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ABSTRACT

The solution to a linear model in which supply and/or demand depends on rational expectations of future prices can involve three parts, which we denote as the fundamental component, the deterministic bubble component, and the stochastic bubble component. This paper explores the properties of these solution components, emphasizing the distinction between deterministic bubbles and stochastic bubbles, for a model of inflation and for a model of the evolution of price and quantity in the market for a storable commodity, such as gold. The analysis focuses on stochastic bubbles as a possibility peculiarly associated with models that involve rational expectations. In both the inflation model and the gold model, although the analysis points to no compelling reason to rule out rational stochastic bubbles a priori, conventional behavioral assumptions imply that any rational bubbles that arise, whether deterministic or stochastic, are explosive. The paper discusses problems of implementing econometric tests for the existence of rational bubbles, and, as an alternative to these tests, suggests "diagnostic checking" of the stationarity properties of time series. Although these diagnostic checks do not constitute definitive hypothesis testing, we conjecture they would provide strong evidence against rational bubbles outside the context of hyperinflation.

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The recent literature includes various examples that illustrate that a linear rational-expectations (RE) model can permit a multiplicity of time paths for market-clearing price--see, for example, Taylor (1977), Shiller (1978), Blanchard (1979), Flood and Garber (1980), and Blanchard and Watson (1982). Burmeister, Flood, and Garber (1982) point out that each of these examples are cases of what they denote as a price "bubble." Obstfeld and Rogoff (1982) demonstrate that maximizing behavior does not preclude rational bubbles in interesting cases.

These examples show that rational bubbles are theoretically possible. Specifically, the solution to a linear RE model can involve three parts, which we denote as the fundamental component (FC), the deterministic bubble component (DBC), and the stochastic bubble component (SBC). However, as Flood and Garber point out, RE, by requiring expectations to be correct on average, places strong and potentially testable restrictions not only on the fundamental component, but also on the form that the bubble components can take. The present paper explores the properties of these solution components for a model of inflation and for a model of the evolution of price and quantity in the market for a storable commodity, such as gold. The critical element in these models is the dependence of supply and/or demand on rational expectations of future prices.

An empirical motivation for studying bubble components is that the volatility exhibited by many time series of prices seems difficult to explain in terms of movements in their FC. Because of this problem, economists frequently suggest casually that in fact FC is sometimes <u>not</u> the only empirically relevant component of price. Recently, some econometric studies have provided evidence that might seem to support this common-sense hypothesis. For example, Shiller (1981) concludes that the variability of common stock prices is many times greater than the apparent variability of their FC. Also, the Salant and Henderson (1978) model of the price of gold, although it includes interesting expectational elements in FC, does not seem able to explain all of the price gyrations between 1968 and 1978 in terms of FC, and surely would not be able to account for price movements since 1978.

The distinction between DBC and SBC is important for a number of reasons. From a theoretical standpoint, SBC is peculiar to RE models whereas DBC can also arise in perfect foresight models. From an empirical standpoint, SBC seems potentially able to explain puzzling qualitative characteristics of observed price paths. For example, SBC implies excess volatility of prices and the dependence of prices on fundamentally irrelevant variables.

In what follows, Section 1 develops the basic theoretical analysis within a linear RE model of inflation. Section 2 extends the analysis to a linear RE model of the market for a storable commodity like gold, that is both currently produced and held in portfolios. Section 3 discusses problems in implementing econometric tests for the existence of bubbles. For empirical analysis of the existence of rational bubbles, we suggest "diagnostic checking" of the properties of time series of observable endogenous variables as an alternative to standard econometric procedures. Section 4 summarizes and concludes the paper.

1. Components of a Rational-Expectation Solution for Market-Clearing Price

To illustrate the idea of rational price bubbles in a familiar context, consider the Cagan model of inflation with rational expectations of inflation replacing Cagan's adaptive expectations. In this model, the current price level satisfies a condition of equality between the real money stock, given by the lhs of equation (1), and the demand for real money balances, given by the rhs of (1):

$$(1) \qquad M_t - P_t = \alpha_t - \beta(E_t P_{t+1} - P_t), \quad \beta > 0,$$

where M_t is the logarithm of the nominal money stock at date t,

P_t is the logarithm of the price level at date t,

α_t represents all of the variables that influence demand other than expected inflation,

β is the semi-elasticity of real money demand with respect to expected inflation,

E_t is an operator that denotes a rational calculation of an expected value, i.e., a calculation consistent with the true model, conditional on information available at date t.

The analysis assumes that M_t and α_t are exogenously determined random variables and that the sequences $\{E_t M_{t+j}\}$ and $\{E_t \alpha_{t+j}\}$ do not grow exponentially with j, for any t. The assumed exogeneity of M_t abstracts from feedback from the price level to the nominal money stock and, given the nature of fiscal processes that involve inflationary financing of public expenditures, is probably unrealistic. The model can be extended along the lines of Section 2 to allow the money stock to be endogenous.

The variables P_t, M_t, and α_t are contemporaneously observed. This assumption means that all market participants have complete knowledge of the current values of relevant variables. A potentially interesting, but ambitious, extension would be to require market participants to form statistical inferences about current events as in RE models with incomplete current information. A further complication would be to assume that market participants have differential information.

The following discussion reveals that a critical property of a linear RE model, which has a number of implications for the characteristics of the model's possible solutions for the time path of price, is whether its eigenvalues are real or imaginary, positive or negative, and lie inside or outside the unit circle. In the present example, rearranging (1) yields the first-order partial difference equation,

(2) $$E_t P_{t+1} = (1+\beta^{-1})P_t - \beta^{-1}(M_t - \alpha_t).$$

Equation (2) is a partial, rather than ordinary, difference equation because $E_t P_{t+j}$ depends on both t and j, and not simply on j. The single eigenvalue of (2) is $1+\beta^{-1}$, which, given that β is positive, is greater than unity. This property of the model reflects an essential aspect of the structure--namely, the inverse relation between the demand for money balances and the expected rate of inflation. Because the eigenvalue is greater than unity, and M_t and α_t do not grow exponentially, a forward-looking particular solution for P_t involves a convergent sum.

If, alternatively, a model involved a first-order equation with an eigenvalue inside the unit circle, a forward-looking particular solution generally would not converge. Consequently, such a model would not imply a meaningful dependence of current price on expected future values of the exogenous variables. In Section 2 below, we analyze a second-order system in which one eigenvalue is greater than unity and the other is positive but less than unity. In this case, the particular solution includes convergent forward-looking and backward-looking terms.

To obtain a forward-looking particular solution for P_t and $E_t P_{t+1}$, operate on both sides of (2) with E_t and use a lag operator--see Sargent (1979, pp. 171-177)--to get

(3) $$E_t P_{t+1} = \beta^{-1} \sum_{i=1}^{\infty} (1+\beta^{-1})^{-i} E_t(M_{t+i} - \alpha_{t+i}).$$

Substituting (3) into (2) yields

(4) $$P_t = (1+\beta)^{-1}[(M_t-\alpha_t) + \sum_{i=1}^{\infty} (1+\beta^{-1})^{-i} E_t(M_{t+i} - \alpha_{t+i})].$$

The particular solution given by (4) represents what, following Flood and Garber (1980), we denote as the fundamental component of price (FC).

In the present example, FC at date t involves the current money stock and demand variables and a sum, with exponentially declining weights, of expected future values of the money stock and demand variables. If the processes generating M_t and α_t are not stochastic, these expectations are trivial and FC is deterministic. If M_t and α_t are constants or follow a random walk, the expression for FC reduces to $M_t - \alpha_t$.

Inspection of (2) reveals that a generalization of the solution for P_t involves adding to the particular solution any terms, denoted by ζ_t, that satisfy

$$(5) \qquad E_t\zeta_{t+1} - (1+\beta^{-1})\zeta_t = 0.$$

The interesting observation in the present context is that (5) can have both a deterministic solution and a stochastic solution. Specifically, the standard analysis of difference equations indicates that the only deterministic solution of (5) is

$$(6) \qquad \zeta_t = c(1+\beta^{-1})^t,$$

where c is a constant to be determined by initial or terminal conditions. The part of the general solution given by (6) represents what we denote as the deterministic bubble component of price (DBC). If c is not equal to zero, because the eigenvalue is greater than unity, DBC as given by (6) is not convergent.

A more intriguing phenomenon than the standard derivation of (6) is that we can also satisfy (5) with solutions to the stochastic difference equation

$$(7) \qquad \zeta_{t+1} - (1+\beta^{-1})\zeta_t = z_{t+1},$$

where z_i is a random variable, representing new information available at date i, that satisfies

$$E_j z_i = \begin{cases} z_i & \text{for} \quad i < j \\ 0 & \text{for} \quad i > j. \end{cases}$$

The key to the relevance of (7) for the general solution is that (5) relates ζ_t to $E_t \zeta_{t+1}$, rather than to ζ_{t+1} itself.

A solution to (7) is

$$(8) \qquad \zeta_t = \sum_{i=1}^{t} (1+\beta^{-1})^{t-i} z_i .$$

Although the eigenvalue is greater than unity, beginning the sum in (8) at date 1 insures that ζ_t is finite for all finite values of t. We discuss below possible empirical interpretations of date 1. The part of the general solution given by (8) represents what we denote as the stochastic bubble component of price (SBC).

Adding together the expressions in (4), (6), and (8) gives the general solution for the time path of the price level,

$$(9) \qquad P_t = (1+\beta)^{-1}[(M_t - \alpha_t) + \sum_{i=1}^{\infty} (1+\beta^{-1})^{-i} E_t(M_{t+i} - \alpha_{t+i})]$$

$$+ c(1+\beta^{-1})^t + \sum_{i=1}^{t} (1+\beta^{-1})^{t-i} z_i ,$$

Updating (9) and operating on both sides with E_t gives

$$(10)\qquad E_tP_{t+1} = \beta^{-1} \sum_{i=1}^{\infty} (1+\beta^{-1})^{-i} E_t(M_{t+i}-\alpha_{t+i})$$

$$+ c(1+\beta^{-1})^{t+1} + \sum_{i=1}^{t} (1+\beta^{-1})^{t+i-1} z_i,$$

because $E_t[E_{t+1}(M_{t+1+i}-\alpha_{t+1+i})] = E_t(M_{t+1+i}-\alpha_{t+1+i})$ by the law of iterated expectations and $E_tz_{t+1} = 0$ by assumption. Substituting into (2) the expression for P_t from (9) and the expression for E_tP_{t+1} from (10) confirms that (9) and (10) satisfy the model given by (1).

The solution for the price level given by (9) contains each of the three components of market-clearing price discussed above. The key elements in the latter two terms, which we denote as the bubble components, are the constant c and the random variable z_i. Note that, given RE, the bubble components can enter the solution only in the self-confirming form given by (6) and (8). The price level at date t can include the bubble components only because the form of these terms implies that the price level at any date $t+j$, $j > 0$, and its RE formed at date t include these same terms multiplied by the relevant eigenvalue raised to the power j. Consequently, given that the eigenvalue, $1+\beta^{-1}$, is greater than unity, the existence of a rational bubble would imply that $\{E_tP_{t+j}\}$ is unbounded, even if, for any t, $\{E_tM_{t+j}\}$ and $\{E_t\alpha_{t+j}\}$ are bounded.

In some optimization models in which real balances appear as an argument of agents' utility functions, the equilibrium price path cannot be explosive in the absence of explosive monetary growth. See Kingston (1982) and Obstfeld and Rogoff (1982) for references, discussion, and derivations relevant to this result. A necessary and sufficient condition for ruling out explosive price paths in these models is a kind of "super Inada" condition imposed on the utility function. This condition, in turn, implies that money is essential to the economy in the sense that real tax revenue from inflation is bounded away from zero, as the growth rate of the money stock tends to infinity. This condition also implies that, given a finite rate of inflation, no finite amount of extra consumption could compensate the agents for reducing their money balances to zero.

These implications are clearly quite restrictive. Moreover, as Kingston (1982) points out, they are not consistent with the Cagan money demand function used in equation (1) above. For this function, real tax revenue from inflation converges to zero as the rate of monetary growth tends to infinity. Equivalently, for the utility functions that are known to deliver the Cagan money demand function the "super Inada" condition does not hold.

These observations imply that in the present model and in general we cannot rule out the existence of rational bubbles, a priori. It is also worth noting that expectations of explosive behavior of the price level are not inconsistent with historical experience. In sum, the question of the existence of rational bubbles during a hyperinflation remains an open empirical question.

The Deterministic Bubble Component of Price

In the present example, DBC at date t equals the product of the eigenvalue raised to the power t and a constant. Although some related literature--for example, the empirical work of Flood and Garber (1980)--is concerned with deterministic bubbles, there are at least three reasons why DBC does not seem to warrant primary attention.

First, as a manifestation of the presence of arbitrary constants in the solution, the term that we have denoted as DBC is not peculiar to stochastic models incorporating expectations. Rather, it involves a phenomenon that arises in the ordinary, as opposed to partial, difference equation systems of deterministic economic models, such as perfect-foresight monetary growth models and growth models with heterogeneous capital goods. In the present model, we can think of FC as a degenerate saddle path, and the possibility of DBC essentially reflects the saddle point instability problem studied by Hahn (1966).

Second, because the imposition of initial conditions can resolve the indeterminacy of the arbitrary constants, the existence of DBC depends on initial conditions. In some cases, especially those in which the market under study has a short history, economic theory gives us little guidance regarding the determination of initial conditions. In other cases, however, it seems likely that actual histories of market-clearing prices would include initial conditions that preclude the existence of DBC. Referring to the solution given by (9), suppose that at any past date, denoted $t = 0$, the price level, P_o, equalled its FC. This initial condition implies that the constant, c, equals zero, and, hence, that DBC equals zero for all dates $t > 0$. In other words, without an unanticipated change in the structure of the model, a possibility that seems inconsistent with the RE concept, DBC cannot exist at any particular date unless it existed at all previous dates.

Third, DBC has no effect on the variance of P_t. Thus, DBC cannot help to explain the tentative observation that market-clearing price in many cases is more variable than its FC. Moreover, without unanticipated structural changes, DBC cannot come and go. In the present example, as noted above, because the eigenvalue is greater than unity, the existence of DBC would imply a direct explosion of P_t. In any case, the time path of DBC cannot exhibit the irregular oscillations with variable periodicity that seem to characterize the cyclical fluctuations of actual prices.

The Stochastic Bubble Component of Price

In the present example, SBC at date t involves an average of new information, represented by the random variable z_i, that became available from date 1 through date t, weighted by powers of the eigenvalue that decrease as i approaches t. The restrictions imposed on z_i, $E_j z_i = z_i$ for $i < j$ and $E_j z_i = 0$ for $i > j$, imply that current and past values of z_i

are known and that z_i is serially independent with mean zero. In the related literature, the work of Taylor (1977), Shiller (1978), and Blanchard and Watson (1982) is concerned with stochastic bubbles.

SBC is especially interesting theoretically because it is a possibility peculiarly associated with models that involve expectations. Specifically, the possible existence of SBC in a solution for the model given by equation (1) required that the difference equation (2), derived from (1), related P_t to $E_t P_{t+1}$ rather than to P_{t+1} itself. As pointed out by Shiller, the essential mathematical property underlying SBC is that an RE model generates a system of n first-order partial difference equations, the general solution to which involves n arbitrary functions on the integers, i.e., n infinite sequences of arbitrary constants, one for each date. Consequently, the imposition of any finite number of initial conditions cannot insure a bubble-free solution to an RE model. In contrast, a system of n ordinary first-order difference equations associated with a deterministic or perfect foresight model involves only n arbitrary constants, such as the constant c, associated with DBC in the present example.

Any information on a new or newly observed phenomenon that satisfies, either itself or through its innovations, the restrictions on z_i can affect P_t in the way prescribed by SBC as long as, beginning at date 1, individuals held expectations of next period's price level that were rational given that the solutions for all subsequent price levels include SBC. Thus, SBC potentially can help to explain cyclical fluctuations in prices and the tentative observation that market-clearing price in many cases is more variable than its FC. Specifically, the existence of SBC implies that constancy of the variables in FC, including the money stock and its expected future values, is not sufficient to insure constancy of the price level. As Shiller puts it,

> ... <u>any</u> unforecastable economic variable or the innovation in any variable [can] enter the solution! If all individuals conclude that the change in the

> Dow Jones average should be used in [(5) as z_i], then they will be rational in assuming so. If they have hunches which can be translated into the variable [z_i], then, if they forecast via [(5)], their hunches will yield rational forecasts (1978, p. 33).

Note that even if FC were deterministic, the existence of SBC would make the solution for P_t stochastic and would make the expectation E_tP_{t+1} nontrivial. In this case, we could say that the existence of SBC makes P_t depend on E_tP_{t+1} at the same time that the dependence of P_t on E_tP_{t+1} makes SBC possible.

The existence of SBC can involve a reaction by market participants to an intrinsically irrelevant variable, i.e., a variable that is not a member of the set of exogenous variables present in FC. Alternatively, it can involve overreaction to a truly relevant variable. For example, the demand variable, α_t, could depend on current or past values of the same variable, z_i, that enters SBC. In this case, the existence of SBC would mean that the effect of the history of the variable z_i on P_t differs from the effect implied by FC. The specific way in which the existence of SBC affects the time series properties of price depends on the process generating the phenomenon represented by z_i, on the associated eigenvalue of the difference equation relating current price and expected future price, and on the role, if any, played by z_i in the other components of price, especially FC.

The random variable z_i need not have a stationary distribution. For example, Blanchard and Watson (1982) propose a form for SBC that implies, in the notation developed above, the following specification of z_i:

$$z_t = \begin{cases} -(1+\beta^{-1})\zeta_{t-1}(1 - \frac{1}{\pi}) + \varepsilon_t & \text{with probability } \pi \\ -(1+\beta^{-1})\zeta_{t-1} + \varepsilon_t & \text{with probability } 1-\pi \end{cases}$$

where $E_{t-1}\varepsilon_t = 0$.

In this model, the determination of z_t involves both the random selection between two populations and a random drawing from the chosen population. The parameters of this process are such that z_i satisfies the condition $E_j z_i = 0$ for $i > j$, but the mean of one of the populations changes through time in such a way that at date t it equals the negative of the value of SBC at date $t-1$. Consequently, the probability that the randomly chosen value of z_t will be large enough to make SBC reverse sign is constant. In other words, this example specifies that bubbles burst instantly with constant probability. The empirical relevance of this formulation depends on whether SBC that exhibit bursting actually exist.

The designation of date 1, the initiation date of SBC as specified in equation (8), would seem to have at least one of three possible empirical counterparts. First, economic history presumably began in the finite past. Specifically, date 1 in all cases could be the point in time at which the market under study was organized. One problem with this interpretation of date 1 is that it would preclude identifying date 0 with equality between P_o and FC.

Second, in many cases date 1 could be the earliest date at which the random event represented by z_i could have occurred. In other words, the history of the variable z_i might include $z_t = 0$ for all $t < 1$. Such an example of a z_i that might be relevant for the German hyperinflation would be troop movements associated with the French occupation of the Ruhr. Another example, which might be relevant for the recent history of markets for gold and foreign exchange, would be oil discoveries, or the unexpected component of oil discoveries, in the North Sea.

Third, in some cases date 1 could be the initial date at which the random event represented by z_i was observed. Such an example could arise whenever a data collecting agency institutes a new survey that generates a new data series. In these cases, the existence of SBC involving z_i would suggest that market participants believed at date 1 that z_i, although previously unobserved, was correlated with the innovations in FC, but that this belief was qualitatively wrong or, at least, quantitatively inaccurate.

2. The Market for a Storable Commodity

In the preceding section, the possibility of rational bubbles involved the price level, i.e., the value of money, and the analysis took the relevant asset quantity, i.e., the nominal money stock, to be exogenous. To see the form that rational bubbles can take in another interesting context, consider the following model of the market for a storable commodity, like gold, that is both currently produced and held in portfolios:

(11) $$S_t + p_t = \alpha + \beta(E_t p_{t+1} - p_t) - \gamma r_t \quad \text{and}$$

(12) $$S_{t+1} - S_t = a + bE_t p_{t+1} - \gamma S_t$$

where S_t is the logarithm of the stock of refined gold at date t,

p_t is the logarithm of the price of refined gold at date t relative to an index of prices of other commodities, and

r_t is an index of real rates of return on other assets (the real interest rate) from date t to date t+1.

The variable r_t is exogenously determined and random. This variable drives the model and makes it stochastic. For simplicity, the analysis treats α and a as positive constants. The coefficients β, γ, b, and δ are nonnegative, and δ is less than unity.

Equation (11) says that the relative price of refined gold satisfies a condition of equality between the existing stock of refined gold, which is predetermined, and the portfolio demand for refined gold. The value of refined gold held in portfolios in terms of other commodities is related positively to the expected real rate of return from holding gold, which is simply the expected rate of change in the relative price of gold, and related negatively to real rates of return on alternative assets. These alternative returns are variable over time, but, for simplicity, are assumed to be contemporaneously observable.

Equation (12) says that current production of refined gold, which involves both extraction and refining, depends positively on the expected relative price of gold and, because the easily accessible gold is mined first, negatively on cumulative extraction. The model ignores final consumption of gold in dentistry and industry. This abstraction seems reasonable because actual consumption of gold is small relative to annual production and seems largely insensitive in the short run to changes in price--see Kettell (1982, pp. 104-122). For simplicity, equations (11) and (12) also specify the periodicity of production adjustment to be the same as the periodicity of price adjustment.

Equation (12) is meaningful only when $S_{t+1} - S_t > 0$ holds. We assume, for simplicity, that this constraint is not binding. This assumption seems consistent with historical experience. If the constraint, in fact, becomes binding at some point in time, then, in the absence of final consumption of gold, equation (12) is replaced by $S_{t+1} - S_t = 0$, and the path of the relative price adjusts to satisfy equation (11).

This model assumes that gold ore in the ground is not a perfect substitute in portfolios for refined gold, but also assumes that, although mining costs increase with cumulative extraction, the quantity of unmined gold is essentially unlimited. These assumptions are convenient, but are not essential for modelling rational bubbles. An alternative model of the gold market would assume that gold ore in the ground is a close substitute for refined gold and that the stock of gold in existence at date t, including both the ore and refined gold, is exogenous and finite. This alternative model would consist of a single equation relating this stock to the demand for it, and, thus, would be formally analogous to the inflation model of the preceding section.

An implicit assumption that is critical for the possible existence of rational bubbles is that the cumulative flow of services generated by refined gold, aggregated over an infinite time horizon, is not finite. To see this point, consider two examples. In the standard model of the market for a truly exhaustible resource, like oil, with given finite initial reserves--see, for example, Dasgupta and Heal (1979; ch. 6)--the time path of the resource price is determined as follows. First, under the assumption of risk neutrality, profit maximization by owners of the resource and/or the portfolio balance requirement ensures that price grows at the rate of interest. Second, substituting for price in the consumers' demand functions and aggregating over time gives cumulative consumption as a function of the initial price, alone. Finally, equating cumulative consumption to the initial reserves determines the initial price, and hence the entire price path, uniquely.

As a second example, consider a resource, such as land, that is not exhaustible. The fundamental component of the price equals the present value of the flow of rental income. Because the cumulative value of this flow is not finite, however, the price of land is not uniquely determined in the same way as the price of oil. Thus, a rational bubble can arise in the price of land in the same way as in the inflation model of Section 1.

The case of gold, in this respect, is like the case of land. Gold jewelry, for example, generates an unending flow of satisfaction or rental income. Thus, if, as price rises through time to maintain portfolio balance, consumption demand in industry and dentistry is choked off before the stock is exhausted, as we are assuming, it would not be irrational for any agent to plan to continue to hold gold jewelry or bullion forever. A rational bubble, of course, would affect the date at which consumption demand is choked off.

The Components of the Price of Gold

Rearranging and combining (11) and (12) yields the following difference equation system:

$$(13)\quad \begin{bmatrix} S_{t+1} \\ E_t p_{t+1} \end{bmatrix} = \begin{bmatrix} 1+b\beta^{-1}-\delta & b(1-\beta^{-1}) \\ \beta^{-1} & 1+\beta^{-1} \end{bmatrix} \begin{bmatrix} S_t \\ p_t \end{bmatrix}$$

$$+ \begin{bmatrix} \gamma b\beta^{-1} r_t + a - \alpha b\beta^{-1} \\ \gamma\beta^{-1} r_t - \alpha\beta^{-1} \end{bmatrix}.$$

The eigenvalues of this system are

$$\lambda_1 = \frac{1}{2}\left(2+\beta^{-1}+b\beta^{-1}-\delta+[(2+\beta^{-1}+b\beta^{-1}-\delta)^2+4(\delta+\beta^{-1}\delta-\beta^{-1}-1)]^{1/2}\right)$$

$$\lambda_2 = \frac{1}{2}\left(2+\beta^{-1}+b\beta^{-1}-\delta-[(2+\beta^{-1}+b\beta^{-1}-\delta)^2+4(\delta+\beta^{-1}\delta-\beta^{-1}-1)]^{1/2}\right).$$

Both eigenvalues are real: λ_1 is greater than unity and λ_2 is between zero and unity.

Let (S_t^*, p_t^*) denote the saddle path, i.e., the path to which all convergent solution paths converge. The general solution of (13) is obtained by adding to (S_t^*, p_t^*) the solutions to the homogeneous system

$$(14)\quad \begin{bmatrix} S_{t+1} \\ E_t p_{t+1} \end{bmatrix} = \begin{bmatrix} 1+b\beta^{-1}-\delta & b(1+\beta^{-1}) \\ \beta^{-1} & 1+\beta^{-1} \end{bmatrix} \begin{bmatrix} S_t \\ p_t \end{bmatrix}.$$

Eigenvectors associated with λ_1 and λ_2, respectively, are

$$V_1 = \begin{bmatrix} 1 \\ [2b(1+\beta^{-1})]^{-1}(\delta+\beta^{-1}-b\beta^{-1}+[(2+\beta^{-1}+b\beta^{-1}-\delta)^2+4(\delta+\beta^{-1}\delta-\beta^{-1}-1)]^{1/2}) \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ [2b(1+\beta^{-1})]^{-1}(\delta+\beta^{-1}-b\beta^{-1}-[(2+\beta^{-1}+b\beta^{-1}-\delta)^2+4(\delta+\beta^{-1}\delta-\beta^{-1}-1)]^{1/2}) \end{bmatrix}.$$

Analogously to the homogeneous equation of the inflation model of Section 1, (14) can have both a deterministic solution and a stochastic solution.

The deterministic solution to (14) is

$$(15) \qquad \begin{bmatrix} S_t \\ p_t \end{bmatrix} = C_1\lambda_1^t V_1 + C_2\lambda_2^t V_2$$

where C_1 and C_2 are constants to be determined by initial or terminal conditions. If C_1 is not equal to zero, because λ_1 is greater than unity, the solution given by (15) is explosive. As in the inflation model, the actual history of price and quantity is likely to include initial conditions that imply that C_1 equals zero. Given that RE precludes unanticipated changes in the structure of the model, a sufficient condition for C_1 equal to zero is that at any past date the system was on a convergent path.

If C_1 is equal to zero, the other constant, C_2, is determined by the initial asset quantity, S_o, as $C_2 = S_o - S_o^*$. Accordingly, we define FC in this model as

$$\begin{bmatrix} S_t \\ p_t \end{bmatrix} = \begin{bmatrix} S_t^* \\ p_t^* \end{bmatrix} + (S_o - S_o^*)\lambda_2^t V_2.$$

This equation specifies the only solution path that starts at the exogenously given stock of gold S_o and converges to the saddle path as t tends to infinity. The difference equation system (13) has the form of the system solved by Blanchard and Kahn (1980, p. 1309), and following their calculations, with appropriate corrections, we can obtain the following explicit expressions for FC of price and quantity:

$$(16) \qquad p_t = [b(1+\beta^{-1})]^{-1}[(\lambda_1-1)^{-1}(\lambda_1 \alpha b\beta^{-1} - a[\lambda_1-1-\beta^{-1}]) + (1+\beta^{-1}-\lambda_1)S_t - \gamma b\beta^{-1}(r_t + \sum_{i=1}^{\infty} \lambda_1^{-i}E_t r_{t+i})]$$

and

$$(17) \qquad S_t = (\alpha b+a)(b+\delta)^{-1} + [S_o - (\alpha b+a)(b+\delta)^{-1}]\lambda_2^t - \gamma b\beta^{-1} \sum_{j=1}^{t} \sum_{i=1}^{\infty} \lambda_2^{j-1}\lambda_1^{-i} E_{t-j}r_{t-j+i}.$$

As given by (16), FC of the relative price of gold depends on the parameters of the portfolio balance equation and the production equation, the existing stock of gold, which is predetermined, the current real interest rate, and a sum, with exponentially declining weights, of expected future interest rates. As given by (17), FC of the stock of gold depends on the relevant parameters, on the initial stock of gold, and on a weighted sum of past expectations of future real interest rates that were formed from date 0 to date t-1. The terms involving past expectations of real interest rates are relevant because the current asset stock reflects the history of production, which, in turn, reflects the history of price expectations, and, hence, the history of interest rate expectations. If r_t is a constant and $S_o = S_o^*$, equations (16) and (17) reduce to $p_t = (b+\delta)^{-1}(-a+\alpha\delta-\gamma\delta r)$ and $S_t = (b+\delta)^{-1}(\alpha b+a-b\gamma r)$.

DBC for this model involves $C_1 \neq 0$ in equation (15). Therefore, solution paths reflecting DBC are nonconvergent. SBC involves stochastic solutions to the homogeneous system (14). Specifically, we can satisfy (14) with solutions to the ordinary stochastic difference equation system

$$(18) \qquad \begin{bmatrix} S_{t+1} \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} 1+b\beta^{-1}-\delta & b(1+\beta^{-1}) \\ \beta^{-1} & 1+\beta^{-1} \end{bmatrix} \begin{bmatrix} S_t \\ p_t \end{bmatrix} + \begin{bmatrix} 0 \\ z_{t+1} \end{bmatrix}$$

where z_i is a random variable that has the same properties it had in the inflation model. Importantly, it represents new information available at date i and satisfies

$$E_j z_i = \begin{cases} z_i & \text{for} \quad i \leq j \\ 0 & \text{for} \quad i > j \end{cases}.$$

The possible empirical interpretations of date 1 are the same as discussed above in the context of the inflation model.

As in the inflation model, the dependence of demand on rational price expectations is essential for the inclusion of the random variable z_i in the solution. Adding this variable to the second equation in (18) does not contradict (13) only because this second equation in (13) involves $E_t p_{t+1}$, rather than merely p_{t+1}. Note that we cannot add a similar variable to the first equation in (18) because the first equation in (13) involves S_{t+1} and not $E_t S_{t+1}$.

The key property that expectations of quantities do not appear in the structure of the model in addition to expectations of prices reflects the proposition that, with markets working to equate quantities demanded and supplied, agents are constrained only by endowments and prices. This observation suggests that models of markets that do not clear, in which both price and quantity expectations are relevant for current demand, could exhibit qualitatively different forms of rational stochastic bubbles.

To solve for SBC, we rewrite (18) in the form

$$[1 - (2+\beta^{-1}+b\beta^{-1}-\delta)L + (1+\beta^{-1}-\delta-\delta\beta^{-1})L^2]S_{t+1} = b(1+\beta^{-1})z_t$$

where L is the lag operator, defined by $L^j S_t = S_{t-j}$. Inverting the polynomial in L by the method of partial fractions--see Sargent (1979, pp. 177-180)--and substituting the resulting solution for S_{t+1} into (18) we get

$$S_{t+1} = b(1+\beta^{-1})(\lambda_1-\lambda_2)^{-1} \sum_{i=1}^{t} (\lambda_1^{t+1-i} - \lambda_2^{t+1-i}) z_i \tag{19}$$

$$p_t = (\lambda_1-\lambda_2)^{-1} \sum_{i=1}^{t} [(\lambda_1-1-b\beta^{-1}+\delta)\lambda_1^{t-i} - (\lambda_2-1-b\beta^{-1}+\delta)\lambda_2^{t-i}] z_i. \tag{20}$$

The solutions given by (19) and (20) represent the stochastic bubble component for this model. In this case, if SBC exists, it arises in both the asset price and the asset stock. The SBC of price at date t, given by (20), involves an average of new information, represented by z_i, that became available from date 1 to date t, weighted by the difference between powers of the eigenvalues that decrease as i approaches t. The SBC of the stock, given by (19), has a similar form, but incorporates new information with a one-period lag.

Because one of the eigenvalues exceeds unity, even though the other one is less than unity, SBC in this model has properties similar to SBC in the inflation model. Specifically, under reasonable assumptions about parameters, rational bubbles in this model are explosive. In contrast to the inflation model, however, because the stock of gold is an endogenous variable, rational bubbles are reflected in both the asset price and the asset stock. These properties suggest what to look for in way of empirical evidence relevant to the existence of rational bubbles in the gold market.

3. Tests for the Existence of Rational Bubbles

This section discusses the formulation of econometric tests for the existence of rational bubble components in asset prices. The hypothesis that rational bubbles exist combines the hypothesis that expectations are rational with the hypothesis that price does not conform to the fundamental component (FC) of the solution for market-clearing price. Consequently, an interesting test of the hypothesis that rational bubbles exist must involve more restrictions on the data and, hence, greater possibilities of rejection than would tests of either one or the other of its component hypotheses. For example, taken alone, results that do not reject market efficiency or results that suggest that price is more variable than its FC, although consistent with the hypothesis that SBC exist, do not provide telling evidence about this hypothesis.

The essential problem involved in testing for the existence of rational bubbles is that we cannot directly observe them separately from the fundamental component of price. Consequently, any test of the hypothesis that an asset price includes bubbles must involve formulation of a joint hypothesis about FC. A relevant criterion for judging the usefulness of a proposed test for the existence of rational bubbles is, therefore, the weakness of the joint hypothesis about FC involved in the test. Specifically, the harder to reject that we judge the joint hypotheses about FC to be, the more convincing is the evidence from the test regarding the existence of SBC.

Direct Estimation of Rational Bubbles

Flood and Garber (1980) and Flood, Garber, and Scott (1982) carry out econometric tests for the existence of rational deterministic bubble components (DBC) in price levels during European hyperinflations. Recall that DBC at date t involves the product of the eigenvalue raised to the power t and a constant, c, in equation (9) above. Flood and Garber assume that the growth rate of the nominal money stock followed an autoregressive process and that the factors other than expected inflation influencing the demand for real money balances followed a random walk.

Using these assumptions relating to FC, they develop two testing procedures. One procedure is to estimate jointly (1) the demand function for real money balances and (2) a solution for the inflation rate consisting of FC and DBC. This procedure uses predicted values from the estimated equation for the inflation rate to measure the rationally expected inflation rate. The other procedure is to estimate jointly (1) a demand function for real money balances in which the rationally expected inflation rate consists of FC and DBC and (2) the autoregressive money process. This procedure uses the estimated money process to generate the expectations of future money growth included in FC. Both procedures yield estimates of the associated constant, c, in equation (9) above, in the supposed DBC. The test results fail to reject the no-bubble hypothesis, i.e., $c = 0$.

As Flood and Garber point out, this strategy of direct estimation of the constant c involves serious technical problems. The jth element of the corresponding regressor is, in our notation, $(1+\beta^{-1})^j$. Since $(1+\beta^{-1}) > 1$, although the estimator of c will be consistent, its asymptotic distribution will be degenerate and confidence intervals cannot be calculated.

To obtain a nondegenerate normal asymptotic distribution, Flood, Garber, and Scott test the hypothesis that a bubble passed through the parallel hyperinflations of the 1920's. Their results reject the no-bubble hypothesis in most cases. Their estimators are consistent and have normal asymptotic distributions (as the number of countries in the sample tends to infinity). However, due to the small number of countries actually involved in the sample, the relevance of these asymptotic properties is questionable.

Burmeister and Wall (1982) extend the direct estimation strategy of Flood and Garber from testing for existence of DBC to testing for both DBC and SBC by using a Kalman Filter. They treat the rational bubble as an unobservable variable whose evolution through time is governed by equation (5) above. The results reject the no-bubble hypothesis in most cases. However, the asymptotic degeneracy problem, pointed out by Flood and Garber, also applies to the estimators obtained using a Kalman Filter.

Indirect Tests for Rational Bubbles

Blanchard and Watson (1982) propose tests for rational bubbles that do not involve direct estimation of the parameters of the solution for price. For stock prices, they assume that the only forcing variable in FC is observable dividends. In the absence of bubbles, the (conditional) variance of the distribution of dividends imposes an upper bound on the (conditional) variance of the distribution of stock prices (or, equivalently, the distribution of excess returns). Blanchard and Watson tighten the bounds derived by Shiller (1981) by using information contained in autocovariances of dividends. They conclude that stock prices violate these bounds and, thus, that bubbles exist.

They also derive implications of the absence of bubbles for cross-covariances of prices and dividends. Specifically, stochastic bubbles are likely to decrease the correlation between prices and dividends. The calculated relations between prices and dividends also suggest the presence of bubbles.

As Blanchard and Watson recognize, the apparent presence of bubbles could be due to other phenomena. In the case of the stock prices, the appeal to rational bubbles as an explanation for excess volatility seems questionable because of similar evidence reported by Shiller (1979) of excess volatility in long-term interest rates. Rational bubbles cannot arise in bond prices with finite maturity because, abstracting from possible default, the prices at the maturity dates are known with certainty. It seems plausible that these same unidentified phenomena that produce excess volatility in bond prices are also present in the stock market.

For the price of gold, Blanchard and Watson assume that some of the important variables affecting FC are unobservable, and they examine the implications of their specification of SBC that burst, discussed above, for the distribution of excess returns. The tests are based on the likely effects of this form of rational bubble for "runs" in excess returns from holding gold and for the coefficient of kurtosis of the distribution of excess returns. The empirical results are not conclusive. Moreover, as Blanchard and Watson recognize, the implications of rational bubbles for the number of runs and coefficient of kurtosis of excess returns are quite sensitive to the particular form of the bubble, to the form of FC, and to the information structure.

Diagnostic Checking for Rational Bubbles

Given the limitations of the above strategies for testing for the existence of rational bubbles, we suggest, as an alternative, the development of diagnostic checks for the stationarity of prices. This strategy involves an assumption about FC that seems quite weak--namely, that the processes generating the variables in FC (perhaps after differencing a few times or removing a deterministic trend) are stationary. Given this assumption, the proposed diagnostic checks can provide evidence about the empirical relevance of rational bubbles because, as the above analysis shows, conventional behavioral assumptions imply that the processes generating rational bubbles are not stationary.

If the observed price sequence in the models analyzed above contains a rational bubble, its nth difference is generated by a nonstationary stochastic process, for any finite n. If the bubble is stochastic, the deviations of price from any deterministic trend are also generated by a nonstationary process. In practice, however, given the finite size of actual samples, we can make any time series look stationary by detrending and/or differencing a sufficient number of times. Consequently, we cannot detrend and difference the observed time series of price, before running our stationarity checks, arbitrarily. The proposed strategy, therefore, is as follows.

First, find stationary stochastic processes that fit the (differenced or detrended) time series of the variables that enter FC. Second, assume that agents' information set consists of current and past values of the relevant variables, and compute the process that generates FC of price (or its nth difference). This process will give some idea of how to detrend the observed time series of price before running diagnostic checks for stationarity.

The third step in the proposed strategy is to carry out such diagnostic checks for stationarity on the time series of price as detrended and/or differenced. There are, of course, no standard statistical tests enabling us to reject, at a specific level of significance, the hypothesis that a given time series is generated by a nonstationary stochastic process. However, as a matter of common practice econometricians use a variety of procedures to make judgments about stationarity.

If we find no evidence of non-stationarity we conclude that no rational bubbles were present. One attraction of these diagnostic checks is that any evidence they provide <u>against</u>

rational bubbles is unambiguous. In contrast, the other testing strategies discussed above are capable only of rejecting the joint hypothesis that no rational bubbles were present and that a particular set of assumptions about FC and the information structure are true.

If, alternatively, we find evidence of nonstationarity, we can draw no definite conclusions. Nonstationarity can mean that rational bubbles, in fact, were present or that our assumptions about FC and the information structure were inappropriate. One possibility, in cases of nonstationarity, would be to difference the time series once more and again carry out diagnostic checks for stationarity. The evidence against rational bubbles would be stronger, the fewer differences necessary to make the series appear stationary.

These diagnostic checks would probably be of little help in investigating the presence of rational bubbles during hyper-inflations, because, in these situations, sample size is small and it is easy to believe that FC itself was generated by a non-stationary process. For more "normal" situations, however, we often have a large number of observations on prices, and differencing or detrending the variables that we think enter FC usually leads to stationary time series. A sequel to this paper will involve the implementation of these diagnostic checks for the price of gold and for the prices of stocks. We conjecture that these checks will provide strong evidence against the empirical relevance of rational bubbles outside the context of hyperinflations.

4. Summary and Conclusions

The first section of the paper developed a linear RE model of inflation in which the price level changes over time to keep the real value of the nominal money stock equal to the demand for real money balances, which depends in turn on rational expectations of inflation. The analysis of this model focuses on stochastic bubbles as a possibility peculiarly associated with RE models. This analysis does not point to any compelling reason to rule out rational stochastic bubbles a priori. An important result, however, is that conventional behavioral assumptions imply parameter values such that any rational bubbles that arise in this model are explosive.

The second section extended the analysis to a linear RE model of the market for a storable commodity, like gold, that is both produced and held in portfolios. In this model, the current and expected future relative prices of the asset change over time to keep the portfolio demand for the asset, which depends on the expected rate of change of its relative price, equal to the existing asset stock, and the asset stock changes over time as a result of net production, which depends on the expected relative price and on cumulative extraction. Again, although there seems to be no compelling reason to rule out rational stochastic bubbles a priori, we find that conventional behavioral assumptions imply that any rational bubbles that arise are explosive. Furthermore, if the stock of the asset is an endogenous variable, rational bubbles are reflected in both the asset price and the asset stock.

The third section discussed the implementation of econometric tests for the existence of rational bubbles. An essential problem is that such tests must involve formulation of a joint hypothesis about FC. Furthermore, because the theoretical analysis suggests that rational bubbles are explosive, any time series that contain rational bubbles probably violate the stationarity assumptions that underlie most existing econometric procedures for hypothesis testing. For these reasons, for empirical analysis of the existence of rational bubbles, we suggest "diagnostic checking" of the stationarity properties of observable time series of price. These checks are based on finding a detrended and/or differenced time series of price that we would expect to be stationary in the absence of rational bubbles and nonstationary in their presence. Although these diagnostic checks do not constitute definitive hypothesis testing, we conjecture that they would provide strong evidence against rational bubbles outside the context of hyperinflations.

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