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ABSTRACT

When equity prices are determined as the discounted sum of current and expected future dividends, Shiller (1981) and LeRoy and Porter (1981) derived a relationship between the variance of the price of equities, p(t), and the variance of the ex post realized discounted sum of current and future dividends: p*(t): Var(p*(t))>= Var(p(t)). The literature has long since recognized that this variance bound is valid only when dividends follow a stationary process. Others, notably West (1988), derive variance bounds that apply when dividends are nonstationary. West shows that the variance in innovations in p(t) must be less than the variance of innovations in a forecast of the discounted sum of current and future dividends constructed by the econometrician, p^(t). Here we derive a new variance bound when dividends are stationary or have a unit root, that sheds light on the discussion in the 1980s of the Shiller variance bound: Var(p(t)-p(t-1)) >= Var(p*(t)-p*(t-1))! We also derive a variance bound related to the West bound: Var(p^(t)-p^(t-1)) >= Var(p(t)-p(t-1)).

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Shiller (1981) and LeRoy and Porter (1981) proposed a test for "excess volatility" of stock prices, when these prices are determined as a discounted sum of current and expected future dividends: the variance of the equity price, p_t, should be less than the variance of the ex-post realized discounted sum of dividends, p_t^*. Subsequently, Marsh and Merton (1986), Kleidon (1986), and Durlauf and Phillips (1988) criticized these tests, arguing that the test requires that the stochastic process for dividends be stationary. Here we demonstrate that if dividends are stationary or have a unit root, $Var(p_t - p_{t-1}) \geq Var(p_t^* - p_{t-1}^*)$. That is, expressing prices in first-differences, the Shiller-LeRoy-Porter inequality is reversed.

In a sense, the profession long ago resolved how to implement variance bounds when dividends are nonstationary. Mankiw, Romer, and Shapiro (1985), and West (1988) introduce volatility bounds that are valid when there is a unit root in the dividend process. The West test involves a forecast of the discounted sum of current and future dividends constructed by the econometrician (a forecast based on a smaller information set than the market's), $\hat{p}_t$. Under the assumption that the econometrician has less information than markets, West shows that the variance in innovations in p_t should be less than the variance in innovations in $\hat{p}_t$. Here we also derive a variance bound that is similar to that of West (1988):

$Var(\hat{p}_t - \hat{p}_{t-1}) \geq Var(p_t - p_{t-1})$.

It has been noted (by Frankel and Stock (1987) and Durlauf and Phillips (1988)) that the variance bound is a weaker restriction than imposed by the standard Euler equation. But it has been argued that variance bounds are nonetheless interesting because they provide some insight into why the Euler condition might fail. For example, Campbell, Lo, and MacKinlay (1997, p.

277) state, "The justification for using a variance-bounds test is not increased power; rather it is that a variance-bounds test helps one to describe the way in which the null hypothesis fails."

In that spirit, there may be some value in re-examining the Shiller-LeRoy-Porter variance bounds test. Shiller (1991) in particular argues for the intuitive appeal of his bound, $Var(p_t^*) \geq Var(p_t)$, by asking readers to examine graphs of p_t^* and p_t. As Shiller says (p. 421), "One is struck by the smoothness and stability of the ex post rational price series p_t^* when compared with the actual price series." Flavin (1983), and especially Kleidon (1986), argued that this interpretation of the graphs was inconclusive. Just because p_t^* appears smoother does not mean it has lower variance, when the dividend process is very persistent. The subsequent exchange between Shiller (1988) and Kleidon (1988) demonstrates that the issue was not fully resolved. The result in this paper formalizes the observation that volatility of p_t compared to p_t^* does not imply the present-value model is violated, in a very simple way. Given the near-random-walk behavior of stock prices, the "volatility" of the stock price is captured by $Var(p_t - p_{t-1})$. But the high volatility of the actual stock price is not inconsistent with the smooth behavior of p_t^* because we show here that the present-value model implies $Var(p_t - p_{t-1}) \geq Var(p_t^* - p_{t-1}^*)$.

This observation should not, however, revive hope for the contention that stock prices are not excessively volatile. Both Mankiw, Romer, and Shapiro (1985), and West (1988) find their variance bounds are violated in data for U.S. stock prices. We shall argue presently that the results of West (1988) should persuade us that the second variance bound derived here, $Var(\hat{p}_t - \hat{p}_{t-1}) \geq Var(p_t - p_{t-1})$, will also fail.

Definitions and assumptions

$p_t^* \equiv \sum_{j=0}^{\infty} b^j d_{t+j}$, $0 < b < 1$. p_t^* is the "perfect foresight" price. d_t are dividends at time *t*.

$p_t \equiv E\left(p_t^* \mid I_t\right)$. I_t is the information set of the market. p_t is the market price.

$\hat{p}_t \equiv E\left(p_t^* \mid H_t\right)$. H_t is an information set, $H_t \subseteq I_t$.

$e_t \equiv p_t - E\left(p_t \mid I_{t-1}\right)$. $f_t \equiv \hat{p}_t - E\left(\hat{p}_t \mid H_{t-1}\right)$.

$\sigma^2 \equiv Var(e_t)$. $\hat{\sigma}^2 = Var(f_t)$

As in West (1988), we assume I_t is a linear space, spanned by the current and past values of a finite number of random variables, and that $I_t \subseteq I_{t+1}$. After *s* differences, all the random variables in I_t jointly follow a stationary ARMA(*q*,*r*) process for finite *s*, *q*, *r*.

Assume that at a minimum, H_t contains current and past values of d_t.

West (1988) shows $p_t^* - p_t = \sum_{j=1}^{\infty} b^j e_{t+j}$ and $p_t^* - \hat{p}_t = \sum_{j=1}^{\infty} b^j f_{t+j}$

Comment: Shiller (1981) shows $Var(p_t^*) \geq Var(p_t)$ when d_t is stationary.

West (1988) shows $\hat{\sigma}^2 \geq \sigma^2$ when d_t is a linear process integrated of any order.

Proposition 1: Suppose d_t is $I(1)$ or $I(0)$, and all of the assumptions above hold. Then $Var(p_t - p_{t-1}) \geq Var(p_t^* - p_{t-1}^*)$.

Proof: $Var(p_t - p_{t-1}) = Var(\{E(p_t \mid I_{t-1}) - p_{t-1}\} + \{p_t - E(p_t \mid I_{t-1})\}) = \Gamma + \sigma^2$, where $\Gamma \equiv \text{var}(E(p_t \mid I_{t-1}) - p_{t-1})$. The last equality holds because $e_t \equiv p_t - E\left(p_t \mid I_{t-1}\right)$ is uncorrelated with *t*-1 information.

$$Var(p_t^* - p_{t-1}^*) = Var(\{E(p_t \mid I_{t-1}) - p_{t-1}\} + \{p_t - E(p_t \mid I_{t-1})\} + \{p_t^* - p_t\} - \{p_{t-1}^* - p_{t-1}\})$$

$$= Var(\{E(p_t \mid I_{t-1}) - p_{t-1}\} + e_t + \sum_{j=1}^{\infty} b^j e_{t+j} - \sum_{j=1}^{\infty} b^j e_{t+j-1})$$

$$= Var(\{E(p_t \mid I_{t-1}) - p_{t-1}\} + (1-b)e_t + (1-b)\sum_{j=1}^{\infty} b^j e_{t+j})$$

$$= \Gamma + (1-b)^2\sigma^2 + \frac{(1-b)^2 b^2}{1-b^2}\sigma^2 = \Gamma + \frac{1-b}{1+b}\sigma^2 \leq Var(p_t - p_{t-1})$$

Comment: Note the surprising relationship to the Shiller (1981) variance bound. Also note that the proposition does not extend to the claim for all $k > 0$, $Var(p_{t+k} - p_t) \geq Var(p_{t+k}^* - p_t^*)$. (See the Appendix for counterexamples for $k > 1$.)

Discussion of Proposition 1

For convenience, define $\Delta p_t \equiv p_t - p_{t-1}$ and $\Delta p_t^* \equiv p_t^* - p_{t-1}^*$. We see from the definitions above that $E(\Delta p_t \mid I_{t-1}) = E(\Delta p_t^* \mid I_{t-1})$. That is, $E(\Delta p_t \mid I_{t-1})$ is an unbiased (relative to the information set I_{t-1}) forecast of both Δp_t and Δp_t^*. So, we can write:

$$Var(p_t^* - p_{t-1}^*) = Var(E(\Delta p_t \mid I_{t-1})) + Var(p_t^* - p_{t-1}^* - E(\Delta p_t \mid I_{t-1})), \text{ and}$$

$$Var(p_t - p_{t-1}) = Var(E(\Delta p_t \mid I_{t-1})) + Var(p_t - p_{t-1} - E(\Delta p_t \mid I_{t-1})).$$

Proposition 1, which states $Var(p_t - p_{t-1}) \geq Var(p_t^* - p_{t-1}^*)$, is equivalent then to the statement,

$$Var(p_t - p_{t-1} - E(\Delta p_t \mid I_{t-1})) \geq Var(p_t^* - p_{t-1}^* - E(\Delta p_t \mid I_{t-1})).$$

That is, the market at time t-1, which has information I_{t-1}, can make a better forecast of Δp_t^* than of Δp_t ! (Here, "better forecast" means a forecast error with lower variance.)

To understand this, first we see that of course the forecast error the market makes for Δp_t is just its forecast error for p_t, since p_{t-1} is in I_{t-1}. That is,

$$Var(p_t - p_{t-1} - E(\Delta p_t \mid I_{t-1})) = Var(p_t - E(p_t \mid I_{t-1})) = Var(e_t) = \sigma^2 .$$

But in forecasting Δp_t^*, we must recognize that neither p_t^* nor p_{t-1}^* are in I_{t-1}. The forecast errors for p_t^* and p_{t-1}^* are correlated – indeed they are perfectly correlated (as we show shortly.) So, while the variance of the market's forecast error of p_t^* is greater than the variance of the market's forecast error of p_t, the variance of the market's forecast error of $p_t^* - p_{t-1}^*$ is much smaller than the variance of its forecast error of p_t^* -- and, as Proposition 1 implies, even smaller than the variance of the forecast error of Δp_t.

To see this, use the fact that, from the definitions above, p_{t-1}^* and p_{t-1} satisfy the following relationships:

$$p_{t-1}^* = d_{t-1} + bp_t^*$$

$$p_{t-1} = d_{t-1} + bE(p_t \mid I_{t-1}) .$$

Subtraction gives us $p_{t-1}^* - p_{t-1} = b(p_t^* - E(p_t \mid I_{t-1}))$. $p_{t-1}^* - p_{t-1}$ is the market's forecast error of p_{t-1}^* at time t-1, and $p_t^* - E(p_t \mid I_{t-1})$ is the market's forecast error of p_t^* at time t-1. This shows the forecast errors of p_{t-1}^* and p_t^* based on I_{t-1} are perfectly correlated.

The variance of the market's forecast error of p_t^* is given by

$$Var(p_t^* - E(p_t \mid I_{t-1})) = \frac{1}{b^2} Var(p_{t-1}^* - p_{t-1}) = \frac{1}{b^2} \sum_{j=1}^{\infty} b^j e_{t+j-1} = \frac{1}{1-b^2} \sigma^2$$

Clearly the variance of the market's forecast error of p_t^* is greater than the variance of the market's forecast error of p_t. But, now consider the variance of the forecast error of Δp_t^*:

$$Var(p_t^* - p_{t-1}^* - E(\Delta p_t \mid I_{t-1})) =$$

$$Var[\{p_t^* - E(p_t \mid I_{t-1})\} - \{p_{t-1}^* - p_{t-1}\}] = Var((1-b)\{p_t^* - E(p_t \mid I_{t-1})\}] = \frac{(1-b)^2}{1-b^2}\sigma^2 = \frac{1-b}{1+b}\sigma^2$$

The variance of the forecast error of Δp_t^* is less than the variance of the forecast error of p_t^* and p_t.

The intuition of Proposition 1 discussed here is in many ways similar to Kleidon's (1986) discussion of why it is misleading to draw inferences from the fact that the graph of p_t^* in Shiller (1981) is smoother than the graph of p_t. However, Kleidon did not consider models in which dividends could follow general $I(1)$ processes and did not examine the variances of differences in prices, so the analogy to that discussion is imperfect.

<u>Proposition 2:</u> Suppose d_t is $I(1)$ or $I(0)$, and all of the above assumptions hold. Then $Var(\hat{p}_t - \hat{p}_{t-1}) \geq Var(p_t - p_{t-1})$.

<u>Proof</u>: Following the same steps as above, but replacing p_t with $\hat{p}_t$, we have

$Var(\hat{p}_t - \hat{p}_{t-1}) = \hat{\Gamma} + \hat{\sigma}^2$, where $\hat{\Gamma} \equiv \text{var}(E(p_t \mid H_{t-1}) - p_{t-1})$, and

$Var(p_t^* - p_{t-1}^*) = \hat{\Gamma} + \frac{1-b}{1+b}\hat{\sigma}^2$. It follows that $\hat{\Gamma} + \frac{1-b}{1+b}\hat{\sigma}^2 = \Gamma + \frac{1-b}{1+b}\sigma^2$.

Then,

$$Var(p_t - p_{t-1}) = \Gamma + \sigma^2 = \Gamma + \frac{1-b}{1+b}\sigma^2 + \frac{2b}{1+b}\sigma^2 \leq \hat{\Gamma} + \frac{1-b}{1+b}\hat{\sigma}^2 + \frac{2b}{1+b}\hat{\sigma}^2 = \hat{\Gamma} + \hat{\sigma}^2 = Var(\hat{p}_t - \hat{p}_{t-1})$$

The inequality in this expression follows because West (1988) shows $\hat{\sigma}^2 \geq \sigma^2$.

<u>Comment</u>: Note the relationship of this variance bound to that of West (1988). At first glance, one might think that the two propositions contain the same result in the special case in which

$d_t = d_{t-1} + e_t$. That is true, but only trivially. Because both Proposition 1 of West (1988) and Proposition 2 here assume H_t includes current and past values of d_t, we have in this case that $\hat{\sigma}^2 = \sigma^2$, and $Var(\hat{p}_t - \hat{p}_{t-1}) = Var(p_t - p_{t-1})$. That is, any information in I_t that is not in H_t is not helpful in forecasting d_{t+1}.

Discussion of Proposition 2

Think of $\hat{p}_t$ as the forecast an econometrician makes of $p_t^* \equiv \sum_{j=0}^{\infty} b^j d_{t+j}$, based on a VAR as in West (1988).

Consider the relationship between the forecast of $\Delta\hat{p}_t \equiv \hat{p}_t - \hat{p}_{t-1}$ and Δp_t. Following the same logic as the Discussion of Proposition 1, we can write

$$Var(p_t - p_{t-1}) = Var(E(\Delta p_t \mid H_{t-1})) + Var(p_t - p_{t-1} - E(\Delta p_t \mid H_{t-1}))$$

$$Var(\hat{p}_t - \hat{p}_{t-1}) = Var(E(\Delta p_t \mid H_{t-1})) + Var(\hat{p}_t - \hat{p}_{t-1} - E(\Delta\hat{p}_t \mid H_{t-1})),$$

where we have used the fact that $E(\Delta p_t \mid H_{t-1}) = E(\Delta\hat{p}_t \mid H_{t-1}) = E(\Delta p_t^* \mid H_{t-1})$. The theorem then implies that

$$Var(p_t - p_{t-1} - E(\Delta p_t \mid H_{t-1})) \le Var(\hat{p}_t - \hat{p}_{t-1} - E(\Delta\hat{p}_t \mid H_{t-1})).$$

Notice the comparison to the West (1988) result. Since p_{t-1} is in I_{t-1} and $\hat{p}_{t-1}$ is in H_{t-1}, we can write West's result that $\hat{\sigma}^2 \ge \sigma^2$ as:

$$Var(p_t - p_{t-1} - E(\Delta p_t \mid I_{t-1})) \le Var(\hat{p}_t - \hat{p}_{t-1} - E(\Delta\hat{p}_t \mid H_{t-1})).$$

Another related paper is that of Engel and West (2004). They show that as $b \to 1$, $Var[(1-b)(\hat{p}_t - \hat{p}_{t-1})] \approx Var[(1-b)(p_t - p_{t-1})]$. Their proof, however, takes a very different tack than the proofs here. They show that as $b \to 1$,

$Var[(1-b)(\hat{p}_t - E(\hat{p}_t \mid H_{t-1}))] \approx Var[(1-b)(p_t - E(p_t \mid I_{t-1}))]$. They then use the result from Engel and West (2005) that as $b \to 1$, $\hat{p}_t - E(\hat{p}_t \mid H_{t-1}) \approx \hat{p}_t - \hat{p}_{t-1}$ and $p_t - E(p_t \mid I_{t-1}) \approx p_t - p_{t-1}$ to conclude that $Var[(1-b)(\hat{p}_t - \hat{p}_{t-1})] \approx Var[(1-b)(p_t - p_{t-1})]$ when b is near one.

Now consider the relationship between the variance of $\Delta\hat{p}_t$ and Δp_t^*. Proposition 2, combined with Proposition 1, give us

$$Var(\hat{p}_t - \hat{p}_{t-1}) \geq Var(p_t - p_{t-1}) \geq Var(p_t^* - p_{t-1}^*).$$

This means that the variance of $\Delta\hat{p}_t$ is an upper bound on the variance of Δp_t^*. Even if the present value model is not how the market prices equities, the econometrician can still calculate an upper bound on the variance of the change in the ex post discounted sum of current and future dividends.

As we have noted, the graphs of Shiller (1981) in essence confirm that the variance bound of Proposition 1 is satisfied. However, the results of West (1988) in essence confirm that the variance bound of Proposition 2 is not satisfied. The near random walk behavior of equity prices means that $Var(p_t - E(p_t \mid I_{t-1}))$ will not be too different than $Var(p_t - p_{t-1})$. Also, West's estimates show that dividends are nearly a random walk, suggesting that $Var(\hat{p}_t - E(\hat{p}_t \mid H_{t-1}))$ is none too different than $Var(\hat{p}_t - \hat{p}_{t-1})$. Given the gross violations of the bound $Var(\hat{p}_t - E(\hat{p}_t \mid H_{t-1})) > Var(p_t - E(p_t \mid I_{t-1}))$ that West reports, we can quite confidently hazard the guess that the bound $Var(\hat{p}_t - \hat{p}_{t-1}) \geq Var(p_t - p_{t-1})$ will also fail.

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Appendix

This appendix shows that we cannot extend Proposition 1 to *k*-differences. Specifically, it is not true that for all $k > 0$, $Var(p_{t+k} - p_t) \geq Var(p^*_{t+k} - p^*_t)$. First, we give a counterexample when dividends are $I(0)$ (specifically, when they are i.i.d.) Then, we show that when dividends follow a random walk, it is true that for all $k > 0$, $Var(p_{t+k} - p_t) \geq Var(p^*_{t+k} - p^*_t)$. However, we then use the i.i.d. example and the random walk example to construct a case in which dividends are $I(1)$, but $Var(p_{t+k} - p_t) < Var(p^*_{t+k} - p^*_t)$.

1. We have $p^*_t \equiv \sum_{j=0}^{\infty} b^j d_{t+j}$, and $p^*_{t+k} \equiv \sum_{j=0}^{\infty} b^j d_{t+k+j}$. Assume, $d_t = v_t$; $v_t \sim$ i.i.d., mean-zero; $\text{var}(v_t) = \upsilon^2$.

Then, $p^*_{t+k} - p^*_t = -\sum_{i=0}^{k-1} b^i v_{t+i} + (1 - b^k) p^*_{t+k}$

$Var(p^*_{t+k} - p^*_t) = \frac{1 - b^{2k}}{1 - b^2} \upsilon^2 + (1 - b^k)^2 Var(p^*_{t+k})$. Since $Var(p^*_{t+k}) = \frac{1}{1 - b^2} \upsilon^2$, we have

$$Var(p^*_{t+k} - p^*_t) = \frac{1 - b^{2k}}{1 - b^2} \upsilon^2 + \frac{(1 - b^k)^2}{1 - b^2} \upsilon^2 = \frac{2(1 - b^k)}{1 - b^2} \upsilon^2 .$$

Recalling that $p_t \equiv E\left(p^*_t \mid I_t\right)$, and assuming that I_t contains d_t (and its past values), we have that $p_t = d_t = v_t$ and $p_{t+k} = v_{t+k}$. So, $p_{t+k} - p_t = v_{t+k} - v_t$, and $Var(p_{t+k} - p_t) = 2\upsilon^2$.

Then $Var(p_{t+k} - p_t) > Var(p^*_{t+k} - p^*_t)$ if and only if $2\upsilon^2 > \frac{2(1 - b^k)}{1 - b^2} \upsilon^2$, or, $b^2 < b^k$. The only positive value of *k* for which this inequality holds is $k = 1$.

2. Now suppose $d_t = d_{t-1} + w_t$, $w_t \sim$ i.i.d., mean-zero; $\text{var}(w_t) = \omega^2$.

Then we can write $p_t^* = \frac{-1}{1-b}\sum_{j=1}^{\infty}(1-b^j)w_{t+j}$, and $p_{t+k}^* = \frac{-1}{1-b}\sum_{j=1}^{\infty}(1-b^j)w_{t+k+j}$. With a bit of work, we can write:

$$p_{t+k}^* - p_t^* = \frac{1}{1-b}\sum_{j=1}^{k-1}(1-b^j)w_{t+j} + \frac{1-b^k}{1-b}\sum_{j=0}^{\infty}b^j w_{t+k+j} .$$

We then get (with more work):

$$Var(p_{t+k}^* - p_t^*) = \omega^2\left\{\frac{1}{(1-b)^2}\left[k-1-2b\left(\frac{1-b^{k-1}}{1-b}\right)+b^2\left(\frac{1-b^{2k-2}}{1-b^2}\right)\right]+\left(\frac{1-b^k}{1-b}\right)^2\frac{1}{1-b^2}\right\}.$$

Simplifying this expression, we get:

$$Var(p_{t+k}^* - p_t^*) = \frac{\omega^2}{(1-b)^2}\left(k-\frac{2b(1-b^k)}{1-b^2}\right).$$

Now assume as before that I_t contains d_t (and its past values), so that we have $p_t = \frac{1}{1-b}d_t$ and $p_{t+k} = \frac{1}{1-b}d_{t+k}$. Then, $p_{t+k} - p_t = \frac{1}{1-b}\sum_{j=1}^{k}w_{t+k}$. Then we have

$$Var(p_{t+k} - p_t) = \frac{\omega^2 k}{(1-b)^2} .$$

Clearly for all $k > 0$, $Var(p_{t+k} - p_t) \geq Var(p_{t+k}^* - p_t^*)$ in this example.

3. Now let’s assume that the dividend process is the sum of two independent components:

$d_t = d_t^1 + d_t^2$. We will assume d_t^1 follows the process of example 1, and d_t^2 follows the process of example 2. Then, we have

$$Var(p_{t+k}^* - p_t^*) = \frac{2(1-b^k)}{1-b^2}\upsilon^2 + \frac{\omega^2}{(1-b)^2}\left(k-\frac{2b(1-b^k)}{1-b^2}\right).$$

Assume that I_t contains current (and past) values of both d_t^1 and d_t^2 . Then

$$Var(p_{t+k} - p_t) = 2\upsilon^2 + \frac{\omega^2 k}{(1-b)^2}$$

We then have:

$$Var(p_{t+k} - p_t) - Var(p^*_{t+k} - p^*_t) = \left(\frac{2}{1-b^2}\right)\left[(b^k - b^2)\upsilon^2 + \frac{b(1-b^k)}{(1-b)^2}\omega^2\right].$$

It follows that

$Var(p_{t+k} - p_t) - Var(p^*_{t+k} - p^*_t) < 0$ when $(b^2 - b^k)\upsilon^2 > \frac{b(1-b^k)}{(1-b)^2}\omega^2$. So we have a

counterexample to the proposition that for all $k > 0$, $Var(p_{t+k} - p_t) \geq Var(p^*_{t+k} - p^*_t)$ in the case in which dividends are $I(1)$.

Note that Gilles and LeRoy (1991) construct an example (one with two sample points, and a specific stochastic process for dividends) in which $Var(p_{t+k} - p_t) \geq Var(p^*_{t+k} - p^*_t)$, but clearly that example does not generalize to all $I(0)$ and $I(1)$ processes.

<u>Comment</u> It is straightforward to generalize Proposition 1 to show that for all $k > 0$, $Var(p_{t+k} - p_t) \geq Var(p^*_{t+k} - p^*_t)$ when I_t contains d_{t+k-1} and all dividends prior to $t+k-1$.