JUMP AND VOLATILITY RISK AND RISK PREMIA:
A NEW MODEL AND LESSONS FROM S&P 500 OPTIONS

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ABSTRACT

We use a novel pricing model to filter times series of diffusive volatility and jump intensity from

S&P 500 index options. These two measures capture the ex-ante risk assessed by investors. We find

that both components of risk vary substantially over time, are quite persistent, and correlate with

each other and with the stock index. Using a simple general equilibrium model with a representative

investor, we translate the filtered measures of ex-ante risk into an ex-ante risk premium. We find

that the average premium that compensates the investor for the risks implicit in option prices, 10.1

percent, is about twice the premium required to compensate the same investor for the realized

volatility, 5.8 percent. Moreover, the ex-ante equity premium that we uncover is highly volatile, with

values between 2 and 32 percent. The component of the premium that corresponds to the jump risk

varies between 0 and 12 percent.

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1 Introduction

This paper uses option prices to estimate the risk of the stock market as it is perceived *ex ante* by investors. We investigate several questions: What is the stock market risk perceived by investors? Is there a difference between the perceived risks and the realized risks? What premium would a "reasonable" investor require as compensation for the perceived risks? And how does that premium compare with the required premium for the realized level of risk?

We consider two types of risk in stock prices: diffusion risk and jump risk.[1] As argued by Merton (1980), diffusion risk can be accurately measured from the quadratic variation of the price process. In contrast, since even very high probability jumps may fail to materialize in sample, the ex-ante jump risk perceived by investors may be quite different from the ex-post realized variation in prices. Therefore, studying measures of realized volatility and realized jumps from the time series of stock prices, will give us a limited picture of the risks feared by investors. Fortunately, since options are priced on the basis of the ex-ante risks, they can give us a privileged view on the risks perceived by investors. Using option data solves the "Peso problem" in measuring the jump risk from realized stock returns.

Our option pricing model allows the volatility of the diffusion risk and the intensity of the jumps to both vary stochastically over time in a potentially interdependent way. When we calibrate the model to a panel data set of S&P 500 index option prices from the beginning of 1996 to the end of 2002, we obtain the time series of the filtered diffusive volatility and jump intensity processes. We find that the innovations to the two risk processes are highly correlated with each other and negatively correlated with the stock returns. Both components of risk vary substantially over time and show a high degree of persistence. The diffusive volatility process varies between 10 and 35 percent per year, which is in line with the level of ex-post risk measured from the time series of stock returns. The jump intensity process shows even wider variation. Some times the probability of a jump is zero, while at other times it is more than 30 percent. The expected jump size is close to negative 30 percent. Interestingly, we do not observe any such large jumps in the time series of the S&P 500 index in our sample, not even around the times when the implied jump intensity is very high. These were therefore cases in which the jumps that were feared did not materialize.

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[1]There is ample empirical evidence for this kind of specification. See for example Jorion (1989), Bakshi, Cao, and Chen (1997), and Bates (2000).

However, these perceived risks are still likely to have impacted the expected return in the stock market.

To investigate the impact of ex-ante risk on expected returns, we solve for the stock market risk premium in a simple exchange economy with a CRRA representative investor. We find that the equilibrium risk premium is a function of both the stochastic volatility and the jump intensity. Given the filtered stochastic volatility and jump intensity processes, together with an assumed coefficient of risk aversion for the representative investor that approximately matches the historic average equity premium, we can estimate the time series of the *ex-ante* equity premium. This is the expected excess return demanded by the investor to hold the entire wealth in the stock market when facing the diffusion and jump risks implicit in option prices. We can also decompose the ex-ante equity premium into compensation for diffusive risk and compensation for jump risk. From the filtered risk series, we find the ex-ante equity premium to be quite variable over time. In our sample, the equity premium demanded by the representative investor varies between as low as 2 percent and as high as 32 percent per year! The compensation for jump risk is on average one third of the total premium. However, in times of crisis, the jump risk may command a premium near 12 percent per year and can be close to two thirds of the total premium.

The ex-ante premium evaluated at the average levels of diffusive volatility and jump intensity implied from the options in our sample is 10.1 percent. In contrast, the same investor would require a premium of only 5.8 percent as compensation for the realized volatility during the same sample period. Therefore, the required compensation for the ex-ante risks is almost twice the compensation for the realized risks! This finding supports the Peso explanation of the equity premium puzzle proposed by Rietz (1988) and Brown, Goetzmann, and Ross (1985).

The option pricing model used in this paper belongs to the family of linear-quadratic jump-diffusion models.[2] It is the first estimated model that allows the jump intensity to follow explicitly its own stochastic process. Most jump-diffusion models impose a constant jump intensity (e.g., Merton (1976) and Bates (1988)) or make it a deterministic function of the diffusive volatility (e.g., Bates (2000), Duffie, Pan, and Singleton (2000), and Pan (2002)). The empirical analysis shows that the jump intensity varies a lot and that, although related to the diffusive volatility, it has its own source of shocks. Our model is quadratic in the state variables. This allows the covariance structure of the shocks to the state variables to

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[2]Cheng and Scaillet (2002) also study linear-quadratic option pricing models.

be unrestricted, which proves to be important since there is substantial correlation in the risk processes that we filter from the data. We are nevertheless still able to solve for the European option prices in a manner similar to the affine case of Duffie, Pan, and Singleton (2000).

The paper closest to ours is Pan (2002).[3] She estimates a jump-diffusion model from both the time series of the S&P 500 index and its options from 1989 to 1996. She uses the pricing model proposed by Bates (2000) which has a square-root process for the diffusive variance and jump intensity proportional to the diffusive variance. The jump risk premium is specified to be linear in the variance. Pan finds a significant jump premium of roughly 3.5 percent, which is of the same order of magnitude of the volatility risk premium of 5.5 percent. The main difference between our paper and hers is that in Pan's framework it is hard to disentangle the diffusion and jump risks and risk premia since they are all driven by the diffusive volatility. Our approach allows us to extract the jump intensity process autonomously from the diffusive volatility process.

Finally, a word of caution. Our analysis relies on option prices and, of course, options may be systematically mispriced. That would bias our ex-ante risk measures. Coval and Shumway (2001) and Driessen and Maenhout (2003) report empirical evidence that some option strategies have unusually high Sharpe ratios, which may indicate mispricing. However, Santa-Clara and Saretto (2004) show that taking into account the Peso problem in the sample of stock returns substantially diminishes the attractiveness of these strategies. In fact, the existence of large (approximate) arbitrage opportunities in the option market does not seem very likely. Even if the presence of jumps prevents the perfect replication of options by dynamically trading in the underlying asset, options can still be approximately replicated with static portfolios of other options, as Carr and Bowie (1994), Derman, Ergener, and Kani (1995), and Carr and Wu (2002) show. Such static option hedges would be easy to implement by investment banks and hedge funds. This cross-option arbitrage is likely to limit the mispricing of options relative to each other. Since the risk components that we extract from option prices are to a large extent driven by the cross section of options, by this argument they should be relatively free from mispricing problems.

The paper proceeds as follows. In section 2, we present the dynamics of the stock market index under the objective and the risk-adjusted probability measure, and we derive an option

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[3]Other related work includes Ait-Sahalia, Wang, and Yared (2001), Bates (2001), Bliss and Panigirtzoglou (2004), Chernov and Ghysels (2000), Engle and Rosenberg (2002), Eraker (2004), and Jackwerth (2000).

pricing formula. In section 3, we discuss the data and the econometric approach. The model estimates and its performance in pricing the options in the sample are covered in section 4. Section 5 contains the main results of the paper, the analysis of the risks implied from option prices and what they imply for the equity premium. Section 6 concludes.

2   The Model

In this section we introduce a new model of the dynamics of the stock market return that displays both stochastic diffusive volatility and jumps with stochastic intensity. We derive the equilibrium stock market risk premium in a simple economy with a representative investor with CRRA utility. This risk premium compensates the investor for both volatility and jump risks. We also obtain the risk-adjusted dynamics of the stock, volatility, and jump intensity processes and use them to price European options.

2.1   Stock Market Dynamics

We model the dynamics of the stock market index (referred to as stock) with two sources of risk: diffusive risk, captured by a Brownian motion, and jump risk, modeled as a Poisson process. The diffusive volatility and the intensity of the jump arrivals are also stochastic and interdependent. We parameterize the processes as:

$$dS = (r + \phi - \lambda\mu_Q)\, S dt + \sqrt{V} S dW_S + Q S dN, \tag{1}$$

$$dV = \left(\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda}\right) dt + \sigma_V\sqrt{V}\, dW_V, \tag{2}$$

$$d\lambda = \left(\frac{1}{4}\sigma_\lambda{}^2 + \kappa_\lambda\sqrt{\lambda} + \kappa_{\lambda\lambda}\lambda + \kappa_{\lambda V}\sqrt{V\lambda}\right) dt + \sigma_\lambda\sqrt{\lambda}\, dW_\lambda, \tag{3}$$

$$\ln(1 + Q) \sim \mathcal{N}\left(\ln(1 + \mu_Q) - \frac{1}{2}\sigma_Q^2, \sigma_Q^2\right), \tag{4}$$

$$\text{Prob}(dN = 1) = \lambda dt, \tag{5}$$

$$\Sigma = \begin{pmatrix} 1 & \rho_{SV} & \rho_{S\lambda} \\ \rho_{SV} & 1 & \rho_{V\lambda} \\ \rho_{S\lambda} & \rho_{V\lambda} & 1 \end{pmatrix}. \tag{6}$$

W_S, W_V, and W_λ are Brownian motions with constant correlation matrix Σ, and N is a Poisson process with arrival intensity λ. Q is the percentage jump size and is assumed to follow a displaced lognormal distribution independently over time. This guarantees that the jump size cannot be less than -1 and therefore that the stock price remains positive at all times. We assume that N and Q are independent of each other and that Q is independent of the Brownian motions. V is the instantaneous variance of stock returns. r is the risk-free interest rate, assumed constant for convenience. We also assume that the stock pays no dividends, although it would be trivial to accommodate them by adding a term in the drift of the stock price. ϕ is the risk premium on the stock, which we show below to be a function of V and λ. Finally, the term $\lambda\mu_Q$ adjusts the drift for the average jump size.

In our model, the stock price, the stochastic volatility, and the jump intensity follow a joint quadratic jump-diffusion process.[4] In fact, without the jump component, our model collapses to a stochastic volatility model very similar to that of Stein and Stein (1991).[5] It can easily be seen that the model does not belong to the affine family of Duffie, Pan, and Singleton (2000), in that the drifts and the covariance terms are not linear in the state variables. For instance, the covariance between dV and $d\lambda$ is $\rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda}$.

Our model belongs to the family of linear-quadratic jump-diffusion models. It is the first model in which the jump intensity λ follows explicitly its own stochastic process. In contrast, existing jump-diffusion models either assume that the jump intensity is constant or make it a deterministic function of other state variables such as the stochastic volatility.[6] For instance, Pan (2002) assumes that λ is a linear function of V. It is of course an empirical issue whether the jump intensity is completely driven by volatility or whether it has its own separate source of uncertainty. The empirical sections will shed some light on this matter.

A major advantage of our model is that it requires no constraints on the covariance matrix of the underlying state variables. In contrast, affine models impose very strict constraints on the covariance matrix. In affine models, the entries in the covariance matrix must be linear in the state variables and, of course, it is required that the covariance matrix be positive

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[4]For intuition, we can think of the stochastic processes of V and λ as the square of linear (Gaussian) processes.

[5]In Stein and Stein (1991), $\sqrt{V}$ follows an Ornstein-Unlenbeck process whereas, in our model, $V = \sqrt{X^2}$ with X following an Ornstein-Unlenbeck process. Since the square-root function is not globally invertible, the two are not the same.

[6]Some of these models can be transformed to allow the jump intensity to evolve separately from the volatility. For example, the two-factor jump-diffusion model in Bates (2000) admits such a transform for extreme values of one of the state variables and for some model parameters.

definite. In particular, the variance terms need to be positive at all times and the implicit correlations need to be less than one in absolute value. Other than the particular covariance matrices of Duffie, Pan, and Singleton (2000) and Pan (2002), it is hard to satisfy these positive definiteness constraints with a covariance matrix that has elements that are linear in the state variables. The quadratic form of the entries in the covariance matrix in our model automatically guarantees that the matrix is always positive definite.[7]

We now turn our attention to finding the risk premium ϕ. Consider a representative investor that has wealth W and allocates it entirely to the stock market. For simplicity, we assume that there is no intermediate consumption so the investor chooses an optimal portfolio to maximize utility of terminal wealth:

$$\max_{w} \mathrm{E}_t\left[u(W_T, T)\right], \tag{7}$$

where E_t is the conditional expectation operator, w is the fraction of wealth invested in the stock, T is the terminal date, and u is the utility function. Define the value function of the investor as:

$$J(W_t, V_t, \lambda_t, t) \equiv \max_{w} \mathrm{E}_t\left[u(W_T, T)\right]. \tag{8}$$

Following Merton (1973) and using subscripts to denote the partial derivative of J, a solution to (7) satisfies the Bellman equation:

$$0 = \max_{w}\left[J_t + \mathcal{L}(J)\right], \tag{9}$$

with:

$$
\begin{aligned}
\mathcal{L}(J) \;=\; & W J_W \left(r + w\phi - w\lambda\mu_Q\right) + J_V\left(\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda}\right) \\
& + J_\lambda\left(\frac{1}{4}\sigma_\lambda^2 + \kappa_\lambda\sqrt{\lambda} + \kappa_{\lambda\lambda}\lambda + \kappa_{\lambda V}\sqrt{V\lambda}\right) + \frac{1}{2}w^2 W^2 J_{WW} V \\
& + \frac{1}{2}J_{VV}\sigma_V{}^2 V + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda{}^2\lambda + wW J_{WV}\rho_{SV}\sigma_V V \\
& + wW J_{W\lambda}\rho_{S\lambda}\sigma_\lambda\sqrt{V\lambda} + J_{V\lambda}\rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda} + \lambda\mathrm{E}\left[\Delta J\right]. \tag{10}
\end{aligned}
$$

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[7] A similar problem occurs in multi-factor affine term structure models. To ensure the positive-definiteness of the covariance matrix, it is typically assumed that the state variables are uncorrelated. Unfortunately, when the models are taken to the data, and the latent variables are filtered, they often turn out to be significantly correlated, which contradicts the assumption.

The term $\Delta J \equiv J(W(1+wQ), V, \lambda, t) - J(W, V, \lambda, t)$ captures jumps in the value function. In equilibrium, the risk-free asset is in zero net supply. Therefore, the representative investor holds all the wealth in the stock market, that is, $w = 1$. Differentiating (9) with respect to w and substituting in $w = 1$, we obtain the risk premium on the stock:

$$\phi = -\frac{J_{WW}}{J_W}WV - \rho_{SV}\sigma_V\frac{J_{WV}}{J_W}V - \rho_{S\lambda}\sigma_\lambda\frac{J_{W\lambda}}{J_W}\sqrt{V}\lambda - \mathrm{E}\left[\frac{\Delta J_W}{J_W}Q\right], \tag{11}$$

where $\Delta J_W \equiv J_W(W(1+Q), V, \lambda, t) - J_W(W, V, \lambda, t)$. The stock risk premium contains four components: the variance of the marginal utility of wealth, and the covariances of the marginal utility of wealth with the diffusive volatility, the jump intensity, and the jump size, respectively.

For tractability, we concentrate our attention to the case of power utility:

$$u = \frac{W_T^{1-\gamma}}{1-\gamma}, \tag{12}$$

where $\gamma > 1$ is the constant relative risk aversion coefficient of the investor. In the Appendix we show that the risk premium on the stock consistent with equilibrium in this economy is a function of V and λ:

$$
\begin{aligned}
\phi(V, \lambda, \tau) &= \gamma V - \rho_{SV}\sigma_V\left(\frac{1}{2}B_V\sqrt{V} + C_{VV}V + C_{V\lambda}\sqrt{V}\lambda\right) \\
&\quad -\rho_{S\lambda}\sigma_\lambda\left(\frac{1}{2}B_\lambda\sqrt{V} + C_{V\lambda}V + C_{\lambda\lambda}\sqrt{V}\lambda\right) \\
&\quad -\left[e^{-\gamma\ln(1+\mu_Q)+\frac{1}{2}\gamma(\gamma-1)\sigma_Q^2}\left(1+\mu_Q-e^{\gamma\sigma_Q^2}\right)-\mu_Q\right]\lambda \qquad (13) \\
&= \gamma V - \frac{1}{2}\left(\rho_{SV}\sigma_V \quad \rho_{S\lambda}\sigma_\lambda\right)B\sqrt{V} - \left(\rho_{SV}\sigma_V \quad \rho_{S\lambda}\sigma_\lambda\right)\begin{pmatrix}C_{VV}\\C_{V\lambda}\end{pmatrix}V \\
&\quad -\left(\rho_{SV}\sigma_V \quad \rho_{S\lambda}\sigma_\lambda\right)\begin{pmatrix}C_{V\lambda}\\C_{\lambda\lambda}\end{pmatrix}\sqrt{V}\lambda \\
&\quad -\left[e^{-\gamma\ln(1+\mu_Q)+\frac{1}{2}\gamma(\gamma-1)\sigma_Q^2}\left(1+\mu_Q-e^{\gamma\sigma_Q^2}\right)-\mu_Q\right]\lambda, \qquad (14)
\end{aligned}
$$

where we define $\tau \equiv T-t$, and $B(\tau) = \left(\begin{smallmatrix}B_V\\B_\lambda\end{smallmatrix}\right)$ is a 2×1 matrix function and $C(\tau) = \left(\begin{smallmatrix}C_{VV} & C_{V\lambda}\\C_{V\lambda} & C_{\lambda\lambda}\end{smallmatrix}\right)$ is a 2×2 symmetric matrix function. B and C solve the following system of ODEs with the

initial conditions $B(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $C(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

$$B' = \frac{1}{2}\left(\Lambda^\top + C\Gamma\right)B + C\Pi, \tag{15}$$

$$C' = \Theta + \frac{1}{2}C\Lambda + \frac{1}{2}\Lambda^\top C + \frac{1}{2}C\Gamma C, \tag{16}$$

where "$\top$" denotes the transpose of a matrix (or the complex transpose in the case of a complex matrix), and the constant matrices Θ, Π, Λ, and Γ are defined as:

$$\Theta \equiv \begin{pmatrix} -\frac{1}{2}\gamma(\gamma - 1) & 0 \\ 0 & e^{-\gamma \ln(1+\mu_Q)+\frac{1}{2}\gamma(\gamma-1)\sigma_Q^2}\left[\gamma(1+\mu_Q) - (\gamma-1)e^{\gamma\sigma_Q^2}\right] - 1 \end{pmatrix}, \tag{17}$$

$$\Pi \equiv \begin{pmatrix} \kappa_V \\ \kappa_\lambda \end{pmatrix}, \tag{18}$$

$$\Lambda \equiv \begin{pmatrix} \kappa_{VV} & \kappa_{V\lambda} \\ \kappa_{\lambda V} & \kappa_{\lambda\lambda} \end{pmatrix}, \tag{19}$$

$$\Gamma \equiv \begin{pmatrix} \sigma_V^2 & \rho_{V\lambda}\sigma_V\sigma_\lambda \\ \rho_{V\lambda}\sigma_V\sigma_\lambda & \sigma_\lambda^2 \end{pmatrix}. \tag{20}$$

For a given value of the risk aversion coefficient γ, the ODEs (15)-(16) can be quickly solved numerically. In the special case where there is no stochastic volatility and jumps, the equity premium (14) collapses to the first term, γV, as shown by Merton (1973). The first three terms in (14) involve V only and thus correspond to compensation for stochastic volatility, and the last term compensates the investor for jump risk as it involves λ only. The interaction between the volatility and jump intensity risks is captured by the cross term involving $V\lambda$.

In a related work, Liu and Pan (2003) derive the optimal portfolio of a CRRA investor who can hold the stock, an option on the stock, and a risk-free asset. In their model, the stock market has stochastic diffusive volatility and jumps of deterministic size with the jump intensity driven by the stochastic volatility. In contrast to our paper, theirs is a partial equilibrium analysis that takes the price of risk as given.

2.2 Option Pricing

We can also price European options in this economy. In the Appendix we show that the risk-adjusted dynamics of the stock price can be written as:[8]

$$dS = \left(r - \lambda^* \mu_Q^*\right) S dt + \sqrt{V} S dW_S^* + Q^* S dN^*, \tag{21}$$

$$dV = \left(\frac{1}{4}\sigma_V^2 + \kappa_V^* \sqrt{V} + \kappa_{VV}^* V + \kappa_{V\lambda}^* \sqrt{V\lambda^*}\right) dt + \sigma_V \sqrt{V} dW_V, \tag{22}$$

$$d\lambda^* = \left(\frac{1}{4}\sigma_\lambda^{*2} + \kappa_\lambda^* \sqrt{\lambda^*} + \kappa_{\lambda\lambda}^* \lambda^* + \kappa_{\lambda V}^* \sqrt{V\lambda^*}\right) dt + \sigma_\lambda^* \sqrt{\lambda^*} dW_\lambda^*, \tag{23}$$

$$\ln(1 + Q^*) \sim \mathcal{N}\left(\ln(1 + \mu_Q^*) - \frac{1}{2}\sigma_Q^2, \sigma_Q^2\right), \tag{24}$$

$$\text{Prob}(dN^* = 1) = \lambda^* dt, \tag{25}$$

$$\Sigma = \begin{pmatrix} 1 & \rho_{SV} & \rho_{S\lambda} \\ \rho_{SV} & 1 & \rho_{V\lambda} \\ \rho_{S\lambda} & \rho_{V\lambda} & 1 \end{pmatrix}, \tag{26}$$

with the following simple relations between the model parameters under the objective and risk-adjusted probability measures:

$$\begin{pmatrix} \kappa_V^* \\ \kappa_\lambda^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{a} \end{pmatrix}\left(\Pi + \frac{1}{2}\Gamma B\right), \tag{27}$$

$$\begin{pmatrix} \kappa_{VV}^* & \kappa_{V\lambda}^* \\ \kappa_{\lambda V}^* & \kappa_{\lambda\lambda}^* \end{pmatrix} = \begin{pmatrix} 1 & 1/\sqrt{a} \\ \sqrt{a} & 1 \end{pmatrix} \circ \left[\Lambda - \gamma\begin{pmatrix} \rho_{SV}\sigma_V & 0 \\ \rho_{S\lambda}\sigma_\lambda & 0 \end{pmatrix} + \Gamma C\right], \tag{28}$$

$$\sigma_\lambda^* = \sqrt{a}\sigma_\lambda, \tag{29}$$

$$\lambda^* = a\lambda, \tag{30}$$

$$\mu_Q^* = (1 + \mu_Q)\, e^{-\gamma \sigma_Q^2} - 1, \tag{31}$$

$$a = (1 + \mu_Q)^{-\gamma} e^{\frac{1}{2}\gamma(\gamma+1)\sigma_Q^2}, \tag{32}$$

where Π, Λ, and Γ are defined as before, and "$\circ$" is the element-by-element product of two matrices. The risk-adjusted coefficients on the left-hand sides of the equations above are related to the coefficients under the objective probability measure by the risk aversion

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[8]In general, all the parameters governing the jump process may change when the probability measure changes. However, in the case of a representative investor with power utility function, the volatility of jump size σ_Q does not change.

coefficient γ. Note that the compensation for the jump risk is reflected in the changed jump intensity as well as the changed distribution of the jump size. In contrast, the compensation for the diffusive risk requires only a change in the drift of the processes. This is caused by the need of compensating the jump risk.

In contrast to the complete market setting of Black and Scholes (1973), the additional sources of uncertainty, in particular, the random jump sizes, introduced in our setting make the market incomplete with respect to the risk-free asset, the underlying stock, and any finite number of option contracts. Consequently, the change of probability measure is not unique. We use the equilibrium with a CRRA representative investor to identify one change of probability measure. It turns out that this particular change of probability measure involves changing the jump size and intensity.

The price f of a European call option with strike price K and maturity date T is:[9]

$$f(S, V, \lambda^*, t; K, T) = S - \frac{e^{-r\tau}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} \frac{K^{ik+1}}{k^2 - ik} e^{-ik(r\tau + \ln S) + A^*(\tau) + B^*(\tau)^\top U^* + U^{*\top} C^*(\tau) U^*} dk, \quad (33)$$

where $i = \sqrt{-1}$, k is the integration variable, $U^* \equiv \begin{pmatrix} \sqrt{V} \\ \sqrt{\lambda^*} \end{pmatrix}$, $A^*(\tau)$ is a scalar function, $B^*(\tau) = \begin{pmatrix} B_V^* \\ B_\lambda^* \end{pmatrix}$ is a 2×1 matrix function, and $C^*(\tau) = \begin{pmatrix} C_{VV}^* & C_{V\lambda}^* \\ C_{V\lambda}^* & C_{\lambda\lambda}^* \end{pmatrix}$ is a 2×2 symmetric matrix function. A^*, B^*, and C^* solve the following system of ODEs with initial conditions $A^*(0) = 0$, $B^*(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $C^*(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

$$A^{*\prime} = \frac{1}{2}\Pi^{*\top} B^* + \frac{1}{8} B^{*\top} \Gamma^* B^* + \frac{1}{4} \text{tr}(\Gamma^* C^*), \quad (34)$$

$$B^{*\prime} = \frac{1}{2}\left(\Lambda^{*\top} + C^* \Gamma^*\right) B^* + C^* \Pi^*, \quad (35)$$

$$C^{*\prime} = \Theta^* + \frac{1}{2} C^* \Lambda^* + \frac{1}{2} \Lambda^{*\top} C^* + \frac{1}{2} C^* \Gamma^* C^*, \quad (36)$$

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[9]Although it contains a complex integral, the result is real.

where "tr" is the trace of a matrix, and the matrices Θ^*, Π^*, Λ^* and Γ^* are defined as:

$$\Theta^* \equiv \begin{pmatrix} -\frac{1}{2}(k^2 - ik) & 0 \\ 0 & ik\mu_Q^* + e^{-ik\ln(1+\mu_Q^*)-\frac{1}{2}(k^2-ik)\sigma_Q^2} - 1 \end{pmatrix}, \tag{37}$$

$$\Pi^* \equiv \begin{pmatrix} \kappa_V^* \\ \kappa_\lambda^* \end{pmatrix}, \tag{38}$$

$$\Lambda^* \equiv \begin{pmatrix} \kappa_{VV}^* - ik\rho_{SV}\sigma_V & \kappa_{V\lambda}^* \\ \kappa_{\lambda V}^* - ik\rho_{S\lambda}\sigma_\lambda & \kappa_{\lambda\lambda}^* \end{pmatrix}, \tag{39}$$

$$\Gamma^* \equiv \begin{pmatrix} \sigma_V^2 & \rho_{V\lambda}\sigma_V\sigma_\lambda^* \\ \rho_{V\lambda}\sigma_V\sigma_\lambda^* & \sigma_\lambda^{*2} \end{pmatrix}. \tag{40}$$

This formula involves the inverse Fourier transform of an exponential of a quadratic form of the state variables, $\sqrt{V}$ and $\sqrt{\lambda^*}$. The ODEs that define A^*, B^*, and C^* can be easily solved numerically. Again, the Appendix presents the gruesome algebra.

In the Appendix we also derive the density function $\varphi(R; V, \lambda^*, \tau)$ of the stock return distribution with horizon τ under the risk-adjusted probability measure:

$$\varphi(R; V, \lambda^*, \tau) = \frac{e^{-r\tau}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-ik(R-r\tau)+A^*(\tau)+B^*(\tau)^\top U^* + U^{*\top} C^*(\tau)U^*} dk, \tag{41}$$

where U^*, A^*, B^*, and C^* are defined as before. This density function can be used to price any European option on the stock. Furthermore, we can plot it to further our understanding of the dynamics of the stock return implied by our model.

3 Estimation

In this section we discuss the data and the econometric method used to estimate the model and filter the time series of diffusive volatility and jump intensity.

3.1 Data

For our calibration exercise, we use the European S&P 500 index options traded on the Chicago Board Options Exchange (CBOE) in the period of January of 1996 to December of

2002 obtained from RiskMetrics. The S&P 500 index and its dividends are obtained from Datastream. The interest rates are LIBOR (middle) rates also obtained from Datastream.

Since the stocks within the S&P 500 index pay dividends whereas our model does not account for payouts, we adjust the index level by the expected future dividends in order to compute the option prices. Realized dividends are used as a proxy for the expected dividends. The dividend-excluded stock price corresponding to the maturity of an option is derived by subtracting the present value of the future realized dividends until maturity from the current index level. Interest rates are interpolated to match the maturities of the options.

We estimate our model at monthly frequency. We collect the index level, interest rates, and option prices on the first trading day of each month. To ensure that the options we use are liquid enough, we choose contracts with maturity shorter than 210 days and moneyness between 0.95 and 1.15. We also exclude options with no trading volume and options with opening interest less than 100 contracts. We only use put options in our study as they are more liquid than call options. For each contract, we use the average of the bid and ask prices as the value of the option. We exclude options with prices less than $1/8 to mitigate market microstructure problems. Finally, we check for no-arbitrage violations in option prices. We end up with 84 trading days and 2,067 option prices in our sample, or roughly 25 options per day.

Table 1 reports the average implied volatility of the options in the sample. Rather than tabulating the option prices, we show the Black-Scholes implied volatilities since they are easier to interpret.[10] We divide all options into six buckets according to moneyness (stock price divided by the strike price) and time to maturity: moneyness less than 1, between 1 and 1.03, and above 1.03; time to maturity less than 30 days, between 30 and 60 days, and greater than 60 days. Note that when moneyness is greater than 1, the put options are out of the money. The average implied volatility across all options in our sample was 22.07 percent. For a fixed maturity, we see that the implied volatilities decrease with the strike price. This is the well-known "volatility smirk". During our sample period, the term structure of implied volatilities was on average flat. The first panel of Figure 1 plots the time series of the implied volatilities of the short-term (maturity less than 30 days) options with three different levels of moneyness. We can see that the implied volatility changes

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[10]Here, we use the Black-Scholes model to invert option prices for implied volatilities. This does not mean that the options are priced in the market according to that model and, indeed, we will use our model with stochastic volatility and jumps to price the options in the empirical section below.

substantially over time and that there are changes in the steepness of the smirk. The spike in the implied volatilities observed in the Fall of 1998 corresponds to the Russian default crisis and Long Term Capital Management debacle. The second panel of Figure 1 plots the time series of the implied volatilities of the at-the-money options with short and long times to maturity. It shows that there is variation in the slope of the term structure through time.

3.2 Econometric Method

We calibrate our model to the data using an approach similar to Bakshi, Cao, and Chen (1997). Denote the vector of parameters under the risk-adjusted probability measure by $\theta^* \equiv (\mu_Q^*, \sigma_Q, \kappa_V^*, \kappa_{VV}^*, \kappa_{V\lambda}^*, \kappa_\lambda^*, \kappa_{\lambda V}^*, \kappa_{\lambda\lambda}^*, \sigma_V, \sigma_\lambda^*, \rho_{SV}, \rho_{S\lambda}, \rho_{V\lambda})$. Bakshi, Cao, and Chen minimize the sum of squared pricing errors (for all strikes and maturities) by choosing the model parameters and state variables in each day of their sample. This method is easy to implement but it is inconsistent with the underlying assumption that the model parameters are constant through time. Instead, we keep the model parameters fixed throughout the sample and allow only the state variables to change. That is, we optimize with respect to a different V_t and λ_t^* in each sample date and a single vector θ^* through the sample.

Our estimation method also differs from Bakshi, Cao, and Chen (1997) in that we minimize errors in implied volatilities, not option prices.[11] We use implied volatilities to increase the robustness of the estimation. Unlike option prices, implied volatilities have similar magnitudes and standard deviations across moneyness and maturity. This ensures that we give the same weight in the estimation to all the options. In contrast, using errors in option prices tends to give more weight to errors in options with larger (and more volatile) prices.

To be specific, let IV_n be the Black-Scholes volatility implied from the market price of the n-th option, and $\widehat{IV}_n(V_t, \lambda_t^*, \theta^*)$ be the Black-Scholes volatility of the same option implied by the price given by the model with parameters θ^*, volatility V_t, and jump intensity λ_t^*. We estimate the model parameters and the time series of the state variables by minimizing the sum of squared errors in implied volatility:

$$\min_{\theta^*, \{V_t\}, \{\lambda_t^*\}} \sum_{t=1}^{84} \sum_{n=1}^{N_t} \left(IV_{t,n} - \widehat{IV}_{t,n}(V_t, \lambda_t^*, \theta^*) \right)^2, \tag{42}$$

---

[11]This method is also used by, for example, Broadie, Chernov, and Johannes (2004).

where N_t is the number of options in the sample date t. The estimation is carried out simultaneously in the entire panel data set. This is a nonlinear least square problem that can be solved numerically. The standard errors of the parameter estimates are computed from the Hessian matrix.

Note that the econometric method we use is not efficient in that it does not take into account the transition density of the state variables between successive sample dates. Instead, the variables are filtered each date to minimize the sum of squared pricing errors on that date, irrespective of what values were filtered for the state variables in the previous date. We could think of using more efficient estimation methods such as those in Bates (2000), Broadie, Chernov, and Johannes (2004), Eraker (2004), and Pan (2002) to estimate the model. Unfortunately, computational constraints prevent us from doing it.

4   Empirical Results

In this section we discuss the empirical results. We present the model estimates and discuss the performance of the model in pricing options.

4.1   Model Estimates

We denote our model of stochastic volatility and stochastic jump intensity by SV-SJ. It contains the pure stochastic volatility model and constant jump intensity model as special cases. In the stochastic volatility model, SV, $\mu_Q^* = \sigma_Q = \kappa_{V\lambda}^* = \kappa_{\lambda}^* = \kappa_{\lambda V}^* = \kappa_{\lambda\lambda}^* = \rho_{S\lambda} = \rho_{V\lambda} = 0$. In the constant jump intensity model, SV-J, $\kappa_{V\lambda}^* = \kappa_{\lambda}^* = \kappa_{\lambda V}^* = \kappa_{\lambda\lambda}^* = \rho_{S\lambda} = \rho_{V\lambda} = 0$ and $\lambda_t^* = \bar{\lambda}^*$ is a constant.

Table 2 reports the estimated parameters for the three models. We can compare the parameter estimates for the SV model with the estimates reported by Bakshi, Cao, and Chen (1997) and Bates (2000). However, notice that their SV model is the square-root model of Heston (1993) whereas ours is the model of Stein and Stein (1991). Also, their sample periods are different from ours. Bakshi, Cao, and Chen use S&P 500 index options data from 1988 to 1991 and Bates uses S&P 500 index futures options data from 1988 to 1993.

In Bakshi, Cao, and Chen (1997) and Bates (2000), the square-root of the long-run mean of V is 18.7 percent and 25.9 percent, respectively. Our estimate of the long-run mean of $\sqrt{V}$, given by κ_V/κ_{VV}, is 25.9 percent, which is similar to Bates' number. The mean-reversion speed is 1.15 and 1.49 in Bakshi, Cao, and Chen (1997) and Bates (2000), whereas it is 0.874 in our paper, implying stronger volatility persistence. The volatility of volatility is 0.39 and 0.74 in Bakshi, Cao, and Chen (1997) and Bates (2000), and it is 0.564 in our paper which is right in the middle of their estimates. The correlation between the stock and volatility processes is -0.64 and -0.57 in Bakshi, Cao, and Chen (1997) and Bates (2000), and it is -0.93 in our paper.

Bakshi, Cao, and Chen (1997) also estimate an SV-J model. In this case, the square-root of their long-run mean of V is 18.7 percent, whereas our estimate of the long-run mean of $\sqrt{V}$ is 23.3 percent. The mean-reversion speed is 0.98 in Bakshi, Cao, and Chen (1997) and 1.007 in our paper. The volatility of volatility is 0.42 by Bakshi, Cao, and Chen (1997) and 0.54 by us. Correlation is -0.76 and -0.90 in their paper and ours respectively. The mean jump size is -0.05 in Bakshi, Cao, and Chen (1997) and -0.185 in our paper. In summary, our estimates for the restricted SV and SV-J models are compatible with the findings in other studies despite the differences in the datasets and models.

Most importantly, models SV and SV-J are both rejected with p-values of zero under a likelihood ratio test based on the sum of squared pricing errors. We therefore concentrate most of our attention on the SV-SJ model. All the coefficients of the model are significant at any conventional level of significance. Table 3 reports summary statistics for the filtered time series of $\sqrt{V_t}$ and λ_t^* which are plotted in Figure 2.

The average level of volatility is 18.3 percent and the average level of jump intensity, loosely speaking the annualized probability of a jump in the next instant, is 16.5 percent. The average jump size is -31.6 percent, which strikes us as quite large relative to the magnitude of jumps observed in the time series of returns.

Both the volatility and jump intensity time series exhibit substantial variation through time. The diffusive volatility varies between 10 and 35 percent. The jump intensity varies from virtually zero at times to almost 67 percent during financial crisis. Interestingly the two risk sources, although correlated, can display very different behavior: from times of high diffusive and jump risks as in the Fall of 1998, to times when diffusive risk is high but jump risk is low as in the beginning of 2001, to times when both risks are low as in the beginning of 1996.

The filtered time series of volatility from the SV-SJ model is quite similar to those from the other models. Of course, the filtered volatility tends to be lower in the SV-SJ model than in the SV model since the stochastic volatility in the latter model needs to account for all the risk, including the jump risk.

Given the coefficient estimates, the drift of the variance process is mean reverting at roughly the same speed for all values of the jump intensity. The drift of the jump intensity is very fast mean reverting when the variance is high but is close to zero for low levels of variance. The filtered time series of stochastic volatility and jump intensity show auto-correlations of 0.537 and 0.721 respectively.

The standard deviation of the variance process estimated from the coefficient σ_V times the square root of the mean of the variance is roughly 2.7 times that mean. This is a very high volatility of volatility. Indeed, it is much too high relative to the standard deviation of the filtered volatility process where the volatility of the increments is only 0.58 times the mean of the variance. Similar puzzling findings are reported by Bakshi, Cao, and Chen (1997) and Bates (2000). A similar calculation of the standard deviation of the jump intensity process implied from the estimated coefficient σ_λ^* shows that the volatility of changes in λ^* is only 0.07 times the average level of λ^*. In contrast, the filtered time series of the jump intensity displays considerably more variation, with a comparable statistic of 0.52. These differences between the volatilities of the state variables implied from the parameter estimates and the volatilities of the filtered time series of the state variables are undoubtedly due to the inefficiency of our estimation method which does not take into account the time series properties of the volatility and jump intensity processes.

The estimated correlation coefficients show that the increments of the diffusive volatility and jump intensity are highly correlated at 0.76. In contrast, a similar correlation computed from the filtered time series of V and λ^* is only 0.11 percent. Increments of the diffusive volatility are highly negatively correlated with stock returns, -0.80, and this is corroborated in the filtered time series. Changes in jump intensity are also negatively correlated with stock returns, albeit with a smaller absolute value, -0.23.

Overall, our results are also consistent with the recent literature on multi-factor variance models (Alizadeh, Brandt, and Diebold (2002), Chacko and Viceira (2003), Chernov, Gallant, Ghysels, and Tauchen (2002a), Engle and Lee (1999), and Ghysels, Santa-Clara, and Valkanov (2004)) which finds reliable support for the existence of two factors driving

the conditional variance. The first factor is found to have high persistence and low volatility, whereas the second factor is transitory and highly volatile. The evidence from estimating jump-diffusions with stochastic volatility points in a similar direction (Jorion (1989), Anderson, Benzoni, and Lund (2002), Chernov, Gallant, Ghysels, and Tauchen (2002a), Chernov, Gallant, Ghysels, and Tauchen (2002b), and Eraker, Johannes, and Polson (2003)). For example, Chernov, Gallant, Ghysels, and Tauchen (2002a) show that the diffusive component is highly persistent and has low variance, whereas the jump component is by definition not persistent and is highly variable.

4.2 Option Pricing Performance

The RMSE (root mean squared error) of the SV-SJ model is 0.675 percent, or roughly two thirds of one unit of the Black-Scholes implied volatility. This pricing error is well within the average bid-ask spread in our sample which is 1.12 percent (with a standard deviation of 0.67 percent), again in units of the Black-Scholes implied volatility. Moreover, allowing the jump intensity to vary stochastically proves to be quite important for options pricing: the RMSE of our model is roughly half the RMSEs of the SV and SV-J models.

Figure 3 plots the market implied volatilities of options with approximately one and a half months to maturity together with the fitted implied volatilities of the three alternative pricing models in four different dates of the sample. We find that the SV-SJ model does a much better job at pricing the cross section of options than the other two. Figure 4 focuses on a single day of the sample, December 1, 1997, and compares the market implied volatilities with the model fitted implied volatilities for options of different maturities and strikes. Again, the gains from having both stochastic volatility and jumps are apparent, especially in fitting the smile of very short-term options.

Having established that our model can accurately capture the time series and cross section properties of option prices, we now try to improve our understanding of the model. In particular, we want to understand the relative roles of the diffusive volatility and jump intensity in pricing options. Figure 5 shows the plots of implied volatility smiles at different maturities produced by our model, using the estimated parameters and for different values of volatility and jump intensity. In the first two cases, the diffusive volatility is fixed at its sample average while the jump intensity is either at its sample average or one standard deviation above and below it. In the next two cases, the jump intensity is fixed at its

sample average while the diffusive volatility is either at its sample average or one standard deviation above and below it. The time to maturity is either 30 days or 90 days. Both volatility and jump intensity impact the level of implied volatilities. The persistence in both risk components guarantees that their effects are felt at long horizons. Jump intensity has a large impact on the prices of short-term out-of-the-money puts (high S/K), thereby affecting the slope of the smile in the short term. The longer the maturity, the flatter of volatility smiles, reflecting mean reversion in the volatility and jump intensity processes.

Figure 6 shows the estimated risk-adjusted probability density function for stock returns with one month horizon. The figure shows how the risk-adjusted density function changes with changes in the diffusive volatility and the jump intensity. The first panel keeps the diffusive volatility at its sample average and displays the density functions for the jump intensity at its sample average and that value plus or minus one standard deviation. The second panel keeps the jump intensity at its sample average and displays the density functions for the diffusive volatility at its sample average and that value plus or minus one standard deviation. Again, we see that both the volatility and jump intensity impact the distribution of stock returns. The higher values of V and λ^* make the stock return distribution more volatile, putting more mass in the tails. The effect of jump intensity is lower around the mean and stronger in the left tail than that of volatility.

5 Option-Implied Risks and the Equity Premium

In this section we study the impact of the diffusive and jump risks on the distribution of stock returns under the objective probability measure. We pay special attention to the equilibrium equity premium implied by the parameter estimates and the filtered state variables.

5.1 From the Risk-Adjusted to the Objective Return Distribution

In order to obtain the distribution of stock returns under the objective probability measure, we fix the risk aversion coefficient of the representative investor at $\gamma = 2$. In an economy without jumps and with constant volatility, Merton (1973) shows that the equity premium demanded by an investor who holds the stock market is equal to γ times the market's variance. Since the realized volatility in our sample was 17 percent, using a risk aversion

coefficient of 2, we obtain an unconditional equity premium of 5.8 percent (2×0.17^2). This premium approximately matches the historic average excess stock market return of between 4 and 9 percent (depending on the sample period) reported by Mehra and Prescott (2003). Note that we are studying the portfolio choice of an investor who derives utility from next period's wealth, not utility from lifetime consumption. In the latter case, it is well know from Mehra and Prescott (2003) and much subsequent work that a much higher level of risk aversion is needed to match the historic equity premium.

In what follows, we keep the horizon of the representative investor at 1 month, $T = 1/12$. The choice of a short horizon abstracts away from hedging demands, making the interpretation of the results simpler. We have tried horizons of up to one year with no significant qualitative change in the results.

Consider our economy with the parameters estimated in Table 2 and the filtered risk processes. In order to change from the risk-adjusted coefficients estimated from option prices to the similar coefficients under the objective probability measure, we use the relations (27) through (32). Table 4 reports the model parameters under both probability measures. The most notable change is in the average jump intensity which is 0.165 under the risk-adjusted probability measure and is half that under the objective probability measure, 0.078. This makes intuitive sense as the risk-adjusted density function puts more mass on bad outcomes. The last two rows of Table 3 show summary statistics of the filtered jump intensity process under the two probability measures, confirming that the level of jump intensity changes by a factor of approximately 2. Most other parameters are either unchanged or change little.

Figure 7 shows the risk-adjusted density function extracted from option prices and the corresponding density function under the objective probability measure for our representative investor. The densities are shown for a horizon of one month and evaluated at the average levels of the volatility and the jump intensity. It can be seen that the risk-adjusted density shifts mass to the tails, and especially to the left tail. Table 5 contains statistics of the excess stock return distribution under the objective and risk-adjusted probability measures for different values of volatility and jump intensity. We focus our attention on the results for the objective probability measure. The risk-adjusted distribution is qualitatively similar, with a mean equal to the risk-free rate and with a jump intensity that is roughly double the intensity under the objective measure. In the base case (the second and fifth rows), the volatility and jump intensity are at their sample averages of 0.183 and 0.078, respectively. In the other cases, either the volatility or the jump intensity are fixed at the sample average

while the other state variable is one standard deviation above or one standard deviation below its sample average. The last row of the table presents the same statistics for the sample of excess stock returns. For the base case, the distribution of excess stock returns under the objective probability measure is more volatile, more left skewed, and more leptokurtic than the sample distribution. In this sense, the distribution implicit in option prices together with the equilibrium conditions is substantially riskier than what was realized in our sample.

Furthermore, it can be seen that the volatility and jump intensity have different impact on the distribution of excess stock returns. The higher the value of volatility or jump intensity, the higher the values of the mean and standard deviation of excess stock returns. The value-at-risk is also higher for higher value of volatility or jump intensity. But higher values of jump intensity lead to higher skewness and kurtosis than with volatility.

We have used the preferences of a "reasonable" representative investor to back out the stock market dynamics under the objective probability measure from the corresponding dynamics under the risk-adjusted probability measure estimated from option prices. Alternatively, we could have estimated the objective dynamics directly from the time series of stock prices. By comparing these objective dynamics with the risk-adjusted dynamics, we could extract the risk premium components and the level of risk aversion of the representative investor. This is essentially the approach taken by Pan (2002) and Bliss and Panigirtzoglou (2004). Unfortunately, that approach requires estimating the expected return on the stock market, which we cannot estimate with any precision given the short length of the time series we have. Additionally, the focus of this paper is that the time series of realized returns may not contain jumps that were nevertheless deemed possible by investors. We have therefore chosen to only use the time series of stock market returns to calibrate the risk aversion coefficient in the informal calculation done above.

5.2   The Equity Premium

Equation (14) gives us the equity premium as a function of the diffusive volatility and jump intensity. With the estimated parameters of the model, we can evaluate the coefficients of that function:

$$\phi = 2V - 0.000\sqrt{V} - 0.028V + 0.007\sqrt{V\lambda} + 0.371\lambda \tag{43}$$

Given the filtered series of the diffusive volatility and jump intensity, we can compute the average of the equity premium in our sample. This gives us an estimate of the unconditional equity premium of 10.1 percent. Note that this is different from putting the average level of the filtered series of the diffusive and jump risks in the above equation. That is the conditional premium evaluated at the average level of the state variables and is reported in Table 5 as 9.6 percent. The difference between the two numbers is due to the nonlinearity of the equity premium in V and λ. Note that this calculation does not match the average excess return of the S&P 500 index in our sample, which was actually negative, -2.2 percent. The reason is that we did not use stock returns in the calculation but only the measures of risk filtered from option prices together with the assumed level of risk aversion.

For comparison, we can calculate the equity premium demanded by an investor in an economy without jumps and with constant volatility. Merton (1973) shows that the equity premium demanded by an investor who holds the stock market in this economy is equal to γ times the variance. Since the realized volatility in our sample was 17 percent, and using again a risk aversion coefficient of 2, we obtain an unconditional equity premium of 5.8 percent (2×0.17^2).

Remember that the premium demanded by an investor with the same preferences in an economy without jumps and with constant volatility was 5.8 percent. This is slightly more than half the unconditional equity premium we computed in our economy with the risk inferred from option prices. Therefore the level of risk perceived by investors far exceeds the realized volatility. The compensation for these perceived risks is correspondingly larger.

These findings have some bearings on the discussion of the equity premium puzzle first investigated by Mehra and Prescott (1985) and recently surveyed in Mehra and Prescott (2003). The equity premium puzzle is typically stated as the historic average stock market return far exceeding the required compensation for its realized risk. It should be noted that the literature on the equity premium puzzle usually measures risk by the covariance of stock market returns with aggregate consumption growth. However, none of our calculations involves consumption and there is no way we can obtain the implied covariances between stock market returns and consumption growth from option prices. What we do show is that the risk premium demanded by an investor with utility for wealth living in an economy with the realized level of market volatility is half the premium demanded by the same investor when we take into account the risks assessed by the option market.

The equity premium puzzle is that the historic stock market premium of, say, 6 percent is much higher than the approximately 1 percent excess return warranted by the covariance of the stock market returns with consumption growth (for reasonable levels of risk aversion). Our point is that the realized covariance of the stock market returns with consumption growth is likely to understate the true risk of the market as much as the realized volatility understates the risk implicit in option prices. In our simple calculation above, we found that the ex-ante risk premium doubles when we use the option implied risks instead of the realized risk. If the same factor were to apply to the consumption based risk measure, the equity premium puzzle would be lessened.

These results indicate that there is a substantial Peso problem when assessing the riskiness of the stock market from the realized volatility. The risks investors perceive ex ante and that are therefore embedded in option prices far exceed the realized variation in stock market returns. If investors price the stock market to deliver returns that compensate them for the perceived level of risk, the equity premium can easily be double what is justified from the realized risk. This is the fundamental idea of Brown, Goetzmann, and Ross (1985): ex-post measured returns include a premium for some bad states of the world that investors deemed probable but that did not materialize in the sample. Similarly, Rietz (1988) proposed a solution for the equity premium puzzle based on a very small probability (about 1 percent) of a very large drop in consumption (25 percent). That is not far from the risks perceived by investors in the option market.

Of course, this discussion only shifts the equity premium puzzle to a puzzling large difference between the level of perceived risk and the level of realized risk: the option market predicted a lot more market crashes than what actually have occurred. For example, given the average jump size and average intensity estimated in Table 4, the stock market should experience market crashes in the magnitude of -29.5 percent once every 12.8 years. This is obviously very different from the observed frequency and magnitude of stock market jumps. The interesting finding is that the puzzlingly high risks implicit in option markets match the puzzlingly high equity premium for very reasonable preferences.

5.3   Time Variation in the Equity Premium

The previous section discussed the unconditional equity premium. We now discuss the time variation in the equity premium. The first panel of Figure 8 plots the filtered time series

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of the diffusive volatility and the jump intensity under the objective probability measure. The second plot shows the time series of the risk premium demanded by the investor in our economy, shown in equation (43).

We further decompose the premium in equation (43) into the compensation for the diffusive volatility which encompasses the first three terms that depend only on V, and the compensation for the jump risk involving the last term that depend only on λ. There is a small term that depends on the product of V and λ which shows up in the total premium but that we do not assign to the components.

It is interesting to find that there were periods of low volatility and low jump intensity (1996), periods of high volatility and high jump intensity (Fall of 1998), and periods of high volatility but low jump intensity (Spring of 2001). This clearly shows that each component of risk is to some degree autonomous. Indeed the correlation between the increments of both series is only 14.6 percent.

The plot of the time series of the equity premium shows high variability. Its standard deviation in our sample is 5.3 percent, roughly half the unconditional premium of 10.1 percent. The premium ranges from 2.3 to 31.9 percent. Furthermore, the first-order serial correlation (at monthly frequency) of the premium is 0.619 which shows some persistence but is far from having a unit root. However, we should note that all the first 10 serial correlations are positive and add up to 1.864. There is therefore memory in the equity premium that is not easily captured by a simple auto-regression.

The jump component is on average 2.9 percent, or a bit less than one third of the total premium. Its standard deviation is of the same order of magnitude, 2.1 percent. The jump premium varies between zero and 11.7 percent and can represent at times as much as near two thirds of the total premium. The jump component of the equity premium is also more persistent than the volatility component, with first-order serial correlations of 0.730 and 0.523, respectively. The sum of the first ten serial correlations is also higher, 3.950 versus 1.702.

6 Conclusion

We filter the times series of diffusive volatility and jump intensity from S&P 500 index options. These are the ex-ante risks in the stock market assessed by option investors. We find that both components of risk vary substantially over time, are quite persistent, and correlate with each other and with the stock index. Using a simple general equilibrium model with a representative investor, we translate the filtered measures of ex-ante risk into an ex-ante risk premium.

We find that the average premium that compensates the investor for the risks implicit in option prices, 10.1 percent, is about twice the premium required to compensate the same investor for the realized volatility in stock market returns, 5.8 percent. These results support the Peso explanation advanced by Brown, Goetzmann, and Ross (1985) and Rietz (1988) for the equity premium puzzle of Mehra and Prescott (1985). We also find that the ex-ante equity premium is highly volatile, taking values between 2 and 32 percent, with the component of the premium that corresponds to the jump risk varying between 0 and 12 percent.

In summary, we are able to partially explain the equity premium puzzle by using measures of risk implied from option prices which far exceed measures of realized risk. We are still left with a puzzle: like Aesop's boy, the option markets cry wolf a lot more often than the wolf actually shows up! However, it is interesting that we can link, using reasonable levels of risk aversion, the puzzlingly high equity premium observed historically with puzzlingly high risks implicit in option markets.

Appendix

Stock Market Risk Premium

First, substitute (11) into (9) and use the fact that in equilibrium $w = 1$ to get the following PDE satisfied by the value function, J:

$$
\begin{aligned}
0 = {} & J_t + rWJ_W - \frac{1}{2}W^2 J_{WW}V - \lambda W \mathrm{E}_Q\left[J_W(W(1+Q), V, \lambda, t)Q\right] \\
& + J_V\left(\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda}\right) + \frac{1}{2}J_{VV}\sigma_V^2 V \\
& + J_\lambda\left(\frac{1}{4}\sigma_\lambda^2 + \kappa_\lambda\sqrt{\lambda} + \kappa_{\lambda V}\sqrt{V\lambda} + \kappa_{\lambda\lambda}\lambda\right) + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2\lambda \\
& + J_{V\lambda}\rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda} + \lambda\mathrm{E}_Q\left[\Delta J\right].
\end{aligned}
\tag{A.1}
$$

In general there is no analytical solutions to this PDE. However, in the case of power utility function we can find one. Next, guess a solution of the following form:

$$
J(W, V, \lambda, t) = e^{r(1-\gamma)\tau}g(V, \lambda, \tau)\frac{W^{1-\gamma}}{1-\gamma},
\tag{A.2}
$$

where $g(V, \lambda, \tau)$ is a function independent of W. Substituting (A.2) into (A.1) to get:

$$
\begin{aligned}
g_\tau = {} & \left(-\frac{1}{2}\gamma(\gamma-1)V + \lambda\mathrm{E}_Q\left[(1+\gamma Q)(1+Q)^{-\gamma} - 1\right]\right)g \\
& + \left(\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda}\right)g_V + \frac{1}{2}\sigma_V^2 V g_{VV} \\
& + \left(\frac{1}{4}\sigma_\lambda^2 + \kappa_\lambda\sqrt{\lambda} + \kappa_{\lambda V}\sqrt{V\lambda} + \kappa_{\lambda\lambda}\lambda\right)g_\lambda + \frac{1}{2}\sigma_\lambda^2\lambda g_{\lambda\lambda} + \rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda}g_{V\lambda},
\end{aligned}
\tag{A.3}
$$

with the initial condition $g(V, \lambda, 0) = 1$. (A.3) is a hyperbolic PDE whose coefficients are quadratic functions of $\sqrt{V}$ and $\sqrt{\lambda}$. Again we make a guess of g of the form:[12]

$$
g(V, \lambda, \tau) = e^{A(\tau) + B(\tau)^\top U + U^\top C(\tau)U},
\tag{A.4}
$$

where we define $U \equiv \begin{pmatrix}\sqrt{V} \\ \sqrt{\lambda}\end{pmatrix}$, and $A(\tau)$ is a function with initial condition $A(0) = 0$.

---

[12]This trick has been frequently used. See for example Ingersoll (1987) and Heston (1993).

Substitute (A.4) into (A.3) and collect terms with the same powers of V and λ. Since (A.3) holds for any values of V and λ, all the coefficients must equal zero. And this leads to the system of ODEs (15)-(16) together with:

$$A' = \frac{1}{2}\Pi^\top B + \frac{1}{8}B^\top \Gamma B + \frac{1}{4}\mathrm{tr}(\Gamma C). \tag{A.5}$$

Equation (14) is then obtained from (11), (A.2), and (A.4).

Option Pricing

Under the risk-adjusted probability measure the price, f, of a European call option with strike price K and maturity date T is a function of the state variables and time, (S, V, λ^*, t). Letting the subscripts of f represent partial derivatives, then $f(S, V, \lambda^*, t)$ satisfies the following PDE:[13]

$$
\begin{aligned}
-f_t &= -rf + \left(r - \lambda^*\mu_Q^*\right)Sf_S + \frac{1}{2}VS^2 f_{SS} + \left(\frac{1}{4}\sigma_V^2 + \kappa_V^*\sqrt{V} + \kappa_{VV}^*V + \kappa_{V\lambda}^*\sqrt{V\lambda^*}\right)f_V \\
&\quad + \frac{1}{2}\sigma_V^2 V f_{VV} + \left(\frac{1}{4}\sigma_\lambda^{*2} + \kappa_\lambda^*\sqrt{\lambda^*} + \kappa_{\lambda V}^*\sqrt{V\lambda^*} + \kappa_{\lambda\lambda}^*\lambda^*\right)f_{\lambda^*} + \frac{1}{2}\sigma_\lambda^{*2}\lambda^* f_{\lambda^*\lambda^*} \\
&\quad + \rho_{SV}\sigma_V V S f_{SV} + \rho_{S\lambda}\sigma_\lambda^*\sqrt{V\lambda^*}Sf_{S\lambda^*} + \rho_{V\lambda}\sigma_V\sigma_\lambda^*\sqrt{V\lambda^*}f_{V\lambda^*} \\
&\quad + \lambda^* \mathrm{E}_{Q^*}\left[f(S(1+Q^*), V, \lambda^*, t) - f(S, V, \lambda^*, t)\right], \tag{A.6}
\end{aligned}
$$

where E_{Q^*} is the expectation with respect to the distribution of Q^* and the boundary condition is:

$$f(S, V, \lambda^*, T) = (S - K)^+. \tag{A.7}$$

To simplify, define $x \equiv \ln S$. Then $f(x, V, \lambda^*, T)$ satisfies:

$$
\begin{aligned}
-f_t &= -rf + \left(r - \lambda^*\mu_Q^* - \frac{1}{2}V\right)f_x + \frac{1}{2}V f_{xx} + \left(\frac{1}{4}\sigma_V^2 + \kappa_V^*\sqrt{V} + \kappa_{VV}^*V + \kappa_{V\lambda}^*\sqrt{V\lambda^*}\right)f_V \\
&\quad + \frac{1}{2}\sigma_V^2 V f_{VV} + \left(\frac{1}{4}\sigma_\lambda^{*2} + \kappa_\lambda^*\sqrt{\lambda^*} + \kappa_{\lambda V}^*\sqrt{V\lambda^*} + \kappa_{\lambda\lambda}^*\lambda^*\right)f_{\lambda^*} + \frac{1}{2}\sigma_\lambda^{*2}\lambda^* f_{\lambda^*\lambda^*} \\
&\quad + \rho_{SV}\sigma_V V f_{xV} + \rho_{S\lambda}\sigma_\lambda^*\sqrt{V}\sqrt{\lambda^*}f_{x\lambda^*} + \rho_{V\lambda}\sigma_V\sigma_\lambda^*\sqrt{V\lambda^*}f_{V\lambda^*} \\
&\quad + \lambda^* \mathrm{E}_{Q^*}\left[f(x + \ln(1+Q^*), V, \lambda^*, t) - f(x, V, \lambda^*, t)\right]. \tag{A.8}
\end{aligned}
$$

---

[13]A version of Ito's lemma for jump-diffusions is used in deriving the PDE. See for example Protter (1990).

We now use the Fourier transform of f to further simplify the above equation.[14] Let $\hat{f}(k, V, \lambda^*, t)$ be the Fourier transform of f with respect to x, that is:

$$\hat{f}(k, V, \lambda^*, t) \equiv \int_{-\infty}^{\infty} e^{ikx} f(x, V, \lambda^*, t) dx. \tag{A.9}$$

The boundary condition $f(x, V, \lambda^*, T) = (e^x - K)^+$ changes to:

$$\hat{f}(k, V, \lambda^*, T) = -\frac{K^{ik+1}}{k^2 - ik}.$$

If we write $k = k_r + ik_i$ where k_r and k_i are the real and imaginary parts of k respectively, then f is recovered via the inverse Fourier transform:

$$f(x, V, \lambda^*, t) = \frac{1}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ikx} \hat{f}(k, V, \lambda^*, t) dk. \tag{A.10}$$

Differentiating (A.9), integrating by parts, and changing the order of expectation and Fourier transform in the last term, (A.8) becomes:

$$
\begin{aligned}
-\hat{f}_t = {} & -(1+ik)r\hat{f} + ik\lambda^*\mu_Q^*\hat{f} - \frac{1}{2}V(k^2 - ik)\hat{f} \tag{A.11} \\
& + \left(\frac{1}{4}\sigma_V^2 + \kappa_V^*\sqrt{V} + \kappa_{VV}^* V - ik\rho_{SV}\sigma_V V + \kappa_{V\lambda}^*\sqrt{V\lambda^*} \right) \hat{f}_V + \frac{1}{2}\sigma_V^2 V \hat{f}_{VV} \\
& + \left(\frac{1}{4}\sigma_\lambda^{*2} + \kappa_\lambda^*\sqrt{\lambda^*} + \kappa_{\lambda V}^*\sqrt{V\lambda^*} - ik\rho_{S\lambda}\sigma_\lambda^*\sqrt{V\lambda^*} + \kappa_{\lambda\lambda}^*\lambda^* \right) \hat{f}_{\lambda^*} + \frac{1}{2}\sigma_\lambda^{*2}\lambda^* \hat{f}_{\lambda^*\lambda^*} \\
& + \rho_{V\lambda}\sigma_V\sigma_\lambda^*\sqrt{V\lambda^*}\hat{f}_{V\lambda^*} + \lambda^* \mathrm{E}_{Q^*}\left[(1 + Q^*)^{-ik} - 1 \right] \hat{f}. \tag{A.12}
\end{aligned}
$$

Notice that the jump variable Q^* is now separated from $\hat{f}$. If we define:

$$h \equiv e^{(1+ik)r\tau}\hat{f},$$

---

[14]This technique is used in Heston (1993), Bates (1996), and Duffie, Pan, and Singleton (2000) among others. The pricing formulas derived in these papers generally involve two integrals. Our approach here is similar to Lewis (2000) in that we only need one integral.

then (A.12) changes to:

$$-h_t = \left[-\frac{1}{2}V(k^2 - ik) + ik\lambda^*\mu_Q^*\right]h \tag{A.13}$$

$$+ \left(\frac{1}{4}\sigma_V^2 + \kappa_V^*\sqrt{V} + \kappa_{VV}^*V - ik\rho_{SV}\sigma_V V + \kappa_{V\lambda}^*\sqrt{V\lambda^*}\right)h_V + \frac{1}{2}\sigma_V^2 V h_{VV}$$

$$+ \left(\frac{1}{4}\sigma_\lambda^{*2} + \kappa_\lambda^*\sqrt{\lambda^*} + \kappa_{\lambda V}^*\sqrt{V\lambda^*} - ik\rho_{S\lambda}\sigma_\lambda^*\sqrt{V\lambda^*} + \kappa_{\lambda\lambda}^*\lambda^*\right)h_{\lambda^*} + \frac{1}{2}\sigma_\lambda^{*2}\lambda^* h_{\lambda^*\lambda^*}$$

$$+ \rho_{V\lambda}\sigma_V\sigma_\lambda^*\sqrt{V\lambda^*}h_{V\lambda^*} + \lambda^* E_{Q^*}\left[(1+Q^*)^{-ik} - 1\right]h, \tag{A.14}$$

with the initial condition:

$$h(k, V, \lambda^*, 0) = -\frac{K^{ik+1}}{k^2 - ik}. \tag{A.15}$$

To solve (A.14) with the initial condition (A.15), it is enough to solve the same equation with the initial value equal to one and then scale the solution by the r.h.s. of (A.15). Given the solution to (A.14) with the initial condition $h(k, V, \lambda^*, 0) = 1$, the option price is:[15]

$$f(S, V, \lambda^*, \tau) = S - \frac{e^{-r\tau}}{2\pi}\int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-ik(r\tau+\ln S)}\frac{K^{ik+1}}{k^2 - ik}h(k, V, \lambda^*, \tau)dk. \tag{A.16}$$

Now the problem is to find a solution to (A.14) with initial value of one. Recognizing the similarity between (A.3) and (A.14), we use the same trick by guessing a solution as:

$$h(V, \lambda^*, \tau) = e^{A^*(\tau)+B^*(\tau)^\top U^* + U^{*\top}C^*(\tau)U^*}. \tag{A.17}$$

Then by a similar calculation as before we can derive the system of ODEs (34)-(36).

Risk-Adjusted Return Density Function

To find the density function of stock returns under the risk-adjusted measure, it is enough to find the corresponding probability function $\Phi = \text{Prob}(S_T \leq K)$. Note that Φ satisfies the same PDE as the option price but with a different boundary condition: $\Phi|_{t=T} = \mathbf{1}_{\{S_T \leq K\}}$ where $\mathbf{1}$ is the indicator function. Under the Fourier transform, this boundary condition

---

[15]This can be shown by using the inverse Fourier transform and the Residue theorem as in Lewis (2000).

becomes $\frac{K^{ik}}{ik}$. Solving the PDE under this new boundary condition, we have:

$$\Phi = \frac{e^{-r\tau}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} \frac{e^{ik\left(\ln\frac{K}{S}-r\tau\right)}}{ik} h \, dk,$$

where $h = e^{A^*(\tau)+B^{*\top}(\tau)^\top U^* + U^{*\top} C^*(\tau) U^*}$ is defined as before. If we denote the stock return between time t and T by R so that $K = Se^R$, then we can differentiate Φ with respect to R to get the density function (41).

Relation Between Probability Measures

Assume that, under the objective probability measure, the option price, $f(S, V, \lambda, t)$, follows the process:

$$df = \left(r + \phi_f - \lambda\mu_{Q_f}\right) f \, dt + \sigma_{f_S} f \, dW_S + \sigma_{f_V} f \, dW_V + \sigma_{f_\lambda} f \, dW_\lambda + Q_f f \, dN, \qquad (A.18)$$

where $Q_f \equiv \left[f(S(1+Q), V, \lambda) - f(S, V, \lambda)\right]/f$ is the percentage jump in the option price and μ_{Q_f} is the average jump size. ϕ_f is the risk premium on the option.

In the presence of the option market, the representative investor allocates his wealth in the stock, the option, and the risk free asset with the portfolio weights denoted by $(w, w_f, 1 - w - w_f)$. Investor's wealth, W, then follows the process:

$$
\begin{aligned}
dW &= (r + w\phi + w_f\phi_f - \lambda\mu_{Q_W}) W \, dt + wW\sqrt{V} \, dW_S \\
&\quad + w_f\sigma_{f_S} W \, dW_S + w_f\sigma_{f_V} W \, dW_V + w_f\sigma_{f_\lambda} W \, dW_\lambda + Q_W W \, dN, \qquad (A.19)
\end{aligned}
$$

where $Q_W = wQ + w_fQ_f$ is the percentage jump in wealth and $\mu_{Q_W} = w\mu_Q + w_f\mu_{Q_f}$ is the average jump size in wealth. The value function J now solves the Bellman equation:

$$0 = \max_{w,w_f} \left[J_t + \mathcal{A}(J)\right], \qquad (A.20)$$

with:

$$
\begin{aligned}
\mathcal{A}(J) &= W J_W \left(r + w\phi + w_f \phi_f - w\lambda\mu_{Q_W} \right) + J_V \left(\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda} \right) \\
&\quad + J_\lambda \left(\frac{1}{4}\sigma_\lambda^2 + \kappa_\lambda\sqrt{\lambda} + \kappa_{\lambda V}\sqrt{V\lambda} + \kappa_{\lambda\lambda}\lambda \right) + \frac{1}{2}w^2 W^2 J_{WW} \Big[w^2 V \\
&\quad + 2w w_f \sqrt{V} \left(\sigma_{fs} + \rho_{SV}\sigma_{fv} + \rho_{S\lambda}\sigma_{f\lambda} \right) + w_f^2 \left(\sigma_{fs}^2 + \sigma_{fv}^2 + \sigma_{f\lambda}^2 \right) \\
&\quad + 2w_f^2 \left(\rho_{SV}\sigma_{fs}\sigma_{fv} + \rho_{S\lambda}\sigma_{fs}\sigma_{f\lambda} + \rho_{V\lambda}\sigma_{fv}\sigma_{f\lambda} \right) \Big] + \frac{1}{2}J_{VV}\sigma_V^2 V + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2\lambda \\
&\quad + W J_{WV}\sigma_V\sqrt{V} \left[w\rho_{SV}\sqrt{V} + w_f \left(\rho_{SV}\sigma_{fs} + \sigma_{fv} + \rho_{S\lambda}\sigma_{f\lambda} \right) \right] \\
&\quad + W J_{W\lambda}\sigma_\lambda\sqrt{\lambda} \left[w\rho_{S\lambda}\sqrt{V} + w_f \left(\rho_{S\lambda}\sigma_{fs} + \rho_{V\lambda}\sigma_{fv} + \sigma_{f\lambda} \right) \right] \\
&\quad + J_{V\lambda}\rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda} + \lambda\mathrm{E}_{Q_W} \left[J(W(1+Q_W),V,\lambda,t) - J(W,V,\lambda,t) \right]. \qquad (A.21)
\end{aligned}
$$

Differentiating (A.20) with respect to w_f and substitute in $w = 1, w_f = 0$, we obtain the risk premium on the option:

$$
\begin{aligned}
\phi_f &= -W\frac{J_{WW}}{J_W}\sqrt{V}\left(\sigma_{fs} + \rho_{SV}\sigma_{fv} + \rho_{S\lambda}\sigma_{f\lambda} \right) - \frac{J_{WV}}{J_W}\sigma_V\sqrt{V}\left(\rho_{SV}\sigma_{fs} + \sigma_{fv} + \rho_{S\lambda}\sigma_{f\lambda} \right) \\
&\quad - \frac{J_{W\lambda}}{J_W}\sigma_\lambda\sqrt{\lambda}\left(\rho_{S\lambda}\sigma_{fs} + \rho_{V\lambda}\sigma_{fv} + \sigma_{f\lambda} \right) - \lambda\mathrm{E}_Q\left[\frac{\Delta J_W}{J_W}Q_f \right]. \qquad (A.22)
\end{aligned}
$$

On the other hand, by Ito's lemma the drift and the diffusion terms of df are:

$$
\begin{aligned}
\phi_f f &= -rf + f_t + \left(r + \phi - \lambda\mu_Q \right)S f_S + \left(\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda} \right)f_V \\
&\quad + \left(\frac{1}{4}\sigma_\lambda^2 + \kappa_\lambda\sqrt{\lambda} + \kappa_{\lambda V}\sqrt{V\lambda} + \kappa_{\lambda\lambda}\lambda \right)f_\lambda + \frac{1}{2}V S^2 f_{SS} + \frac{1}{2}\sigma_V^2 V f_{VV} + \frac{1}{2}\sigma_\lambda^2\lambda f_{\lambda\lambda} \\
&\quad + \rho_{SV}\sigma_V V S f_{SV} + \rho_{S\lambda}\sigma_\lambda\sqrt{V\lambda}S f_{S\lambda} + \rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda}f_{V\lambda} \\
&\quad + \lambda\mathrm{E}_Q\left[f(S(1+Q),V,\lambda,t) - f(S,V,\lambda,t) \right], \qquad (A.23) \\
\sigma_{fs} &= S\sqrt{V}f_S/f, \qquad\qquad (A.24) \\
\sigma_{fv} &= \sigma_V\sqrt{V}f_V/f, \qquad\qquad (A.25) \\
\sigma_{f\lambda} &= \sigma_\lambda\sqrt{\lambda}f_\lambda/f. \qquad\qquad (A.26)
\end{aligned}
$$

Combining equations (A.22)-(A.26) leads to the following PDE satisfied by the option price:

$$
\begin{aligned}
-f_t =\ & -rf + \left(r - \lambda \mathrm{E}_Q\left[\frac{J_W^*}{J_W}Q\right]\right)Sf_S + \left(\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda}\right. \\
& + W\frac{J_{WW}}{J_W}\rho_{SV}\sigma_V V + \frac{J_{WV}}{J_W}\sigma_V^2 V + \frac{J_{W\lambda}}{J_W}\rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda}\right)f_V + \left(\frac{1}{4}\sigma_\lambda^2 + \kappa_\lambda\sqrt{\lambda}\right. \\
& + \kappa_{\lambda V}\sqrt{V\lambda} + \kappa_{\lambda\lambda}\lambda + W\frac{J_{WW}}{J_W}\rho_{S\lambda}\sigma_\lambda\sqrt{V\lambda} + \frac{J_{WV}}{J_W}\rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda} + \frac{J_{W\lambda}}{J_W}\sigma_\lambda^2\lambda\right)f_\lambda \\
& + \frac{1}{2}VS^2 f_{SS} + \frac{1}{2}\sigma_V^2 V f_{VV} + \frac{1}{2}\sigma_\lambda^2\lambda f_{\lambda\lambda} + \rho_{SV}\sigma_V VSf_{SV} + \rho_{S\lambda}\sigma_\lambda\sqrt{V\lambda}Sf_{S\lambda} \\
& + \rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda}f_{V\lambda} + \lambda f \mathrm{E}_Q\left[\frac{J_W^*}{J_W}Q_f\right],
\end{aligned}
\tag{A.27}
$$

where we define $J_W^* \equiv J_W(W(1+Q), V, \lambda, t)$. In the case of power utility function, the value function J has an analytical solution given by (A.2) and (A.4). Substitute this solution into (A.27) to get:

$$
\begin{aligned}
-f_t =\ & -rf + \left(r - \lambda \mathrm{E}_Q\left[(1+Q)^{-\gamma}Q\right]\right)Sf_S + \left[\frac{1}{4}\sigma_V^2 + \kappa_V\sqrt{V} + \kappa_{VV}V + \kappa_{V\lambda}\sqrt{V\lambda}\right. \\
& - \gamma\rho_{SV}\sigma_V V + \sigma_V^2\sqrt{V}\left(\frac{1}{2}B_V + C_{VV}\sqrt{V} + C_{V\lambda}\sqrt{\lambda}\right) \\
& + \rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V}\left(\frac{1}{2}B_\lambda + C_{\lambda\lambda}\sqrt{\lambda} + C_{V\lambda}\sqrt{V}\right)\right]f_V + \left[\frac{1}{4}\sigma_\lambda^2 + \kappa_\lambda\sqrt{\lambda} + \kappa_{\lambda V}\sqrt{V\lambda}\right. \\
& + \kappa_{\lambda\lambda}\lambda - \gamma\rho_{S\lambda}\sigma_\lambda\sqrt{V\lambda} + \rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{\lambda}\left(\frac{1}{2}B_V + C_{VV}\sqrt{V} + C_{V\lambda}\sqrt{\lambda}\right) \\
& + \sigma_\lambda^2\sqrt{\lambda}\left(\frac{1}{2}B_\lambda + C_{\lambda\lambda}\sqrt{\lambda} + C_{V\lambda}\sqrt{V}\right)\right]f_\lambda + \frac{1}{2}VS^2 f_{SS} \\
& + \frac{1}{2}\sigma_V^2 V f_{VV} + \frac{1}{2}\sigma_\lambda^2\lambda f_{\lambda\lambda} + \rho_{SV}\sigma_V VSf_{SV} \\
& + \rho_{S\lambda}\sigma_\lambda\sqrt{V\lambda}Sf_{S\lambda} + \rho_{V\lambda}\sigma_V\sigma_\lambda\sqrt{V\lambda}f_{V\lambda} + \lambda f \mathrm{E}_Q\left[(1+Q)^{-\gamma}Q_f\right].
\end{aligned}
\tag{A.28}
$$

On the other hand, under the risk-adjusted probability measure, $f(S, V, \lambda, t)$ satisfies equation (A.6). Then relations (27)-(32) are verified by substituting them into (A.6) to get (A.28).

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Table 1: **Implied Volatilities of S&P 500 Index Options**

This table reports the summary statistics of the Black-Scholes implied volatilities of the S&P 500 index options traded on CBOE that are used in the econometric analysis. The sample consists of beginning-of-the-month put options with time to maturity less than 210 days and moneyness between 0.95 and 1.15 in the period of January of 1996 to December of 2002. The options are divided into six buckets according to moneyness (S/K) and time to maturity (T). We report average implied volatility, the standard deviation of implied volatilities (in parentheses), and the number of options (in brackets) within each moneyness-maturity bucket.

	Days to Expiration		
Moneyness	$T \leq 30$	$30 < T \leq 60$	$T > 60$
$S/K < 1$	20.06	20.25	19.97
	(4.66)	(4.04)	(4.36)
	[340]	[186]	[57]
$1 \leq S/K < 1.03$	21.83	21.26	21.57
	(4.41)	(4.42)	(4.82)
	[342]	[209]	[165]
$S/K \geq 1.03$	24.30	23.86	23.93
	(4.83)	(5.00)	(4.98)
	[178]	[206]	[384]

Table 2: **Estimated Parameters**

This table reports the estimated parameters and the standard errors (in parentheses) under the risk-adjusted probability measure for the three models: stochastic volatility model (SV), constant jump intensity model (SV-J), and stochastic jump intensity model (SV-SJ). The parameters are estimated by minimizing the sum of squared implied volatility errors – the difference between the market implied volatility and the model-determined implied volatility.

	SV	SV-J	SV-SJ
κ_V^*	0.226	0.235	0.621
	(0.026)	(0.017)	(0.007)
κ_{VV}^*	-0.874	-1.007	-5.791
	(0.08)	(0.015)	(0.018)
$\kappa_{V\lambda}^*$	-	-	1.614
	-	-	(0.027)
σ_V	0.564	0.541	0.517
	(0.007)	(0.013)	(0.007)
κ_λ^*	-	-	0.033
	-	-	(0.007)
$\kappa_{\lambda V}^*$	-	-	-13.567
	-	-	(0.133)
$\kappa_{\lambda\lambda}^*$	-	-	2.097
	-	-	(0.001)
σ_λ^*	-	-	0.027
	-	-	(0.003)
μ_Q^*	-	-0.185	-0.316
	-	(0.014)	(0.005)
σ_Q	-	0.035	0.123
	-	(0.009)	(0.007)
ρ_{SV}	-0.929	-0.898	-0.801
	(0.002)	(0.001)	(0.001)
$\rho_{S\lambda}$	-	-	-0.226
	-	-	(0.003)
$\rho_{V\lambda}$	-	-	0.760
	-	-	(0.002)
RMSE	1.394%	1.390%	0.675%

Table 3: **Filtered Diffusive Volatility and Jump Intensity**

This table reports the summary statistics of the filtered diffusive volatility ($\sqrt{V_t}$) and jump intensity under the risk-adjusted and objective probability measures (λ_t^* and λ_t respectively). $\sqrt{V_t}$ and λ_t^* are obtained simultaneously with the model parameters from the optimization described in Table 2. λ_t is derived from λ_t^* by using equations (30) and (32) for fixed risk aversion coefficient $\gamma = 2$.

Model		Mean	Std.	Skew.	Kurt.	Max.	Min.	Autocorr.	Corr $\left(\sqrt{V_t}, \lambda_t\right)$
SV	$\sqrt{V_t}$	0.200	0.053	1.020	4.857	0.396	0.092	0.675	-
	λ_t^*	-	-	-	-	-	-	-	-
	λ_t	-	-	-	-	-	-	-	-
SV-J	$\sqrt{V_t}$	0.198	0.053	0.973	4.781	0.393	0.085	0.677	-
	λ_t^*	0.049	-	-	-	-	-	-	-
	λ_t	0.032	-	-	-	-	-	-	-
SV-SJ	$\sqrt{V_t}$	0.183	0.050	1.300	4.678	0.345	0.108	0.537	0.283
	λ_t^*	0.165	0.119	1.382	6.127	0.666	0.000	0.721	-
	λ_t	0.078	0.056	1.382	6.127	0.316	0.000	0.721	-

Table 4: **Estimated Parameters Under Both Probability Measures**
This table reports the estimated parameters for the SV-SJ model under both risk-adjusted
and objective probability measures. The risk-adjusted estimated parameters are identical
to those reported in Table 2, and the estimated parameters under the objective probability
measure are derived from the risk-adjusted estimated parameters using equations (27)-(32)
for fixed risk aversion coefficient $\gamma = 2$. We also report the average filtered jump intensity
under both risk-adjusted and objective probability measures ($\bar{\lambda}^*$ and $\bar{\lambda}$ respectively).

Risk-Adjusted Probability Measure		Objective Probability Measure	
κ_V^*	0.621	κ_V	0.622
κ_{VV}^*	-5.791	κ_{VV}	-6.603
$\kappa_{V\lambda}^*$	1.614	$\kappa_{V\lambda}$	2.342
σ_V	0.517	σ_V	0.517
κ_λ^*	0.033	κ_λ	0.023
$\kappa_{\lambda V}^*$	-13.567	$\kappa_{\lambda V}$	-9.356
$\kappa_{\lambda\lambda}^*$	2.097	$\kappa_{\lambda\lambda}$	2.097
σ_λ	0.027	σ_λ	0.019
μ_Q^*	-0.316	μ_Q	-0.295
σ_Q	0.123	σ_Q	0.123
ρ_{SV}	-0.801	ρ_{SV}	-0.801
$\rho_{S\lambda}$	-0.226	$\rho_{S\lambda}$	-0.226
$\rho_{V\lambda}$	0.760	$\rho_{V\lambda}$	0.760
$\bar{\lambda}^*$	0.165	$\bar{\lambda}$	0.078

Table 5: **Distribution of Excess Stock Returns**

This table reports the statistics of the one-month excess stock return distribution for different values of volatility and jump intensity under the risk-adjusted and objective probability measures. The estimated parameters under the two probability measures are those reported in Table 4. The moments and Value-at-Risk (VaR) of the risk-adjusted distribution are computed using the density function given by equation (41) while the corresponding statistics of the objective distribution are computed using a similar formula. The last row reports the corresponding statistics of the sample monthly excess returns of the S&P 500 index in the period of January of 1996 to December of 2002.

$\sqrt{V}$	λ^*	Mean	Std.	Skew.	Kurt.	VaR$_{99\%}$	VaR$_{95\%}$
		Risk-Adjusted Probability Measure					
0.183	0.046	0	0.190	-1.016	2.183	-0.145	-0.090
0.183	0.165	0	0.214	-1.828	5.856	-0.180	-0.095
0.183	0.284	0	0.236	-2.183	7.333	-0.270	-0.095
0.133	0.165	0	0.180	-2.928	13.06	-0.160	-0.075
0.183	0.165	0	0.214	-1.828	5.856	-0.180	-0.095
0.234	0.165	0	0.252	-1.165	2.777	-0.200	-0.110
$\sqrt{V}$	λ	Objective Probability Measure					
0.183	0.022	0.075	0.180	-0.663	1.125	-0.130	-0.080
0.183	0.078	0.096	0.193	-1.086	2.898	-0.140	-0.080
0.183	0.135	0.117	0.205	-1.359	4.228	-0.155	-0.080
0.133	0.078	0.064	0.156	-1.829	7.380	-0.120	-0.065
0.183	0.078	0.096	0.193	-1.086	2.898	-0.140	-0.080
0.234	0.078	0.137	0.233	-0.681	1.345	-0.160	-0.095
Sample		-0.022	0.168	-0.196	2.556	-0.107	-0.081

Figure 1: **Time Series of Implied Volatilities**
The first panel plots the time series of the Black-Scholes implied volatilities of the short-term (time to maturity less than 30 days) options with three different levels of moneyness: in-the-money ($S/K = 0.975$), at-the-money ($S/K = 1$), and out-of-the-money ($S/K = 1.025$). The sample consists of beginning-of-the-month S&P 500 index put options in the period of January of 1996 to December of 2002. In each sample date, three short-term options are chosen so that their moneyness are closest to 0.975, 1, and 1.025 respectively. The second panel plots the time series of the Black-Scholes implied volatilities of the at-the-money options with two maturities: short term (time to maturity less than 30 days) and long term (time to maturity more than but closest to 60 days).

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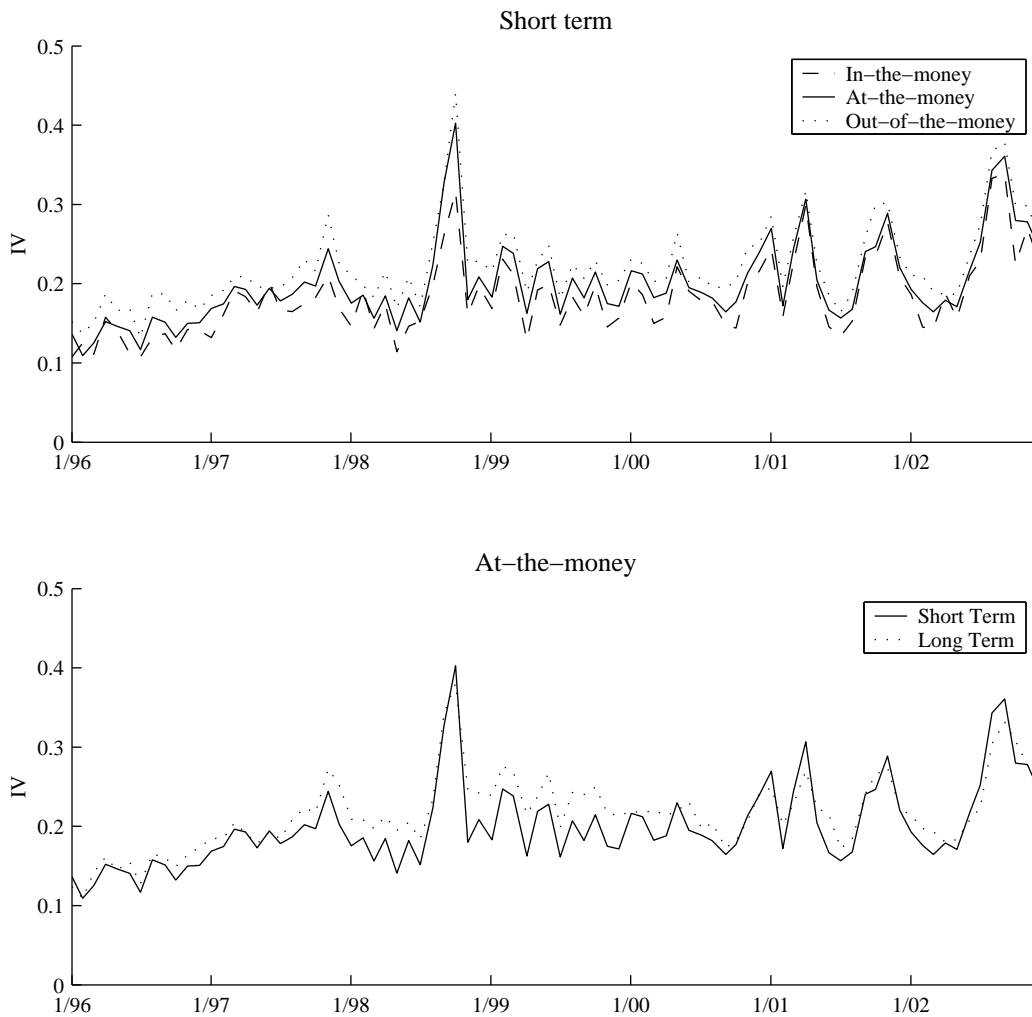
Figure 2: **Filtered Diffusive Volatility and Jump Intensity under the Risk-Adjusted Probability Measure**

The first panel plots the time series of the filtered diffusive volatility ($\sqrt{V_t}$) for the three models: SV, SV-J, and SV-SJ. The second panel plots the time series of the filtered jump intensity under the risk-adjusted probability measure (λ_t^*) for the SV-SJ model.
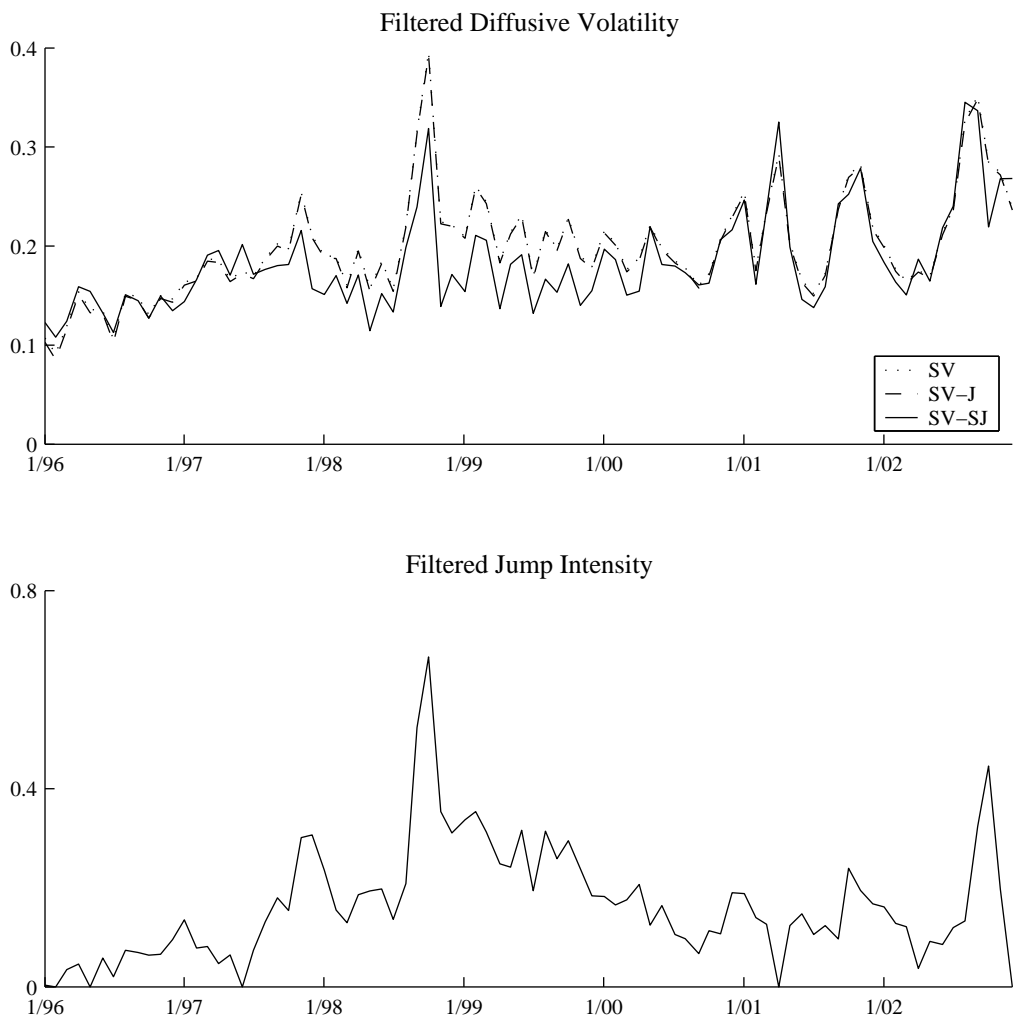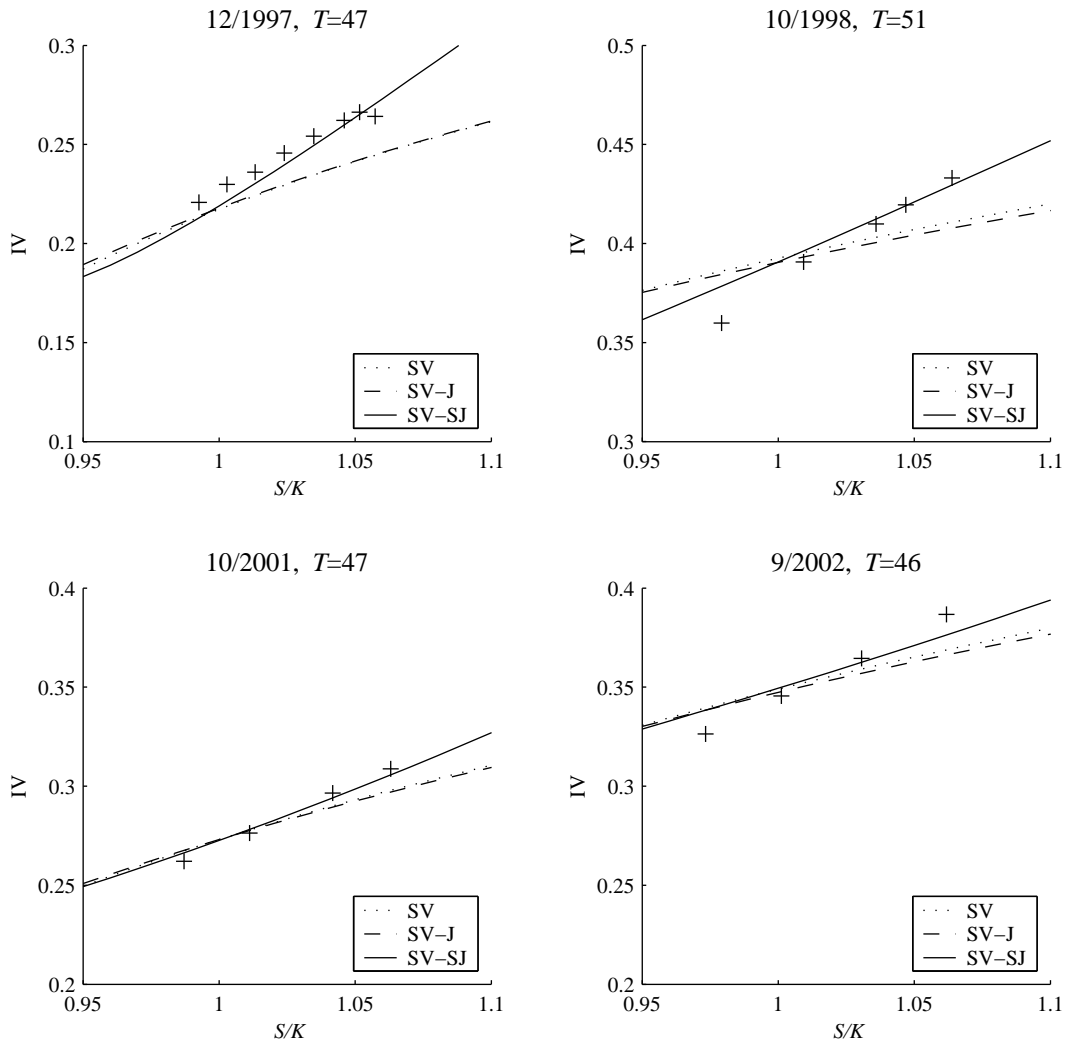

Filtered Diffusive Volatility

Filtered Jump Intensity

Figure 3: **Market and Fitted Implied Volatilities**
The four panels show the plots of the market implied volatilities as a function of moneyness for the S&P 500 index options with approximately one and a half months to maturity together with the fitted implied volatilities of the three alternative pricing models in four different dates of the sample: the first trading days in December of 1997, October of 1998, October of 2001, and September of 2002, respectively. We use the estimated parameters of the three models reported in Table 2 to compute the fitted implied volatilities. The plus signs "+" represent the market implied volatilities. The fitted implied volatilities of the SV, SV-J, and SV-SJ models are represented by the dotted, dashed, and solid lines respectively.

Figure 4: **Market and Fitted Implied Volatilities on December 1, 1997**
The two panels show the plots of the market implied volatilities as a function of moneyness for the S&P 500 index options with time to maturity less than a month and almost four months respectively together with the fitted implied volatilities of the three alternative pricing models on December 1, 1997. We use the estimated parameters of the three models reported in Table 2 to compute the fitted implied volatilities. The plus signs "+" represent the market implied volatilities. The fitted implied volatilities of the SV, SV-J, and SV-SJ models are represented by the dotted, dashed, and solid lines respectively.
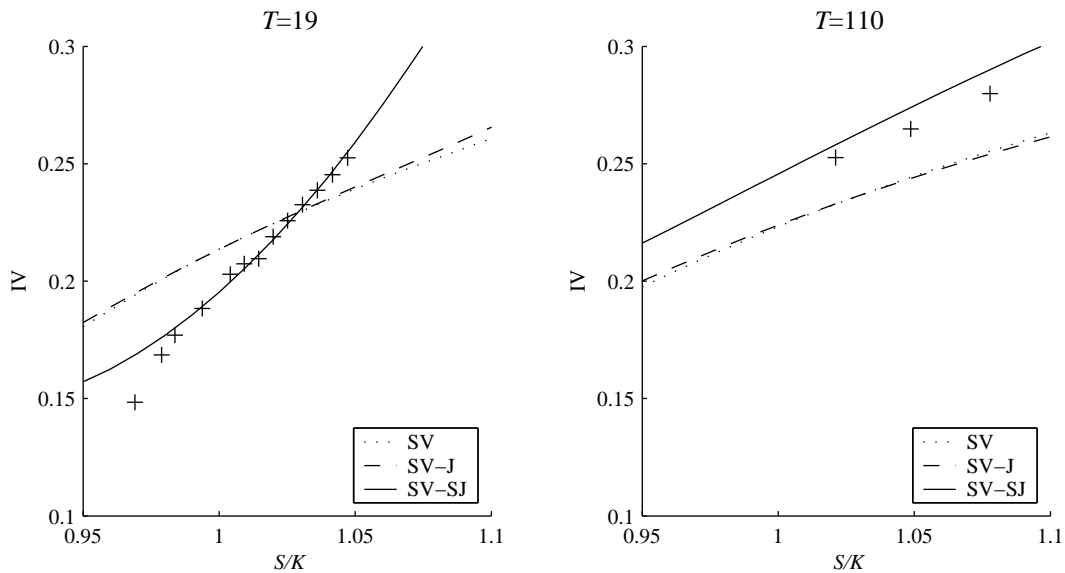
Figure 5: **Volatility Smile of the SV-SJ Model**

The four panels show the plots of the Black-Scholes implied volatility smiles at different maturities produced by the SV-SJ model, using the estimated parameters reported in Table 2 and for different values of volatility ($\sqrt{V_t}$) and jump intensity (λ_t^*). In the top two panels, $\sqrt{V_t}$ is fixed at its sample average of 0.183 while λ_t^* is at its sample average and that value plus or minus one standard deviation. In the bottom two panels, λ_t^* is fixed at its sample average of 0.165 while $\sqrt{V_t}$ is at its sample average and that value plus or minus one standard deviation. The maturities are 30 and 90 days for the left and right panels respectively.
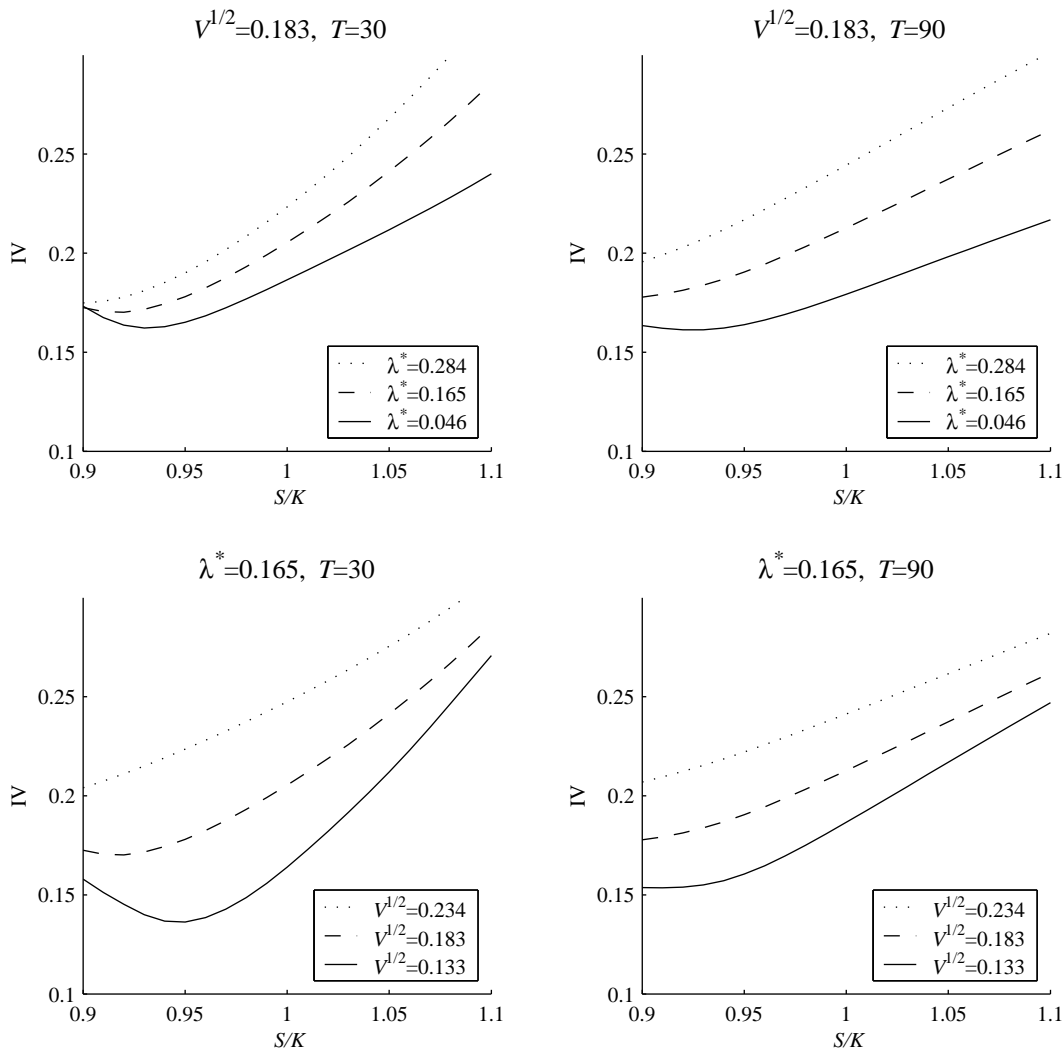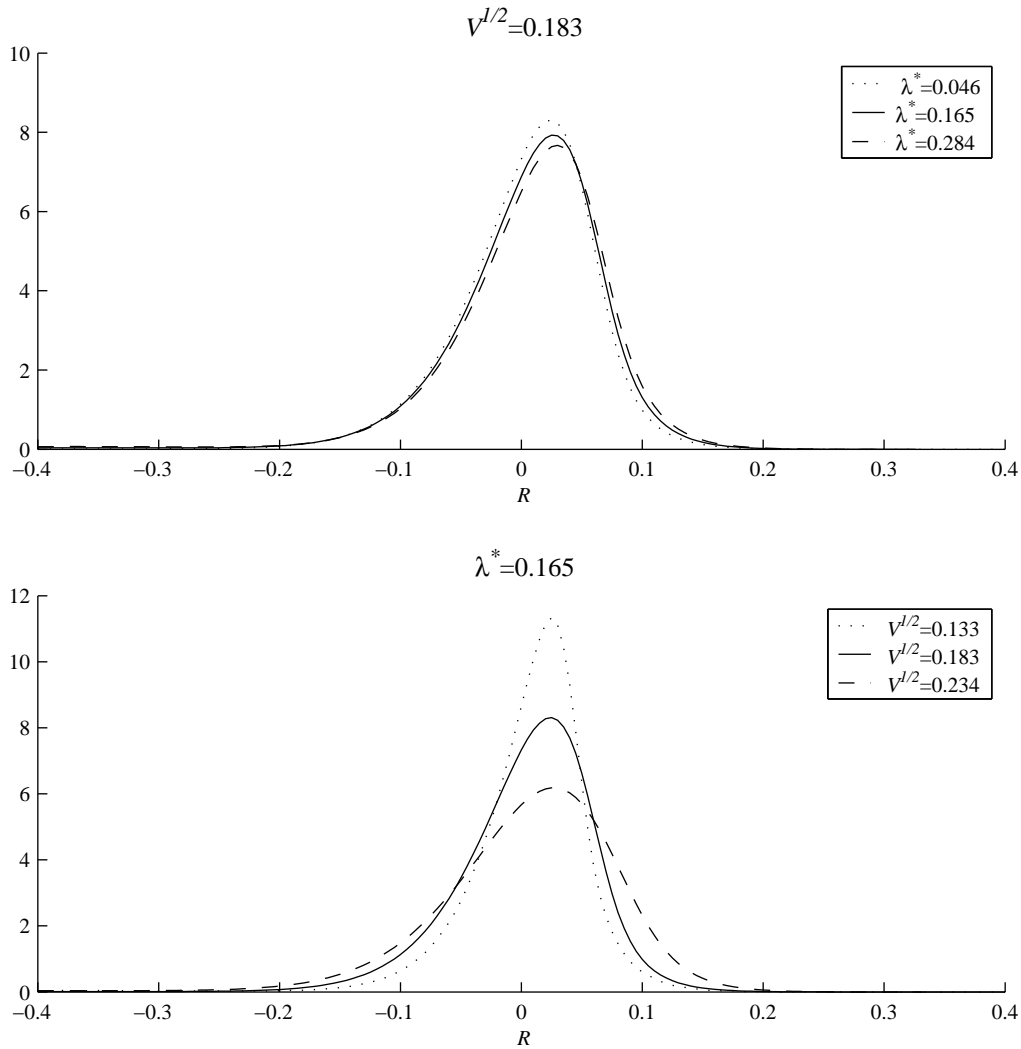
Figure 6: **Risk-Adjusted Density Function of Stock Returns**
The two panels show the plots of the estimated risk-adjusted density function of stock returns
with one month horizon under the SV-SJ model, using the estimated parameters reported in
Table 2 and for different values of volatility ($\sqrt{V_t}$) and jump intensity (λ_t^*). The first panel
keeps $\sqrt{V_t}$ at its sample average of 0.183 and displays the density functions for the jump
intensity at its sample average and that value plus or minus one standard deviation. The
second panel keeps λ_t^* at its sample average of 0.165 and displays the density functions for
$\sqrt{V_t}$ at its sample average and that value plus or minus one standard deviation.

Figure 7: **Risk-Adjusted and Objective Density Functions of Stock Returns**
This figure shows the plots of the estimated risk-adjusted density function of stock returns
with one month horizon under the SV-SJ model and the corresponding density function under
the objective probability measure for our representative investor with fixed risk aversion
coefficient $\gamma = 2$, using the estimated parameters under the two probability measures
reported in Table 4. The volatility and jump intensity are chosen at their sample averages,
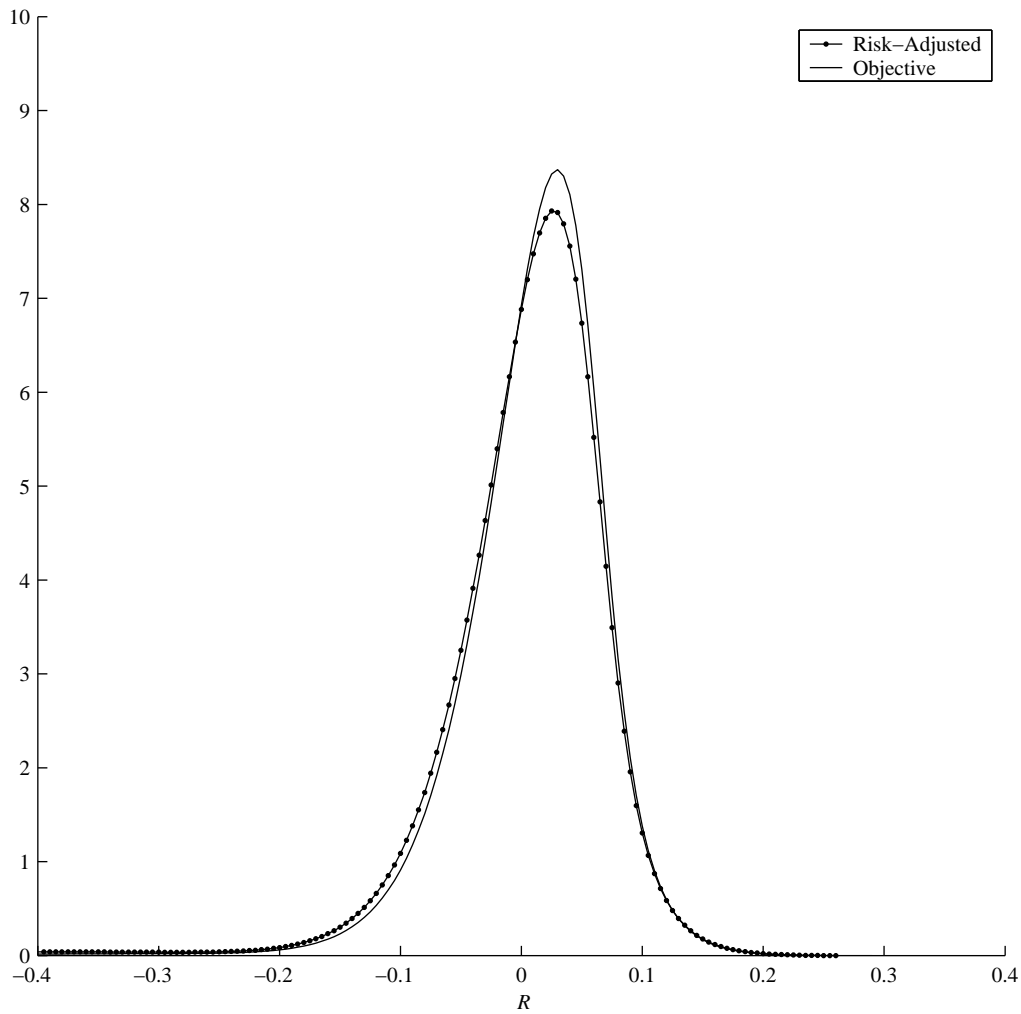$\sqrt{V} = 0.183$ and $\lambda^* = 0.165$ ($\lambda = 0.078$).

Figure 8: **Equity Premium**

The first panel shows the plots of the filtered time series of volatility (V) and jump intensity (λ) under the objective probability measure for our representative investor with fixed risk aversion coefficient $\gamma = 2$. The second panel shows the plot of the time series of the total risk premium demanded by the investor shown in equation (43) together with the plots of the time series of the volatility and jump components of the risk premium. The volatility component encompasses the first three terms in equation (43) that depend only on V, and the jump component involves the last term in equation (43) that depends only on λ.