NBER WORKING PAPER SERIES

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Working Paper 10921
http://www.nber.org/papers/w10921

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
November 2004

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NBER Working Paper No. 10921
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JEL No. E1, O4

ABSTRACT

This brief note revisits the proof of the Steady-State Growth Theorem, first provided by Uzawa (1961). We provide a clear statement of the theorem and a new version of Uzawa's proof that makes the intuition underlying the result more apparent.

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1. INTRODUCTION

The Steady-State Growth Theorem says that if a neoclassical growth model exhibits steady-state growth, then technical change must be labor augmenting, at least in steady state.[1] It did not escape the attention of economists, either in the 1960s or more recently, that this is a very restrictive theorem. We often want our models to exhibit steady-state growth, but why should technical change be purely labor-augmenting? The induced-innvoation literature associated with Fellner (1961), Kennedy (1964), Samuelson (1965), and Drandakis and Phelps (1966) explicitly pondered this question without achieving a clear answer. Recently, Acemoglu (2003) and Jones (2004) have returned to this puzzle.

Perhaps surprisingly, then, given its importance in the growth literature, we have been unable to find a clear statement and proof of the theorem. Uzawa (1961) is typically credited with the proof of the result,[2] and there is no doubt that he proved the theorem. However, Uzawa is primarily concerned with showing the equivalence of *Harrod-neutral* technical change (i.e. technical change that leaves the capital share unchanged if the interest rate is constant) and labor-augmenting technical change, formalizing the graphical analysis of Robinson (1938). It is of course, only a small and well-known step to show that steady-state growth requires technical change to be Harrod neutral. But again, the modern reader of Uzawa will be struck by the absence of a statement and direct proof of the steady-state growth theorem.

Barro and Sala-i-Martin (1995, Chapter 2) come closest to providing a clear statement and proof of the theorem. However, their statement of the

[1]It is sometimes added that an alternative is for the production function to be Cobb-Douglas, at least in steady state. But this is really subsumed in the original version of the theorem since technical change can always be written in the labor-augmenting form in steady state if the production function is Cobb-Douglas.

[2]For example, see Barro and Sala-i-Martin (1995) and Solow (1999).

theorem is more restrictive: if technical change is factor augmenting at a constant exponential rate, then steady-state growth requires it to be labor-augmenting. This leaves open the door, however slightly, to the possibility that there might be some perverse non-factor augmenting twist of technical change that could be consistent with steady-state growth.

This comment fills the gap in the literature. We provide a clear statement and proof of the steady-state growth theorem. The inspiration for the proof is Uzawa (1961), but we present the crucial steps in a slightly different way that allows the economic intuition for the proof to come through.

2. STATING THE THEOREM

The steady-state growth theorem applies to the one-sector neoclassical growth model. We begin by defining the model precisely and then defining a balanced growth path.

DEFINITION 2.1. A neoclassical growth model is given by the following economic environment:

$$Y_t = F(K_t, L_t; t), \tag{1}$$

$$\dot{K}_t = Y_t - C_t - \delta K_t, \;\; K_0 > 0, \;\; \delta \geq 0, \tag{2}$$

and

$$L_t = L_0 e^{nt}, \;\; L_0 > 0, \;\; n \geq 0. \tag{3}$$

The production function F satisfies the standard neoclassical properties: constant returns to scale in K and L, positive and diminishing marginal products of K and L, and the Inada conditions that the marginal product of a factor input goes to zero as that input goes to infinity and goes to infinity as the input goes to zero.

A balanced growth path in the neoclassical growth model is defined as a situation in which all quantities grow at constant exponential rates (possibly zero) forever. We follow the usual convention of also refering to this as a steady state.

Finally, we will define $F_K K/Y$ to be the capital share and $F_L L/Y$ to be the labor share. As usual, the two factor shares sum to a value of unity, by Euler's theorem. We follow standard notation in denoting $y \equiv Y/L$ and $k \equiv K/L$, and we will use an asterisk superscript to denote a variable along the steady-state path.

With these definitions, we can now present the Steady-State Growth Theorem:

THEOREM 2.1 (The Steady-State Growth Theorem, Uzawa 1961). *If a neoclassical growth model possesses a steady state with constant, nonzero factor shares and $\dot{y}_t^*/y_t^* = g > 0$, then it must be possible along the steady-state path to write the production function as $Y_t^* = G(K_t^*, A_t L_t)$, where $\dot{A}_t/A_t = g$ and where G is a neoclassical production function.*

Before presenting the theorem, we pause to make several remarks. First, the restriction to the case of positive factor shares is primarily intended to rule out "AK" style models. Second, as is well-known, in the case of Cobb-Douglas production, capital- and labor-augmenting technical change are equivalent. One sometimes sees the theorem interpreted as saying that technical change must be labor-augmenting or the production function must be Cobb-Douglas. This is equivalent to the statement of the theorem as given.

3. PROVING THE THEOREM

This proof largely follows Uzawa (1961) in spirit. It differs in that we provide more economic intuition, highlight the key steps of the proof more clearly, and fill in some details.[3]

The capital-output ratio is a key variable throughout the proof, so we define $x \equiv K/Y$. We also make the standard definition $f(k;t) \equiv F(k,1;t)$. The proof now follows.

1. The first step of the proof is to rewrite the production function in terms of the capital-output ratio: $y_t = \phi(x_t;t)$. Intuitively, this step is readily understood by drawing the production function in (k,y) space: for each ray through the origin — that is for each capital-output ratio — there is a unique level of output per worker on that ray.[4]

2. Next, we note that the elasticity of y_t with respect to x_t satisfies a familiar property:

$$\frac{\partial \log y_t}{\partial \log x_t} = \frac{\alpha(x_t;t)}{1-\alpha(x_t;t)} \tag{4}$$

[3]The only substantive innovation in the proof is in writing the key differential equation in (4) below in terms of the elasticity of output with respect to the capital-output ratio. This produces a familiar equation in a way that Uzawa's consideration of the marginal product of capital does not.

[4]Formally, we can use the inverse function theorem to justify this step. The capital-output ratio depends only on k_t and t, since y_t is a function of k_t and t: $x_t = k_t/y_t = k/f(k_t;t) \equiv h(k_t;t)$. We can apply the inverse function theorem to show that this function can be inverted:

$$\begin{aligned}\frac{\partial h(k_t;t)}{\partial k_t} &= \frac{1}{f(k;t)} - \frac{k_t f_k(k_t;t)}{f(k_t;t)^2} \\ &= \frac{1}{f(k_t;t)}\left(1 - \frac{f_k(k_t;t)k_t}{f(k_t;t)}\right) \\ &\neq 0 \ \forall k_t,\end{aligned}$$

where the last step follows from the fact that the labor share is strictly between zero and one. Therefore, by the inverse function theorem, $h^{-1}(\cdot\,;t)$ exists, and we can write $k_t = h^{-1}(x_t;t)$. Finally, we can substitute this result into the production function to get $y_t = f(k_t;t) = f(h^{-1}(x_t;t),t) \equiv \phi(x_t;t)$.

where $\alpha(x_t;t) \equiv f_k k_t/y_t$ is the capital share. This equation says that the elasticity of output with respect to the capital-output ratio is equal to the ratio of the capital and labor shares. Such an equation is well-known in the case of Cobb-Douglas production, where it has been exploited by Mankiw, Romer and Weil (1992), Klenow and Rodriguez-Clare (1997), and Hall and Jones (1999), among others. Equation (4) shows that this property holds more generally.[5]

3. Now comes the key step of the proof. From this point on, we assume the economy is on a balanced growth path. Because the capital share is constant in steady state, the right side of equation (4) is invariant over time. Then, since $y_t = \phi(x_t;t)$, we can write this equation as

$$\frac{\partial \log \phi(x^*;t)}{\partial \log x^*} = \frac{\alpha(x^*)}{1-\alpha(x^*)}, \tag{5}$$

where we use an asterisk to indicate a quantity along a balanced growth path.

Because the right-hand side of this equation does not depend on time, this partial differential equation can be solved to yield[6]

$$\log \phi(x^*;t) = a(t) + \int \frac{\alpha(x^*)}{1-\alpha(x^*)} \frac{dx^*}{x^*} \tag{6}$$

for some function $a(t)$. And therefore

$$y_t^* = \phi(x^*;t) = A(t)\psi(x^*), \tag{7}$$

where $A(t) \equiv \exp(a(t)) > 0$ and $\psi(x^*) \equiv \exp\left(\int \frac{\alpha(x^*)}{1-\alpha(x^*)} \frac{dx^*}{x^*}\right)$.

This is the crucial result. We've shown that the effects of t and x^* can be separated. This implies, for example, the familiar result that $y_t^*/A_t = x^*$ is

[5]To derive this equation, begin with $k_t = y_t x_t$. This implies that $dk_t/k_t = dy_t/y_t + dx_t/x_t$. Multiply through by y/dy to get $\frac{dk_t y_t}{dy_t k_t} = 1 + \frac{dx_t y_t}{dy_t x_t}$, which can be rearranged to yield the desired result.

[6]This can be readily verified by differentiating the solution.

constant along a balanced growth path, where $A_t \equiv A(t)$. Since y_t^* grows at rate g by assumption, it must therefore be the case that $\dot{A}_t/A_t = g$ as well.

4. To conclude the proof, note that $k = xy$, so that

$$\frac{k_t^*}{A_t} = \frac{y_t^*}{A_t}\psi^{-1}\left(\frac{y_t^*}{A_t}\right) \equiv \tilde{G}^{-1}\left(\frac{y_t^*}{A_t}\right). \tag{8}$$

Inverting[7], we have

$$\frac{y_t^*}{A_t} = \tilde{G}\left(\frac{k_t^*}{A_t}\right) \tag{9}$$

and therefore

$$Y_t^* = A_t L_t \tilde{G}\left(\frac{K_t^*}{A_t L_t}\right) \equiv G(K_t^*, A_t L_t). \tag{10}$$

And this proves the key result: technical change is labor-augmenting along the balanced growth path. Finally, since $Y_t^* = F(K_t^*, L_t; t) = G(K_t^*, A_t L_t)$, it must be the case that G satisfies the standard neoclassical properties as well.

4. DISCUSSION

Here's the one paragraph version of the proof. The crux of the proof is step 3 above. To begin, we notice that the familiar Cobb-Douglas property also holds more generally: the elasticity of output per worker with respect

[7]To show invertibility, differentiate:

$$\begin{aligned}
\tilde{G}^{-1}(z) &= z\psi^{-1}(z) \\
\Rightarrow \frac{d\tilde{G}^{-1}}{dz} &= \psi^{-1}(z) + z\frac{d\psi^{-1}(z)}{dz} \\
&= \psi^{-1} + \frac{z}{\left(\frac{d\psi}{dx}\right)} \\
&> 0 \quad \forall z > 0
\end{aligned}$$

as $\psi = y/A$ is always positive and $d\psi/dx$ is also always positive. So $\tilde{G}(\cdot)$ exists.

to the capital-output ratio is $\alpha(x;t)/(1-\alpha(x;t))$. Then, the fact that the capital share must be constant in steady state means that the production function must be factorable. That is, it must be possible, at least in steady state, to write the production function as $y_t = A(t)\psi(x)$. But this means that y/A and k/A must be constant as well, and one can really just look at the production function $y/A = F(k/A, 1/A; t)$ to see that this requires technical change to be labor augmenting. The intuition that is closest to the spirit of this proof, then, is that technical change must be labor-augmenting in order to keep the capital share constant.

A related intuition can be obtained by looking at the marginal product of capital. Because both the capital-output ratio and the capital share must be constant in steady state, we know the marginal product of capital must be constant as well. This marginal product is $F_1(K, L; t) = F_1(k/y, 1/y; t)$ since the marginal product is homogeneous of degree zero in the factor inputs. Since k/y is constant in steady state and y grows at a constant exponential rate, technical change must exactly offset the fact that "effective labor" is falling at the rate of growth of y. That is, technical change must be labor-augmenting. If this were not the case, then the marginal product of capital would trend over time.

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