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A CONTINUOUS-TIME AGENCY MODEL OF OPTIMAL CONTRACTING AND CAPITAL STRUCTURE

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ABSTRACT

We consider a principal-agent model in which the agent needs to raise capital from the principal to finance a project. Our model is based on DeMarzo and Fishman (2003), except that the agent's cash flows are given by a Brownian motion with drift in continuous time. The difficulty in writing an appropriate financial contract in this setting is that the agent can conceal and divert cash flows for his own consumption rather than pay back the principal. Alternatively, the agent may reduce the mean of cash flows by not putting in effort. To give the agent incentives to provide effort and repay the principal, a long-term contract specifies the agent's wage and can force termination of the project. Using techniques from stochastic calculus similar to Sannikov (2003), we characterize the optimal contract by a differential equation. We show that this contract is equivalent to the limiting case of a discrete time model with binomial cash flows. The optimal contract can be interpreted as a combination of equity, a credit line, and either long-term debt or a compensating balance requirement (i.e., a cash position). The project is terminated if the agent exhausts the credit line and defaults. Once the credit line is paid off, excess cash flows are used to pay dividends. The agent is compensated with equity alone. Unlike the discrete time setting, our differential equation for the continuous-time model allows us to compute contracts easily, as well as compute comparative statics. The model provides a simple dynamic theory of security design and optimal capital structure.

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1. Introduction

In this paper, we consider a dynamic contracting environment in which a risk-neutral agent or entrepreneur with limited resources manages an investment activity. While the investment is profitable, it is also risky, and in the short-run can generate large losses. The agent will need outside financial support to cover these losses and continue the project. The difficulty is that while the distribution of the cash flows is publicly known, the agent may distort these cash flows by taking a hidden action that leads to a private benefit. Specifically, the agent may (i) conceal and divert cash flows for his own consumption, and/or (ii) stop providing costly effort, which reduces the mean of the cash flows. Therefore, from the perspective of the principal or investors funding the project, there is the concern that a low cash flow realization may be a result of the agent's actions, rather than the project fundamentals. To provide the agent with appropriate incentives, investors control the agent's wage, and may withdraw their financial support for the project and force its early termination. We seek to characterize an optimal contract in this framework and relate it to the firm's choice of capital structure.

A discrete-time model of this sort is considered by DeMarzo and Fishman (2003), hereafter denoted DF. Here we extend their analysis to a continuous-time setting in which the cumulative cash flows generated by the investment follow a Brownian motion with a positive drift. With Brownian motion, the losses on the project over any interval of time can be arbitrarily large. An optimal contract must specify the level of losses that investors will tolerate before terminating their support. A key advantage of the stationary, continuous-time model considered here is that the optimal contracts and payoffs can be characterized in terms of an ordinary differential equation, making the solution and comparative statics simpler to quantify. It also makes the model easier to calibrate and embed within other standard finance models.

Another important contribution of the paper is methodological. We solve the model in two ways. First, we represent the project cash flows as a discrete-time binomial tree. The agency problem is that the agent may report low cash flows when they are really high, and/or stop providing costly effort, which affects the probability of the high cash flow. Given the discrete-time setting, we can apply the results of DF to describe the solution. We show that the limit of this solution as the time increments vanish leads to an ordinary differential equation that characterizes equilibrium payoffs.[1] Second, we formulate the model directly in continuous time. Using techniques similar to those introduced by Sannikov (2003), we again characterize the solution in terms of an ordinary differential equation, and show that it coincides with the limit of the discrete case. These techniques are quite powerful and may prove useful in other dynamic contracting models.

In the discrete-time setting, DF demonstrate that the optimal contract can be implemented using a combination of standard securities: equity, long-term debt, and a credit line. Dividends are paid when cash flows exceed long-term debt payments and the credit line

[1] While using a binary-tree to approximate Brownian motion is natural when modeling payoffs, in the context of an agency problem there is no guarantee that the optimal contract in the binary case will approximate the optimal contract in continuous time. See, for example, Hellwig and Schmidt (2002) for a discussion of the difficulties of this approach in the Holmstrom-Milgrom (1987) continuous-time principal-agent model.

is paid off. If long-term debt payments are not made or the credit line is overdrawn, the project is terminated with a probability that depends on the size of the cash shortfall.

In the setting of this paper we obtain a similar implementation with some distinctions. First, termination is no longer stochastic, but occurs the moment the credit line is overdrawn or there is a default on the long-term debt. Another distinction is that because the project can generate large short-term losses, projects that are very risky will not use long-term debt but instead require a compensating balance with the credit line. (A compensating balance is a cash deposit that the firm must hold with the lender to maintain the credit line.) The compensating balance serves two roles. First, it allows for a larger credit line, which is valuable given the risk of the project. Second, it provides an inflow of interest payments to the project that can be used to somewhat offset operating losses. The model therefore provides an explanation for why firms might hold substantial cash balances at low interest rates while simultaneously borrowing at higher rates.

After characterizing the implementation of the optimal contract, we compute a number of comparative statics as well as determine the dynamics of security prices. In both cases, our differential equation characterization proves very useful for the analysis.

For the bulk of our analysis, we focus on the case in which the agent can conceal and divert cash flows. We show in Section 5 that the characterization of the optimal contract is unchanged if the agent makes a hidden binary effort choice. We also consider the possibility of contract renegotiation in Section 6, and characterize the optimal renegotiation-proof contract.

1.1. A Simple Example

We illustrate briefly the nature of our results with a simple example. Suppose the cumulative cash flows of the project follow a Brownian motion with a mean of 10 and volatility of 20 per period. Figure 1 illustrates a possible sample path.

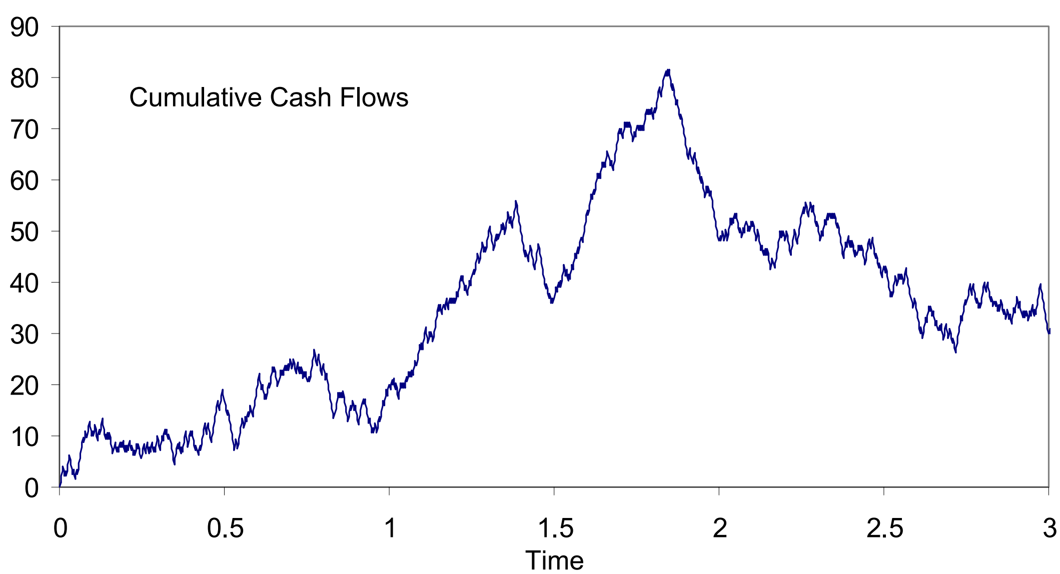


Figure 1: Sample Path of Cumulative Project Cash Flows

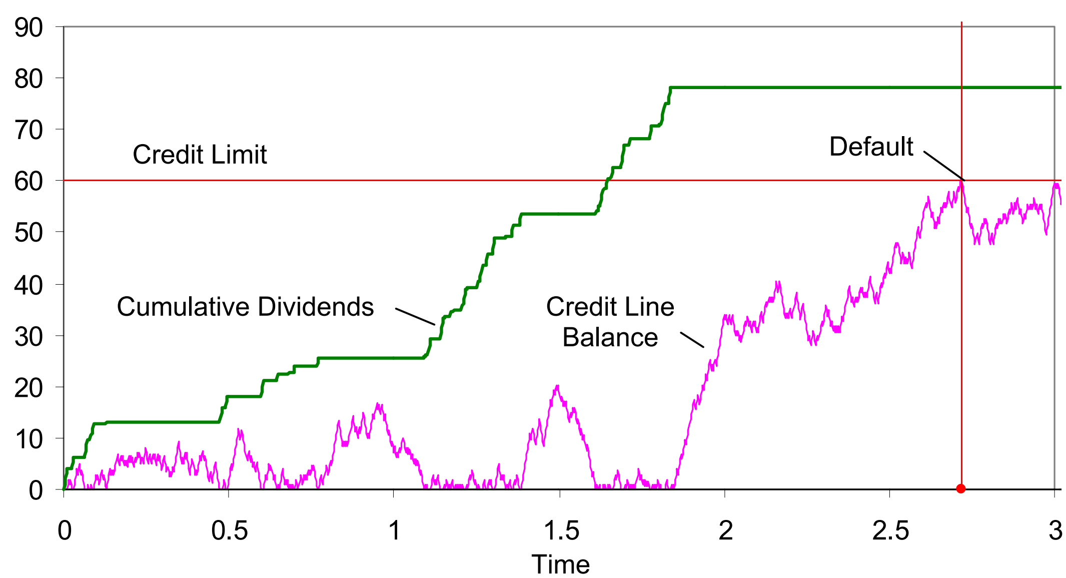
Suppose the market interest rate is 10%. Then the first best value of the project is 10/10% = 100. If the assets are worth 65 if liquidated, the project should be run forever.

Now introduce the agency problem. Suppose the agent who works for the firm can divert cash flows at a cost of 20% of the amount diverted. Or suppose that the agent can shirk and earn a private benefit of 8 per period, but that shirking reduces the mean of the cash flows from 10 to 0 per period. Assume that the agent's subjective discount rate is 15%. Then, by the methods of our paper, the following combination of securities solves the agency problem optimally:[2]

> Long-term Debt: The firm issues debt with face value of 10 and coupon rate of 10%.
>
> Credit line: The firm opens a credit line with a credit limit of 60 and an interest rate of 15%.
>
> Equity: The firm issues 20% of outside equity. The remaining 80% is held by the agent.

For the sample path shown in Figure 1, the draw on the credit line and the cumulative dividends evolve as in Figure 2. Dividends are paid only if the credit line balance is zero. Otherwise operating profits are used to pay down the credit line. In this example, the firm exhausts its credit line and defaults after 2.7 periods. In default, the creditors recover the liquidation value of 65 on total debt (long-term plus credit line) of 70.

Figure 2: Dividends and Credit line balance under the optimal contract for the sample path of Figure 1



Intuitively, because the interest on the credit line equals the agent's subjective discount rate, the agent will not draw from the credit line more than necessary to cover short-term operating losses. If the credit line is fully repaid, operating profits are used to pay dividends. The agent will not divert cash flows because he receives 80% of dividends. If

[2] We emphasize that the use of debt and equity is not assumed, but shown to be optimal.

the credit line is exhausted, the firm defaults and the project terminates. Assuming the long-term debt has seniority, in this example the long-term debt is riskless and has a market value of 10 because the liquidation value of 65 is greater than the face value of the debt. We can compute that the providers of the credit line expect profit of 3 and 20% of equity is worth 15. With these securities the agent raises 28 in external capital without drawing on the credit line.

Our analysis will demonstrate the optimality and incentive compatibility of the above set of securities for this example and show how the choice between credit line and long-term debt is determined. We will see, for example, that whether the firm issues risky debt, riskless debt, or holds a compensating cash balance will depend on the risk of the project and its liquidation value.

1.2. Related Literature.

Our paper is part of a growing literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Phelan and Townsend (1991), and Atkeson (1991) among others. (See, for example, the text by Ljungqvist and Sargent (2000) for a description of many of these models.) As previously mentioned, this paper builds directly on the model of DeMarzo and Fishman (2003). Other recent work developing optimal dynamic agency models of the firm includes Albuquerque and Hopenhayn (2001), Clementi and Hopenhayn (2000), DeMarzo and Fishman (2003b), and Quadrini (2001). With the exception of DeMarzo and Fishman (2003), these papers do not share our focus on an optimal capital structure. In addition, none of these models are formulated in continuous time.

While discrete time models are adequate conceptually, in many cases a continuous-time setting may prove to be much simpler and more convenient analytically. An important example of this is the principal-agent model of Holmstrom and Milgrom (1987), hereafter HM, in which the optimal continuous-time contract is shown to be linear. Schattler and Sung (1993) develop a more general mathematical framework for analyzing agency problems of this sort in continuous time, and Sung (1995) allows the agent to control volatility as well. Hellwig and Schmidt (2002) look at the conditions for a discrete-time principal-agent model to converge to the HM solution. See also Bolton and Harris (2001), Ou-yang (2003), Detemple, Govindaraj and Loewenstein (2001), Cadenillas, Cvitannic and Zapatero (2003) for further generalization and analysis of the HM setting.

Several features distinguish our model from the HM problem: the investor's ability to terminate the project, the agent's consumption while the project is running, and the nature of the agency problem. In HM, the agent runs the project until date *T*, and then receives compensation. In our model, the agent receives compensation many times while the project is running, until the contract calls for the agent’s termination. Also, HM analyze a setting in which the agent takes hidden actions. In our main setting the agent observes private payoff-relevant information; we also consider the possibility of a binary hidden action choice. Unlike HM, the termination decision is a key feature of the optimal contract in our setting. Here, as in DF, we demonstrate how this decision can be implemented through bankruptcy.[3]

Sannikov (2003) and Williams (2004) analyze principal-agent models, in which the principal and the agent interact dynamically. Their interaction is characterized by evolving state variables. In their models, the agent continuously chooses actions (e.g. hidden effort) that are not directly observable to the principal, and the principal takes actions (e.g. payments to the agent) that affect the agent's payoff. Besides having a dynamic nature in the spirit of Sannikov (2003) and Williams (2004), our paper develops a new method to deal with the problem of private observations in continuous time. Also, unlike in Sannikov (2003) and Williams (2004), hidden savings do not pose any additional difficulties in our model. We derive an optimal contract in a setting without hidden savings, and verify that it remains incentive compatible even when the agent can save secretly.

In contemporaneous work, Biais et al. (2004) consider a dynamic principal-agent problem in which the agent's effort choice is binary (work or shirk). While they do not formulate the problem in continuous time, they do exam the continuous limit of the discrete-time model and focus on the implications for the firm's balance sheet. As we show in Section 5, their setting is a special case of our model and our characterization of the optimal contract applies.

This paper is organized as follows. Section 2 presents a discrete-time model with binary cash flows, summarizes the optimal contract that was found by DF, and derives the form of the contract in the limit as cash flows arrive more frequently. Section 3 presents a continuous-time model, in which cash flows arrive via a Brownian motion with a positive drift. The optimal contract in the continuous-time setting is then derived, and shown to coincide with the contract in the limit of the discrete-time settings. Section 4 discusses the implementation of the optimal contract in terms of familiar securities: credit line, debt and equity. The pricing of these securities and comparative statics results are also considered. Sections 5 shows the optimality of our contract with hidden binary effort and Section 6 considers several extensions, including renegotiation-proof contracts. Section 7 concludes the paper.

2. The Discrete Time Model

There is an agent and investors. Investors are risk neutral, have unlimited capital, and value a cash flow stream $\{dC_t\}$ as $E \sum_t e^{-rt} dC_t$, where r is the riskless interest rate. The agent is also risk neutral, has limited capital, and values a cash flow stream $\{dC_t\}$ as $E \sum_t e^{-\gamma t} dC_t$, where $\gamma > r$ is the agent's subjective discount rate.[4]

The agent has a risky project that requires capital K. The agent has initial wealth $Y_0 \geq 0$. If $K > Y_0$, the agent must borrow to finance the project. Alternatively, even if $Y_0 \geq K$,

[3] Spear and Wang (2002) also analyze the decision of when to fire an agent in a discrete-time model. They do not consider the implementation of the decision through standard securities.

[4] The case $\gamma < r$ is not appropriate given an infinite horizon, as the agent could earn unbounded utility by saving at the riskless rate. If there exists a dispersion of subjective discount rates in the population, then in equilibrium investors will be the most patient types. See also the discussion at the end of Section 2.1.

given $\gamma > r$ the agent would like to borrow for consumption purposes. If the project is funded, it produces cash flows at interval dt. The cash flow at date t is given by the random variable dY_t. We assume that the cash flows $\{dY_t\}$ are i.i.d. with distribution

$$dY_t = \begin{cases} y_1 \equiv \mu\, dt + \sigma\sqrt{\frac{1-q}{q}\, dt} & \text{with probability } q \\ y_0 \equiv \mu\, dt - \sigma\sqrt{\frac{q}{1-q}\, dt} & \text{with probability } 1-q \end{cases}$$

That is, each cash flow has a Bernoulli distribution with mean $\mu\ dt$ and variance $\sigma^2\ dt$. Note that y_0 may be negative. In this case the firm must have cash, or established credit, of at least $-y_0$ at the start of each period for the project to continue.

At the end of each period, the project may be terminated. If it is terminated, the agent receives a reservation payoff $R \geq 0$, and the assets of the firm can be liquidated for $L \leq K$. We also assume that the investment is efficient, so that $rK + \gamma R < \mu$, and therefore that termination is inefficient.

We assume that cash flows up to y_0 are observable and collectible by investors, but investors do not know whether y_0 or y_1 has occurred. Specifically, the agent privately observes the realization of dY_t. Investors must rely on the agent to report this realization. Of course, the agent may lie about the cash flow in order to cheat investors. If the cash flow in period t is y_1, the agent may conceal the excess $y_1 - y_0$. The diversion of cash flows may be costly: the agent obtains only a fraction $\lambda \in (0,1]$ of concealed cash flows. When $\lambda = 1$, diversion is costless. As $\lambda \to 0$, diversion becomes impossible and the agency problem disappears. The diverted cash may be consumed immediately or saved at interest rate $\rho \leq r$. The agent's savings are unobservable to the principal, and can be used for future consumption or to exaggerate future cash flows.[5]

In contrast to the operating cash flows, liquidation of the assets is observable and contractible. This modeling reflects the idea that the agent can divert the profits but not the assets. The agent's reservation payoff R is a private benefit that is not contractible; at any time the agent may quit and terminate the project in order to receive R.

Suppose investors fund the agent. Investors do not observe the actual cash flows or their diversion, and do not observe the agent's consumption or any savings. Investors only observe the agent's payments and reports. A contract therefore specifies payments made from investors to the agent as a function only of messages sent and past payments made by the agent to investors. The contract can also specify circumstances under which control of the project passes from the agent to investors, who then terminate the project. It is this threat of termination of the project that induces the agent to pay investors some share of the cash flows. Finally, we assume the contract signed at date 0 remains in force for the life of the project. That is, the agent and investors can commit not to renegotiate. We discuss the consequences of renegotiation in Section 6.

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[5] We can also allow the agent to *costlessly* conceal and save cash flows *within the firm* and use them to exaggerate future cash flows. That is, the proportional cost of diversion, $(1-\lambda)$, need only be borne if the agent diverts the funds for personal consumption. This possibility does not alter the form of an optimal contract, and is formally considered in Section 3.1 (see Proposition 5).

2.1. The Optimal Contract

DeMarzo and Fishman (2003), or DF, describe a recursive method for determining the optimal contract in a general discrete-time setting that includes the model described above.[6] Because the future cash flows of the project are history independent, the optimal contract at any date t depends only on the promised payoff to the agent, W_t. That is, W_t is a sufficient statistic for the history of the interaction, and so is the only state variable necessary to describe an optimal contract.

There are three regions that govern the behavior of the optimal contract, determined by a liquidation boundary $W^L \geq R$ and a dividend boundary $W^1 \geq W^L$. For $W_t \in [R, W^L)$, the project is (stochastically) terminated. For $W_t > W^1$, the agent receives compensation $W_t - W^1$ in the form of cash dividends. For $W \in [W^L, W^1]$, all cash flows are paid to investors, and the agent is rewarded through the promise of future payoffs only. Since the agent is only compensated through the future payoff, the expected future payoff must be higher than the agent's current payoff to account for the agent's discount rate, γ. In addition, in order to maintain incentive compatibility, the agent's payoff must increase by λ for each dollar paid to investors. Thus, the agent's promised payoff evolves according to

$$W_{t+dt} = e^{\gamma dt} W_t + \lambda(dY_t - \mu\, dt) \tag{1}$$

The investor's future payoff can then be given in terms of the agent's through the continuation function,

$b(W)$ = maximal investor payoff given agent earns payoff $W \in [\mathrm{R}, \infty)$.

DF demonstrate that this continuation function is concave and, in the region $W \in [W^L, W^1]$, satisfies

$$b(W_t) = e^{-rdt}\left(\mu\, dt + E[b(W_{t+dt})]\right) \tag{2}$$

The intuition for equation (2) is straightforward: the investors' current payoff is the present value (at discount rate r) of this period's expected cash flow plus their expected future payoff. The future payoff is also described by the continuation function, evaluated at the agent's future payoff.

DF show that the dividend boundary W^1 is determined by the lowest payoff for the agent such that $b'(W^1) = -1$. That is, for $W < W^1$, $b'(W) > -1$, so that it is cheaper to compensate the agent using future promises than with cash. On the other hand, to provide the agent with payoff $W > W^1$, it is optimal to give the agent an immediate cash transfer from the investors of $W - W^1$. That is, in the dividend region, $W \geq W^1$, the continuation function is linear with $b'(W) = -1$. Since immediate compensation is better than deferred compensation, this implies that in this region,

$$b(W_t) \geq e^{-rdt}\left(\mu\, dt + E[b(W_{t+dt})]\right) \tag{3}$$

[6] In particular, DF do not restrict the distribution of the cash flows, or require stationarity, etc.

Finally, in the liquidation region $W \le W^L$, the project is terminated with probability $(W^L - W)/(W^L - R)$, or continued with the agent receiving payoff W^L. Thus, $b(R) = L$ and the continuation function is linear with slope $b'(W) = l \equiv \frac{b(W^L)-L}{W^L-R}$ until W^L. Since stochastic termination is better than continuing, equation (3) holds for this region as well.

DF show that when $\gamma = r$, the dividend boundary is such that no dividends are paid until the firm becomes self-financing (i.e., it can meet all future commitments solely from operating cash flows). Since in our setting operating losses over any interval are unbounded, if $\gamma = r$ it would be optimal to delay dividends and payments to the agent indefinitely. It is for this reason that we must restrict attention to the case $\gamma > r$.

2.2. The Continuous Limit

Using the characterization of the optimal contract in discrete time, we now decrease the length of the period to determine the characterization for the continuous-time limit. As $dt \to 0$, the binomial-tree process for the cash flows converges to Brownian motion with mean μ and volatility σ. In this case, equation (2) becomes a second-order differential equation for the optimal continuation function (see (4)). The liquidation and dividend boundaries determine the boundary conditions for this differential equation. Liquidation occurs if and only if the agent's payoff equals the outside option R, so that the liquidation boundary becomes $b(R) = L$. The dividend boundary is determined implicitly (see(5)).

PROPOSITION 1. *Let b be the limit of the optimal continuation function as $dt \to 0$. Then b is concave and twice continuously differentiable. The liquidation boundary $W^L \to R$ and $b(R) = L$. In the region $W \in [R, W^1]$,*

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W) \tag{4}$$

and $dW = \gamma W\,dt + \lambda\left(dY_t - \mu\,dt\right)$. Finally, W^1 satisfies

$$b'(W^1) = -1 \text{ and } rb(W^1) + \gamma W^1 = \mu\,. \tag{5}$$

For $W \ge W^1$, the agent receives an immediate cash payment of $W - W^1$ and $b(W) = b(W^1) - (W - W^1)$.

SKETCH OF PROOF: First, dW follows immediately from (1). For b, the limit of concave functions is concave. Since the agent's future continuation payoff is noisy ($\lambda^2\sigma^2 > 0$), b' must be continuous since otherwise there would be no way to achieve the payoff at a "kink." First we show that (4) holds in the region $W \in [W^L, W^1]$. Using Taylor expansions and ignoring terms that are $o(dt)$, we can rewrite (2) as follows:

$$b(W) = (1 - r\,dt)\left(\mu\,dt + E\left[b(W) + b'(W)dW + \tfrac{1}{2}b''(W)dW^2\right]\right),$$

Since $E[dW] = \gamma W\,dt$ and $E[dW^2] = \lambda^2\,\sigma^2\,dt + o(dt)$,

$$b(W) = (1 - r\,dt)\left(\mu\,dt + b(W) + \gamma W b'(W)\,dt + \tfrac{1}{2}\lambda^2\sigma^2 b''(W)\,dt\right)$$

which reduces to (4) on elimination of dt^2 terms and dividing by dt. Note that (4) also implies that b'' is continuous on (W^L, W^1).

Next we verify (5). For $W > W^1$, we can use (3) and Taylor expansions to derive

$$b(W) \geq (1 - r\,dt)\left(\mu\,dt + b(W) + \gamma W b'(W)\,dt + \tfrac{1}{2}\lambda^2\sigma^2 b''(W)\,dt\right)$$
$$= (1 - r\,dt)\left(\mu\,dt + b(W) + \gamma W b'(W)\,dt\right)$$

where we use the fact that b is linear in this region. Collecting terms and dividing by dt yields

$$rb(W) - \gamma W b'(W) \geq \mu. \tag{6}$$

On the other hand, for $W \leq W^1$, from (4),

$$rb(W) - \gamma W b'(W) = \mu + \tfrac{1}{2}\lambda^2\sigma^2 b''(W) \leq \mu \tag{7}$$

Since $b'(W^1) = -1$ by definition, together these imply $b''(W^1) = 0$, or equivalently (5).

Finally, we verify the liquidation boundary $W^L = R$. If $R \leq W < W^L$, (6) also holds. But since $b'(W) = l \geq -1$, this implies that $rb(W) + \gamma W \geq \mu$. But this contradicts the fact that liquidation is inefficient and therefore that $rb(R) + \gamma R = rL + \gamma R < \mu$. ◆

The intuition for (4) is as follows: To receive b, investors must earn total return rb. They earn this return by receiving the expected cash flow μ, less the cost of paying the agent his required return, $\gamma W b'$, less the incentive cost associated with the agent's risk, $\tfrac{1}{2}\lambda^2\sigma^2 b''$. The boundary conditions (5) are "smooth pasting" and "super contact" conditions for the optimality of W^1; at the dividend boundary both first and second derivatives are matched. Alternatively, we can interpret the condition $rb(W^1) + \gamma\, W^1 = \mu$ as the point at which satisfying the agent's and investor's rents just exhausts the expected cash flows. Finally, note that unlike the discrete-time model of DF, in continuous-time termination is no longer stochastic. Stochastic termination is required in discrete-time in order to maintain incentives when the agent's promised payoff is too close to R, since if the project is continued the agent can at a minimum steal next period's cash flow and then receive R. This is not an issue when the decision to terminate can be made continuously. An example of the optimal continuation function is shown in Figure 3.

Proposition 1 provides a characterization of the optimal contract and payoff dynamics for the limit of the discrete-time model. In this limit, cumulative cash flows follow a Brownian motion with drift μ and volatility σ. Before discussing further the properties of this solution, we first formulate the problem directly in continuous-time, and show that the characterization of the optimal contract is preserved.

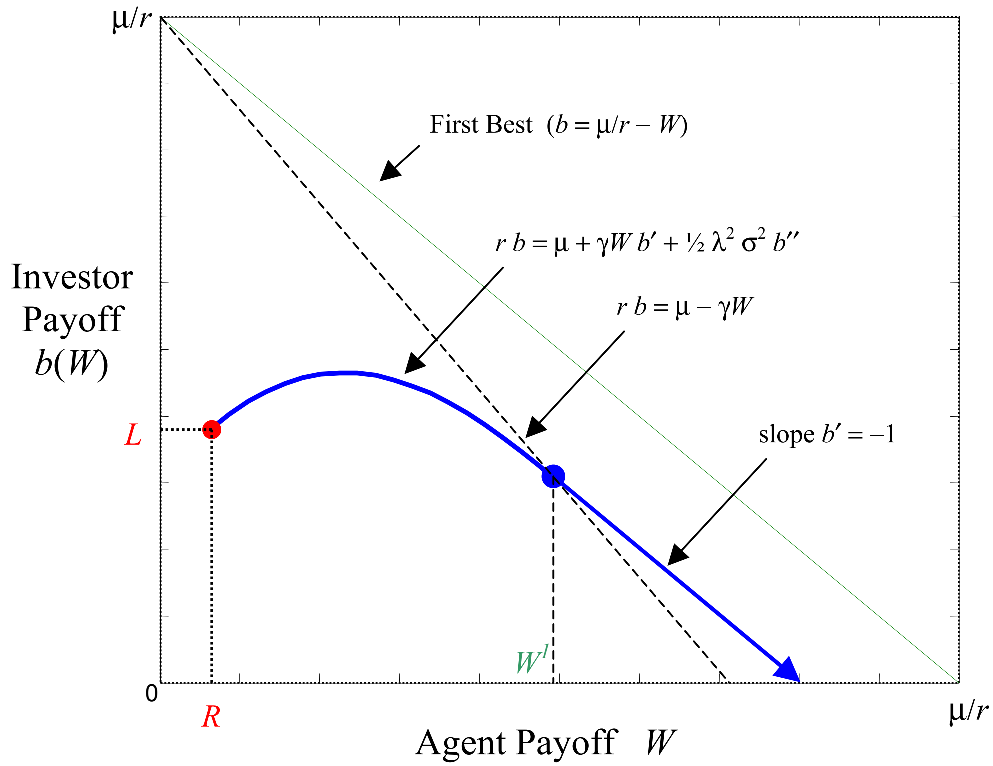

Figure 3: The Optimal Continuation Function b

3. The Continuous-Time Model

In this section we develop a continuous-time formulation of the contracting problem. There are two important reasons for this. First, the discrete-time method of the previous section relies on the considerable machinery developed in DeMarzo and Fishman (2003). Here we propose a methodology that can be used directly to analyze the continuous-time model, which will prove useful in extensions of this analysis. Second, while the previous section approximates the optimal contract for each model in the sequence of discrete time models, there is the possibility that the continuous-time setting may introduce superior contracting possibilities not available in discrete time. Here we develop a continuous-time framework and show that the convenient characterization of an optimal contract described in Section 2.2 holds precisely in continuous time.

In the continuous-time model, the agent manages a project that generates a stochastic stream of cash flows, given by

$$dY_t = \mu\, dt + \sigma\, dZ_t,$$

where Z is a standard Brownian motion on a probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ with an augmented filtration $\mathcal{F}_t$, $0 \le t \le \infty$ generated by the Brownian motion. The cash flows Y from the project are observable only by the agent and not the principal. The agent makes a report $\{\hat{Y}_t; t \ge 0\}$ of the realized cash flows to the principal. The principal does not know whether the agent is lying or telling the truth. The principal extracts the reported cash flows $d\hat{Y}_t$ from the agent and gives him back transfers of dI_t that are based on the agent's reports. Formally, the agent's income process I_t is non-decreasing and $\hat{Y}$-measurable. If the agent underreports realized cash flows, he steals the difference. Stealing may be costly: the agent is able to enjoy only a fraction $\lambda \in (0,1]$ of what he steals. Also, the agent can overreport and put his own money back into the project. As a result, the agent receives a flow of income of[7]

$$[dY_t - d\hat{Y}_t]^{\lambda} + dI_t, \quad \text{where } [dY_t - d\hat{Y}_t]^{\lambda} \equiv \lambda(dY_t - d\hat{Y}_t)^{+} - (dY_t - d\hat{Y}_t)^{-} \tag{8}$$

To make sure that the agent does not receive income of minus infinity, we assume that process $\hat{Y}_t - Y_t$ has to have bounded variation.

The agent is risk-neutral and discounts his consumption at rate γ. This continues until a termination time τ that is contractually specified by the principal. During this time, the agent chooses a nondecreasing consumption process $C = \{C_t;\ 0 \le t \le \tau\}$. He also maintains a private savings account, from which he consumes and into which he deposits his income. The principal cannot observe the balance of the agent's savings account. The agent's balance S_t grows at interest rate $\rho < \gamma$:

$$dS_t = \rho S_t dt + [dY_t - d\hat{Y}_t]^{\lambda} + dI_t - dC_t \tag{9}$$

The agent must maintain a nonnegative balance on his account.

Once the contract is terminated, the agent receives payoff R from an outside option available for him. Assume that $R \ge 0$. Therefore, the agent's total expected payoff from the contract at date 0 is given by[8]

$$W_0 = E\left[\int_0^{\tau} e^{-\gamma s} dC_s + e^{-\gamma\tau} R\right]. \tag{10}$$

The principal discounts cash flows at rate r, such that $\gamma > r \ge \rho$. Once the contract is terminated, he receives expected liquidation payoff $L \ge 0$. The principal's total expected profit at date 0 is

$$b_0 = E\left[\int_0^{\tau} e^{-rs}(d\hat{Y}_s - dI_s) + e^{-r\tau} L\right].$$

The project requires an investment of $K \ge L$ in order to be started. The agent has initial wealth Y_0. The principal specifies a contract before date 0. A contract (τ, I) specifies a termination time τ and payments $\{I_t;\ 0 \le t \le \tau\}$ that are based on reports $\hat{Y}$. Formally, I is a $\hat{Y}$-measurable continuous process, and τ is a $\hat{Y}$-measurable stopping time.

[7] Note that (8) implies that the agent pays a proportional cost $(1-\lambda)$ to divert funds, but does not recover the cost if the funds are put back into the firm. We could also allow the agent to conceal and save funds within the firm, avoiding the cost $(1-\lambda)$ if the funds are ultimately used to boost future reported cash flows (i.e., the cost is only paid if the funds are diverted to the agent's personal account). As we show in Section 3.1, this does not change the results in any way.

[8] We can ignore consumption beyond date τ because $\gamma \ge r$ implies it is optimal for the agent to consume all savings at termination (i.e., $S_\tau = 0$).

In response to a contract (τ, I), the agent chooses a strategy. A feasible strategy is a pair of processes (C, $\hat{Y}$) adapted to Y, such that

(i) process $Y_t - \hat{Y}_t$ has bounded variation,

(ii) process C_t is nondecreasing, and

(iii) the savings process, defined by (9), stays nonnegative.

The agent chooses a feasible strategy to maximize his expected payoff. Therefore, the agent's strategy (C, $\hat{Y}$) is *incentive compatible* if it maximizes his total expected payoff W_0 given a contract (τ, I).

We have not explicitly modeled the agent's option to quit and receive the outside option R at any time. We could incorporate this by including an individual rationality constraint requiring that the agent's future payoff from continuing at date t, W_t, is no worse than his outside option R for all t. However, in our setting this is not necessary as the individual rationality constraint will never bind. The agent can always under-report and steal at rate of $\gamma R/\lambda$, consuming γR until termination, and this strategy gives the agent a payoff of R under *any* contract. Thus any incentive compatible strategy yields the agent at least R.

Thus, the optimal contracting problem is to find a contract (τ, I) and an incentive-compatible response strategy (C, $\hat{Y}$) that maximize the principal's profit subject to delivering to the agent an initial required payoff W_0. By varying W_0 we can use this solution to consider different divisions of bargaining power between the agent and the investors.

3.1. Solving the Continuous-Time Model

We solve the problem of finding an optimal contract in several steps. First, we show that it is sufficient to look for an optimal contract within a smaller class of contracts, namely contracts in which the agent chooses to report cash flows truthfully and maintain zero savings. After that, we consider a relaxed problem by ignoring the possibility that the agent can save secretly. We show how to conveniently represent the truth-telling conditions in continuous time and prove that the contract characterized in Section 2 is optimal when the agent cannot save. Finally, we show that the contract is fully incentive compatible even when the agent can save secretly.

We begin with a revelation principle type of result:

PROPOSITION 2. *There exists an optimal contract in which the agent chooses to tell the truth, and maintains zero savings.*

PROOF: See Appendix. ◆

The intuition for this result is straightforward – it is inefficient for the agent to conceal and divert cash flows ($\lambda \leq 1$) or to save them ($\rho \leq r$). We can improve the contract by having the investors save and make direct payments to the agent. Thus, we can look for an optimal contract in which truth telling and zero savings is incentive compatible.

Note that if the agent could not save, then he would not be able to over-report cash flows and would consume all income as it is received. Thus,

$$dC_t = dI_t + \lambda(dY_t - d\hat{Y}_t). \tag{11}$$

We can relax the problem by restricting the agent's savings so that (11) holds. After we find an optimal contract for the relaxed problem, we show that it remains incentive-compatible even if the agent can save secretly.

One difficulty with working in a dynamic setting is the complexity of the contract space. The contract can depend on the entire path of reported cash flows $\hat{Y}$, making it difficult to evaluate the agent's incentives in a tractable way. Our first task is to find a convenient representation for the agent's incentives. To do so, define V_t as the total expected payoff the agent receives, from transfers and termination utility, if he tells the truth:[9]

$$V_t \equiv E\left[\int_0^\tau e^{-\gamma s}\, dI_s + e^{-\gamma\tau} R \,\middle|\, \mathcal{F}_t\right]$$

The following result provides a useful representation for V_t.

LEMMA A. *There is a Y-measurable process* $\{\beta_t;\ 0 \le t \le \tau\}$ *such that*

$$V_t = V_0 + \int_0^t e^{-\gamma s}\beta_s \underbrace{\frac{dY_s - \mu\, ds}{\sigma}}_{dZ_s} \tag{12}$$

PROOF: The process V_t is a martingale. By the Martingale Representation Theorem, there is a process β such that $dV_t = e^{-\gamma t}\beta_t\, dZ_t$ and thus (12) holds. ◆

When the agent reports truthfully, then the agent's payoff at termination is given by

$$V_\tau = \int_0^\tau e^{-\gamma t}\, dI_t + e^{-\gamma\tau} R = V_0 + \int_0^\tau e^{-\gamma t}\beta_t \frac{dY_t - \mu\, dt}{\sigma}$$

Because I and τ depend exclusively on the agent's report, when the agent reports $\hat{Y}$ then he gets utility

$$\begin{aligned} W_0 &= E\left[\underbrace{V_0 + \int_0^\tau e^{-\gamma t}\beta_t \frac{d\hat{Y}_t - \mu\, dt}{\sigma}}_{V_\tau = \text{payoff from contract}} + \underbrace{\int_0^\tau e^{-\gamma t}\lambda(dY_t - d\hat{Y}_t)}_{\text{payoff from stealing}}\right] \\ &= E\left[V_0 + \int_0^\tau e^{-\gamma t}\beta_t \frac{dY_t - \mu\, dt}{\sigma} + \int_0^\tau e^{-\gamma t}\underbrace{\left(\lambda - \frac{\beta_t}{\sigma}\right)}_{\text{incentives}}(dY_t - d\hat{Y}_t)\right] \end{aligned} \tag{13}$$

This representation allows us to formulate our incentive compatibility condition:

[9] In the analysis that follows we assume that the contract does not use additional randomization (beyond that in the agent's report). A remark after the proof of Proposition 4 shows that randomization would not improve the principal's profit, so that this assumption is not restrictive.

PROPOSITION 3. *If the agent cannot save, truth-telling is incentive compatible if and only if* $\beta_t \geq \lambda\sigma$.

PROOF: If $\beta_t \geq \lambda\sigma$ for all t then (13) is maximized when the agent chooses $d\hat{Y}_t = dY_t$, since the agent cannot over-report cash flows. If $\beta_t < \lambda\sigma$ on a set of positive measure, then the agent is better off underreporting on this set than always telling the truth. ◆

In order to construct an optimal contract and verify that no other contract does better, it is convenient to work with the agent's continuation value

$$W_t = E\left[\int_t^\tau e^{-\gamma(s-t)}\, dI_s + e^{-\gamma(\tau-t)} R \,\middle|\, \mathcal{F}_t\right].$$

Because $V_t = \int_0^t e^{-\gamma s}\, dI_s + e^{-\gamma t} W_t$, using (12) we have that W_t evolves according to

$$dW_t = \gamma W_t\, dt - dI_t + \beta_t\, dZ_t \tag{14}$$

The Optimal Contract

We now show that the principal's profit in continuous time is the same as in the limit of the discrete model described in Proposition 1 and illustrated in Figure 3. Specifically, the principal's profit $b(W)$ is concave, twice continuously differentiable, and there is a "dividend point" W^1 such that

$$rb(W) \geq \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W) \tag{15}$$

with equality if and only if $W \in [R, W^1]$. At the boundaries of this interval, $b(R) = L$ and $rb(W^1) = \mu - \gamma W^1$, and in the dividend region, $W \geq W^1$, $b'(W) = -1$. Intuitively, this reflects the fact that the principal compensates the agent solely through promises of future payments for $W < W^1$, and pays the agent immediately for $W > W^1$.

This leads to the following characterization of the optimal contract:

PROPOSITION 4. *A contract that maximizes the principal's profit and delivers to the agent value* $W_0 \in [R, W^1]$ *takes the following form:* W_t *evolves as*

$$dW_t = \gamma W_t\, dt - dI_t + \lambda(d\hat{Y}_t - \mu\, dt)$$

When $W_t \in [R, W^1)$, $dI_t = 0$. *When* $W_t = W^1$, *payments* dI_t *cause* W_t *to reflect at* W^1. *If* $W_0 > W^1$, *an immediate payment* $W_0 - W^1$ *is made. The contract is terminated at time* τ *when* W_t *hits* R. *The principal's expected payoff at any point is given by* $b(W_t)$.

PROOF: Define

$$G_t \equiv \int_0^t e^{-rs}(dY_s - dI_s) + e^{-rt} b(W_t).$$

Under an arbitrary incentive-compatible contract, W_t evolves according to (14). Then from Ito's lemma,

$$e^{rt}dG_t = \underbrace{\left(\mu + \gamma W_t b'(W_t) + \tfrac{1}{2}\beta_t^2 b''(W_t) - rb(W_t)\right)dt}_{\leq 0} \underbrace{-(1+b'(W_t))dI_t}_{\leq 0} + (\sigma + \beta_t b'(W_t))dZ_t$$

From (15) and the fact that $b'(W_t) \geq -1$, G_t is a supermartingale. It is a martingale if and only if $\beta_t = \lambda\sigma$, $W_t \leq W^1$ for $t > 0$, and I_t is increasing only when $W_t \geq W^1$.

We can now evaluate the principal's payoff for an arbitrary incentive compatible contract. Note that $b(W_\tau) = L$. For all $t < \infty$,

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L\right] = E\left[G_{t\wedge\tau} + 1_{t\leq\tau}\left(\int_t^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L - e^{-rt}b(W_t)\right)\right]$$

$$= \underbrace{E\left[G_{t\wedge\tau}\right]}_{\leq G_0 = b(W_0)} + e^{-rt}E\left[1_{t\leq\tau}\left(\underbrace{\int_t^\tau e^{-r(s-t)}(dY_s - dI_s) + e^{-r(\tau-t)}L}_{\leq \mu/r - W_t = \text{ first best}} - b(W_t)\right)\right]$$

Now, since $b'(W) \geq -1$, $\mu/r - W - b(W) \leq \mu/r - R - L$. Therefore, letting $t \to \infty$,

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L\right] \leq b(W_0)$$

Finally, for a contract that satisfies the conditions of the proposition, G_t is a martingale until time τ because $b(W_t)$ stays bounded. Therefore, the payoff $b(W_0)$ is achieved with equality. ◆

Remark. It is easy to modify this proof to show that the principal cannot improve his profit by adding additional randomization. Such randomization would add an extra term to the expression for dG_t, but the process G_t would still be a supermartingale since $b(W)$ is a concave function.

Hidden Savings

Thus far, we have restricted the agent from saving. We now show that the contract of Proposition 4 remains incentive compatible even when we relax this restriction. The intuition for the result is that because the marginal benefit to the agent of reporting or consuming cash is constant over time, and since private savings grow at rate $\rho < \gamma$, there is no incentive to delay reporting or consumption. In fact, in the proof we show that this result holds even if the agent can save within the firm without paying the diversion cost.

PROPOSITION 5. *Suppose W_t solves*

$$dW_t = \gamma W_t dt - dI_t + \lambda\left(d\hat{Y}_t - \mu\, dt\right) \tag{16}$$

until stopping time $\tau = \min\{t \mid W_t = R\}$. Then the agent earns payoff of at most W_0 from any feasible strategy in response to a contract (τ, I). Furthermore, if W_t is bounded above, the payoff W_0 is attained if the agent reports truthfully and maintains zero savings.

PROOF: See Appendix. ◆

This result confirms that the even with savings, the contract characterized in Proposition 1 and Proposition 4 remains optimal even if the agent has access to hidden savings.

Thus, our continuous-time solution corresponds to the limit of the discrete time model as the period between cash flows shortens.

4. Optimal Capital Structure

DeMarzo and Fishman (2003) demonstrate that the optimal discrete-time contract can be implemented using standard securities: equity, long-term debt, and a credit line. Specifically, the firm has long-term debt with a predetermined coupon payment due each period, a credit line with a fixed credit limit and interest rate γ, and outside equity for a fraction $(1 - \lambda)$ of the firm. The agent is compensated solely through holding the remaining fraction λ of the firm's equity. The agent uses the firm's cash flows to pay the debt coupons and credit line first. Once the credit line is fully repaid, cash flows are paid out as dividends to equity holders. If the credit limit is exceeded, stochastic termination results.

We extend this implementation to the continuous-time setting. First we describe the securities:

Long-term Debt. In our infinite-horizon, stationary setting, long-term debt is represented by a consol bond with required coupon payments $x\,dt$. We normalize the coupon rate on the debt to be r, so that the face value of debt is $D = x/r$. If the coupon payment is not made, the firm defaults, the contract is terminated, and the face value of the debt D is due.

Credit Line. The credit line has a fixed credit limit C^L and charges interest rate γ. The firm may draw on this credit line at any time. Once the credit line is exhausted, the firm defaults, the contract is terminated, and the face value of the credit line C^L is due.

Equity. At the agent's discretion, the firm may pay dividends to the equity holders. Excess cash flows not used to pay dividends or the credit line can accumulate in the firm earning interest rate r.

The agent holds the fraction λ of the equity and receives dividends on this equity stake prior to default. In the event of default, the liquidation value L is used to pay back the debt and the credit line first. If any cash remains, the outside equity holders receive a liquidating dividend of $\max(0, L - D - C^L)$.[10] Note that the agent does *not* receive any liquidating dividend. That is, the agent is compensated with an equity stake, together with a zero interest loan from the firm with a face value, due upon termination, that equals or exceeds the agent's share of any liquidating dividend.[11]

To complete this implementation, the remaining parameters are the amount D of long-term debt and the size C^L of the credit line. Note that with this implementation, the agent has discretion regarding when to use the credit line and when to pay dividends (in

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[10] The firm could have cash in addition to L at the time of default. But it is optimal for equity holders to use this cash to pay down the credit line and avoid default and/or pay a dividend. Thus we can ignore this possibility.

[11] Specifically, when $\lambda < 1$ the loan has face value of at least $\lambda \max(0, L - D - C^L)/(1-\lambda)$. Lemma E in the appendix shows that $L - D - C^L < 0$ in the optimal contract when $\lambda = 1$, so the loan is not needed unless the agent's equity stake is sufficiently small.

addition to stealing cash flows directly from the firm). We begin with the following result, which describes a family of incentive compatible contracts:

PROPOSITION 6. *Suppose the firm's capital structure satisfies*

$$rD + \gamma C^L = \mu - \gamma R/\lambda \tag{17}$$

Then it is incentive compatible for the agent to refrain from stealing, and to use the project cash flows to pay the debt coupons and credit line before issuing dividends. Once the credit line is fully repaid, all excess cash flows are issued as dividends. The agent's continuation payoff is determined by the current draw M_t on the credit line:

$$W_t = R + \lambda\left(C^L - M_t\right) \tag{18}$$

PROOF: See Appendix. ◆

The fact that the agent does not steal is not surprising. Given a fraction λ of the equity, the agent is indifferent between stealing and paying dividends. The problem is that the agent may pay dividends prematurely (exploiting creditors) or delay them (to build an additional cash reserve). The proposition states that for the agent to pay dividends appropriately, incentive compatibility requires an appropriate balance between the level of long-term debt and the credit line. The intuition for condition (17) is that at the credit limit, the agent's "share" of expected cash flows after paying interest on debt, $\lambda(\mu - rD - \gamma C^L)$, is just equal to the flow he would earn on his outside option, γR. Therefore, this is the maximum possible liability such that the agent has an interest in continuing rather than quitting.

Note that by (18), the credit line balance M_t serves as a state variable that tracks the agent's continuation payoff W_t at each point in time. While (17) determines an incentive compatible capital structure, it need not be optimal. An optimal capital structure will choose a credit line with a limit high enough to provide flexibility, but not so high as to delay dividend payments too long. That is, the range of M_t should correspond to the range of W_t in the optimal contract. This requires that W^1 correspond to $M_t = 0$ in equation (18), leading to the following result:

PROPOSITION 7. *The optimal contract of Section 3 is implemented by choosing the capital structure:*

$$C^L = \lambda^{-1}(W^1 - R) \quad \text{and} \quad D = \tfrac{1}{r}(\mu - \lambda^{-1}\gamma W^1) \tag{19}$$

PROOF: See Appendix. ◆

The intuition for this result is as follows. The credit line balance evolves as follows:

$$dM_t = \gamma M_t\, dt + x\, dt + dDiv_t - d\hat{Y}_t,$$

where $dDiv$ are excess cash flows paid out as dividends when the credit line balance is zero. Equation (18) that relates the agent's value and the balance on the credit line, and equation (19) relates the lengths of the credit line and the interval $[R, W^1]$, on which W_t evolves for the duration of the contract. These two equations imply that

$$dW_t = \gamma W_t\, dt - \lambda\, dDiv_t + \lambda\, (d\hat{Y}_t - \mu\, dt),$$

where $\lambda \, dDiv_t = dI_t$ is the income process that makes W_t reflect at W^1. Thus, the evolution of the credit line induces the optimal dynamics for the agent's continuation payoff.

The optimal capital structure has a particularly simple interpretation when the there is no diversion cost ($\lambda = 1$). Note that with $\lambda = 1$, the book value of the debt is $D + M_t$ (face value of long-term debt plus the draw on the credit line), and the total market value of the long-term debt and the credit line is $b(W_t)$, the principal's expected payoff. In that case, given the boundary condition (5) that $rb(W^1) = \mu - \gamma W^1$, we have the following:

COROLLARY. *If $\lambda = 1$, the optimal capital structure has $C^L = W^1 - R$ and $D = b(W^1)$. Thus, at the dividend point the market and book value of the firm's aggregate debt coincides.*

Of course, our implementation is not unique in the following sense: any combination of securities that in aggregate provide for the same payments is also optimal. (For example, the long-term debt could be "stripped" into zero-coupon bonds, or the face value and coupon rate could altered in a way that leaves the coupon payment x unchanged.) We believe this implementation is quite natural however, and corresponds well to observed securities.[12]

Finally, note that this implementation corresponds to the continuous-time limit of the implementation in DeMarzo and Fishman (2003). In the stationary version of their model,

$$C^L = \lambda^{-1}(W^1 - W^L) \quad \text{and} \quad x = rD = \mu + \lambda^{-1}(1 - e^{\gamma dt})W^1$$

which corresponds to (19) as $dt \to 0$ and $W^L \to R$.

4.1. Security Market Values

We now consider the market values of the credit line, long-term debt and equity that implement the optimal contract. For this we need to make an assumption regarding the prioritization of the debt in default. We assume that the long-term debt is senior to the credit line; similar calculations could be performed for different assumptions regarding seniority.[13] With this assumption, the long-term debtholders get $L_D = \min(L, D)$ upon termination. The market value of long-term debt is therefore

$$V_D(M) = E\left[\int_0^\tau e^{-rt} x \, dt + e^{-r\tau} L_D \,\middle|\, M \right]$$

Note that we compute the expected discounted payoff for the debt conditional on the current draw M on the credit line, which measures the firm's "distance to default" in our implementation.

[12] An alternative implementation is given in Biais et al. (2004). They suppose the firm retains cash, and that the coupon payment on the debt varies contractually with the level of the cash reserves.

[13] Recall that only the aggregate payments to investors matter for incentives; the division of the payments between the securities is only relevant for pricing.

Until termination, the equity holders get dividends $dDiv_t$, of which the agent owns fraction λ. At termination, the *outside* equity holders receive the remaining part of liquidation value, $\max(0, L - D - C^L)$, after debt and credit line have been paid off. Thus, when $\lambda < 1$, on a per share basis outside equity holders receive[14]

$$L_E = \frac{1}{1-\lambda}\max(0, L - D - C^L) .$$

The value of equity (per share) to outside equity holders is then

$$V_E(M) = E\left[\int_0^\tau e^{-rt}\, dDiv_t + e^{-r\tau}L_E \,\middle|\, M\right]$$

Finally, the market value of the credit line is

$$V_C(M) = E\left[\int_0^\tau e^{-rt}\,(dY_t - x\,dt - dDiv_t) + e^{-r\tau}L_C \,\middle|\, M\right]$$

where $L_C = \min(C^L, L - L_D)$. Note that for the optimal capital structure, the aggregate value of the outside securities equals the principal's continuation payoff. That is, from (18),

$$b(R + \lambda(C^L - M)) = V_D(M) + V_C(M) + (1-\lambda)\, V_E(M).$$

We show in the appendix how to represent these market values in terms of an ordinary differential equation, so that they may be computed easily. See Figure 4 for an example. In this example, $L < D$ so that the long-term debt is risky. Note that the market value of debt is decreasing towards L as the balance on the credit line increases towards the credit limit. Similarly the value of equity declines to 0 at the point of default. The figure also shows that the initial value of the credit line is positive – the lender earns a profit by charging interest rate $\gamma > r$. However, as the distance to default diminishes, additional draws on the credit line result in losses for the lender (for each dollar drawn, the value of the credit line goes up by less than one dollar, and eventually declines).

[14] Lemma E in the Appendix shows that $L < D + C^L$ when $\lambda = 1$ and there are no outside equity holders, so in that case we can set $L_E = 0$ to compute the "shadow price" of outside equity.

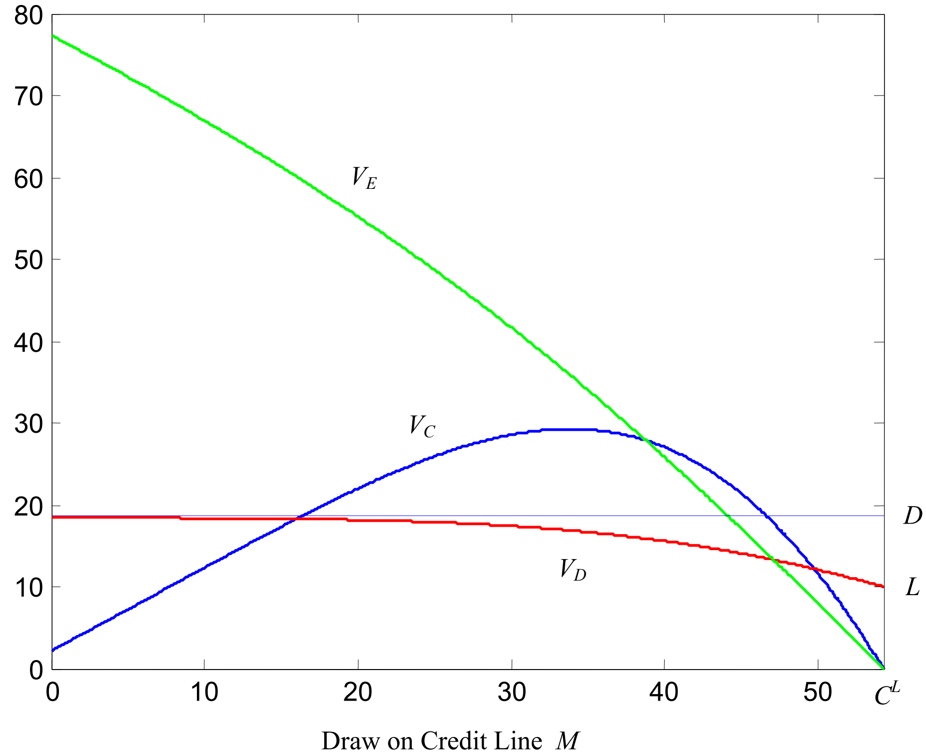

Figure 4: Market Values of Securities for $\mu = 10$, $\sigma = 10$, $\lambda = 50\%$, $r = 10\%$, $\gamma = 15\%$, $L = 10$, $R = 0$

Figure 4 also illustrates several other interesting properties of the security values. Note, for example, that the leverage ratio of the firm is not constant over time. When cash flows are high, the firm will payoff the credit line and its leverage ratio will decline. On the other hand, during times of low profitability, the firm increases its leverage. This pattern is broadly consistent with the empirical behavior of leverage.

One surprising observation from Figure 4: the value of equity is concave in the credit line balance, which implies that the value of equity would decline if the cash flow volatility were to increase. In fact, we can show:

PROPOSITION 8. *When debt is risky ($L < D$), for the optimal capital structure the value of equity decreases if cash flow volatility increases. Thus, equity holders would prefer to reduce volatility.*

PROOF: See appendix. ◆

This is counter to the usual presumption that risky debt implies that equity holders benefit from an increase in volatility due to their option to default. That is, in our setting, there is no "asset substitution problem" related to leverage. Note also that the agent's payoff is linear in the credit line balance, so that the agent is indifferent regarding changes to volatility.

4.2. Initiating the Contract

The defining features of an optimal contract are the credit limit and the coupon rate on long-term debt. These features of an optimal contract are uniquely defined by the model's parameters: the liquidation value L, the agent's outside option R, mean and volatility of cash flows μ and σ, and the agent's discount rate γ.

Note that the optimal capital structure does not depend on the agent's and the principal's respective bargaining powers. Bargaining power will affect the agent's payoff, which we will see manifests itself as the firm paying an initial dividend or starting with a draw on the credit line. To focus our discussion we will assume that the agent has no capital and investors are perfectly competitive, so that the agent's payoff W_0 is highest payoff such that investors break-even if they provide the required capital K; that is, it is the largest solution to $b(W_0) = K$.

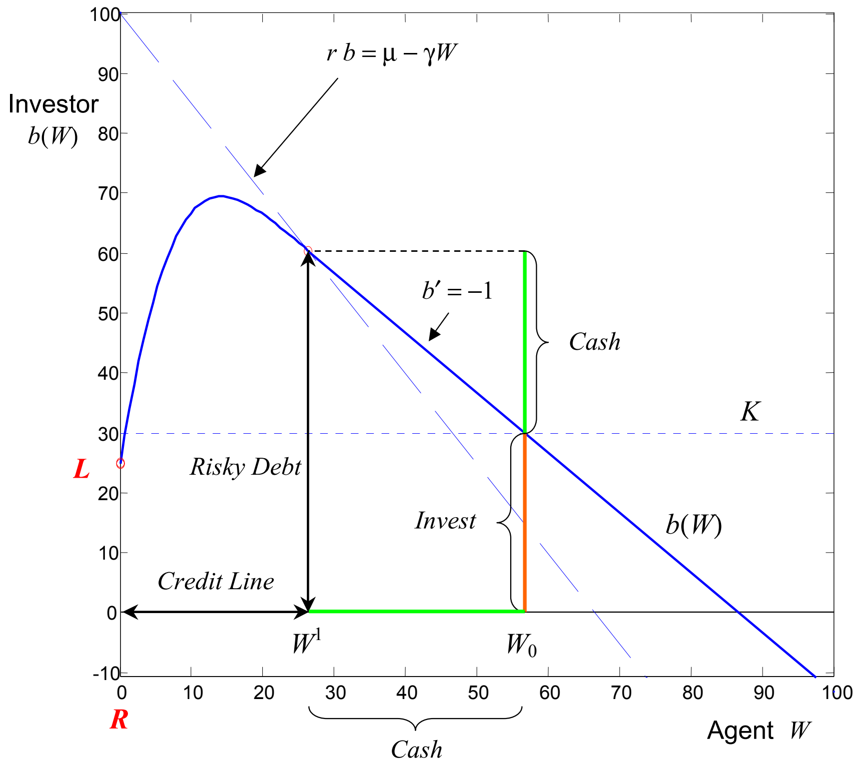

Figure 5: Initiating the Contract ($L = 25$, $R = 0$, $\mu = 10$, $\sigma = 5$, $r = 10\%$, $\gamma = 15\%$, $\lambda = 1$)

Figure 5 shows an example with $\lambda = 1$, illustrating the size of the credit line ($C^L = W^1 - R$) and the debt ($D = b(W^1)$) for a small volatility of cash flows ($\sigma = 5$). In the figure, $D > L$, so the debt is risky. The firm requires $K = 30$ in external capital, which implies the agent's initial payoff is $W_0 > W^1$. This payoff is achieved by giving the agent $W_0 - W^1$ in cash, and starting the firm with zero balance on the credit line (providing the agent with continuation payoff W^1). The start-up capital and the agent's initial consumption

$$K + W_0 - W^1 = b(W^1)$$

are raised by issuing the debt and initiating the credit line. Given the high interest rate γ on the credit line, the lender earns an expected profit from the credit line, and so will pay this to the firm upfront. This payment offsets the discount on the debt due to credit risk.

Recall that the optimal credit line results from the following trade-off: a large credit line delays the agent's consumption, but also gives more flexibility to delay termination. Payments on debt are chosen to give the agent incentives to report truthfully: if payments on debt were too burdensome, the agent would draw down the credit line immediately and quit the firm; if they were too small, the agent would delay termination by saving excess cash flows when the credit line is paid off.

Let us illustrate how these intuitive considerations affect the optimal contract for different values of σ. In Figure 6, we change the parameters of the example in Figure 5 by increasing σ. When volatility rises to 12.5 (left panel), the principal's profit drops. In this case, $b(W^1) \in (0,K)$. Riskier cash flows require more financial flexibility, so the credit line becomes longer. As the principal can extract less cash through coupon payments, the debt shrinks, and is now riskless ($D < L$). The agent's payoff W_0 is lower than before. In this case, the project cannot raise initial capital through debt only; the agent will also have to draw on the credit line to raise K. Because $b' > -1$ on (W_0, W_1), the agent must draw more than $K - D$ to fund the difference. This can be interpreted as an initial fee charged by the lender to open the credit line with this initial balance; this fee compensates the lender for the negative NPV of the credit line due to the firm's greater credit risk.

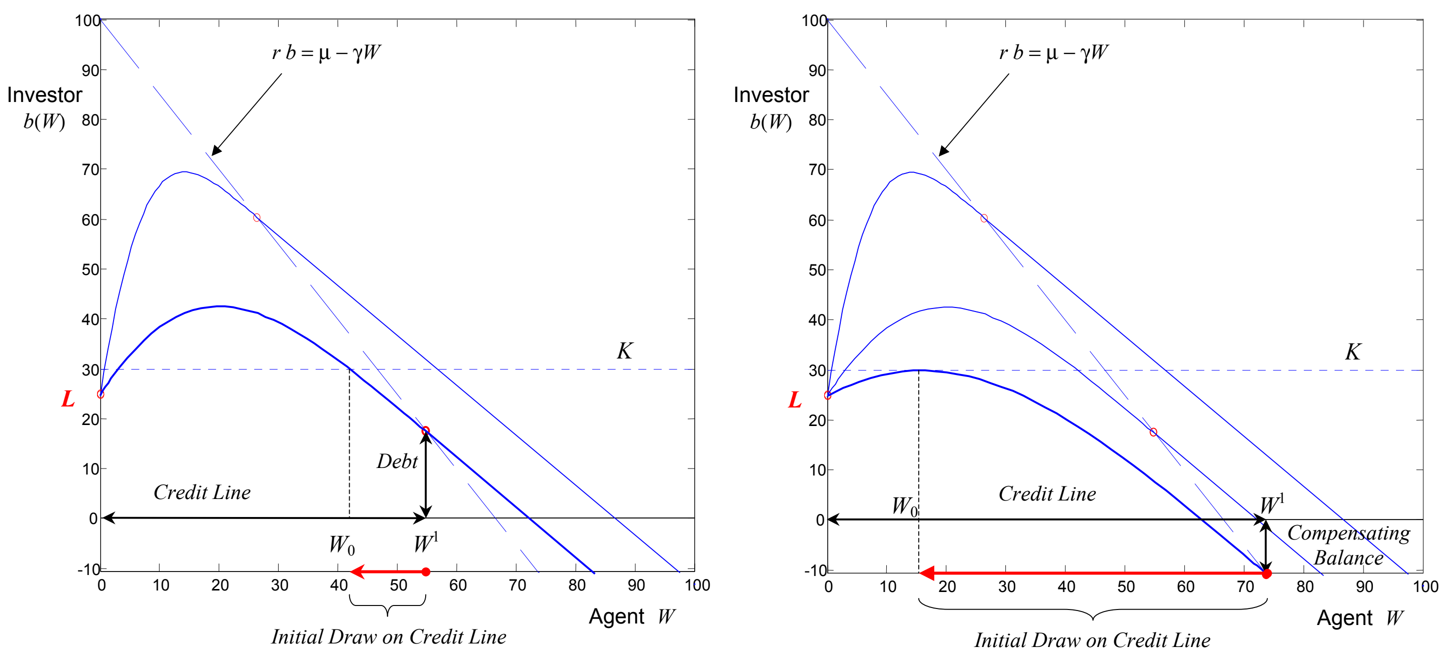

Figure 6: The Effect of Higher Volatility ($\sigma = 12.5$, $\sigma = 19.07$)

If σ increases further to 19.07 as shown in the right panel of Figure 6, the principal's profit falls further. This very risky project requires a very long credit line. Note that $D = b(W^1) < 0$. We interpret $D < 0$ as a compensating balance requirement – that is, the firm must hold cash in the bank equal to $-D$ as a condition of the credit line. Both the required capital K and the compensating balance $-D$ are funded through a large initial draw on the credit line. Thus the agent usually runs the project for a long time before he can consume.

The compensating balance provides additional operating income of rD to the firm. This income increases the attractiveness of the project to the agent, preventing the agent from drawing the entire credit line and running away. By funding the compensating balance upfront, investors are committed to providing the firm with income rD even when the credit line is paid off. This commitment is necessary since investors' continuation payoff at W^1 is negative, which would violate their limited liability. The compensating balance therefore serves to tie the agent and the investors to the firm in an optimal way.

Note that for $\sigma > 19.07$, the maximal profit for the principal falls below K. Thus, while such a project is positive NPV, it cannot be financed due to the incentive constraints.

Remark. While we have focused on the case in which investors are competitive, other possibilities are straightforward to consider. For example, if the principal were a monopolist hiring the agent to run the firm, the contract would be initiated at the value W^* that maximizes the principal's payoff $b(W^*)$. This would not change the optimal debt and credit limit, but in this case the firm would always start with a draw on the credit line. Interestingly, as can be seen in Figure 6, while higher volatility decreases $b(W^*)$, the effect on the agent's payoff W^* is not monotonic. Thus the agent might prefer to manage a higher risk project.

4.3. Comparative Statics

Figure 7 shows more general comparative statics for our model. As we saw above, the credit limit increases and the debt decreases with volatility (σ). Note in particular that the credit line goes to 0 as $\sigma \to 0$. In this case the project becomes riskless and no flexibility is need; indeed, the debt is equal to the first best value of the project.

How about the other parameters of the model? Let us focus on the case where $\lambda = 1$ and the project is profitable even if the agent does not have any initial cash, which implies that $b'(R) > 0$. If the liquidation value L increases, the credit line shortens because termination becomes less costly. This reduces the agent's temptation to draw the entire credit line and default, so the principal can extract greater coupon payments on debt. If the agent's outside option R increases, the agent becomes more tempted to draw down the credit line. The length of the credit line decreases to reduce this temptation, and payments on debt decrease to make it more attractive for the agent to run the project, as opposed to taking the outside option. If the mean of cash flows μ increases, the credit line increases to delay termination and debt increases because the principal can extract more cash flows from the agent. If the agent's discount rate γ increases, then the credit line decreases because it becomes costlier to delay the agent's consumption. The effect on debt is ambiguous. On one hand, a shorter credit line reduces the agent's temptation to draw the entire credit line and default. On the other hand, because the agent becomes more impatient, this temptation is stronger. As a result the optimal level of debt increases and then decreases with γ. The figure also shows the extreme sensitivity of the optimal capital structure near $\gamma = r$.

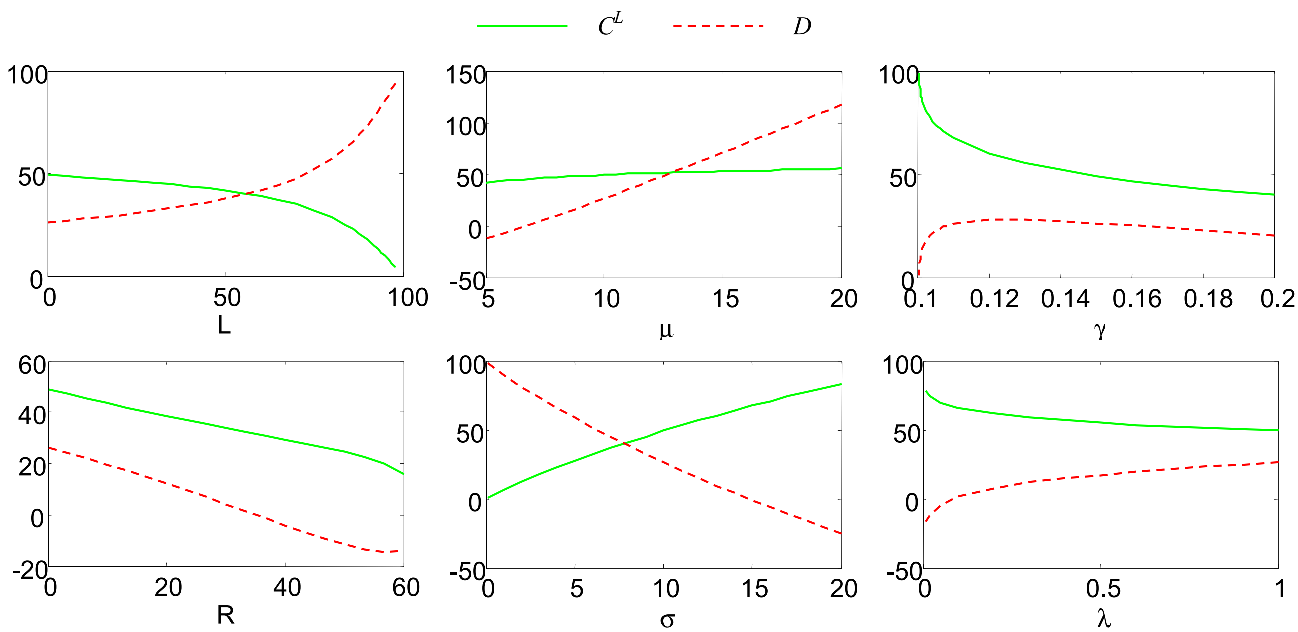

Figure 7: Comparative Statics (base case: $L = 0$, $R = 0$, $\mu = 10$, $\sigma = 10$, $r = 10\%$, $\gamma = 15\%$, $\lambda = 1$)

Finally, what is the effect of λ, which measures the degree of the agency problem in our model? When $\lambda = 1$, the agent can steal cash flows costlessly, and the agency problem is most severe. As λ gets close to zero, the agent receives little benefit from stealing cash flows, and the agency problem diminishes.

In our implementation, the degree of agency manifests itself directly through the size of the agent's equity stake. It also has an indirect effect on the amount of debt and the credit line. For example, if $R > 0$, (17) implies that the maximum total interest payments declines as λ declines. The following result shows that the effect on the optimal debt structure of changing λ is equivalent to the effect of an appropriate change of the termination payoffs.[15]

PROPOSITION 9. *The optimal debt and credit line with agency parameter and termination payoffs* (λ, R, L) *are the same as with parameters* $(1, R^\lambda, L^\lambda)$ *where*

$$R^\lambda = \tfrac{1}{\lambda} R \quad \textit{and} \quad L^\lambda = \tfrac{1}{\lambda} L + (1 - \tfrac{1}{\lambda}) \tfrac{\mu}{r}.$$

PROOF: See Appendix. ◆

[15] DeMarzo and Fishman (2003) demonstrate this result as well in discrete time.

Thus, we can focus our discussion on the case λ = 1, and rely on Proposition 9 to translate the results to other settings. For example, the effect of raising λ when R = 0 (see Figure 7) follows from Proposition 9: increasing λ raises L^{λ}, and thus lowers the credit line and raises debt (if $R > 0$, the effect on debt is ambiguous).

Using the characterization of an optimal contract by a differential equation, we can derive these and many other comparative statics results analytically. In the appendix, we describe a new methodology for explicitly calculating comparatives statics. First, we derive the effect of parameters on the principal's profit. We differentiate equation (4) that describes the principal's profit with respect to parameters and apply a generalization of the Feynman-Kac formula to write the solution as an expectation. In particular, we can represent the sensitivity of the principal's profit to the underlying parameters in terms of the following three functions:

$$G_{\tau}(W) = E\left[e^{-r\tau} \middle| W_0 = W\right], \quad G_1(W) = E\left[\int_0^{\tau} e^{-rt} W_t b'(W_t) dt \middle| W_0 = W\right],$$

$$\text{and } G_2(W) = E\left[\int_0^{\tau} e^{-rt} b''(W_t) dt \middle| W_0 = W\right]$$

Once we know the effect of parameters on the principal's profit, we deduce their effect on debt and credit line from condition $rb(W^1) = \mu - \gamma W^1$, and on the agent's starting value from $b(W_0) = K$. Using these techniques, we have the following results:[16]

	$dC^L /$	$dD /$	$dW_0 /$	$db(W^*) /$
dL	$-\frac{rG_{\tau}(W^1)}{\gamma - r} < 0$	$\frac{\gamma G_{\tau}(W^1)}{\gamma - r} > 0$	$-\frac{G_{\tau}(W_0)}{b'(W_0)} > 0$	$G_{\tau}(W^*) > 0$
dR	$\left(\frac{rb'(R)G_{\tau}(W^1)}{\gamma - r} - 1 < 0\right)$	$-\frac{\gamma b'(R)G_{\tau}(W^1)}{\gamma - r} < 0$	$\frac{b'(R)G_{\tau}(W_0)}{b'(W_0)} < 0$	$-b'(R)G_{\tau}(W^*) < 0$
$d\gamma$	$-\frac{W^1 + rG_1(W^1)}{\gamma - r} < 0$	$\frac{W^1 + \gamma G_1(W^1)}{\gamma - r} ? 0$	$\left(-\frac{G_1(W_0)}{b'(W_0)} < 0\right)$	$\left(G_1(W^*) > 0\right)$
$d\mu$	$\frac{G_{\tau}(W^1)}{\gamma - r} > 0$	$\left(\frac{1}{r} - \frac{\gamma G_{\tau}(W^1)}{r(\gamma - r)} > 0\right)$	$-\frac{1 - G_{\tau}(W_0)}{rb'(W_0)} > 0$	$\frac{1 - G_{\tau}(W^*)}{r} > 0$
$d\sigma^2$	$-\frac{rG_2(W^1)}{\gamma - r} > 0$	$\frac{\gamma G_2(W^1)}{\gamma - r} < 0$	$-\frac{G_2(W_0)}{b'(W_0)} < 0$	$G_2(W^*) < 0$

Figure 8: Comparative Statics for the Optimal Contract

When we sign these comparative statics, we focus on the case when the project is profitable even when the agent does not have any initial wealth, which implies that

[16] Recall that $b(W^*)$ is the maximum payoff the principal can achieve, and so represents the maximum amount of external capital the firm can raise.

$b'(R)>0$. Many signs are immediate from the facts that $G_\tau \in (0,1)$, $G_1 > -W^1/r$ and $G_2 < 0$. The less obvious cases are in parentheses, and are proved in the appendix.

5. Hidden Effort

Throughout our analysis we have concentrated on the setting in which the cash flows are privately observed, and the agent may divert them for his own consumption. In this section we discuss the relationship between this model and a standard principal-agent model in which the agent makes a hidden binary effort choice. This model is also studied by Biais et al. (2004) in contemporaneous work. Our main result is that, subject to natural parameter restrictions, the solutions are identical for both models. Thus, all of our results apply to both settings.

In the standard principal-agent model with hidden effort, the principal observes the cash flows. Based on the cash flows, the principal decides how to compensate the agent, and whether to continue the project. Thus, there are only two key changes to our model. First, since cash flows are observed, there is no issue of the agent saving and using the savings to over-report future cash flows. Second, we assume that at each point in time, the agent can choose to shirk or work. Depending on this decision, the resulting cash flow process is

$$d\hat{Y}_t = dY_t + a\,dt\,, \text{ where } a = \begin{cases} 0 & \text{if the agent works} \\ -A & \text{if the agent shirks} \end{cases}$$

We assume that working is costly for the agent, or equivalently that shirking results in a private benefit.[17] Specifically, we suppose the agent receives an additional flow of utility equal to $\lambda A\ dt$ if he shirks. The agent cannot misreport the cash flows, since $r < \gamma$ the agent will consume all payments immediately. Thus, if the agent shirks,

$$dC_t = dI_t + \lambda A\,dt\,.$$

Again, λ parameterizes the cost of effort and therefore the degree of the moral hazard problem. We assume $\lambda \leq 1$ so that working is efficient.

Our first result establishes the equivalence between this setting and our prior model:

PROPOSITION 10. *The optimal Principal-Agent contract implementing high effort is the optimal contract of Section 3.*

PROOF: The incentive compatibility condition in **PROPOSITION 3** is unchanged: to implement high effort at all times, we must have $\beta_t \geq \lambda\,\sigma$. But then Proposition 4 shows that our contract is the optimal contract subject to this constraint. ◆

It is not surprising that our original contract is incentive compatible in this setting, since shirking is equivalent stealing cash flows at a fixed rate. What is perhaps more surprising is that the additional flexibility the agent has in the cash flow diversion model does not require a "stricter" contract.

[17] The difference between the two interpretations amounts to shifting the agent's utility by a constant.

Of course, **PROPOSITION 10** does assume that implementing high effort at all times is optimal. Under what circumstances is this the case? If a contract were to call for the agent to shirk at some point, this would result in diminished cash flows, but would have the benefit that we would no longer need to provide the agent with incentives.[18] Thus, in these states the agent's continuation payoff would no longer need to be sensitive to the realized cash flows, so that

$$dW_t = \begin{cases} \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - \mu\, dt) & \text{if } a = 0 \\ \gamma W_t dt - (dI_t + \lambda A\, dt) & \text{if } a = -A \end{cases}$$

Because the principal's continuation function is concave, this reduction in the volatility of W_t could be beneficial. For that not to be the case, and for high effort to remain optimal, it must be that for all W,

$$rb(W) \geq (\mu - A) + (\gamma W - \lambda A)b'(W) \tag{20}$$

Intuitively, this equation states that the principal's flow payoff from having the agent shirk would be less than under our existing contract.[19] Define

$$w^s = \lambda A/\gamma \text{ and } b^s = (\mu - A)/r = (\mu - \gamma w^s/\lambda)/r,$$

the agent and principal's payoff if the agent shirks forever and receives no other payment. Then we have the following necessary and sufficient condition, as well as a simple sufficient condition, for high effort to remain optimal at all times:

PROPOSITION 11. *Implementing high effort at all times is optimal in the Principal-Agent setting if and only if* $b^s \leq f\left(w^s\right)$ *where* $f(z) \equiv \min_w b(w) + \frac{\gamma}{r}(z-w)b'(w)$. *A simpler sufficient condition is*

$$b^s \leq \frac{\gamma}{r} b\left(w^s\right) + \left(1 - \frac{\gamma}{r}\right) b(W^*) \tag{21}$$

Given λ*, both of these conditions imply a lower bound on A, or equivalently,* w^s.

PROOF: See Appendix. ◆

We can interpret Proposition 11 as follows. The point (w^s, b^s) represents the agent's and principal's payoff if the agent shirks forever. Thus, shirking is never optimal if and only if this point lies below the function f. The function f is concave and below b, with equality only at the maximum. This is illustrated in Figure 9. The factor γ/r increases the steepness of f relative to b; when $\gamma = r$, f and b coincide. As can be seen from the figure, Proposition 11 puts a lower bound on w^s, or equivalently on A, the magnitude of the cash flow impact of shirking. For example, in Figure 9, if $w^s \geq \underline{w}^s$, then high effort is always optimal. This is the case for (w^s_1, b^s_1).

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[18] Specifically, in Proposition 3 we can set $\beta_t = 0$ in states where the contract called for the agent to shirk.

[19] Formally, condition (20) is required in the proof of Proposition 4 for G_t to remain a supermartingale for either effort choice.

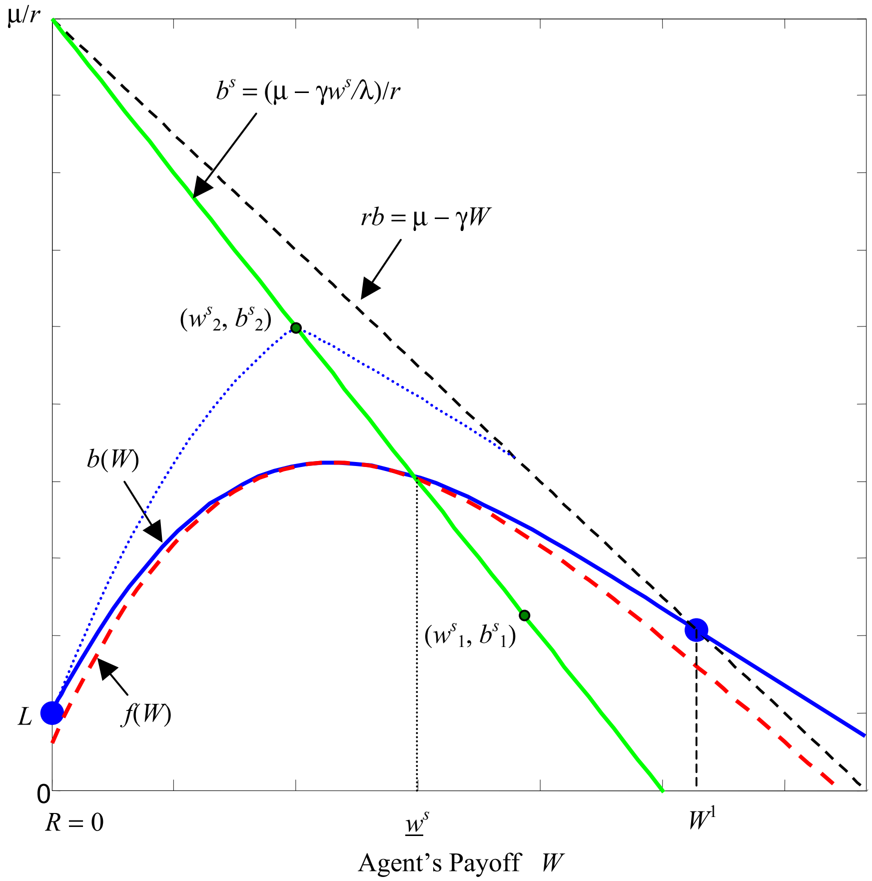

Figure 9: Example showing Optimality of High Effort

On the other hand, if A is too small so that $w^s < \underline{w}^s$, then the optimal principal-agent contract will involve shirking after some histories. However, in some cases the optimal contracting techniques of this paper may still apply. For example, see (w^s_2, b^s_2) in Figure 9. In this case, high effort is optimal until the point (w^s_2, b^s_2) is reached, in which case it is optimal to shirk forever. Thus, the optimal contract is again as in our model, but with shirking forever replacing termination so that $(R, L) = (w^s_2, b^s_2)$.[20]

Remark. We can also consider a hybrid model, in which the agent can both divert cash flows and choose whether to work or shirk. In this case, let λ_d parameterize the benefit the agent receives from diverting cash flows, and let λ_a represent the benefit from shirking. Then we can show that the optimal contract implementing high effort is the optimal contract of Section 3 with $\lambda = \max(\lambda_d, \lambda_a)$.

[20] This will be the case whenever shirking yields the highest possible payoff for the investors; i.e., when A is sufficiently small. For intermediate values of A, an optimal contract calls for shirking only temporarily. In that case, a more complicated contract than the one described in this paper will be necessary to achieve optimality.

6. Further Extensions of the Model

In this section we consider various extensions of the basic model. First, we allow the termination payoffs (R, L) to be determined endogenously by either the principal's option to hire a new agent or the agent's option to start a new project. Second, we consider the construction of an optimal renegotiation-proof contract. Third, we consider the case in which the agent and principal disagree about key parameters of the model, such as the project's profitability, or the agent's impatience.

6.1. Endogenous Termination

Thus far, we have treated the termination payoffs (R, L) as exogenous. Suppose, however, that they are endogenously determined as in the following to examples:

Unique Assets, Replaceable Agent: Suppose the assets of the firm are unique, but the agent can be fired and replaced at cost c_a to the principal/investors. The agent's termination payoff if fired is R, but the investor's payoff on firing the agent is

$$L = b(W^*) - c_a \tag{22}$$

Unique Agent, Replaceable Assets: Suppose the agent can quit the firm a start a new firm by raising external capital $K = L$ from new investors. If the agent quits, the old investors liquidate and receive L, while the agent receives

$$R = e^{-\gamma \Delta t} W_0 \tag{23}$$

where Δt is the time required to start a new firm and W_0 satisfies $b(W_0) = L$.[21]

The optimal contract in either case takes exactly the same form as described in Proposition 4. The only change is that now the boundary condition (22) or (23) replaces $b(R) = L$. The solution is illustrated in Figure 10. Because $db(W^*)/dL = G_\tau\,(W^*) < 1$, when assets are unique the liquidation value L is decreasing in c_a with $dL/dc_a = G_\tau(W^*) - 1$. From the results of Section 4.3, the credit line increases and the debt decreases in c_a. This is intuitive, because the project requires more financial flexibility when it is more difficult to replace the agent. Similarly, when the agent is unique, as Δt falls and it becomes easier for the agent to start a new firm, R rises. This leads to a decrease in both the credit line and in debt. Note that as $\Delta t \to \infty$ and starting a new firm becomes impossible, $R \to 0$, and as $\Delta t \to 0$ and restarting is costless, $R \to R^*$, the point at which $b'(R^*) = 0$.

[21] This setting is similar to Hart and Moore's (1994) notion of "inalienable human capital" and its relationship to optimal debt structure.

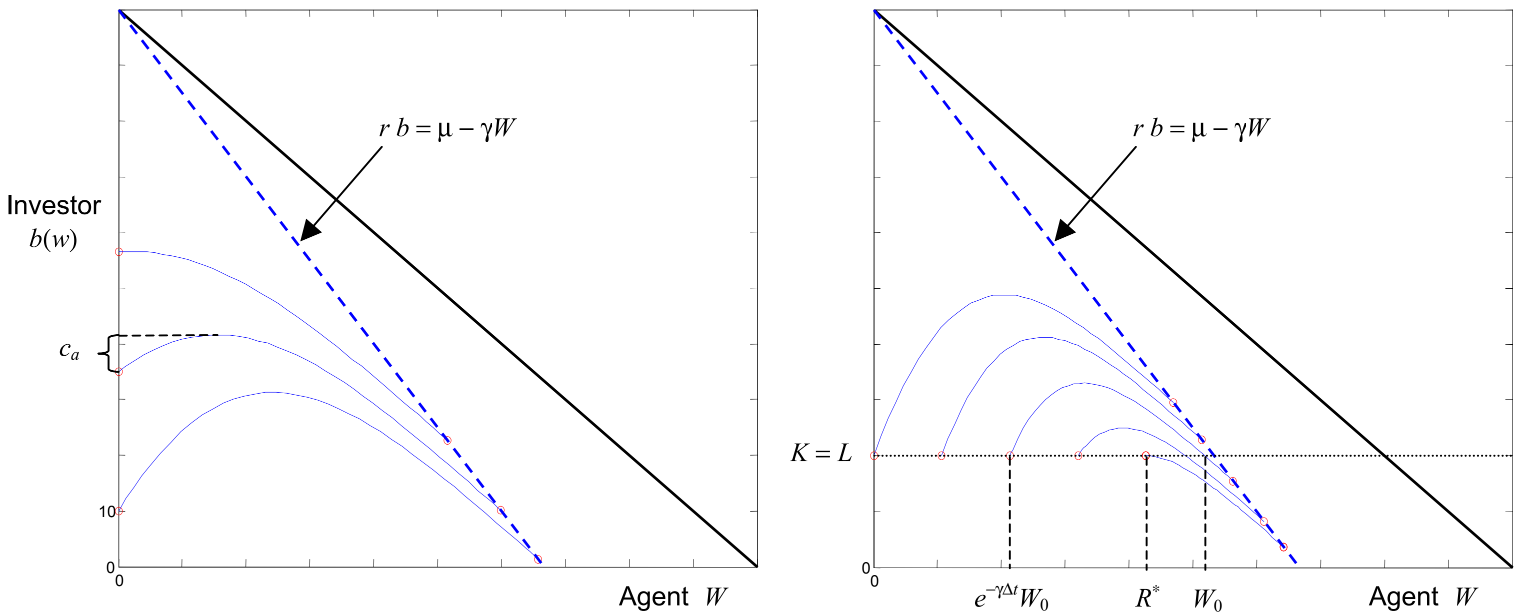

Figure 10: Determining L or R endogenously

6.2. Renegotiation-Proofness

Note that the optimal contracts in the basic model are generally not renegotiation-proof. When $b'(R) > 0$, then both the principal and the agent would like to renegotiate at termination time. Instead of termination, which gives the agent and the principal payoffs of R and L, they could renegotiate by restarting the contract from with the agent's value $W > R$, which gives the principal profit $b(W) > L$.

To be renegotiation-proof, the principal's profit function $b(W)$ cannot have positive slope anywhere. To find this function we must solve the optimality equation from boundary conditions $b'(W^1) = -1$ and $rb(W^1) = \mu - \gamma W^1$ for an appropriate choice of W^1, such that the maximum of the resulting function is $b(R^*) = L$. This is equivalent to the case in Section 6.1 of a unique agent that can restart the firm immediately ($\Delta t = 0$). Let us set $b(W) = L$ on the interval $[R, R^*]$.

A renegotiation-proof contract, under which the principal breaks even, exists only if the agent has initial wealth $Y_0 \geq K - L$. In that case, the agent's continuation value W_0 is such that $b(W_0) = K - Y_0$. Until termination the agent's continuation value W_0 evolves in the interval $[R^*, W^1]$ as

$$dW_t = \gamma W_t\,dt + \lambda(d\hat{Y}_t - \mu\,dt) - dI_t + dP_t,$$

where processes I and P reflect W_t at endpoints W^1 and R^* respectively. The project is terminated stochastically whenever W_t is reflected at R^*. The probability that the project continues at time t is

$$\Pr(\tau \geq t) = \exp\left(\frac{-P_t}{R^* - R}\right).$$

Then W_t is the agent's true expected future payoff. Indeed, whenever W_t hits R^* and dP_t is added to the agent's continuation value, the project is terminated with probability $dP_t / (R^* - R)$ to account for this increment to the agent's value.

The implementation of a renegotiation-proof contract involves a credit line and debt as in the optimal contract of Proposition 7 with R^* in place of R. Since $R^* > R$, both the credit line and debt decrease. This is intuitive, because renegotiation-proofness reduces the profitability of the project.[22]

6.3. Private Benefits and Differing Opinions

Suppose the agent receives private benefits of control from running the project. Specifically, suppose that prior to termination the agent earns additional utility at rate $\gamma\omega$. In this case, the agent's continuation value should evolve according to

$$dW_t = \gamma(W_t - \omega)\,dt - dI_t + \lambda(d\hat{Y}_t - \mu\,dt)$$

How does this alter the form of the optimal contract? Interestingly, as the following result shows, this is equivalent to simply adjusting the agent's payoff under the contract by ω.

PROPOSITION 12. *Suppose the agent earns private benefits at rate $\gamma\omega$ while running the project. Then the optimal contract is the same as the optimal contract without private benefits and termination payoff for the agent of $R - \omega$. That is, while the project is running, the principal accounts for the agent's payoff through state variable $\hat{W}_t$ that evolves as*

$$d\hat{W}_t = \gamma\hat{W}_t\,dt - dI_t + \lambda(d\hat{Y}_t - \mu\,dt)$$

in the interval $[R, W^1]$. Under this contract, given a value of the state variable $\hat{W}_t$, the agent's total payoff including private benefits is $W_t = \hat{W}_t + \omega$.

PROOF: See Appendix. ◆

Thus, using our comparative statics results for R from Section 4.3, increasing the agent's private benefits increases the credit limit and amount of debt in the optimal capital structure. Intuitively, the potential threat of losing the private benefits in termination enhances the agent's incentives and hence increases the debt capacity of the firm. Moreover, $\hat{W}_0$ rises, so that the agent's total payoff rises by more than a dollar for each dollar of private benefits, all else equal.

A similar result follows if the agent and the investor have different beliefs about the mean of the cash flows, μ. For example, suppose the agent believes the mean is $\mu + \delta$. Holding these beliefs fixed, the agent's continuation payoff should evolve according to

$$\begin{aligned} dW_t &= \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - (\mu + \delta)\,dt) \\ &= \gamma(W_t - \lambda\delta/\gamma)\,dt - dI_t + \lambda(d\hat{Y}_t - \mu\,dt) \end{aligned}$$

[22] Gromb (1999) also considers renegotiation-proofness in a related discrete-time model. While not providing a complete characterization, he does show that in an infinite-horizon stationary setting the maximum external capital the firm can raise is the liquidation value L.

Thus, a discrepancy δ between the agent's and investor's beliefs is therefore equivalent to a private benefit of magnitude $\omega = \lambda\delta/\gamma$.

6.4. Misestimating γ

What happens if an agent whose subjective discount factor is $\gamma' \neq \gamma$ receives contract (I, τ), which is optimal for the agent with discount factor γ? How will the agent behave? It depends on whether γ' is greater or less than γ. We focus on the case when λ = 1.

If $\gamma' < \gamma$, then when $W_t < W^1$ then the agent will deposit all cash flows from the project onto the credit line, and have balance zero in his savings account. The agent does not steal because a dollar on the credit line earns a higher rate of interest than the agent's subjective discount rate. Also the agent pays the credit line before saving because the credit line has a higher interest rate, and the agent is free to draw from the credit line at any time.

What happens when the balance on the credit line is paid off, i.e. $W_t = W^1$? It can be shown that the agent will choose to save at interest rate ρ as long as $S_t < S^1$ for some critical value S^l. When $S_t = S^1$, the agent consumes all excess cash flows. Intuitively, for $S_t < S^1$, a dollar saved gives the agent more than a dollar of utility because it makes it less necessary to draw on the high-interest credit line to cover potential future losses. When $S_t = S^1$, the agent is on the margin indifferent between saving and consuming because the savings account earns an interest rate lower than the agent's subjective discount factor. We show in the following proposition that the principal earns the same profit as if the agent had discount factor γ, i.e. $b_\gamma(W_0)$. When γ is close to γ', then $b_\gamma(W_0)$ is close to $b_{\gamma'}(W_0)$, so the mistake of overestimating the agent's discount factor is not too costly for the principal.

The following result summarizes this, and shows that the agent's behavior is drastically different if $\gamma' > \gamma$:

PROPOSITION 13. *Suppose that the principal offers a contract designed for the agent with discount rate γ to an agent whose true discount rate is $\gamma' < \gamma$. Then this agent would derive utility greater than W_0, and the principal would receive profit of exactly $b_\gamma(W_0)$.*

If the agent's true discount factor γ' is greater than γ for which the contract is designed, then the agent will draw the entire credit line and default immediately. The agent earns utility W_0, whereas the principal earns $L - (W_0 - R)$.

PROOF: See Appendix. ◆

We conclude that underestimating the agent's subjective discount factor is a very costly mistake for the principal. This drastic difference raises the issue of robustness: what contract is optimal if the principal is uncertain about the agent's rate of time preference?

7. Conclusion

We analyzed a situation in which an agent or entrepreneur needs to raise external capital to (i) start-up a profitable project, (ii) cover future operating losses that may occur, and (iii) consume. In our setting, the agent can divert cash flows from the project for personal consumption without the investor's knowledge. To enforce payments, the investors can threaten to withhold future funding and terminate the project. We analyze an optimal contract between the investors and the agent in this setting.

An optimal contract takes a similar form both in a discrete-time setting of DeMarzo and Fishman (2003) and in our continuous-time setting. The contract involves a credit line, debt and equity. Debt, outside equity, and possibly the credit line provide the funds for start-up capital and initial consumption for the agent. For the duration of the project, the credit line provides the flexibility to cover possible operating losses. The agent has incentives to pay interest and not consume from the credit line because in case of default he has to surrender the project to investors. The agent holds an equity stake and has discretion over the payment of dividends. The agent's equity stake is sufficiently large that he does not divert excess cash flows for personal consumption, but pays them out as dividends appropriately.

The continuous-time setting of our paper has several advantages. First, the features of an optimal contract are cleaner. Unlike in discrete time, an optimal contract in continuous time does not require stochastic termination. Second, some of the analysis is simplified. Because time is continuous, we do not have to consider problems associated with different points within each time period separately. Most importantly, a continuous time model provides a convenient characterization of the optimal contract, which involves an ordinary differential equation. With this characterization we can say a great deal about how the optimal capital structure is determined by the specific features of the project. Also, we are able to compute the values of securities that are involved in the implementation of an optimal contract. Finally, we can easily analyze extensions. For example, we show how our contract also solves a standard principal-agent setting with costly effort. Other extensions are considered; in many cases the solution only involves finding the appropriate boundary conditions for the differential equation that defines an optimal contract.

Our results open several thought-provoking questions for future research. Here are only a few examples. First, we discover that in our setting there is no asset substitution problem. That is, increased variance of cash flows does not benefit equity holders because it makes agent's incentive problem more difficult. Second, we find that it may be very important for the principal to assess the characteristics of the project correctly. If an investor makes a mistake, the agent may draw the entire credit line for personal consumption and default immediately. This raises the question of robust contract design. Finally, the simplicity of our characterization opens the possibility of embedding our model within other standard finance models. For example, we can consider extending our framework to allow for dynamic investment decisions or project choice, and determine how the dynamics of these other decisions relate to cash flows and optimal capital structure.

8. Appendix

Proof of Proposition 2: Follows from Lemmas B and C below. ◆

Lemma B. *Consider any incentive-compatible contract. Then there is another incentive-compatible contract, which gives the same profit to the agent and the same or greater payoff to the principal, under which the agent chooses to reveal cash flows truthfully.*

Proof: Our argument has a flavor of the revelation principle. However, the revelation principle does not apply directly, because the agent's payoff depends not only on the principal's action but also directly on his report (see (8)).

Consider an incentive-compatible contract with transfer process is $I(\hat{Y}(\cdot))$ and termination time $\tau(\hat{Y}(\cdot))$. We would like to define a new contract such that

i) the agent gets the same payoff as under the old contract

ii) the agent chooses to reveal cash flows truthfully

iii) the principal gets the same or greater profit as under the old contract

Given the agent's report Y', define the transfer process I' under the new contract to be such that

$$dI'_t(Y') = [\, dY'_t - d\hat{Y}_t(Y')\,]^{\lambda} + dI_t(\hat{Y}(Y')),$$

where $\hat{Y}(Y')$ is the report generated by the agent under the old contract, when he observes Y'. Also, define the termination time under the new contract as $\tau(\hat{Y}(Y'))$. It is easy to see that in the new contract, transfer process I' and termination time $\tau(\hat{Y}(Y'))$ are measurable with respect to the agent's report Y'.

First, if the agent tells the truth, then he receives the same stream of income as if he reported $\hat{Y}$ under the old contract. Second, if the agent lies and says Y', he receives less income than he would by telling $\hat{Y}(Y')$ under the old contract, because

$$[dY_t - dY'_t]^{\lambda} + dI'_t(Y') = [dY_t - dY'_t]^{\lambda} + [\, dY'_t - d\hat{Y}_t(Y')\,]^{\lambda} + dI_t(\hat{Y}(Y'))$$
$$\leq [dY_t - d\hat{Y}_t(Y')\,]^{\lambda} + dI_t(\hat{Y}(Y'))$$

Because the agent found it optimal to report $\hat{Y}$ under the old contract, he prefers to tell the truth under the new contract. Third, because

$$dY_t - dI'_t(Y) = dY_t - [dY_t - d\hat{Y}_t(Y)\,]^{\lambda} - dI_t(\hat{Y}(Y)) \geq d\hat{Y}_t - dI_t(\hat{Y}),$$

the principal's profit under the new contract is the same as or greater than under the old contract. Therefore, the new contract that we constructed satisfies conditions (i), (ii) and (iii), as required. ◆

Lemma C. *Consider any incentive-compatible contract (τ, I), under which the agent reports truthfully and consumes C. Then there new contract (τ, I') with an alternative payment process I', under which the agent chooses to maintain zero savings (since the principal does savings for the agent). This new contract gives the agent the same payoff as before; the principal receives the same or higher profit.*

PROOF: Let

$$S_t(Y)=\int_0^t e^{\rho(t-s)}(dI_s(Y)-dC_s(Y))ds\geq 0$$

be the savings process under the old contract (τ, I). For any report Ŷ define

$I'_t(\hat{Y}) = C_t(\hat{Y})$

If the agent tells the truth and consumes C_t under the new contract (τ, I′), then he maintains zero savings. The agent's total expected payoff under the new contract is

$$W_0'=E\left[\int_0^\tau e^{-\gamma t}dC_t+e^{-\gamma\tau}R\right]$$

which is the same as under the old contract.

The principal's profit under the new contract is greater or equal than his profit under the old contract. Indeed, when the principal does savings for the agent and the principal's interest rate r is greater than the agent's interest rate ρ, then the principal's expected profit improves by

$$E\left[\int_0^\tau e^{-rt}(r-\rho)S_t dt\right]$$

Lastly, we need to show that the new contract is incentive-compatible. Incentive compatibility follows if we show that the new contract does not allow any new feasible strategies for the agent: let us show that if any alternative strategy (Ĉ, Ŷ) is feasible in response to (τ, I′), then it is also feasible in response to (τ, I). A strategy is feasible if it generates a nonnegative savings process. We have

$$\underbrace{\int_0^t e^{\rho(t-s)}([dY_s-\hat{Y}_s]^\lambda+dI_s(\hat{Y})-d\hat{C}_s(Y))ds}_{\text{savings for }(\hat{C},\hat{Y})\text{ in response to }(\tau,I)}=\underbrace{\int_0^t e^{\rho(t-s)}([dY_s-\hat{Y}_s]^\lambda+dI_s'(\hat{Y})-d\hat{C}_s(Y))ds}_{\geq 0,\text{ savings for }(\hat{C},\hat{Y})\text{ in response to }(\tau,I')}-$$

$$\underbrace{\int_0^t e^{\rho(t-s)}(dI_s'(\hat{Y})-dC_s(\hat{Y}))ds}_{=0\text{ by definition of }I'}+\underbrace{\int_0^t e^{\rho(t-s)}(dI_s(\hat{Y})-dC_s(\hat{Y}))ds}_{=S_t(\hat{Y})\geq 0}\geq 0$$

This completes the proof that the new contract is incentive-compatible. ◆

PROOF OF PROPOSITION 5: Recall that the rate of return on savings is $\rho \leq r$. We consider the case $\rho = r$ in which savings is most attractive without loss of generality. We also generalize the setting to allow the agent to save within the firm and on his own account (this will be useful in our implementation of the optimal contract). Savings within the firm are represented by S_t^f and evolve according to

$$dS_t^f = rS_t^f\,dt+(dY_t-d\hat{Y}_t)-dQ_t$$

Here, dQ_t represents the agent's diversion of the cash flows to his own account, which evolves according to

$$dS_t = rS_t\,dt + [dQ_t]^{\lambda} + dI_t - dC_t$$

Note that the agent bears the cost of diversion when moving funds outside the firm. Both accounts must maintain non-negative balances. We show that for an arbitrary feasible strategy $(C,\hat{Y})$ of the agent,

$$\hat{V}_t = \int_0^t e^{-\gamma s}\,dC_s + e^{-\gamma t}\left(S_t + \lambda S_t^f + W_t\right)$$

is a supermartingale. Now,

$$e^{\gamma t} d\hat{V}_t = dC_t + dS_t - \gamma S_t\,dt + \lambda(dS_t^f - \gamma S_t^f\,dt) + dW_t - \gamma W_t\,dt$$

Using (16) and the definitions of dS_t and dS_t^f,

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= [dQ_t]^{\lambda} - \lambda dQ_t - (\gamma - r)(S_t + \lambda S_t^f)\,dt + \lambda(dY_t - \mu\,dt) \\ &= -(1-\lambda)dQ_t^- - (\gamma - r)(S_t + \lambda S_t^f)\,dt + \lambda\sigma\,dZ_t \end{aligned}$$

Because $\lambda \leq 1$, dQ_t^- is non-decreasing, $\gamma > r$, and the savings balances are non-negative, $\hat{V}$ is a supermartingale until time τ because W_t is bounded below. If W_t is bounded above and there is no savings, $S_t = S_t^f = 0$, and the agent reports truthfully so that $d\hat{Y}_t = dY_t$ and $dQ_t = 0$, then $\hat{V}$ is a martingale. Thus,

$$W_0 = \hat{V}_0 \geq E\left[\hat{V}_\tau\right] = E\left[\int_0^\tau e^{-\gamma s}\,dC_s + e^{-\gamma\tau}\left(S_\tau + \lambda S_\tau^f + R\right)\right]$$

with equality if the agent maintains zero savings and reports truthfully. ◆

PROOFS OF PROPOSITION 6 AND PROPOSITION 7: Let us prove Proposition 6 first. Let Div_t be an increasing process representing the cumulative dividends paid by the firm. Then the credit line balance evolves according to

$$dM_t = \gamma M_t\,dt + x\,dt + dDiv_t - d\hat{Y}_t.$$

where we can assume $dDiv_t$ and $d\hat{Y}_t$ are such that $M_t \geq 0$. Then from (17), $\lambda x = \lambda\mu - \gamma(R + \lambda C^L)$ and

$$\begin{aligned} dW_t &= -\lambda\,dM_t = -\lambda\gamma M_t\,dt - \lambda x\,dt - \lambda\,dDiv_t + \lambda d\hat{Y}_t \\ &= \gamma(W_t - (R + \lambda C^L))dt - (\lambda\mu - \gamma(R + \lambda C^L))dt - \lambda\,dDiv_t + \lambda d\hat{Y}_t \\ &= \gamma W_t dt - \lambda\,dDiv_t + \lambda(d\hat{Y}_t - \mu\,dt) \end{aligned}$$

Letting $dI_t = \lambda\, dDiv_t$, the result of Proposition 6 follows from Proposition 5.

To prove Proposition 7, note that $dI_t = 0$ unless $M_t = 0$ (that is, $W_t = W^1$ under assumptions of Proposition 7). Since dividends cause M_t to reflect at 0, dI_t causes dW_t to reflect at W^1. Therefore, we have implemented the optimal contract described in Proposition 4. ◆

Market Values of Securities:

The following lemma is useful for computation of market values of securities and for comparative statics:

LEMMA D. *Suppose W_t evolves as*

$$dW_t=\gamma W_t\, dt - dI_t + \lambda\, (d\hat{Y}_t - \mu\, dt)$$

in the interval $[R, W^1]$ until time τ when W_t hits R, where I_t is a nondecreasing process that reflects W_t at W^1. Let k be a real number, and $g:[R, W^1] \to \Re$ a bounded function. Then the same function $G:[R, W^1] \to \Re$ both solves equation

$$rG(W) = g(W) + \gamma W\, G'(W) + 1/2\, \lambda^2\sigma^2 G''(W) \tag{24}$$

with boundary conditions $G(R) = L$ and $G'(W^1)= -k$ and satisfies

$$G(W_0) = E\left[\int_0^\tau e^{-rt} g(W_t)dt - k\int_0^\tau e^{-rt} dI_t + e^{-r\tau} L\right] \tag{25}$$

PROOF: Suppose that G solves (24), and let us show that it satisfies (25). Define

$$H_t = \int_0^\tau e^{-rt} g(W_t)dt - k\int_0^\tau e^{-rt} dI_t + e^{-r\tau} G(W_t)$$

Then using Ito's lemma,

$$e^{rt}dH_t = \left(g(W_t) + \gamma W_t G'(W_t) + \frac{1}{2}\lambda^2\sigma^2 G''(W_t) - rG(W_t) \right)dt - (k + G'(W_t))dI_t + G'(W_t)\lambda\sigma dZ_t$$

From equation (24), condition $G'(W^1)= -k$, and the fact that I increases only when $W_t = W^1$, H is a martingale. Because G is bounded, H is a martingale until time τ, so

$$G(W_0) = H_0 = E[H_\tau] = E\left[\int_0^\tau e^{-rt} g(W_t)dt - k\int_0^\tau e^{-rt} dI_t + e^{-r\tau} L\right]$$

This completes the proof. ◆

The values of credit line, debt and equity can be expressed in terms the following functions, which can be computed by Lemma D:

$$G_\tau(W) = E\left[e^{-r\tau} \mid W_0 = W\right] \quad \text{and} \quad G_I(W) = E\left[\int_0^\tau e^{-rt} dI_t \mid W_0 = W\right]$$

By Lemma D, both of these functions solve differential equation

$$rG(W) = \gamma W\, G'(W) + 1/2\, \lambda^2\sigma^2 G''(W) \tag{26}$$

with boundary conditions $G_\tau(R) = 1$, $G_\tau'(W^1)=0$ and $G_I(R) = 0$, $G_I'(W^1)=1$. Functions G_τ and G_I can be easily computed. To evaluate market values of securities, we also use the fact that

$$E\left[\int_0^\tau e^{-rt}dt \mid W_0 = W\right] = \frac{1-G_\tau(W)}{r}$$

Then, expressed as functions of the agent's continuation value W_t, market values the credit line, debt and equity are

$$V_c(W) = E\left[\int_0^\tau e^{-rt}\left(dY_t - xdt - \frac{dI_t}{\lambda}\right) + e^{-r\tau}L_C \mid W_0 = W\right]$$

$$= \frac{\gamma W^1}{\lambda}\frac{1-G_\tau(W)}{r} - \frac{G_I(W)}{\lambda} + L_C G_\tau(W)$$

$$V_D(W) = E\left[\int_0^\tau e^{-rt}xdt + e^{-r\tau}L_D \mid W_0 = W\right] = x\frac{1-G_\tau(W)}{r} + L_D G_\tau(W)$$

$$V_E(W) = E\left[\int_0^\tau e^{-rt}\frac{dI_t}{\lambda} + e^{-r\tau}L_E \mid W_0 = W\right] = \frac{G_I(W)}{\lambda} + L_E G_\tau(W)$$

PROOF OF PROPOSITION 8: When $L < D$, then $L_E = 0$. Then, to demonstrate that equity holders prefer less volatility, we need to prove that G_I is concave. From the stochastic representation, we see that G_I is an increasing function. From (26),

$$1/2\ \lambda^2\sigma^2 G_I''(R) = -\gamma R G_I'(R) < 0.$$

Suppose that G_I were not concave everywhere on $[R,W^1]$, and let $V = \inf\{G_I''(W) > 0\}$. Then $V>R$ and $G_I''(V) = 0$ by continuity of G_I''. But then from (26)

$$1/2\ \lambda^2\sigma^2 G'''(V)=(r-\gamma)\ G'(V) - \gamma V\ G''(V)= (r-\gamma)\ G'(V) < 0,$$

so $G_I''(V+\varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$, contradiction. ◆

The following lemma tells us that when there are no outside equity holders, then no funds remain after debt and credit line holders are paid from the liquidation value.

LEMMA E. *If $\lambda = 1$, then in the optimal contract, $L < D + C^L$.*

PROOF: When $\lambda = 1$, $D + C^L = b(W^1) + W^1 - R$. Since $b'(W) > -1$ for $W \in (R,W^1)$, $b(W^1) + W^1 > b(R) + R = L + R$. Thus, $D + C^L > L$. ◆

PROOF OF PROPOSITION 9: Let b be the optimal continuation function for parameters $(1, R^\lambda, L^\lambda)$ and define

$$b^\lambda(W) = \lambda b(W/\lambda) + (1-\lambda)(\mu/r)$$

We claim that b^λ is the optimal continuation function with parameters (λ, R, L). To see this, we can easily check that $b^\lambda(R) = L$. Since

$$b^{\lambda}{}'(W) = b'(W/\lambda) \quad \text{and} \quad b^{\lambda}{}''(W) = \tfrac{1}{\lambda} b''(W/\lambda),$$

then $b^{\lambda}{}'(W^1) = -1 \Rightarrow b'(W^1/\lambda) = -1$ and $b''(W^1/\lambda) = 0 \Rightarrow b^{\lambda}{}''(W^1) = 0$. Thus, b^{λ} satisfies both boundary conditions at W^1. In addition, for $W \in [R, W^1]$,

$$\begin{aligned} rb^{\lambda}(W) &= \lambda rb(W/\lambda) + (1-\lambda)\mu \\ &= \lambda\left[\mu + \gamma(W/\lambda)b'(W/\lambda) + \tfrac{1}{2}\sigma^2 b''(W/\lambda)\right] + (1-\lambda)\mu \\ &= \mu + \gamma W b^{\lambda}{}'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b^{\lambda}{}''(W) \end{aligned}$$

so that b^{λ} satisfies (4) and hence is the optimal continuation function. Thus, W^1 is the dividend boundary for parameters (λ, R, L) if and only if W^1/λ is the dividend boundary for (1, R^{λ}, L^{λ}). From (19), this implies that the optimal capital structure is unchanged. ◆

Comparative Statics Results:

LEMMA F. *Suppose* θ *is one of parameters* L, μ, γ *or* σ^2 *and denote by* $b_\theta(W)$ *the optimal continuation function for that parameter value. Then*

$$\frac{\partial b_\theta(W)}{\partial \theta} = E\left[\int_0^\tau e^{-rt}\left(\frac{\partial \mu}{\partial \theta} + \frac{\partial \gamma}{\partial \theta} W_t b'_\theta(W_t) + \frac{1}{2}\frac{\partial \sigma^2}{\partial \theta} b''_\theta(W_t)\right)dt + e^{-r\tau}\frac{\partial L}{\partial \theta} \,|\, W_0 = W\right]$$

PROOF: Consider a value of W^1 and a corresponding incentive-compatible contract of Proposition 5: one in which process I reflects W_t at W^1. Then the principal's profit under this contract is

$$b_{\theta,W^1}(W) = E\left[\int_0^\tau e^{-rt}\mu dt - \int_0^\tau e^{-rt} dI_t + e^{-r\tau} L \,|\, W_0 = W\right]$$

By Lemma D, $b_{\theta,W^1}(W)$ solves equation

$$rb_{\theta,W^1}(W) = \mu + \gamma W b'_{\theta,W^1}(W) + \tfrac{1}{2}\sigma^2 b''_{\theta,W^1}(W) \tag{27}$$

with boundary conditions $b_{\theta,W^1}(R) = L$ and $b'_{\theta,W^1}(W^1) = -1$. Denote by $W^1(\theta)$ the choice of W^1 that maximizes the principal's profit $b_{\theta,W^1}(W_0)$ for a given value of parameter θ. Then $b_\theta(W) = b_{\theta,W^1(\theta)}(W)$. By the Envelope Theorem,

$$\frac{\partial b_\theta(W)}{\partial \theta} = \left.\frac{\partial b_{\theta,W^1}(W)}{\partial \theta}\right|_{W^1 = W^1(\theta)} \tag{28}$$

Differentiating (27) with respect to θ at $W^1 = W^1(\theta)$ and using (28) we find that $\frac{\partial b_\theta(W)}{\partial \theta}$ satisfies equation

$$r\frac{\partial b_\theta(W)}{\partial\theta} = \frac{\partial\mu}{\partial\theta} + \frac{\partial\gamma}{\partial\theta}Wb'_\theta(W) + \gamma W\frac{\partial}{\partial W}\frac{\partial b_\theta(W)}{\partial\theta} + \frac{1}{2}\frac{\partial\sigma^2}{\partial\theta}b''_\theta(W) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial W^2}\frac{\partial b_\theta(W)}{\partial\theta}$$

with boundary conditions $\frac{\partial b_\theta(R)}{\partial\theta} = \frac{\partial L}{\partial\theta}$ and $\frac{\partial}{\partial W}\frac{\partial b_\theta(W^1)}{\partial\theta} = 0$. The conclusion of the lemma follows from Lemma D. ◆

COROLLARY. *From Lemma F and we obtain that*

$$\frac{\partial b(W)}{\partial L} = G_\tau(W), \qquad \frac{\partial b(W)}{\partial\gamma} = G_1(W), \qquad \frac{\partial b(W)}{\partial\mu} = \frac{1-G_\tau(W)}{r}, \textit{ and}$$

$$\frac{\partial b(W)}{\partial\sigma^2} = G_2(W), \tag{29}$$

where

$$G_\tau(W) = E\left[e^{-r\tau} \mid W_0 = W\right], \qquad G_1(W) = E\left[\int_0^\tau e^{-rt}W_t b'(W_t)dt \mid W_0 = W\right],$$

$$\textit{and } G_2(W) = E\left[\int_0^\tau e^{-rt}b''(W_t)dt \mid W_0 = W\right]. \tag{30}$$

Additionally, because the principal's profit remains the same if the agent's outside option increases by dR and liquidation value decreases by $b'(R)dR$, the effect of a change in R on the principal's profit is captured by

$$\frac{\partial b(W)}{\partial R} = -b'(R)G_\tau(W).$$

To find how parameters affect the optimal choice of W^1, note that

$$b_\theta(W^1(\theta)) = \frac{\mu - \gamma W^1(\theta)}{r} \Rightarrow \frac{\partial W^1(\theta)}{\partial\theta} = \frac{r}{\gamma - r}\frac{\partial}{\partial\theta}\left(\frac{\mu-\gamma W^1}{r} - b_\theta(W^1)\right)\Bigg|_{W^1=W^1(\theta)} \tag{31}$$

We can then compute the derivatives of W^1 with respect to parameters using the Corollary of Lemma F. Similarly, the derivative of W_0 with respect to the parameters can be found by differentiating $b_\theta(W_0(\theta)) = K$ with respect to θ and using the Corollary. We obtain comparative statics results summarized in Figure 8 in the paper. In Figure 8, we still need to sign the non-obvious entries in parentheses. The following Lemma allows us to compare the principal's profit for different γ's and to sign two entries that involve $G_1(W)$.

LEMMA G. *Suppose that the principal offers a contract designed for the agent with discount rate* γ *to an agent whose true discount rate is* $\gamma' < \gamma$. *Then this agent would derive utility greater than* W_0, *and the principal would receive profit of exactly* $b(W_0)$.

PROOF: Let us investigate how an agent with discount rate γ' responds to a contract created for an agent with discount rate γ. First, let us interpret the contract. The agent's value W_t can be interpreted as the agent's balance on a high-interest savings account. It evolves as

$$dW_t = \gamma W_t\, dt + (d\hat{Y}_t - \mu\, dt)$$

where $d\hat{Y}_t - \mu\, dt$ is the flow of deposits. The high-interest account has a cap of W^1. The agent's consumption is

$$dC_t = dY_t - \mu\, dt - (d\hat{Y}_t - \mu\, dt) - dQ_t,$$

where $\mu\, dt$ is a tax and dQ_t is the flow of deposits onto the low-interest savings account. The balance on that account is

$$dS_t = \rho S_t\, dt + dQ_t$$

When the agent manages these two accounts, it is optimal to never have positive balance on the low-interest savings account, unless the high-interest savings account is full (i.e. $W_t = W^1$). Also, it is optimal to deposit all cash flows onto the high-interest savings account and not consume when $W_t < W^1$, because those cash flows can earn a higher interest rate than the agent's own discount rate. The agent consumes only when $W_t = W^1$ and the balance on his own savings account is positive. This sort of strategy gives the agent of value higher than W_t, which he could get by simply drawing the credit line to the end and defaulting immediately.

Let us show that the principal still gets profit $b(W_t)$ when the agent follows any such strategy. When $W_t < W^1$, the agent deposits all cash flows onto the credit line, just like an agent with discount factor γ would do. The only difficulty can come from the fact that when $W_t = W^1$, the agent may manage his own savings account with cash flows from the project, and keep the balance on the credit line at 0 by paying the principal a flow $\mu - \gamma W^1$ of coupon payments on long-term debt. This modification in the agent's strategy does not alter the principal's profit because $\mu - \gamma W^1 = rb(W^1)$, which is exactly what the principal needs to get to realize a profit of $b(W^1)$. ◆

Note that the contract in Lemma G is not optimal for agent γ'. An optimal contract would give the principal higher profit for the same value of the agent. Therefore, to every point $(W, b_\gamma(W))$ with $W \geq W^*(\gamma)$, there is a point $(W', b_{\gamma'}(W')) > (W, b_\gamma(W))$. We conclude that $b_\gamma(W)$ must be increasing as γ falls for all $W \geq W^*(\gamma)$, so $G_1(W) < 0$.

COROLLARY. $-\dfrac{G_1(W_0)}{b'(W_0)} < 0$ *and* $G_1(W^*) > 0$.

For the remaining two entries of Figure 8, we need to relate $b'(W)$ and $G_\tau(W)$.

LEMMA H. *The following inequality holds for all* $W < W^1$:

$$b'(W) < \frac{(\gamma - r)G_\tau(W)}{rG_\tau(W^1)} - \frac{\gamma}{r}. \tag{32}$$

PROOF: Differentiating equation (33) with respect to *W* we find that *b'(W)* satisfies

$$(r-\gamma)b'(W) = \gamma W b''(W) + \frac{\sigma^2}{2} b'''(W) \tag{34}$$

with boundary conditions $b'(W^1)=-1$ and $b''(W^1)=0$. Denote the right hand side of (32) by $g(W)-\gamma/r$. From (30), we know that $g(W)$ satisfies

$$\begin{aligned} rg(W) &= \gamma W g'(W) + \frac{\sigma^2}{2} g''(W) \Rightarrow \\ (r-\gamma)(g(W)-\frac{\gamma}{r}) + (r-\gamma)\frac{\gamma}{r} + \gamma g(W) &= \gamma W g'(W) + \frac{\sigma^2}{2} g''(W) \end{aligned} \tag{35}$$

with boundary conditions $g(W^1)=(\gamma-r)/r$ and $g'(W^1)=0$. Denote $f(W) = g(W)-\gamma/r - b'(W)$. To prove the lemma, we need to show that $f(W)>0$ for all $W<W^1$. Since $f(W^1)=0$, this property follows if we show that $f'(W)<0$ for all $W<W^1$. Subtracting (34) from (35), we find that

$$\frac{\sigma^2}{2} f''(W) = (r-\gamma)f(W) + (r-\gamma)\frac{\gamma}{r} + \gamma g(W) - \gamma W f'(W) \tag{36}$$

with boundary conditions $f(W^1)=0$ and $f'(W^1)=0$. From (36) we find that

$$\frac{\sigma^2}{2} f''(W^1) = (r-\gamma)\frac{\gamma}{r} + \gamma\frac{\gamma - r}{r} = 0$$

$$\frac{\sigma^2}{2} f'''(W^1) = (r-2\gamma)f'(W^1) + \gamma g'(W^1) + \gamma W^1 f''(W^1) = 0 \text{, and}$$

$$\frac{\sigma^2}{2} f^{(4)}(W^1) = (r-3\gamma)f''(W^1) + \gamma g''(W^1) + \gamma W^1 f'''(W^1) > 0$$

Therefore, $f'(W)<0$ for $W<W^1$ in a small neighborhood of W^1. If $f'(W)<0$ fails for some $W<W^1$, there has to be a largest point *V* at which it fails. Then $f'(V)=0$ and $f(W)$ is positive and decreasing on $[V,W^1)$. But then from (36)

$$\frac{\sigma^2}{2} f''(V) = (r-\gamma)f(V) + (r-\gamma)\frac{\gamma}{r} + \gamma g(V) > 0, \text{ since } g(V) > \frac{\gamma - r}{r}.$$

We conclude that $f'(V+\varepsilon)>0$, which contradicts our definition of V as the largest point at which $f'(V)\geq 0$. We conclude that $f'(W)<0$ and $f(W)>0$ for $W<W^1$, so (32) holds. ◆

Now we can sign the remaining two fields in Figure 8.

COROLLARY. *Applying (32) at W=R, we have*

$$\frac{rb'(R)G_\tau(W^1)}{\gamma - r} - 1 < -\frac{\gamma G_\tau(W^1)}{\gamma - r} < 0 \quad and \quad 1 - \frac{\gamma G_\tau(W^1)}{\gamma - r} > \frac{rb'(R)G_\tau(W^1)}{\gamma - r} > 0.$$

Hidden Effort and Extensions:

PROOF OF PROPOSITION 11: Let $w^s = \lambda A/\gamma$ and $b^s = (\mu - A)/r$. We can rewrite (20) as $b^s \le b(W) + \frac{\gamma}{r}\left(w^s - W\right)b'(W)$, and this must hold for all W, leading to the condition

$$b^s \le f(w^s) = \min_W b(W) + \tfrac{\gamma}{r}\left(w^s - W\right)b'(W). \tag{37}$$

To prove that condition (21) of Proposition 11 guarantees (37), it is sufficient to show that for all w,

$$b\left(w^s\right) - \tfrac{\gamma - r}{r}\left(b(W^*) - b\left(w^s\right)\right) \le b\left(W\right) + \tfrac{\gamma}{r}(w^s - W)b'(W) \tag{38}$$

Note that since b is concave and $\gamma > r$,

$$b(w^s) \le b(W) + \left(w^s - W\right)b'(W) \le b(W) + \tfrac{\gamma}{r}\left(w^s - W\right)b'(W)$$

if $(w^s - W)b'(W) > 0$, which implies (38) for W not between w^s and W^*. For W between w^s and W^*, note that

$$\begin{aligned} b(w^s) - \tfrac{\gamma - r}{r}(b(W^*) - b(w^s)) &\le b(w^s) - \tfrac{\gamma - r}{r}(b(W) - b(w^s)) \\ &\le b(w^s) - \tfrac{\gamma - r}{r}(W - w^s)b'(W) \\ &\le b(W) + (w^s - W)b'(W) - \tfrac{\gamma - r}{r}(W - w^s)b'(W) \\ &= b(W) + \tfrac{\gamma}{r}(w^s - W)b'(W) \end{aligned}$$

so that (38) again holds, verifying the sufficiency of condition (21).

Note that $f'(w^s) = \gamma/r\ b'(W) \ge -\gamma/r$, whereas $\partial b^s/\partial w^s = -(\gamma/r)/\lambda$. Thus, both (37) and (21) imply a lower bound on w^s (or equivalently A).

Finally, we note the following properties of f described in the paper: Setting $W = w^s$ in (37) implies $f(w) \le b(w)$. Also, since f is the lower envelope of linear functions it is concave. Finally, (21) implies that $f(W^*) = b(W^*)$. ◆

PROOF OF PROPOSITION 12: Let b be the optimal continuation function given boundary condition $b(R-\omega) = L$. Then define $b^*(W) = b(W - \omega)$. Then $b^*(R) = L$ and

$$\begin{aligned} rb^*(W) &= rb(W - \omega) = \mu + \gamma(W - \omega)b'(W - \omega) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W - \omega) \\ &= \mu + + \gamma(W - \omega)b^{*}{}'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b^{*}{}''(W) \end{aligned}$$

Finally, $b'(W^1) = -1$ implies $b^{*\prime}(W^1 + \omega) = -1$ and $b^{*\prime\prime}(W^1 + \omega) = 0$. Thus, by the same arguments as in the proof of Proposition 4, b^* is the optimal continuation function for the setting with private benefits. ◆

PROOF OF PROPOSITION 13: The first result holds by Lemma G. Next, suppose the agent's true discount factor γ' is greater than γ. The process

$$\hat{V}_t = \int_0^t e^{-\gamma' s} dC_s + e^{-\gamma' t}(S_t + W_t)$$

is a strict supermartingale. Indeed,

$$e^{\gamma t} d\hat{V}_t = -(1-\lambda)(dY_t - d\hat{Y}_t)^- - (\gamma' - \gamma) W_t dt - (\gamma' - \rho) S_t dt + \lambda \sigma dZ_t,$$

so $\hat{V}$ has a negative drift. Since W_t and S_t are bounded from below, $\hat{V}$ is strict supermartingale until time τ. If the agent draws the entire credit line and defaults at time 0, then he gets a payoff of W_0. If he follows any other strategy, then τ > 0 and the agent's payoff is

$$E\left[\int_0^\tau e^{-\gamma' s} dC_s + e^{-\gamma' \tau}(S_\tau + W_\tau)\right] = E\left[\hat{V}_\tau\right] < \hat{V}_0 = W_0$$

Therefore, the agent will draw the entire credit line immediately if $\gamma' > \gamma$. ◆

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