac{V_{Ki}}{s_i V_{Kw}} - \frac{V_{Li}}{s_i V_{Lw}}. \tag{15}$$

The top panel of figure 3 plots equation (14) and the bottom panel plots equation (15). Since the Vanek prediction performs well for both labour and human capital, it is no surprise that equation (14) fits well. The surprise is that equation (15) in the bottom panel fits so well: although the Vanek prediction performs horribly for physical capital, equation (15) performs wonderfully.

Table 2 (columns 4 and 5) provide additional statistics about equations (14) and (15). In particular, compare the fit of $F_{Ki} = V_{Ki} - s_i V_{Kw}$ in column 3 to physical capital less labour (equation 15) in column 5. Column 5 shows that the Spearman correlation is 0.80, much higher than 0.18 in column 3. Likewise, the sign statistic has risen to 0.80, up from its column 3 value of 0.59. Note that this improved fit for physical capital has nothing to do with scaling by $s_i V_{Kw}$. The Spearman correlation between $F_{Ki}/(s_i V_{Kw})$ and $V_{Ki}/(s_i V_{Kw}) - 1$ is 0.22 for physical capital. Summarizing, the Vanek prediction in differenced form performs remarkably well.

8.3. A Less Structured Relationship Between Trade and Endowments

Inferences such as equations (14) and (15) that are based strictly on the Vanek equation can place blinkers on a researcher who wants to explore the data. It is of interest to see other, less theoretically motivated data displays of the factor content of trade. One particularly striking display involves the factor content of world exports. This is given by

$$\begin{bmatrix} F_{Li}^X \\ F_{Hi}^X \\ F_{Ki}^X \end{bmatrix} \equiv (0 \cdots D_i \cdots 0)\,(I - B)^{-1} \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}. \tag{16}$$

The top panel of figure 1 in the introduction plots F_{Hi}^X/F_{Li}^X against V_{Hi}/V_{Li}. The Spearman correlation is 0.84, the slope is 1.14 and the R^2 statistic is 0.85. These results are very striking. The bottom panel of figure 1 plots F_{Ki}^X/F_{Li}^X against V_{Ki}/V_{Li}. Again the results are very striking, with a Spearman correlation of 0.90, a slope of 1.02 and an R^2 of 0.80.

Figure 1 makes it clear that the Vanek prediction is not the only approach to thinking about the relationship between endowments and the factor content of trade. Less structured approaches also offer insights.

9. Conclusions

Consumption similarity is both necessary and sufficient for a robust Vanek prediction. It thus highlights the central assumption of the Heckscher-Ohlin-Vanek model: excess factor supplies must be absorbed only through trade, not domestic consumption. Integral to the proof of necessity and sufficiency is the first appearance of a Vanek-consistent expression for the factor content of trade in settings with intermediate inputs and international differences in choice of techniques. Our new factor content expression is very different from what has been implemented empirically. Indeed, we showed that a number of prominent empirical papers have used incorrect definitions of the factor content of trade.

The theoretical results of this paper have both strengths and weaknesses. The consumption condition allows for international differences in factor prices and technology, imperfect competition, scale returns, and externalities. It is thus quite general. On the other hand, the interpretation of the consumption similarity condition is couched in terms of equilibrium quantities (C_{ij}, s_i and C_{wj}) rather than in terms of restrictions on technology, preferences and endowments. This tension reflects how our necessary and sufficient condition complements rather than displaces previous research such as Helpman and Krugman (1985). Previous research starts with assumptions about unobservables (technology and preferences) and ends with predictions about observables (the Vanek prediction). Such research provides clearly interpretable assumptions, but an empirical prediction that is consistent with a large class of models whose limit has until now been unknown. In contrast, the present paper starts with the Vanek prediction, but ends with restrictions on observables rather than technology and preferences. Thus, our theoretical results complement previous work.

We empirically assessed the Vanek prediction using a unique dataset that contains input-output tables for 41 developed and developing countries. The factor content of trade was calculated using our Vanek-consistent definition. In figure 2, we showed that the Vanek prediction performs superbly for labour and human capital, but horribly for physical capital. We then looked at the Vanek prediction in differenced form i.e., human capital less labour and physical capital less labour. As shown in figure 3, this prediction did very well for both differences. Finally, we found evidence strongly supporting a less-structured relationship between endowments and the factor content of world exports. This was shown in figure 1. Overall, these empirical results leave us much more impressed than before with the role of endowments as a source of comparative advantage.

A. Appendix: Proofs

A.1. Alternative Proof of Theorem 1

Proof. There is a more elegant proof of theorem 1 which does not rely on the computational device of treating equation (3) as a derived demand for inputs. From the perspective of the world as a whole, Q is either used as intermediates (BQ) or final goods (C). The difference between Q and $BQ + C$ is interregional shipments $T = (I - B)\,Q - C$. See equation (8). Further, by standard input-output logic, delivery of an $NG \times 1$ vector Z of final demand requires $(I - B)^{-1} Z$ units of gross output and thus $D\,(I - B)^{-1} Z$ units of primary factors.[21] Thus, $D\,(I - B)^{-1} T$ is the factor content of interregional shipments. This concludes our discussion of what F_i means. ■

A.2. Proof of Lemma 1

Proof. Recall that T_i is the ith column of T. Let $\overline{Q}_i$ be the ith column of Q. Let C_i be the ith column of C. From equation (8), $T_i = (I - B)\,\overline{Q}_i - C_i$. Using this and $A \equiv D(I - B)^{-1}$, the definition of F_i in equation (7) implies $F_i = D\overline{Q}_i - AC_i$. Since $D\overline{Q}_i = D_i Q_i = V_i$ (the second equality follows from equation 4) and A is partitioned as $[A_1\ A_2\ \cdots\ A_N]$,

$$F_i = V_i - \Sigma_j A_j C_{ij}. \tag{17}$$

In the next we show that $V_w = \Sigma_j A_j C_{wj}$.

Combining equations (1)-(3) yields $Q_j = C_{wj} + \Sigma_i B_{ji} Q_i$. Stacking this result gives

$$\begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} = \begin{bmatrix} C_{w1} \\ \vdots \\ C_{wN} \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & B_{1N} \\ \vdots & \ddots & \vdots \\ B_{N1} & \cdots & B_{NN} \end{bmatrix} \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} = (I - B)^{-1} \begin{bmatrix} C_{w1} \\ \vdots \\ C_{wN} \end{bmatrix}. \tag{18}$$

Pre-multiplying equation (18) by $D \equiv [D_1\ D_2\ \cdots\ D_N]$, then using equation (4) and $A = [A_1\ A_2\ \cdots\ A_N]$ yields $\Sigma_j V_j = \Sigma_j A_j C_{wj}$. Since $V_w \equiv \Sigma_j V_j$,

$$V_w = \Sigma_j A_j C_{wj}. \tag{19}$$

Multiplying equation (19) by s_i and subtracting the result from equation (17) yields

$$F_i = (V_i - s_i V_w) - \Sigma_j A_j \left(C_{ij} - s_i C_{wj}\right).$$

■

[21] How much gross output Z^G is needed to deliver final output Z? Since $Z^G = BZ^G + Z$, the answer is $Z^G = (I - B)^{-1} Z$.

A.3. Preliminaries to the Proof of Theorem 3

Define $R \equiv \left\{(\omega_i, q_k) : \omega_i \geq 0, ||\omega_i|| = 1, \underline{q} \leq q_k \leq \bar{q}\right\}$ for finite constants $\underline{q} > 0$ and $\bar{q}$. Let $\mathcal{K}(g,i)$ be the set of firms producing good g in country i.

Lemma 2. *Assume Assumption 1. Fix $\delta > 0$ and $k \in \mathcal{K}(g,i)$. For each $K \times 1$ vector of constants d'_k satisfying $\omega_i(\pi) d'_k = \omega_i(\pi) d_k(\pi)$, $d'_k > 0$, and $||d'_k - d_k(\pi)|| < \delta$, there exists a π' (i.e., a $c_k(\cdot\,|\pi')$ on R) such that*

$$d'_k = (1/q_k(\pi))\, \partial c_k(\omega_i, q_k(\pi)|\pi')/\partial\omega_i \quad \text{evaluated at } \omega_i = \omega_i(\pi) \tag{20}$$

$$c_k(\omega_i(\pi), \cdot\,|\pi') = c_k(\omega_i(\pi), \cdot\,|\pi). \tag{21}$$

Proof. Define

$$c_k(\omega_i, q_k|\pi') \equiv c_k(\omega_i, q_k|\pi) + \omega_i(d'_k - d_k(\pi))\, q_k(\pi) \quad \forall (\omega_i, q_k) \in R. \tag{22}$$

We first show that since $c_k(\cdot\,|\pi)$ is a cost function on R, so is $c_k(\cdot\,|\pi')$. $c_k(\cdot\,|\pi)$ and hence $c_k(\cdot\,|\pi')$ are differentiable (Assumption *1 (iii)*), increasing in q_k, concave in ω_i, and linearly homogeneous in ω_i. Differentiating equation (22),

$$\frac{\partial c_k(\omega_i, q_k|\pi')}{\partial\omega_i} = \frac{\partial c_k(\omega_i, q_k|\pi)}{\partial\omega_i} + (d'_k - d_k(\pi))\, q_k(\pi). \tag{23}$$

Since $c_k(\cdot\,|\pi)$ is increasing in ω_i, $\partial c_k(\cdot\,|\pi)/\partial\omega_i$ is bounded away from zero on the compact set R. Since $||d'_k - d_k(\pi)|| < \delta$ one can choose δ such that the right-hand side of equation (23) is positive. Thus, $c_k(\cdot\,|\pi')$ is increasing in ω_i. From Diewert (1982, theorem 2 and corollary 1.1), this establishes that $c_k(\cdot|\pi')$ is a cost function on R.[22] Equation (20) follows from equations (9) and (23) evaluated at $(\omega_i(\pi), q_k(\pi))$. Further, by hypothesis, $\omega_i(\pi)(d'_k - d_k(\pi)) = 0$. Hence, equation (21) follows from equation (22) with $\omega_i = \omega_i(\pi)$. ■

We define the set of factor demand perturbations implied by Π:

$$\mathcal{P}(\pi, \varepsilon) \equiv \{D' : D'_i Q_i(\pi) = V_i \;\forall\, i,\; D' > 0, \quad \omega_i(\pi) D'_i = \omega_i(\pi) D_i(\pi) \;\forall\, i,\; ||D' - D(\pi)|| < \varepsilon\} \tag{24}$$

where $D' \equiv (D'_1, \ldots, D'_N)$ and $D' > 0$ means that D' is non-negative with at least one positive element.

Let K_i be the number of factors available in country i (i.e., non-zero elements of V_i) and let G_i be the number of goods produced in country i. Lemma 3 (*ii*) implies that $\mathcal{P}(\pi, \varepsilon)$ is non-empty whenever there is at least one country positively endowed with at least two factors ($K_i > 1$) and producing at least two goods ($G_i > 1$).

[22] Diewert lists four other regularity conditions on c_k that are easily verified. One can allow for $c_k(\cdot|\pi')$ to be non-decreasing and also deal with $q_k = 0$ (Diewert's II(*ii*)) by allowing d'_k to be a function on R rather than a constant.

Lemma 3. *Assume Assumption 1. (i)* $\mathcal{P}(\pi,\varepsilon) = D(\Pi(\pi,\varepsilon))$. *(ii)* $\mathcal{P}(\pi,\varepsilon)$ *is a convex set with* $\dim(\mathcal{P}) \geq \Sigma_{i=1}^{N}(K_i - 1)(G_i - 1)$.

Proof. Recall that $\mathcal{K}(g,i)$ is the set of firms that produce good g in country i. Let Q_{gi} be the gth element of Q_i and let D_{gi} be the gth column of D_i. Industry g output Q_{gi} is the sum of firm-level outputs q_k: $Q_{gi} = \sum_{k\in\mathcal{K}(g,i)} q_k$. Industry factor demands $D_{gi}Q_{gi}$ are the sum of firm-level factor demands $d_k q_k$:

$$D_{gi}Q_{gi} = \sum\nolimits_{k\in\mathcal{K}(g,i)} d_k q_k. \tag{25}$$

For part (i) consider a $D' \in \mathcal{P}$. For each column D'_{gi} of D' it is tedious but straightforward to verify the following. There exists a $d' \equiv \{d'_k\}_{k\in\mathcal{K}(g,i)}$ satisfying the conditions of lemma 2 and

$$\sum\nolimits_{k\in\mathcal{K}(g,i)} d'_k q_k(\pi) = D'_{gi}Q_{gi}(\pi) \ \ \forall\ k,\ g \text{ and } i. \tag{26}$$

This equation states that the industry-level D'_{gi} are derivable from the firm-level d'_k.[23]

An outcome is a list O of all the endogenous variables. We next show that outcome $O' \equiv (d', D', \omega(\pi), E(\pi))$ satisfies equations (4), (9) and (25) when the equations are evaluated at $(\pi', E(\pi))$ i.e., O' is consistent with competitive factor market clearing. Recall that E is a list that includes p_k, q_k as well as Q_i and its gth element Q_{gi}. Equation (4) follows from $D' \in \mathcal{P}$ and the definition of $\mathcal{P}$ i.e., competitive factor demand $D'_i Q_i(\pi)$ equals exogenous supply V_i. Equation (9) follows from equation (20) evaluated at $E(\pi)$ i.e., d'_k is cost minimizing. Equation (25) follows from equation (26).

This result together with equation (21) imply that $D' = D(\pi')$. From the definitions of $\mathcal{P}$ and Π, this establishes that if $D' \in \mathcal{P}$ then there is a $\pi' \in \Pi(\pi,\varepsilon)$ such that $D' = D(\pi')$. Restated, $\mathcal{P} \subseteq D(\pi)$. The definitions of Π and $\mathcal{P}$ imply that if $\pi' \in \Pi$ then $D(\pi') \in \mathcal{P}$ i.e., $D(\Pi) \subseteq \mathcal{P}$. This establishes $\mathcal{P} = D(\Pi)$ and part (i) of lemma 3.

For part (ii) consider the equation system $D'_i Q_i = V_i$ and $\omega_i D'_i = \omega_i D_i\ \forall\ i$. The unknowns $\{D'_i\}_{i=1}^{N}$ have $\Sigma_i K_i G_i$ elements that need not be zero. As shown in the proof of lemma 5 below, this equation system has at least one linearly dependent equation per country or at most $\Sigma_i(K_i + G_i - 1)$ linearly independent equations. Since the solution set is non-empty ($D'_i = D_i\ \forall i$ is a solution), the solution set dimension is at least $\Sigma_i K_i G_i - \Sigma_i(K_i + G_i - 1) = \Sigma_i(K_i - 1)(G_i - 1)$. To guarantee that $\mathcal{P}$ and Π are not degenerate, we assume that there is a country that has at least two factors and produces at least two goods. ■

[23] The case where $\mathcal{K}(g,i)$ has only one firm and the case where every firm in $\mathcal{K}(g,i)$ has a $d_k(\pi)$ with only one positive element must be treated separately from the general case because of the degeneracy of one or more of the conditions $\omega_i(\pi)\, d'_k = \omega_i(\pi)\, d_k(\pi)$, $\omega_i(\pi)\, D'_{gi} = \omega_i(\pi)\, D_{gi}(\pi)$, and $\sum_{k\in\mathcal{K}(g,i)} d_k(\pi)\, q_k(\pi) = D_{gi}(\pi)\, Q_{gi}(\pi)$.

A.4. Proof of Theorem 3

Define

$$C_i \equiv \begin{bmatrix} C_{i1} \\ \vdots \\ C_{iN} \end{bmatrix}, C_w \equiv \begin{bmatrix} C_{w1} \\ \vdots \\ C_{wN} \end{bmatrix} = \Sigma_i C_i.$$

Then $\Sigma_j A_j (C_{ij} - s_i C_{wj})$ of lemma 1 can be written more compactly as $A(C_i - s_i C_w)$. By lemma 1 and $A \equiv D(I-B)^{-1}$, if the Vanek prediction holds for D' then

$$D'(I-B)^{-1}(C_i - s_i C_w) = 0 \;\forall i. \tag{27}$$

In the next we show that if the Vanek prediction holds for all D' in $\mathcal{P}(\pi, \varepsilon)$, then equation (27) implies $C_{ij} = s_i C_{wj}$ for all i and j.

The definitions of $\Pi(\pi, \varepsilon)$ and E imply that $\omega_i, Q_i, D_i, B, s_i$, and $C_{ij} - s_i C_{wj}$ are constant on Π. We therefore treat them as fixed parameters. $D' = (D'_1, ..., D'_N) \in \mathcal{P}(\pi, \varepsilon)$ implies

$$\begin{aligned} D'_i Q_i &= V_i \;\forall i & (28)\\ \omega_i D'_i &= \omega_i D_i \;\forall i. & (29) \end{aligned}$$

Let $\mathcal{L}_{ij}$ be a $G \times G$ matrix that satisfies

$$(I-B)^{-1} = \begin{pmatrix} \mathcal{L}_{11} & \cdots & \mathcal{L}_{1N} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{N1} & \cdots & \mathcal{L}_{NN} \end{pmatrix}. \tag{30}$$

From equation (27),

$$\Sigma_j \left[D'_j \Sigma_{j'} \mathcal{L}_{jj'} (C_{ij'} - s_i C_{wj'}) \right] = 0 \;\forall i. \tag{31}$$

Equations (28), (29) and (31) are all linear in the $K \times G$ matrices D'_i. Let $x' \equiv vec(D')$ be an $KGN \times 1$ vector formed from the elements of D'. Then equations (28), (29) and (31) can be represented in terms of x':

$$\begin{aligned} \Psi x' &= \psi \quad \text{where } \Psi \text{ is } KN \times KGN \text{ and } \psi \text{ is } KN \times 1, & (32)\\ \Phi x' &= \phi \quad \text{where } \Phi \text{ is } GN \times KGN \text{ and } \phi \text{ is } GN \times 1, & (33)\\ \Gamma x' &= 0_{KN} \quad \text{where } \Gamma \text{ is } KN \times KGN, & (34) \end{aligned}$$

and 0_{KN} is an $KN \times 1$ vector of zeros. Define

$$\mathcal{M}_\Gamma \equiv \begin{bmatrix} \Psi \\ \Phi \\ \Gamma \end{bmatrix}, m_\Gamma \equiv \begin{bmatrix} \psi \\ \phi \\ 0_{KN} \end{bmatrix}, \mathcal{M} \equiv \begin{bmatrix} \Psi \\ \Phi \end{bmatrix}, \text{and } m \equiv \begin{bmatrix} \psi \\ \phi \end{bmatrix}$$

so that equations (32)-(33) become $\mathcal{M}x' = m$ and equations (32)-(34) become $\mathcal{M}_\Gamma x' = m_\Gamma$.

Lemma 4. $\{F_i(\pi') = V_i - s_i V_w\}_{i=1}^N$ *for all* π' *in* $\Pi(\pi,\varepsilon) \Longrightarrow rank(\mathcal{M}_\Gamma) \le KN + GN - N$.

Proof. By lemma 3 (i), $\{F_i(\pi') = V_i - s_i V_w\}_{i=1}^N$ for all π' in $\Pi(\pi,\varepsilon)$ if and only if $\{F_i(D') = V_i - s_i V_w\}_{i=1}^N$ for all D' in $\mathcal{P}(\pi,\varepsilon)$ where $F_i(\cdot)$ indicates how F_i depends on D' via equation (7) and $A \equiv D(I-B)^{-1}$. If D' is in $\mathcal{P}$ then D' solves equations (28)-(29) or equivalently, $x' \equiv vec(D')$ solves $\mathcal{M}x' = m$. If in addition $\{F_i(D') = V_i - s_i V_w\}_{i=1}^N$ then by lemma 1, D' solves equation (31) and x' solves $\Gamma x' = 0_{KN}$. Thus $\{F_i(D') = V_i - s_i V_w\}_{i=1}^N$ for all D' in $\mathcal{P}$ implies that all solutions x' of $\mathcal{M}x' = m$ also solve $\mathcal{M}_\Gamma x' = m_\Gamma$. It follows that $rank(\mathcal{M}_\Gamma) = rank(\mathcal{M})$.

Consider the equations underlying $\mathcal{M}$. Equation (29) implies $\omega_i D_i' Q_i = \omega_i D_i Q_i\ \forall i$. From equation (4), $\omega_i D_i' Q_i = \omega_i V_i$ for $i = 1,...,N$. But this is also implied by pre-multiplying equation (28) by ω_i. Hence there are at least N linearly dependent rows in $\mathcal{M}$. Since $\mathcal{M}$ is $(KN+GN) \times KGN$, $rank(\mathcal{M}) \le \min(KN+GN-N, KGN) \le KN+GN-N$. Hence $rank(\mathcal{M}_\Gamma) \le KN+GN-N$. ■

Let Q_{gi}, ω_{fi}, and D'_{fgi} be typical elements of Q_i, ω_i, and D_i', respectively. Let $\mathcal{L}_{gij}$ be the gth row of the $G \times G$ matrix $\mathcal{L}_{ij}$. Define

$$\gamma_{gij} \equiv \Sigma_{j'} \mathcal{L}_{gjj'} (C_{ij'} - s_i C_{wj'}) . \tag{35}$$

Then equation (31) can be rewritten as $\Sigma_j \Sigma_g D'_{fgj} \gamma_{gij} = 0\ \forall f$ and i. Equivalently, $\Sigma_i \Sigma_g D'_{fgi} \gamma_{gji} = 0\ \forall f$ and j. Since each country j produces at least one good, for each j there is a good $h(j)$ such that $Q_{h(j),j} > 0$.

Lemma 5. $rank(\mathcal{M}_\Gamma) \le KN + GN - N \Longrightarrow$
$\Sigma_{j'} \mathcal{L}_{gjj'} (C_{ij'} - s_i C_{wj'}) - \left[\Sigma_{j'} \mathcal{L}_{h(j),j,j'} (C_{ij'} - s_i C_{wj'})\right] Q_{gj} / Q_{h(j),j} = 0\ \forall g$, *$i$ and j.*

Proof. Since $rank(\mathcal{M}_\Gamma) \le KN+GN-N$, every $KN+GN-N+1$ square sub-matrix of $\mathcal{M}_\Gamma$ has a zero determinant. Figure 4 illustrates one such sub-matrix that is particularly useful. It is partitioned into 9 blocks. The three top blocks correspond to the KN equations grouped in equation (28) or (32) which in non-matrix form is $\Sigma_g D'_{fgi} Q_{gi} = V_{fi}$ $\forall f$ and i. Thus, the element in row (f', i') and column (f, g, i) is the coefficient on D'_{fgi} in the (f', i')-th equation. If $(f', i') \ne (f, i)$ the coefficient is zero. The three blocks in the middle row correspond to a $(G-1)N$ subset of the GN equations grouped in equation (29) or (33) which in non-matrix form is $\Sigma_f \omega_{fi} D'_{fgi} = \Sigma_f \omega_{fi} D_{fgi}\ \forall g$ and i. Thus, the element in row (g', i') and column (f, g, i) is the coefficient on D'_{fgi} in the (g', i')-th equation. If $(g', i') \ne (g, i)$ the coefficient is zero. The three blocks in the bottom row correspond to one of the KN equations grouped in equation (31) or (34) which in non-matrix form is $\Sigma_i \Sigma_g D'_{fgi} \gamma_{gji} = 0\ \forall f$ and j. In the figure, $f > 1$.

Using obvious notation, the partitioned matrix in figure 4 can be rewritten as

$$\mathcal{H}(f, j; f, g, i) \equiv \begin{bmatrix} \Psi_1 & \Psi_2 & \Psi_{fgi} \\ 0_1 & \Phi_2 & \Phi_{fgi} \\ \Gamma_{fj} & 0_2 & \gamma_{gji} \end{bmatrix} \text{ for } f > 1.$$

Note from figure 4 that g only appears in conjunction with country indices, that is, in (g, i) pairs. Thus without loss of generality, let each country have its own goods index. Choose these so that $h(j) = 1\ \forall j$. ($h(j)$ was defined prior to lemma 5.) Then $Q_{1j} > 0\ \forall j$ and Ψ_1 is invertible. By Assumption 1 (*iv*), Φ_2 is invertible. Since $\mathcal{H}$ is an $KN + GN - N + 1$ sub-matrix of $\mathcal{M}_\Gamma$, $\mathcal{H}$ has a zero determinant. Applying partitioned matrix rules for determinants and inverses (Hendry, 1995, equations A1.9-A1.10), for $f > 1$

$$\begin{aligned} |\mathcal{H}(f,j;f,g,i)| &= |\Psi_1||\Phi_2| \left(\gamma_{gji} - \begin{bmatrix} \Gamma_{fj} & 0_2 \end{bmatrix} \begin{bmatrix} \Psi_1 & \Psi_2 \\ 0_1 & \Phi_2 \end{bmatrix}^{-1} \begin{bmatrix} \Psi_{fgi} \\ \Phi_{fgi} \end{bmatrix} \right) \\ &= |\Psi_1||\Phi_2| \left(\gamma_{gji} - \begin{bmatrix} \Gamma_{fj} & 0_2 \end{bmatrix} \begin{bmatrix} \Psi_1^{-1} & -\Psi_1^{-1}\Psi_2\Phi_2^{-1} \\ 0_1\Psi_1^{-1} & \Phi_2^{-1} \end{bmatrix} \begin{bmatrix} \Psi_{fgi} \\ \Phi_{fgi} \end{bmatrix} \right) \\ &= |\Psi_1||\Phi_2| \left(\gamma_{gji} - \Gamma_{fj}\Psi_1^{-1}\Psi_{fgi} + \Gamma_{fj}\Psi_1^{-1}\Psi_2\Phi_2^{-1}\Phi_{fgi} \right) \\ &= 0. \end{aligned} \tag{36}$$

From figure 4, $|\Psi_1| > 0$, $|\Phi_2| > 0$, $\Gamma_{fj}\Psi_1^{-1}\Psi_{fgi} = \gamma_{1ji}Q_{gi}/Q_{1i}$, and $\Gamma_{fj}\Psi_1^{-1}\Psi_2\Phi_2^{-1}\Phi_{fgi} = 0$ for $f > 1$. Hence equation (36) implies $\gamma_{gji} - \gamma_{1ji}Q_{gi}/Q_{1i} = 0\ \forall g,\ i,\ j$. Switching i and j indices and recalling that $h(j) = 1\ \forall j$, $\gamma_{gij} - \gamma_{h(j),i,j}Q_{gj}/Q_{h(j),j} = 0$. From equation (35), it follows that

$$\Sigma_{j'}\mathcal{L}_{gjj'}(C_{ij'} - s_iC_{wj'}) - \left[\Sigma_{j'}\mathcal{L}_{h(j),j,j'}(C_{ij'} - s_iC_{wj'})\right] Q_{gj}/Q_{h(j),j} = 0\ \forall g, i \text{ and } j. \tag{37}$$

■

Lemma 6. *If equation (37) holds then* $C_{ij} = s_iC_{wj}\ \forall i$ *and* j.

Proof. We prove the lemma separately for each (i,j). Fix i and j. Then $h(j)$ is fixed so that without loss of generality let $h(j) = 1$.

Let

$$\Upsilon \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -Q_{2j}/Q_{1j} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -Q_{Gj}/Q_{1j} & 0 & \cdots & 1 \end{bmatrix} \text{ and } z \equiv \begin{bmatrix} \sum_{j'}\mathcal{L}_{1jj'}(C_{ij'} - s_iC_{wj'}) \\ \sum_{j'}\mathcal{L}_{2jj'}(C_{ij'} - s_iC_{wj'}) \\ \vdots \\ \sum_{j'}\mathcal{L}_{Gjj'}(C_{ij'} - s_iC_{wj'}) \end{bmatrix}.$$

Stacking equation (37) yields $\Upsilon z = 0_G$, where 0_G is a $G \times 1$ vector of zeros.

The solution set of $\Upsilon z = 0_G$ is $\{z : z = \alpha Q_j, \alpha \in \mathcal{R}\}$. Hence from the definition of z, $\Sigma_{j'}\mathcal{L}_{gjj'}(C_{ij'} - s_iC_{wj'}) = \alpha Q_{gj}\ \forall g, i$ and j. Equivalently, $\Sigma_{j'}\mathcal{L}_{jj'}(C_{ij'} - s_iC_{wj'}) = \alpha Q_j$ $\forall i$ and j. Summing this over i yields $\Sigma_{j'}[\mathcal{L}_{jj'}\Sigma_i(C_{ij'} - s_iC_{wj'})] = \alpha NQ_j\ \forall j$. Because $\Sigma_i(C_{ij'} - s_iC_{wj'}) = 0$ and $Q_{1j} > 0\ \forall j$, it follows that $\alpha = 0$. Thus, for all i and j, $\Sigma_{j'}[\mathcal{L}_{jj'}(C_{ij'} - s_iC_{wj'})] = 0$. Stacking this and using equation (30) yield

$$(I - B)^{-1} \begin{bmatrix} C_{11} - s_1C_{w1} & \cdots & C_{N1} - s_NC_{w1} \\ \vdots & \ddots & \vdots \\ C_{1N} - s_1C_{wN} & \cdots & C_{NN} - s_NC_{wN} \end{bmatrix} = 0.$$

Since $(I - B)^{-1}$ exists, it follows that $C_{ij} = s_iC_{wj}\ \forall i$ and j. ■

B. Appendix: Data

Data on endowments V_i and direct factor usage by industry D_i are from various sources. Capital stock is constructed as follows. We use the latest capital stock data from the Penn World Table 5.6 (PWT 5.6) and update the data to 1997 by applying Leamer's (1984) double declining balance method to investment. The real gross domestic investment series come from the Penn World Table 6.1 (PWT 6.1). Let $V_{Ki}(t_0)$ be capital stock in country i in year t_0 (the latest year available) from PWT 5.6 (in 1985 international prices).[24] Let $I_i(t)$ be the investment series for year t from PWT 6.1 (in 1996 international prices).[25] Let $PI^{PWT5.6}(t_0)$ and $PI^{PWT6.1}(t_0)$ be the price level of investment for year t_0 from PWT 5.6 and PWT 6.1, respectively. Assuming a typical asset life of 15 years, the depreciation rate is $\delta = 13.3\%$. Then country i's capital stock V_{Ki} at the beginning of 1997 (in 1996 international prices) is defined as

$$V_{Ki} \equiv (1-\delta)^{1996-t_0} V_{Ki}(t_0)\, PI^{PWT6.1}(t_0)/PI^{PWT5.6}(t_0) + \Sigma_{t=t_0+1}^{1996} (1-\delta)^{1996-t} I_i(t).$$

Direct usage of capital by industry is generated by assuming that industry capital stocks are proportional to industry payments to capital. This will be the case in steady state under the assumption of constant depreciation rates. Data on capital payments are from the GTAP (version 5) input-output accounts.

Turning next to labour, let L_{gi} and P_{gi} be labour employment and payroll of industry g in country i. Data are from the OECD STAN database for OECD countries, the UNIDO data base for manufacturing in non-OECD countries and from the ILO for non-manufacturing in non-OECD countries. The endowment of labour, $V_{Li} \equiv \Sigma_g L_{gi}$, is scaled so that it sums to the PWT 6.1 workforce totals in 1997. Direct usage of labour by industry (D_{fgi}) is calculated as L_{gi}/Q_{gi} where Q_{gi} is output of industry g in country i. Q_{gi} is from GTAP.

The endowment of human capital is defined as the number of grade-12 equivalent workers in the economy. It was generated as follows. Let $\omega_i(e)$ and $L_i(e)$ be the annual earnings and national employment of country i workers with e years of schooling. $L_i(e)$ is from Barro and Lee (2000). The Barro-Lee dataset provides educational attainment at 7 levels: no education ($e = 0$), primary entered ($e = 3$), primary completed ($e = 6$), secondary entered ($e = 9$), secondary completed ($e = 12$), post-secondary entered ($e = 13.5$), and post-secondary completed ($e = 16$). Let ρ_i be the returns to schooling in country i from Psacharopoulos and Patrinos (2002, table A2, the most recent year). We assume that (*1*) national payroll $P_i = \Sigma_g P_{gi}$ is the sum of the earnings of each education class: $P_i = \Sigma_e \omega_i(e) L_i(e)$; and that (*2*) wages are generated by a Mincerian equation $\omega_i(e) \equiv (1+\rho_i)^e \omega_i(0)$ where $\omega_i(0)$ is the wage rate of unskilled workers. The employment of human capital is defined as the number of high-school graduates that could be hired

[24] $V_{Ki}(t_0) \equiv KAPW_i(t_0) \times RGDPCH_i(t_0) \times POP_i(t_0)/RGDPW_i(t_0)$ where $KAPW_i$ is country i's capital per worker, $RGDPW_i$ is i's real GDP per worker using the chain index, $RGDPCH_i$ is i's real GDP per capita using the chain index, and POP is i's population.

[25] $I_i(t) \equiv RGDPL_i(t) \times KI_i(t) \times POP_i(t)$ where $RGDPL_i$ is country i's real GDP per capita using the Laspeyres index, KI_i is i's share of real gross domestic investment in $RGDPL_i$, and POP is i's population.

for an amount P_{gi}: $H_{gi} \equiv P_{gi}/\omega_i(12)$. It follows that the employment of human capital can be calculated as[26]

$$H_{gi} = (P_{gi}/P_i)\,\Sigma_e(1+\rho_i)^{e-12}L_i(e)$$

where $L_i\,(e)$ is scaled so that $\Sigma_e L_i\,(e)$ is equal to the PWT 6.1 workforce totals for country i in 1997.

Finally, the endowment of human capital is simply $V_{Hi} \equiv \Sigma_g H_{gi}$. Direct usage of human capital by industry is calculated as H_{gi}/Q_{gi}.

Data on input-output tables $\overline{B}_i$ and trade flows X_i and M_{ij} are from GTAP (version 5). The B matrix is imputed using equation (13) combined with equations (10) and (11).

Consumption shares s_i are defined as $(GDP_i - TB_i)\,/\Sigma_j GDP_j$ where GDP_i is country i's real GDP in 1997 and TB_i is i's trade balance. Data on GDP_i come from the PWT 6.1.[27]

In order to match the classification of industries in D with those in B we aggregated industries up to 24 ISIC (rev. 2) industries. The industries are: 110-130 (Agriculture, hunting, forestry and fishing); 200 (Mining and quarrying); 311+312 (Food); 313+314 (Beverages, Tobacco); 321 (Textiles); 322 (Apparel); 323+324 (Leather products, Footwear); 331+332 (Wood products, Furniture); 341+342 (Paper products, Printing and publishing); 353+354 (Petroleum refineries, Misc. petro and coal products); 351+352+355+356 (Industrial chemicals, Other chemicals, Rubber products, Plastic products); 361+362+369 (Pottery, Glass, Other non-metallic mineral products); 371 (Iron and steel); 372 (Nonferrous metals); 381 (Fabricated metal products); 384 (Transport equipment); 382+383+385 (Non-electrical machinery, Electric machinery, Instruments); 390 (Misc. manufacturing); 400 (Electricity, gas, and water); 500 (Construction); 600 (Wholesale and retail trade and restaurants and hotels); 700 (Transport, storage and communication); 800 (Financing, insurance, real estate and business services); and 900 (Community , social and personal services). Davis and Weinstein (2001) have 35 ISIC (rev. 2) industries. Our use of data for developing countries has prevented us from being quite as disaggregated as them.

[26]Plugging $w_i(e) \equiv (1+\rho_i)^e w_i(0)$ into $P_i = \Sigma_e w_i(e)L_i(e)$ yields $w_i\,(0) = P_i/\Sigma_e(1+\rho_i)^e L_i(e)$. Thus, $w_i\,(12) = P_i/[\Sigma_e(1+\rho_i)^{e-12}L_i(e)]$.

[27]$GDP_i \equiv RGDPC_i \times POP_i$ where $RGDPC_i$ is country i's real GDP per capita using the chain index (in 1996 international price) and POP_i is i's population.

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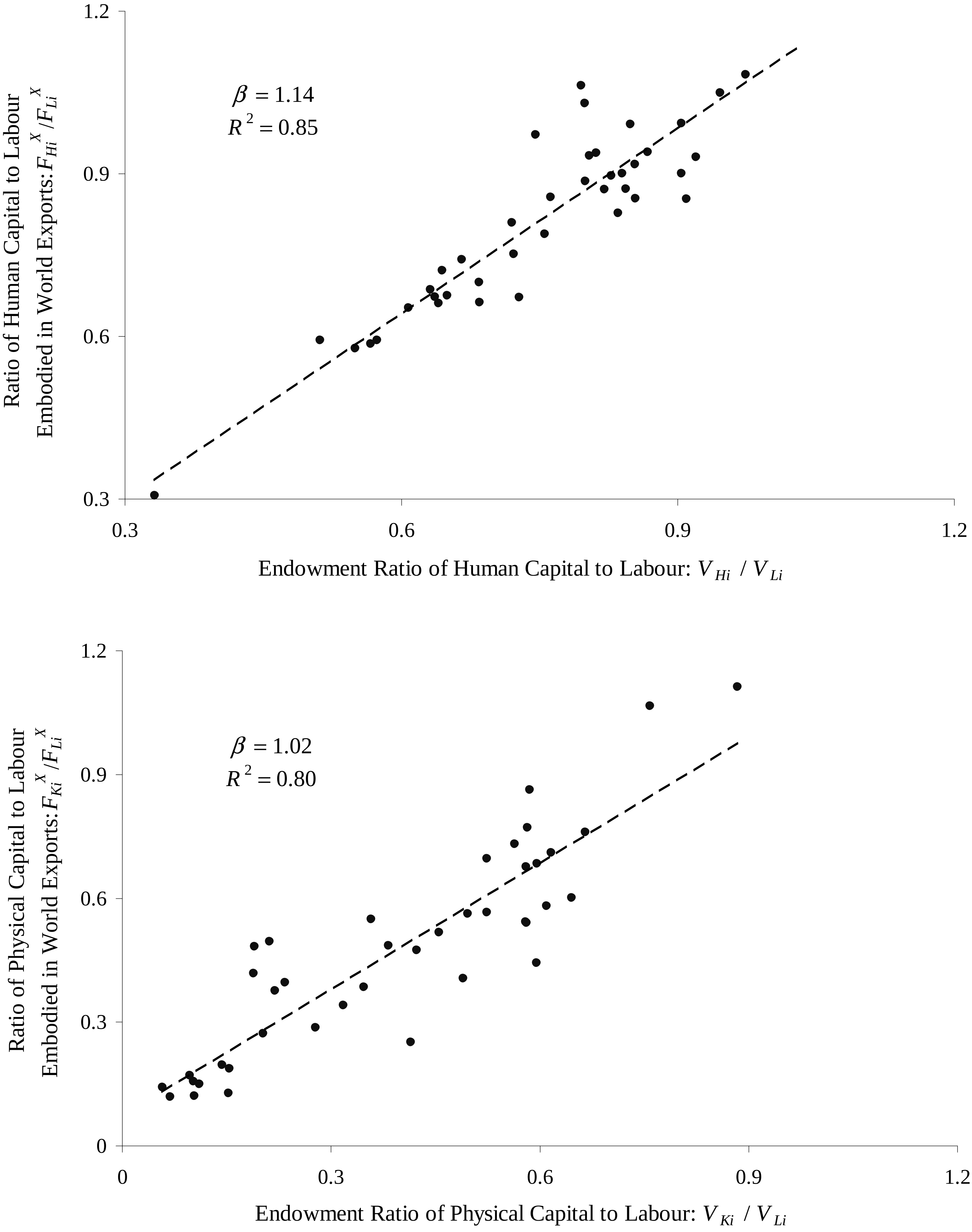


Notes: See section 8.3 and equation (16) for details.

Figure 1. Factor Content of World Exports and Factor Abundance

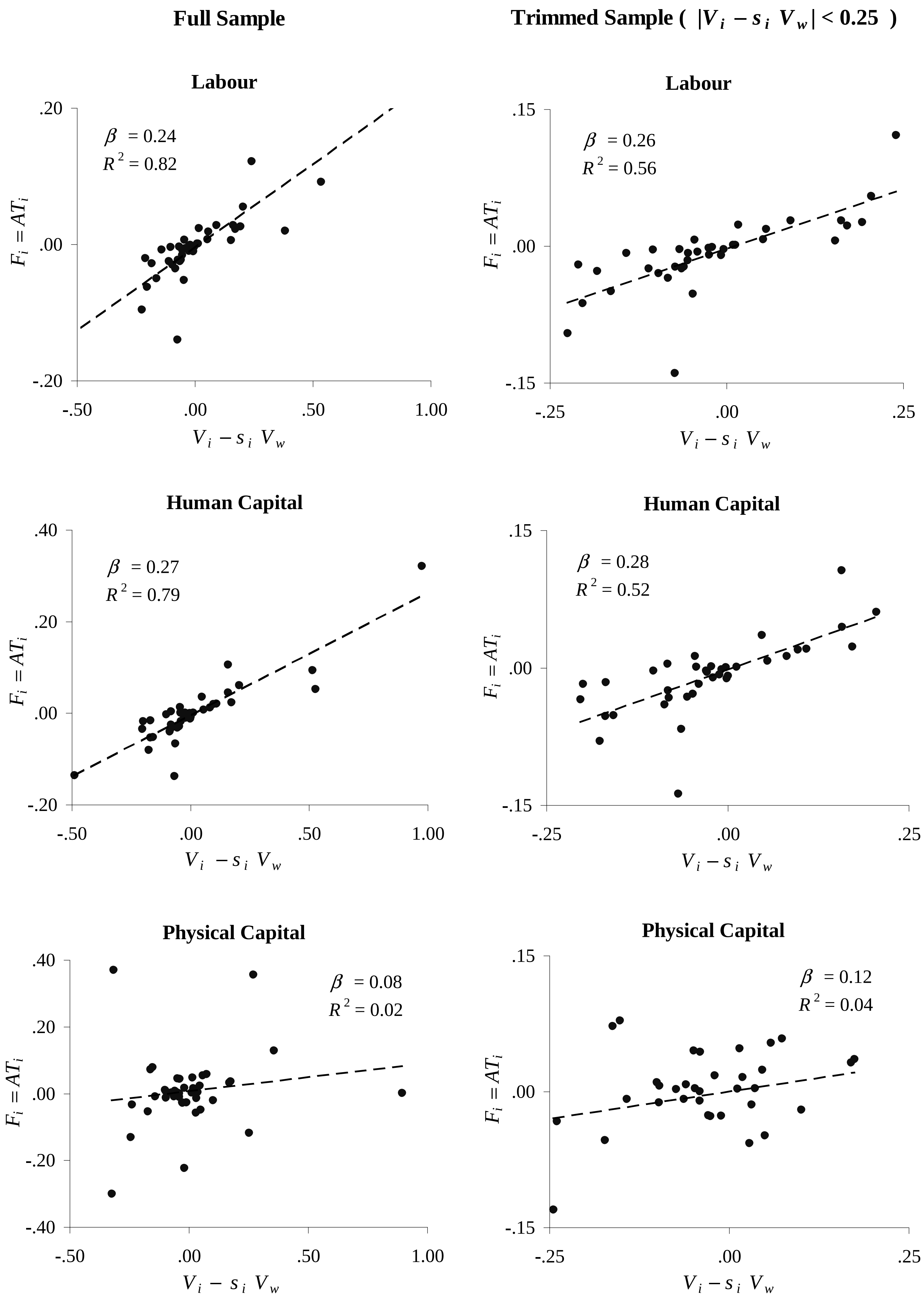
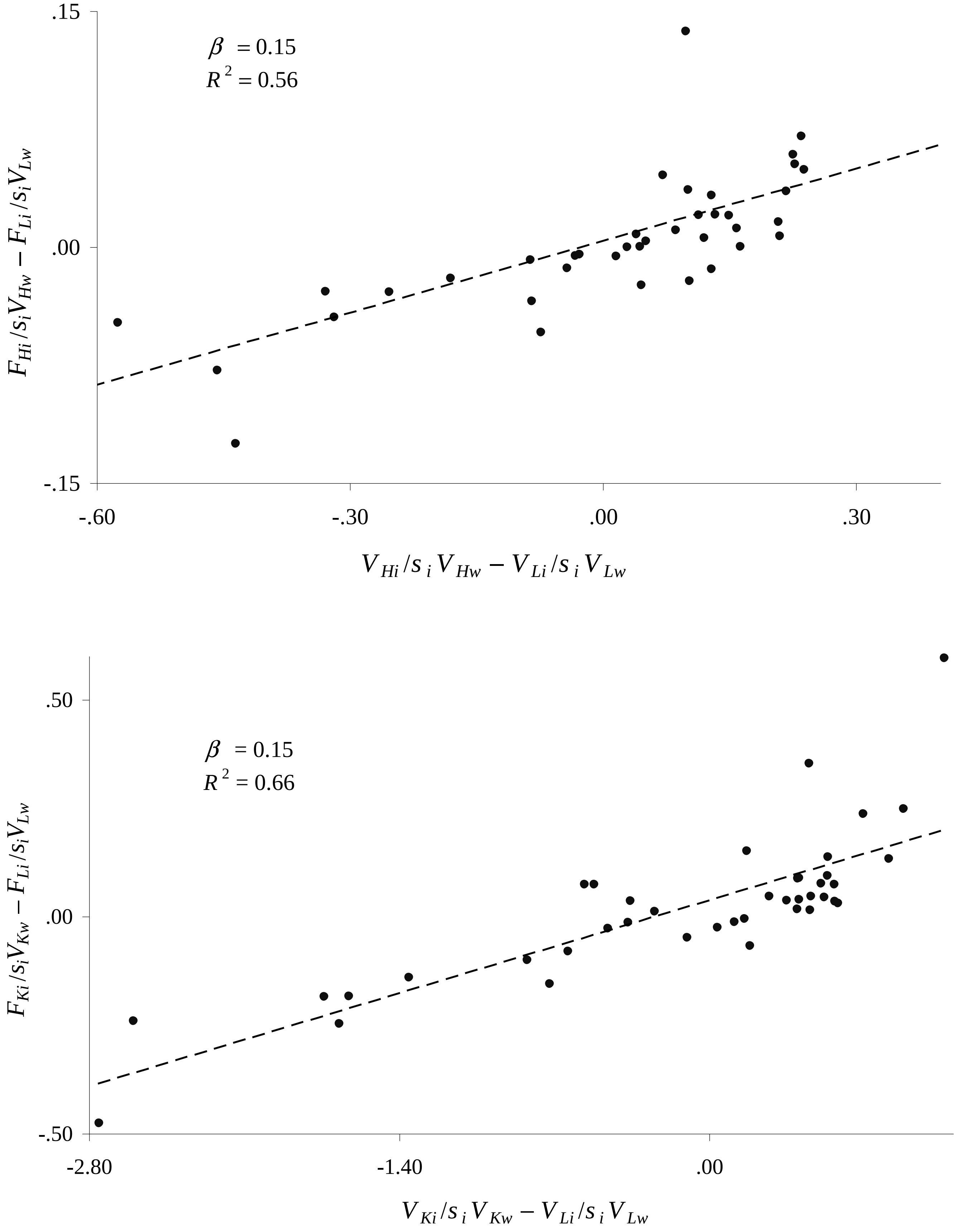


Figure 2. The Vanek Prediction for Labour, Human Capital, and Physical Capital



Notes: See section 8.2 and equations (14) and (15) for details.

Figure 3. Relative Factor Content of Trade and Relative Factor Abundance

Figure 4. The $\mathcal{H}$ Matrix

(f, g, i) indices for coefficients on D'_{fgi}

		$(1,1,1) \cdots (1,1,i) \cdots (1,1,N)$	$(f,1,1) \cdots (f,1,i) \cdots (f,1,N)$	$(K,1,1) \cdots (K,1,i) \cdots (K,1,N)$	$(1,2,1) \cdots (1,2,i) \cdots (1,2,N)$	$(1,g,1) \cdots (1,g,i) \cdots (1,g,N)$	$(1,G,1) \cdots (1,G,i) \cdots (1,G,N)$	(f,g,i)
Eqn. (28)	$(1, 1)$	$Q_{11} \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$Q_{21} \cdots 0 \cdots 0$	$Q_{g1} \cdots 0 \cdots 0$	$Q_{G1} \cdots 0 \cdots 0$	0
$(f, i) =$	$(1, i)$	$0 \cdots Q_{1i} \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$0 \cdots Q_{2i} \cdots 0$	$\cdots 0 \cdots Q_{gi} \cdots 0$	$\cdots 0 \cdots Q_{Gi} \cdots 0$	0
	$(1, N)$	$0 \cdots 0 \cdots Q_{1N}$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots Q_{2N}$	$0 \cdots 0 \cdots Q_{gN}$	$0 \cdots 0 \cdots Q_{GN}$	0
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(f, 1)$	$0 \cdots 0 \cdots 0$	$Q_{11} \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	0
	(f, i)	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots Q_{1i} \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	Q_{gi}
	(f, N)	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots Q_{1N}$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	0
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(K, 1)$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$Q_{11} \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	0
	(K, i)	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots Q_{1i} \cdots 0$	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	0
	(K, N)	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots Q_{1N}$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	0
		− − − − −	− − − − −	− − − − −	− − − − −	− − − − −	− − − − −	−
Eqn. (29)	$(2, 1)$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$\omega_{11} \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	0
$(g, i) =$	$(2, i)$	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$0 \cdots \omega_{1i} \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	0
	$(2, N)$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots \omega_{1N}$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	0
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(g, 1)$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$\omega_{11} \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	0
	(g, i)	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots \omega_{1i} \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	ω_{fi}
	(g, N)	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots \omega_{1N}$	$0 \cdots 0 \cdots 0$	0
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$(G, 1)$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$\omega_{11} \cdots 0 \cdots 0$	0
	(G, i)	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$\cdots 0 \cdots 0 \cdots 0$	$\cdots 0 \; \omega_{1i} \cdots 0$	0
	(G, N)	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots \omega_{1N}$	0
		− − − − −	− − − − −	− − − − −	− − − − −	− − − − −	− − − − −	−
Eqn. (31)	(f, j)	$0 \cdots 0 \cdots 0$	$\gamma_{1j1} \cdots \gamma_{1ji} \cdots \gamma_{1jN}$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	$0 \cdots 0 \cdots 0$	γ_{gji}