FINANCIAL CLAUSTROPHOBIA:
ASSET PRICING IN ILLIQUID MARKETS

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ABSTRACT

There are many examples of markets where an agent who wants to get out of an investment position quickly may find himself trapped and forced to remain in that position because of a lack of liquidity. What are the asset-pricing implications when agents cannot always buy and sell assets immediately? We study this issue in a multi-asset exchange economy with heterogeneous agents. In this model, agents can trade initially, but then cannot trade again until after a trading "blackout" period. The more liquid the market, the sooner agents can trade again. Faced with illiquidity, agents abandon diversification and choose highly polarized portfolios. Risky assets are held primarily by the less-patient short-horizon agents in the economy. Polarization causes the usual risk-return tradeo. to break down and an asset's price may have more to do with the demographics of who owns it than with the riskiness of its cash flows. Risky assets are generally more valuable in an illiquid market than in a liquid market. Market illiquidity can also have large effects on the equity premium.

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1. INTRODUCTION

Traditional asset-pricing theory is founded on the assumption that financial markets are always liquid and that agents can trade whenever they please. In reality, however, many financial markets are not perfectly liquid and immediacy is not always available. What happens when agents face illiquid markets in which they cannot always buy and sell assets immediately?

This issue is of fundamental importance because this type of illiquidity is pervasive throughout many financial markets. A large percentage of the wealth of the typical household is in the form of human capital, business interests such as sole proprietorships, partnerships, and equity in other closely-held firms, deferred compensation, pension plans, tax-deferred retirement accounts, savings bonds, annuities, trusts, inheritances, and residential real estate. The markets for these types of assets are generally very illiquid and months or even years may be required to sell an asset. On the institutional side, an increasing amount of wealth is being allocated to illiquid asset classes such as private equity, emerging markets, venture capital, commercial real estate, and the rapidly-growing hedge-fund sector. In these asset markets, the ability to buy or sell immediately is the rare exception rather than the rule.

This issue takes on additional importance given the increasingly common phenomenon of liquidity crises in even well-established securities markets. As one example, the Russian default in 1998 triggered a liquidity "meltdown" that led directly to the demise of Long Term Capital Management and left many other Wall Street traders trapped in risky positions which they could not unwind, producing what might be termed "financial claustrophobia."

> "The scramble to unload almost any kind of risky investment has been so urgent that some markets, particularly for riskier bonds, are paralyzed, leaving firms holding far more of them than they want. . . . Those holding Russian securities were stuck. There was no trading. No bids, no offers."
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> — *Wall Street Journal*, September 22, 1998.

Other recent examples include "flights to quality" in which major institutional investors such as Steinhardt Partners, Askin Capital, MKP, Ellington Capital, and Beacon Hill Asset Management were unable to sell assets rapidly enough to avoid financial distress.

Market illiquidity has many potential implications for asset pricing and raises a number of fundamental issues. How do equilibrium asset prices in an illiquid market differ from those in a liquid market? Current portfolio decisions take on a degree or permanence or irreversibility when agents may have to wait before trading again. How does this added dimension influence optimal portfolio choice? Who bears the risks created by market illiquidity? What are the welfare implications of market illiquidity?

To address these issues, this paper examines the asset-pricing implications of market illiquidity within a continuous-time exchange economy with multiple assets and heterogenous agents. In this framework, agents have logarithmic preferences, but differ in their subjective time discount factors. The agents may trade their endowments initially, but are then unable to trade again until after a trading "blackout" period. The more liquid the market, the more quickly agents can trade again. Although admittedly very stylized, this feature of the model does have the advantage of capturing in a simple way the intuitive notion of market illiquidity as an absence of immediacy. Furthermore, this feature reflects the key role that liquidity plays in determining the fundamental nature of financial markets. This is because, in this framework, illiquidity changes what is otherwise a complete market into a dynamically incomplete market. Varying the length of the "blackout"

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period allows us to study asset pricing through a full spectrum of markets ranging from completely liquid to completely illiquid.

This analysis provides a number of important new insights about the effects of illiquidity in financial markets. When markets are liquid and the agents can always trade, we obtain the standard result that it is optimal for agents to diversify by holding the market portfolio. When the market is illiquid, however, the optimal portfolio behavior of agents changes dramatically. Surprisingly, the agents no longer choose to hold diversified portfolios. Rather, they select very polarized portfolios with the less-patient investors often holding primarily risky assets and the more-patient investors owning the remaining safer assets. The reason for this counterintuitive portfolio behavior is that in illiquid markets, agents must consider not only the instantaneous risk of their portfolios, but also the long-term risk of the assets they hold. This introduces a demand for risk sharing over time that does not exist when agents can always trade. Because less-patient agents with high subjective discount rates are better able to bear the long-term risk of assets in the economy, they gravitate towards riskier undiversified portfolios in equilibrium. Thus, the familiar intuition that investors should hold diversified portfolios is predicated on the assumption that agents face liquid markets. If markets are illiquid, then agents can do better by specializing portfolios based on whether they have a shorter or longer horizon than other agents in the market. This insight may help explain a number of well-known puzzles about the limited participation of households in securities markets (for example, see Mankiw and Zeldes (1991)).

Portfolio polarization has direct implications for the equilibrium prices of assets. When markets are liquid, prices are determined by the distribution of wealth among agents and the usual risk and return tradeoffs. When markets are illiquid and agents hold undiversified portfolios, however, the covariances of their individual marginal utilities with the dividend streams are very different from those that would exist in a complete market and the usual risk and return relation breaks down. In fact, when agents hold extremely polarized portfolios, the riskiness of dividends becomes nearly irrelevant in pricing assets. Thus, prices are determined primarily by the distribution of wealth. This means that the value of an asset in an illiquid market may have more to do with its current owner than with its future cash flows. This result introduces an institutional or demographic dimension into asset pricing which may be useful in rationalizing stylized facts often ascribed to behavioral biases. Because asset pricing is driven by different factors when agents cannot always trade, prices in an illiquid market can be very different from those in a liquid market. We show that risky assets are often worth more in an illiquid market, although the opposite is also possible.

Market illiquidity also has important implications for the equity premium. We show that as a market becomes more liquid, the equity premium can either increase, decrease, or remain the same depending on the distribution of wealth in the economy. In one example where the less-patient agent holds all of the risky asset initially, we show that increasing the liquidity of the market increases the equity premium by more than two percent on an annualized basis.

Since the illiquidity of the market affects the relative valuation of assets, it impacts the value of each agent's endowment and can significantly alter the distribution of wealth in the economy. Often, illiquidity has the effect of redistributing wealth from more-patient to less-patient agents. Despite this, all agents suffer welfare losses as the market becomes illiquid. The incidence of welfare losses is far from symmetric, however, and in some cases, the welfare losses for one set of agents may be vanishingly small. Further, these agents may own virtually all of the assets in the market. Thus, there may be little incentive in aggregate to increase the amount of liquidity available in the market.

This paper contributes to a rapidly growing literature on the effects of liquidity on asset prices. Important theoretical papers in this area include Mayers (1972, 1973, 1976), Amihud and Mendelson (1986), Lippman and McCall (1986), Grossman and Laroque (1990), Boudoukh and Whitelaw (1993), Longstaff (1995, 2001), Eisfeldt (1999), Vayanos (1998, 2003), Holmström and Tirole (2001), Huang (2002), Liu, Longstaff, and Kahl (2003), Duffie, Gârleanu, and Pederson (2003a, b), O'Hara (2003), Weill (2003), and Vayanos and Wang (2003). Empirical evidence about the valuation effects of illiquidity includes Cornell and Shapiro (1990), Amihud and Mendelson (1991), Boudoukh and Whitelaw (1991), Silber (1992), Kamara (1994), Brenner, Eldor, and Hauser (2001), Chordia, Roll, and Subrahmanyam (2002), Pastor and Stambaugh (2003), Longstaff (2003), Wang (2003), and Acharya and Pederson (2003).

2. THE MODEL

In exploring the asset-pricing implications of market illiquidity, our goal is to provide a modeling framework rich enough to capture agents' incentives for trading assets over time, yet simple enough to make the intuition behind the results clear. To this end, we develop a two-asset version of a standard Lucas (1978) pure exchange economy with heterogeneous agents.

The basic structure of the model closely parallels that of Cochrane, Longstaff, and Santa-Clara (2003). There are two assets or "trees" in this economy. Each asset produces a stream of dividends in the form of the single consumption good. Let X_t and Y_t denote the dividends generated by the assets. The dividends follow simple i.i.d. geometric Brownian motions

$$\frac{dX}{X} = \mu_X \ dt \ + \ \sigma_X \ dZ_X, \tag{1}$$

$$\frac{dY}{Y} = \mu_Y \ dt \ + \ \sigma_Y \ dZ_Y, \tag{2}$$

where the correlation between dZ_X and dZ_Y is $\rho \ dt$. Although we refer to the assets as risky throughout the discussion, nothing prevents one of the assets from being a riskless bond since σ_X or σ_Y can equal zero.[1] We normalize the number of shares of each asset in the economy to be one. To keep notation as simple as possible, expectations and variables without time subscripts (such as X and Y) will denote initial or time-zero values.

There are two agents in this model. The first agent is endowed with w and v shares of the first and second assets, respectively. Thus, the second agent is endowed with $1 - w$ and $1 - v$ shares of the two assets. Denote the agents' consumption streams by C_t and D_t, respectively. Preferences at time zero are given by

$$\ln(C) + E\left[\int_0^\infty e^{-\beta t} \ln(C_t) \ dt\right], \tag{3}$$

$$\ln(D) + E\left[\int_0^\infty e^{-\delta t} \ln(D_t) \ dt\right], \tag{4}$$

where the subjective time discount rates β and δ may differ, implying that one agent is less patient that the other. For concreteness, assume that the first agent is less patient than the second, $\beta > \delta$. These preferences imply that consumption at time zero takes the form of a discrete "gulp," while consumption at all future times occurs as a flow. This feature allows the model to be well specified even when portfolio decisions cannot be made continuously.[2]

In this framework, the agents make optimal consumption decisions and are allowed to trade their endowments at time zero. After this initial round of trading, however, the agents cannot trade again until after T periods, where T may range from zero to infinity. This feature captures the intuitive notion of

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[1]More specifically, one of the assets can be a riskless consol bond in positive net supply by setting either σ_X or σ_Y equal to zero.

[2]Since consumption at time zero is in the form of a "gulp" with no flow component, the range of integration in Equations (3) and (4) does not actually include zero. Since all of the integrands will be bounded at zero, however, none of the results are affected by using zero as the lower limit in the integrals. We adopt this convention for notational convenience.

market illiquidity as a lack of immediacy; an agent who buys shares at time zero has to wait T periods before being able to resell them. After the "blackout" period has elapsed, agents can again buy and sell shares without restriction. We denote the number of shares of the two assets held by the first agent by N_t and M_t, respectively. Market clearing implies that the second agent holds $1 - N_t$ and $1 - M_t$ shares of the two assets.

To illustrate the asset-pricing implications of market illiquidity, we present a series of cases in the next several sections. First, we consider the benchmark case of a completely liquid market where $T = 0$ and there is no "blackout" period. Next, we consider the polar extreme case of an completely illiquid market where $T = \infty$ and agents cannot ever resell their shares. Finally, we consider the general case of a partially liquid or "thin" market where $0 < T < \infty$.

3. THE LIQUID-MARKET CASE

In this section, we present the liquid-market case which will be used as a benchmark for comparison to the illiquid cases in later sections. In doing this, we focus primarily on providing asset-pricing results as of time zero, since this is the only time at which market prices will always be defined in the illiquid cases. More general results for the liquid-market case are given in the Appendix.

Let P_t and Q_t denote the equilibrium prices for the first and second assets. At time zero, the Euler conditions for the first agent imply

$$P = E\left[\int_0^\infty e^{-\beta t}\left(\frac{C}{C_t}\right) X_t\ dt\right],\tag{5}$$

$$Q = E\left[\int_0^\infty e^{-\beta t}\left(\frac{C}{C_t}\right) Y_t\ dt\right].\tag{6}$$

Similarly, the Euler conditions for the second agent imply

$$P = E\left[\int_0^\infty e^{-\delta t}\left(\frac{X + Y - C}{X_t + Y_t - C_t}\right) X_t\ dt\right],\tag{7}$$

$$Q = E\left[\int_0^\infty e^{-\delta t}\left(\frac{X + Y - C}{X_t + Y_t - C_t}\right) Y_t\ dt\right],\tag{8}$$

after substituting in the market clearing condition $D_t = X_t + Y_t - C_t$.

Because the two assets can always be traded in this case, the market is dynamically complete. The Appendix shows that equilibrium consumption is given by

$$C = \frac{wX + vY + (wA(1,1,\delta) + vB(1,1,\delta))\,(X + Y)}{1 + 1/\beta - w\,(A(1,1,\beta) - A(1,1,\delta)) - v\,(B(1,1,\beta) - B(1,1,\delta))},\tag{9}$$

$$C_t = \frac{e^{-\beta t}\ C}{e^{-\beta t}\ C\ +\ e^{-\delta t}\ (X + Y - C)}\ (X_t + Y_t).\tag{10}$$

where we define the functions $A(a,b,c)$ and $B(a,b,c)$ as

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$$A(a,b,c) = k_1 \ (X/Y) \ F\left(1, \ 1-\gamma; \ 2-\gamma; \ -\tfrac{aX}{bY}\right) + \ k_2 \ F\left(1, \ \theta; \ 1+\theta; \ -\tfrac{bY}{aX}\right), \tag{11}$$

$$B(a,b,c) = k_3 \ (Y/X) \ F\left(1, \ 1+\theta; \ 2+\theta; \ -\tfrac{bY}{aX}\right) - \ k_4 \ F\left(1, \ -\gamma; \ 1-\gamma; \ -\tfrac{aX}{bY}\right), \tag{12}$$

and where k_1, k_2, k_3, k_4, γ, and θ are constants defined in the appendix. The function $F(a,b;c;z)$ is the standard hypergeometric function (see Abramowitz and Stegum (1970) Chapter 15). The hypergeometric function is defined by the power series

$$F(a,b;c;z) = 1 + \frac{a \cdot b}{c \cdot 1} z + \frac{a(a+1) \cdot b(b+1)}{c(c+1) \cdot 1 \cdot 2} z^2 + \frac{a(a+1)(a+2) \cdot b(b+1)(b+2)}{c(c+1)(c+2) \cdot 1 \cdot 2 \cdot 3} z^3 + \dots \tag{13}$$

The hypergeometric function has an integral representation, which can be used for numerical evaluation and as an analytic continuation beyond $\|z\| < 1$,

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 w^{b-1}(1-w)^{c-b-1}(1-wz)^{-a} \ dw; \quad Re(c) > Re(b) > 0. \tag{14}$$

From Equation (9), optimal initial consumption depends directly on the distribution of wealth in the economy. In particular, the first agent has an endowment of w shares of the first asset and v shares of the second asset. The values of v and w appear directly in the expression for C. As shown in Equation (10), the effects of the initial distribution of wealth on consumption propagate through time since future consumption is an explicit function of the initial level of consumption, which in turn depends on the initial endowments and dividends.

Similarly, the optimal portfolio for the first agent is given by

$$N_t = M_t = \frac{\delta \ e^{-\beta t} \ C}{\delta \ e^{-\beta t} \ C \ + \ \beta \ e^{-\delta t} \ (X+Y-C)}. \tag{15}$$

This optimal portfolio rule has several important aspects. First, each agent holds a portfolio that has the same number of shares of each of the two assets. Since there are equal numbers of shares of the two asset in the economy, however, this means that the optimal portfolio for each agent is simply the market portfolio. Thus, a one-fund separation result holds; agents would be indifferent between trading the assets individually or trading the shares of a stock index fund. Second, trading occurs in equilibrium since the number of shares held by the two agents changes over time. In particular, since the first agent is less patient than the second, the first agent systematically sells his portfolio to the second agent over time. This enables the first agent to consume more than the total dividends he receives initially. Over time, however, the share of dividends consumed by the first agent declines while the share of dividends consumed by the second agent increases. Third, the portfolio rule is a deterministic function of time; the portfolio rule does not vary with changes in the state variables X_t and Y_t. Finally, the optimal portfolio rule is also affected by the initial distribution of wealth through the dependence on the initial consumption level.

Asset prices can now be obtained by substituting the optimal consumption process into the Euler equations and evaluating the expectations. The Appendix shows that the time-zero prices for the two assets are given by

$$P = C \ A(1, \ 1, \ \beta) + \ (X+Y-C) \ A(1, \ 1, \ \delta), \tag{16}$$

$$Q = C \ B(1, \ 1, \ \beta) + \ (X+Y-C) \ B(1, \ 1, \ \delta). \tag{17}$$

Interestingly, the functions $A(a, b, c)$ and $B(a, b, c)$ have intuitive interpretations as the equilibrium prices for the first and second assets (normalized by total consumption) that would exist in a representative-agent version of this economy where there were a and b shares of each asset in aggregate, and where the subjective time discount rate for the representative agent's logarithmic utility function was c. Thus, the price functions P and Q in Equations (16) and (17) are simple consumption-weighted averages of the prices in a representative-agent economy with a subjective discount rate of β and δ, respectively.

4. THE ILLIQUID-MARKET CASE

In this section, we consider the case where agents can trade assets at time zero, but not thereafter. This is the case where the trading "blackout" period is infinite. By contrasting this case with the benchmark liquid-market case of the previous section, we can examine directly how asset prices and portfolio choices are affected by the liquidity of the market.

In this setting, agents choose an initial consumption level and portfolio of assets at time zero. Once the portfolio is chosen, however, it cannot be rebalanced since no trading occurs after time zero. Thus, $N_t = N$ and $M_t = M$ for all t. Each agent's initial consumption consists of their endowment of shares with accrued dividends less the value of the portfolio they choose. Subsequent consumption equals the dividends on their portfolio. Thus,

$$C = w(P + X) + v(Q + Y) - NP - MQ, \tag{18}$$

$$C_t = NX_t + MY_t. \tag{19}$$

Because consumption is determined by N and M in this setting, the optimal consumption and portfolio decision for the first agent reduces to the problem of making an optimal choice of N and M at time zero, and similarly for the second agent. At time zero, the Euler conditions for the first agent imply

$$P = E\left[\int_0^\infty e^{-\beta t} \left(\frac{C}{NX_t + MY_t}\right) X_t \, dt\right], \tag{20}$$

$$Q = E\left[\int_0^\infty e^{-\beta t} \left(\frac{C}{NX_t + MY_t}\right) Y_t \, dt\right]. \tag{21}$$

Similarly, the Euler conditions for the second agent imply

$$P = E\left[\int_0^\infty e^{-\delta t} \left(\frac{X + Y - C}{(1-N)X_t + (1-M)Y_t}\right) X_t \, dt\right], \tag{22}$$

$$Q = E\left[\int_0^\infty e^{-\delta t} \left(\frac{X + Y - C}{(1-N)X_t + (1-M)Y_t}\right) Y_t \, dt\right]. \tag{23}$$

We turn now to the issue of how agents choose their optimal portfolios in this setting. A key difference between the liquid-market benchmark case in the previous section and the illiquid-market case is that the illiquid market is no longer dynamically complete. Specifically, while both assets can be traded at time zero, the agents' portfolios cannot be rebalanced once the trading "blackout" period begins. Thus, portfolio

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choices take on a "real options" element of irreversibility. Because of this illiquidity-induced incompleteness, an ordinarily myopic agent with logarithmic preferences must now consider the long-run implications of portfolio choices.

One immediate consequence is that the agents will never take a short position in either asset. To see the intuition behind this result, imagine that one of the agents takes a short position in, say, the first asset. Thus, the agent must pay the other agent a stream of dividends equal to $\mid N \mid X_t$ over time. Now imagine that the dividend on the other asset declines to the point that the amount of dividends being received from the second asset equals the amount of dividends that the agent must pay the other for the short position. In this situation, the agent would have zero consumption, and his expected utility goes to negative infinity. Thus, no agent would prefer a portfolio with a short position to his initial endowment.

Taking the analysis one step further, observe that expected utility for an agent is well defined and finite when, say, $N = 0$ and $0 < M < 1$, since his consumption is strictly positive. However, for any value of N less than zero, expected utility equals negative infinity. Thus, expected utility is not a globally differentiable function of the portfolio holdings N and M. An important implication of this is that an unconstrained equilibrium may not always be possible. This is consistent with Diamond (1967), Geanakoplos and Polemarchakis (1986), Geanakoplos, Magill, Quizii, and Dreze (1990), and others who show that an unconstrained Pareto optimal equilibrium need not exist in an incomplete market.

Intuitively, the reason why an unconstrained equilibrium may not exist in this illiquid-markets case is that it is possible for an agent to be at a corner and hold zero shares of one of the assets and still have a lower marginal valuation for the asset than the other agent. That this can occur is a consequence of the fact that the agents have different subjective discount rates. Thus, an agent with a high subjective discount rate may place a lower value of an asset even if he holds less of it than the other agent who has a low subjective discount rate. If portfolio decisions were continuously reversible, the agent could improve his welfare by *temporarily* shorting the asset to the point where the marginal valuations for the asset are equalized across agents, and then covering the short position later if consumption begins to approach zero. In an illiquid market, however, there is no price which would induce any agent to take even a temporarily irreversible short position because of the positive probability of consumption reaching zero.

When an unconstrained equilibrium exists, the Appendix shows that the first-order conditions can be solved to yield the following explicit expressions for P and Q:

$$P = \frac{(wX + vY)A(N, M, \beta)}{1 - (w - N)A(N, M, \beta) - (v - M)B(N, M, \beta)}, \tag{24}$$

$$Q = \frac{(wX + vY)B(N, M, \beta)}{1 - (w - N)A(N, M, \beta) - (v - M)B(N, M, \beta)}, \tag{25}$$

$$P = \frac{((1 - w)X + (1 - v)Y)A(1 - N, 1 - M, \delta)}{1 - (N - w)A(1 - N, 1 - M, \delta) - (M - v)B(1 - N, 1 - M, \delta)}, \tag{26}$$

$$Q = \frac{((1 - w)X + (1 - v)Y)B(1 - N, 1 - M, \delta)}{1 - (N - w)A(1 - N, 1 - M, \delta) - (M - v)B(1 - N, 1 - M, \delta)}. \tag{27}$$

Setting Equations (24) and (26) equal to each other, and similarly for Equations (25) and (27), results in a system of two equations in N and M. Despite the nonlinear form of the hypergeometric functions defining $A(a, b, c)$ and $B(a, b, c)$, it is straightforward to solve these two equations for N and M numerically. Once the optimal values of N and M are determined, the equilibrium values of P and Q are given explicitly by substituting N and M back into Equations (24) and (25), or alternatively, Equations (26) and (27).

When one agent is at a corner and only a constrained equilibrium exists, the agent's first-order condition for the corresponding asset is not required to hold. In this case, the Appendix shows that the equilibrium

portfolio holdings for the other asset and the prices P and Q can be determined from the remaining first-order conditions. This is equivalent to requiring that the equilibrium price of the constrained asset equal the higher valuation placed on it by the unconstrained agent. If the valuation for the constrained asset were anything less than this value, then this would not represent an equilibrium if the unconstrained agent consisted of a set of identical competing agents; these agents would compete and bid up the price of the constrained asset until its price equaled their marginal valuation.[3]

5. THE GENERAL CASE

Having solved the model for the polar extremes of the liquid-market and illiquid-market cases, it is now straightforward to solve for the general case where the trading "blackout" lasts for T periods, $0 < T < \infty$. Again, designating the number of shares chosen by the first agent by N and M, the Euler conditions for the agents imply

$$P = E\left[\int_0^T e^{-\beta t}\left(\frac{C}{NX_t + MY_t}\right) X_t\ dt\ +\ e^{-\beta T}\left(\frac{C}{C_T}\right) P_T\right], \tag{28}$$

$$Q = E\left[\int_0^T e^{-\beta t}\left(\frac{C}{NX_t + MY_t}\right) Y_t\ dt\ +\ e^{-\beta T}\left(\frac{C}{C_T}\right) Q_T\right], \tag{29}$$

$$P = E\left[\int_0^T e^{-\delta t}\left(\frac{X + Y - C}{(1-N)X_t + (1-M)Y_t}\right) X_t\ dt + e^{-\delta T}\left(\frac{X + Y - C}{X_T + Y_T - C_T}\right) P_T\right], \tag{30}$$

$$Q = E\left[\int_0^T e^{-\delta t}\left(\frac{X + Y - C}{(1-N)X_t + (1-M)Y_t}\right) Y_t\ dt + e^{-\delta T}\left(\frac{X + Y - C}{X_T + Y_T - C_T}\right) Q_T\right], \tag{31}$$

where the value of each asset equals the present value of its dividends from time zero to time T plus the present value of its market value at time T.

The key to solving this case is that because agents are able to trade again at time T, the model then reverts to the liquid-market case. Thus, the equilibrium values of P_T, Q_T, and C_T can be obtained by solving the liquid-market case from the perspective of agents at time T. By a simple extensions of the results in Section 3, the Appendix shows that asset values at time T are given by

$$P_T = C_T\ A(1,1,\beta;\ T) + (X_T + Y_T - C_T)\ A(1,1,\delta;\ T), \tag{32}$$

$$Q_T = C_T\ B(1,1,\beta;\ T) + (X_T + Y_T - C_T)\ B(1,1,\delta;\ T), \tag{33}$$

where the argument T of the functions $A(a,b,c;\ T)$ and $B(a,b,c;\ T)$ indicates that they are evaluated using X_T and Y_T. These expressions are simply the price functions in Equations (16) and (17) for the liquid-market case, but evaluated at values for time T. Similarly, consumption is given by

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[3]I am grateful to Tony Bernardo for this insight.

$$C = w(P + X) + v(Q + Y) - NP - MQ, \tag{34}$$

$$C_t = NX_t + MY_t, \tag{35}$$

$$C_T = \frac{(NA(1,1,\delta;\ T) + MB(1,1,\delta;\ T))\ (X_T + Y_T)}{1/\beta - N(A(1,1,\beta;\ T) - A(1,1,\delta;\ T)) - M(B(1,1,\beta;\ T) - B(1,1,\delta;\ T))}, \tag{36}$$

where $0 < t < T$. The first two of these expressions are identical to those in Equations (16) and (17) for the illiquid-market case. The expression for C_T in Equation (36), which is derived in the Appendix, is essentially the expression for C in Equation (9) for the liquid-market case, but evaluated at X_T and Y_T, and using the values N and M as the endowments at time T.[4]

Because of the structure of the general case, the model resembles the illiquid-market case until time T, after which it reduces to the liquid-market case. Again, agents will not take short positions in either asset at time zero for the same reasons as before. Because the market is dynamically incomplete from time zero to time T, an unconstrained equilibrium will again not always be possible.

Substituting the expressions for P_T, Q_T, C, and C_T into the first-order conditions results in a system of four equations in P, Q, N, and M. Since C is linear in P and Q, the model can again be solved explicitly for P and Q, leaving a system of two equations in N and M to solve. These equations in N and M are easily solved numerically. Substituting the values N and M into the explicit expressions for P and Q completes the solution. When there is only a constrained equilibrium, we follow the same procedure described in the previous section.

6. ASSET-PRICING IMPLICATIONS

In this section, we explore the asset-pricing implications of market illiquidity. To make the intuition as clear as possible, we illustrate these implications using results for a range of examples. Although realistic, these examples are intended simply as illustrations and not as calibrated models.

It is useful to view the two assets as representing distinct sectors of the market with different degrees of risk. In these examples, we assume that the first asset is riskier with cash flow volatility of 50 percent, and that the second asset is less risky with cash flow volatility of 20 percent. To reflect the heterogeneous behavior of investors in actual markets, we allow the agents in the model to have different subjective discount rates, $\beta = 0.20$ and $\delta = 0.01$. Thus, the first agent is much less patient than the second; the first agent has a shorter effective horizon than the second agent.[5]

6.1 Optimal Portfolio Choice.

Table 1 reports the equilibrium portfolio holdings N and M for the first agent for different illiquidity horizons ranging from zero (the liquid-market case) to infinity (the illiquid-market case). To illustrate the effects of the key variables in the model, we tabulate results for varying assumptions about the current dividends and the distribution of wealth among the agents. Specifically, we consider cases where $X = 0.25, 0.50$, or 0.75, and $Y = 1.00 - X$. In addition, each agent is endowed with a total of one share, but where the first agent

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[4]The other minor differences between Equation (9) and Equation (36) arise because there is no consumption "gulp" at time T.

[5]The remaining parameters used in these examples are $\mu_X = \mu_Y = .02$ and $\rho = 0$. Varying these parameters has little qualitative effect on the results.

has either $w = 1.00$ and $v = 0.00$ shares of the two assets, $w = 0.50$ and $v = 0.50$ shares of the two assets, or $w = 0.00$ and $v = 1.00$ shares of the two assets.

Table 1 shows that when the market is fully liquid, it is optimal for the agents to hold an equal number of shares of the two assets. Because there are an equal number of shares of the two assets in this economy, this means that each agent holds the market portfolio in the liquid-market equilibrium. This is consistent with traditional asset-pricing theory. The amount of the market portfolio held by the first agent depends on both his initial endowment and the difference in subjective discount rates. This can be seen clearly when the first agent is endowed with 50 percent of the assets in the liquid-market case. In this situation, the first agent immediately sells off part of his holdings to the second agent to finance a higher initial level of consumption, leaving him holding significantly less than 50 percent of the assets.

When agents face illiquid markets, however, their optimal portfolio behavior becomes surprisingly different. Table 1 shows that it is no longer optimal for the agents to hold the market portfolio. In general, the two agents now find it optimal to hold highly polarized or undiversified portfolios; the agents abandon diversification as a portfolio strategy. To illustrate, the first two rows of Table 1 give results when dividends for the risky first asset represent 25 percent of all dividends, the first agent is endowed with all the shares of the first asset, and the second agent is endowed with all the shares of the second asset. In the absence of liquidity restrictions, the first agent would hold 0.071 shares each of the two assets. With liquidity restrictions, however, his optimal portfolio now consists of roughly 80 percent of the shares of the first asset and virtually none of the second asset. This represents a portfolio that is almost as polarized or undiversified as it is possible to achieve in this example. By symmetry, the same is also true for the second agent. Similar results hold throughout all of the other examples shown in Table 1.

A particularly striking aspect of these results is that polarized portfolios are optimal whenever agents face market illiquidity of any duration, no matter how short. This is illustrated in Table 1 which shows that optimal portfolios are extremely polarized even when the length of the illiquidity horizon is only one week. In fact, the degree of polarization is actually somewhat higher when agents face shorter horizons of market illiquidity. These results demonstrate that portfolio polarization is an inherent aspect of market illiquidity.

What is the reason for the dramatic change in portfolio behavior when agents face illiquid markets? Intuitively, the effect of illiquidity is to change the horizon over which investment decisions affect the risk of an agent's portfolio. When markets are perfectly liquid, portfolio decisions can be revisited and revised continuously. Thus, the effect of an investment decision on the risk of the portfolio only lasts for an instant. Because both agents have logarithmic preferences in this framework, their attitudes towards risk over the next instant are the same. Consequently, they choose identical portfolios.

When markets are illiquid, however, portfolio decisions become at least temporarily irreversible. Thus, agents must now consider the longer-term effects of portfolio decisions on the risk of their consumption streams. In contrast to their attitudes towards instantaneous risk, however, the agents view their longer-term portfolio risk very differently. This is because the less-patient short-horizon agent discounts the utility of future consumption more heavily. Thus, the effect of risk on the expected future utility is dampened, making the less-patient first agent better able to bear longer-term portfolio risk. As a result, the first agent ends up holding most, if not all, of the riskier first asset in equilibrium, and vice versa.

The key lesson here is that illiquidity fundamentally changes the way agents behave towards risk. Agents who behave in one way when decisions have only short-term consequences for risk can behave in a completely different way when consequences are longer lasting. This property has many important implications for financial economics since it may help explain a number of well-known puzzles about the way actual agents choose investment portfolios. For example, Mankiw and Zeldes (1991) find evidence that the equity market participation of the typical U.S. household is far less than predicted by classical models of portfolio choice (Merton (1969, 1971)). Our results suggest that if households view equity markets as even slightly less liquid than markets for other financial assets, say for cash, demand deposits, or Government bonds, it may be rational for them to hold less stock than other investors such as financial institutions which presumably have shorter horizons.

6.2 Asset Prices.

To provide a benchmark for examining the asset-pricing implications of market illiquidity, Table 2 presents the equilibrium price-dividend ratios for the two assets in the liquid-market case. Table 2 also reports the percentage differences between the asset prices in the indicated illiquid markets and their corresponding prices in the liquid-market benchmark.

As can be seen, market illiquidity can have very large effects on the equilibrium prices of the assets. In some of the examples shown, the difference between the liquid-market and illiquid-market prices exceeds 30 percent. These are huge differences given that the underlying cash flows generated by the assets are identical across the cases. Observe that the differences can be both positive and negative in sign.

Furthermore, there are significant asset-pricing effects even when the period of illiquidity is as short as one week. The differences in asset prices for the one-week illiquidity case can be as large as 17 basis points, which annualizes to 8.84 percent. Similarly, the case in which the period of illiquidity last for three months results in valuation differences that annualize to values ranging from -3.20 to 8.80 percent.

Because asset prices revert to their liquid-market values after the liquidity restriction period lapses, the percentage differences shown map directly into returns. For example, if the price of the first asset is five percent higher when there is a one-year illiquidity horizon, then an agent who purchases shares of the asset and holds them for one year experiences a return that is five percent less than if the market had been liquid. Table 2 shows that the price differences for the one-year illiquidity horizon implies return differentials ranging from 3.11 to -8.04 percent. Clearly, these effects on holding-period returns are economically significant compared with historical expected returns in many equity markets. Again, these effects on holding period returns are independent of the cash flows generated by the assets and, thus, would be difficult to reconcile within classical asset-pricing models that assume that continuous trading is possible.

Another very counterintuitive feature of the results reported in Table 2 is that the riskier asset is typically worth more when agents face illiquid markets, while the opposite is true for the less-risky asset. This result is particularly surprising given that one might expect the riskier asset to trade at a lower price when investors may have to bear its risk for extended periods of time.

The reason why the riskier asset has a higher price and the less-risky asset a lower price when agents face illiquid markets is directly related to the polarization of agents' portfolios. To make the intuition as simple as possible, consider the extreme situation of the illiquid-market case when the first agent holds all of the first asset and none of the second. Thus, the second agent holds none of the first asset and all of the second. By substituting $M = 0$ into the first-order condition for P in Equation (20), X_t drops out of the equation and the term under the integral is no longer stochastic. This means that the equilibrium price P cannot depend of the second moments of X_t and Y_t. A similar result holds for the first-order condition for Q in Equation (23). Thus, when portfolios are completely polarized, equilibrium prices are not affected by the riskiness of the cash flows of the assets.

As another way of thinking about these results, observe that when portfolios are highly polarized, the consumption streams of the two agents in these examples are essentially uncorrelated. Thus, the marginal utilities of the two agents are also nearly uncorrelated. Recall that Euler conditions can be rearranged to express an asset's price in terms of the covariance (consumption beta) of its cash flows with marginal utility. In a representative-agent or complete-markets setting, an asset need only satisfy a single "consumption CAPM" expression. For the Euler conditions of both agents to be satisfied in this illiquid market, however, the price of an asset has to simultaneously satisfy two conflicting "consumption CAPMs," with betas based on uncorrelated marginal utilities. To resolve the tension between these conflicting conditions, equilibrium prices must be largely unrelated to the riskiness of the assets' cash flows.

In a liquid market, the first asset's value is heavily discounted because of the riskiness of its cash flows, and conversely for the less-risky second asset. In an illiquid market, however, conventional risk and return relations break down, and the price of the risky asset is not as heavily discounted as in a liquid market. Thus, as shown in Table 2, the price of the risky first asset is generally higher when markets are illiquid, and

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vice versa for the second asset. There are sets of parameters and values of the state variables, however, for which the opposite is true.

6.3 The Equity Premium.

Adding together the values of the individual assets allows us to examine the effect of illiquidity on total market capitalization. Table 3 reports the market price-earnings ratio for the benchmark liquid-market case as well as the percentage difference in market capitalization resulting from an illiquid market of the indicated duration.

Table 3 demonstrates that illiquidity can have important effects on the value of the market and, therefore, on the equity premium (interpreting the assets as stocks) even for short illiquidity horizons. For example, when the length of the illiquidity horizon is one year, market capitalization ranges from 2.00 percent less to 1.34 percent more than in the liquid-market case. Since the market becomes fully liquid at the end of the one-year period, these results mean that the equity premium over the next year can differ by as much as 2 percent from the equity premium that would hold in a liquid market. Effects of this size are definitely on the same order of magnitude as historical estimates of the equity premium.

The key determinant of the effect of market liquidity on the equity premium is clearly the initial distribution of asset ownership. In the special case where each agent is endowed with the market portfolio (consisting of an equal number of shares of each asset), market illiquidity has no effect on the equity premium. Conversely, the largest effects on the equity premium occur when the initial distribution of asset ownership is heavily polarized. Specifically, the effect on the equity premium is largest when the the less-patient first agent initially holds all of the risky asset.

As the length of the illiquidity horizon increases, the effect on market capitalization is correspondingly greater. In the illiquid-market case, the capitalization of the market ranges from 22 percent less to nearly 9 percent more than in the liquid-market case.

These results are important since they indicate that market liquidity can have first-order effects on the equity risk premium. An immediate implication of this is that the equity premium can vary significantly over time in response to changes in market liquidity. For example, in the case where the less-patient agent initially holds all of the risky asset, an increase in market liquidity results in positive returns. Thus, these results are consistent with the widely-held view that much of the bull market during the latter part of the 1990s may have been due to increases in liquidity rather than to fundamental changes in cash flows.

A further implication of these results is that the demographics of asset ownership can have large effects on the equity premium when markets are not fully liquid. Table 3 shows that as the market moves from the case where less-patient investors hold all of the risky asset to the case where the risky asset is held more broadly, the capitalization of the market increases. Again, this is broadly consistent with the view that the bull market of the 1990s may have been fueled in part by the increasing participation by households in the equity market either directly or through pension and retirement plans.

6.4 Optimal Consumption.

To illustrate the effect of illiquid markets on consumption decisions, Table 4 reports the optimal consumption level C for the first agent at time zero. Also reported is the subsequent consumption level C^+ immediately after time zero resulting from the portfolio choice made at time zero. Recall that when markets are illiquid and assets cannot be purchased or sold, agents' consumption equals the dividends they receive from their investment portfolios.

When the market is fully liquid, the less-patient first agent accelerates his consumption by selling off his portfolio over time to the second agent. Because of this, the first agent's consumption represents a declining percentage of the total market dividends. From Equation (10), this percentage is a continuous deterministic function of time. Thus, the subsequent consumption level C^+ is identical to the level of consumption at time zero C.

In contrast, when markets are illiquid, the first agent cannot sell assets continuously to accelerate his consumption. Rather market illiquidity forces the first agent to be a buy-and-hold investor during the period of illiquidity. The effect of this on his consumption decision appears complex. For example, Table 4 shows that the consumption level chosen at time zero can be greater than, equal to, or less than that for the liquid-market case. In actuality, it can be shown that the consumption at time zero is a deterministic fraction of the agent's wealth. Thus, the first agent's initial level of consumption increases, remains the same, or declines as his wealth is affected by the changes in asset values resulting from the effects of illiquidity.

Table 4 also shows that when the market is illiquid, the level of subsequent consumption C^+ chosen by the first agent is much lower than in the liquid-market case. Intuitively, this is because in the liquid-market case, the first agent starts with a high level of consumption at time zero, but ends with a lower level of consumption after T periods of selling off his assets. Thus, the average level of consumption from time zero to time T is lower than C^+. When the market is illiquid, however, the level of consumption cannot be adjusted during the period of illiquidity. Thus, the value C^+ chosen when markets are illiquid represents the first agent's attempts to approximate the average level of consumption he would have from time zero to time T if the market were liquid.

6.5 Welfare Effects.

To make the welfare effects of market illiquidity as clear as possible, we contrast the agents' expected utility in the liquid-market case with their expected utility in the illiquid-market case. For these two cases, we derive analytical solutions for the expected utility of each agent when optimal portfolio and consumption policies are followed. We present these analytical solutions in the Appendix. To illustrate the welfare loss for each agent resulting from market illiquidity, Table 5 reports the percentage decrease in wealth that would make an investor in the liquid-market case as well off as in the illiquid-market case.

Table 5 shows that the welfare loss from market illiquidity can be huge. In some cases, the agents would be willing to forego more than 50 percent of their wealth to avoid the portfolio restrictions imposed by market illiquidity. It is important to stress that the agents suffer welfare losses even when their personal wealth increases as the market becomes less liquid.

The welfare effects of illiquidity, however, are far from uniform across the agents. In many of the examples shown, the first agent suffers a much larger welfare loss than the second agent. The welfare loss for the second agent is only 2.59 percent in the first example shown in Table 5. Furthermore, it is straightforward to find other examples in which the welfare loss of the second agent is arbitrarily small. This means that there are scenarios where the more-patient second agent would be unwilling to pay even a small amount to obtain immediacy; the welfare costs of market illiquidity are borne almost exclusively by the less-patient first agent. These results have important implications for the incentives of agents to cooperate in creating liquid markets.

7. CONCLUSION

We examine the asset-pricing implications of market illiquidity using a multi-asset heterogeneous agent model in which agents may have to wait before being able to trade. Although there are many possible types of illiquidity, the admittedly stylized framework we consider has the advantage of capturing the intuitive notion of illiquidity as the inability to trade in a timely way. In this framework, illiquidity is a feature of the entire market, rather than a security-specific attribute. Thus, our definition of illiquidity has more of a "market-macrostructure" flavor that differentiates it from much of the market-microstructure literature that assumes immediacy is always available at a cost.

The results provide a number of interesting new insights about the effects of illiquidity on asset pricing. We show that agents facing illiquid markets abandon the strategy of holding diversified portfolios. Generally, the agents with the highest subjective discount rate and, therefore, the shortest horizon, concentrate their portfolio holdings in the riskiest asset. Thus, more-patient, longer-horizon agents may hold very little of

the risky asset. This aspect of the model has interesting parallels with the empirical evidence of limited participation by many households in the U.S. equity markets.

We also find that when agents must wait before they can trade again, asset prices can differ significantly from their liquid-market values. Thus, illiquidity can have first-order effects on asset prices. Furthermore, the results indicate that risky assets are often worth more when agents face illiquid markets, and vice versa for less-risky assets. This is an artifact of the tendency for agents to hold polarized rather than diversified portfolios when they face illiquid markets. A further implication is that prices have little relation with the risk of dividends, but are more affected by demographics of asset ownership. Finally, we show that market illiquidity can also have large effects on the overall capitalization of the market and, hence, on the equity premium.

APPENDIX

1. The Liquid-Market Case.

Setting the expressions for P in Equations (5) and (7) equal to each other, and similarly for the expressions for Q in Equations (6) and (8), implies the following pair of equations:

$$0 = E\left[\int_0^\infty \left\{e^{-\beta t}\left(\frac{C}{C_t}\right) - e^{-\delta t}\left(\frac{X + Y - C}{X_t + Y_t - C_t}\right)\right\} X_t \ dt\right], \qquad (A1)$$

$$0 = E\left[\int_0^\infty \left\{e^{-\beta t}\left(\frac{C}{C_t}\right) - e^{-\delta t}\left(\frac{X + Y - C}{X_t + Y_t - C_t}\right)\right\} Y_t \ dt\right]. \qquad (A2)$$

As an aside, we note that since consumption at time zero is in the form of a discrete "gulp" with no flow component, the lower limit of the integral defining the agents' expected utility in Equations (3) and (4) should actually be written as 0^+ rather than 0. In determining the first-order conditions, we use the correct lower limits for the integrals. However, since the agents' consumption and marginal utilities at time zero are finite, making zero the lower limit does not change the value of any of the integrals. Thus, without any loss of generality, we set the lower limit of all integrals in the paper to zero.

Both of the above expressions are satisfied by requiring that

$$e^{-\beta t}\left(\frac{C}{C_t}\right) = e^{-\delta t}\left(\frac{X + Y - C}{X_t + Y_t - C_t}\right), \qquad (A3)$$

holds for all t, X_t, and Y_t. Solving this expression for C_t implies

$$C_t = \frac{e^{-\beta t}C}{e^{-\beta t}C + e^{-\delta t}(X + Y - C)} (X_t + Y_t), \qquad (A4)$$

which is Equation (10). Dividing C by C_t and rearranging gives

$$\left(\frac{C}{C_t}\right) = \frac{C + (X + Y - C)e^{(\beta - \delta)t}}{X_t + Y_t}. \qquad (A5)$$

Substituting this into Equations (5) and (6) and rearranging yields

$$P = C \ E\left[\int_0^\infty e^{-\beta t}\left(\frac{X_t}{X_t + Y_t}\right) dt\right] + (X + Y - C) \ E\left[\int_0^\infty e^{-\delta t}\left(\frac{X_t}{X_t + Y_t}\right) dt\right], \qquad (A6)$$

$$Q = C \ E\left[\int_0^\infty e^{-\beta t}\left(\frac{Y_t}{X_t + Y_t}\right) dt\right] + (X + Y - C) \ E\left[\int_0^\infty e^{-\delta t}\left(\frac{Y_t}{X_t + Y_t}\right) dt\right]. \qquad (A7)$$

Section 4 of this Appendix shows that these equations can be reexpressed as

$$P = C \ A(1,1,\beta) \ + \ (X + Y - C) \ A(1,1,\delta), \qquad (A8)$$
$$Q = C \ B(1,1,\beta) \ + \ (X + Y - C) \ B(1,1,\delta), \qquad (A9)$$

which are Equations (16) and (17). Finally, the same analysis can be applied to the agents' Euler conditions for any $t > 0$ to give

$$P_t = C_t \ A(1, 1, \beta; \ t) \ + \ (X_t + Y_t - C_t) \ A(1, 1, \delta; \ t), \tag{A10}$$

$$Q_t = C_t \ B(1, 1, \beta; \ t) + (X_t + Y_t - C_t) \ B(1, 1, \delta; \ t), \tag{A11}$$

To solve for C, note that after consuming at time zero, the first agent's wealth equals $w(P+X) + v(Q+Y) - C$, where the first two terms represent the value of the agent's endowment (with dividends). Setting the value of the agent's wealth equal to the present value of his future consumption stream gives

$$w(P+X) + v(Q+Y) - C = E\left[\int_0^\infty e^{-\beta t} \left(\frac{C}{C_t}\right) C_t \ dt\right], \tag{A12}$$

$$= C/\beta, \tag{A13}$$

which implies

$$C = \frac{w(P+X) + v(Q+Y)}{1 + 1/\beta}. \tag{A14}$$

Substituting in the expressions for P and Q in Equations (A8) and (A9), and then solving for C gives Equation (9).

From (A4), optimal consumption is homogeneous of degree one in total dividends $X_t + Y_t$. Based on this, we conjecture (and later verify) that the dynamic portfolio strategy that generates C_t consists of equal numbers of shares of the two assets, $N_t = M_t$, where N_t is a differentiable function of time. By definition, consumption equals the sum of dividends received minus net purchases of assets. Thus,

$$C_t = N_t(X_t + Y_t) - (P_t + Q_t)N_t', \tag{A15}$$

where N_t' denotes a derivative. From Equations (A10) and (A11) and the results in Section 4 of this appendix, it follows that

$$P_t + Q_t = \frac{C_t}{\beta} + \frac{X_t + Y_t - C_t}{\delta}. \tag{A16}$$

Substituting this and the expression for C_t in Equation (A4) into Equation (A15) gives the ordinary differential equation

$$N_t' - \left(\frac{\beta\delta(Ce^{\delta t} + (X+Y-C)e^{\beta t})}{C\delta e^{\delta t} + \beta(X+Y-C)e^{\beta t}}\right) N_t = \frac{-C\beta\delta e^{\delta t}}{C\delta e^{\delta t} + \beta(X+Y-C)e^{\beta t}}. \tag{A17}$$

This is a standard first-order linear differential equation which can be solved directly by an integration. The initial value of N is determined by imposing the condition that $N(P+Q)$ equals the first agent's initial wealth after time-zero consumption. From Spiegel (1967), the solution to this differential equation is the expression given in Equation (15). This verifies the conjecture and also establishes that the consumption strategy identified in Equation (A4) is feasible.

2. The Illiquid-Market Case.

Substituting C from Equation (18) into Equations (20) through (23) and applying the results from Section 4 of this appendix allows us to rewrite Equations (20) through (23) as

$$P = ((w - N)P + (v - M)Q + wX + vY) \ A(N, M, \beta), \qquad (A18)$$

$$Q = ((w - N)P + (v - M)Q + wX + vY) \ B(N, M, \beta), \qquad (A19)$$

$$P = ((N - w)P + (M - v)Q + (1 - w)X + (1 - v)Y) \ A(1 - N, 1 - M, \delta), \qquad (A20)$$

$$Q = ((N - w)P + (M - v)Q + (1 - w)X + (1 - v)Y) \ B(1 - N, 1 - M, \delta). \qquad (A21)$$

Consider first the case where there is an unconstrained equilibrium in which all four of the above first-order conditions are satisfied. The first two equations (A18) and (A19) represent a linear system in P and Q. The solution to this system is given in Equations (24) and (25). Similarly, the second two equations (A20) and (A21) also represent a linear system in P and Q. The solution to this system appears as Equations (26) and (27). Setting the resulting solutions for P equal to each other, and similarly for the solutions for Q, gives a set of two equations in N and M from which P and Q have been eliminated. Standard numerical techniques can now be used to solve these two equations for the optimal N and M.

Now consider the situation where the first agent has a lower valuation for the first asset when $N = 0$ and $0 < M < 1$ than has the second agent. In this situation, the first agent is not willing to buy additional shares from the second agent. Alternatively, the first agent is not willing to sell shares to the second agent despite the fact that the second agent is willing to pay more for the shares than the shares are worth to the first agent. This follows since the first agent's expected utility goes to $-\infty$ whenever $N < 0$. Because of this, the first agent's first-condition for the first asset cannot be satisfied and only a constrained equilibrium is possible. To solve for the equilibrium in this situation, we can again solve the linear system in (A20) and (A21) for P and Q by requiring that the first-order conditions for the second agent are satisfied. The resulting expressions for P and Q are then substituted into the first agents first-order condition for the second asset in (A19). Since $N = 0$, (A19) can then be solved numerically for M. The prices are then given explicitly by substituting the values of N and M back into the corresponding expressions for P and Q in Equations (A20) and (A21). A similar procedure can be used to solve for asset prices and optimal portfolio choices if a constrained equilibrium of one of the following types occurs: a) $0 < N < 1$, $M = 0$, b) $N = 1$, $0 < M < 1$, c) $0 < N < 1$, $M = 1$, d) $N = 0$, $M = 1$, and e) $N = 1$, $M = 0$.

3. The General Case.

At time T, the agents' first-order conditions imply

$$P = E_T \left[\int_0^\infty e^{-\beta s} \left(\frac{C_T}{C_{T+s}} \right) X_{T+s} \ ds \right], \qquad (A22)$$

$$Q = E_T \left[\int_0^\infty e^{-\beta s} \left(\frac{C_T}{C_{T+s}} \right) Y_{T+s} \ ds \right], \qquad (A23)$$

$$P = E_T \left[\int_0^\infty e^{-\delta s} \left(\frac{X_T + Y_T - C_T}{X_{T+s} + Y_{T+s} - C_{T+s}} \right) X_{T+s} \ ds \right], \qquad (A24)$$

$$Q = E_T \left[\int_0^\infty e^{-\delta s} \left(\frac{X_T + Y_T - C_T}{X_{T+s} + Y_{T+s} - C_{T+s}} \right) Y_{T+s} \ ds \right]. \qquad (A25)$$

As in Section 1 of this Appendix, these first-order conditions are satisfied for all $T+t$, X_{T+s}, and Y_{T+s} when

$$C_{T+s} = \frac{e^{-\beta s} C_T (X_{T+s} + Y_{T+s})}{e^{-\beta s} C_T + e^{-\delta s}(X_T + Y_T - C_T)}, \qquad (A26)$$

where $s > 0$. Substituting this expression back into Equations (A22) and (A23) and using the results in Section 4 of this Appendix gives the closed-form expressions for P_T and Q_T in Equations (32) and (33). To solve for C_T, we set the value of the first agents' portfolio at time T equal to the present value of his future consumption,

$$NP_T + MQ_T = E_T \left[\int_0^\infty e^{\beta s} \left(\frac{C_T}{C_{T+s}} \right) C_{T+s} \ ds \right], \qquad (A27)$$

$$= C_T / \beta. \qquad (A28)$$

Substituting in for P_T and Q_T and solving the above expression for C_T gives Equation (36).

4. The A(a, b, c) and B(a, b, c) Functions.

We define $A(a, b, c)$ as the expectation

$$A(a, b, c) = E \left[\int_0^\infty e^{-ct} \left(\frac{X_t}{aX_t + bY_t} \right) \ dt \right], \qquad (A29)$$

that appears in various forms in the agents' first-order conditions. This can be rewritten as

$$\frac{1}{a} \ E \left[\int_0^\infty e^{-ct} \left(\frac{1}{1 + qe^u} \right) \ dt \right], \qquad (A30)$$

where $q = \frac{bY}{aX}$, and u is a normally distributed random variable with mean μt and variance $\sigma^2 t$, where

$$\mu = \mu_Y - \mu_X - \sigma_Y^2/2 + \sigma_X^2/2, \qquad (A31)$$
$$\sigma^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y. \qquad (A32)$$

Introducing the density for u into the above expectation gives

$$\int_0^\infty \int_{-\infty}^\infty e^{-ct} \frac{1}{\sqrt{2\pi\sigma^2 t}} \frac{1}{1 + qe^u} \exp\left(\frac{-(u - \mu t)^2}{2\sigma^2 t} \right) \ du \ dt. \qquad (A33)$$

Interchanging the order of integration and collecting terms in t gives,

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{1 + qe^u} \exp\left(\frac{\mu u}{\sigma^2} \right) \int_0^\infty t^{-1/2} \exp\left(-\frac{u^2}{2\sigma^2} \frac{1}{t} - \frac{\mu^2 + 2c\sigma^2}{2\sigma^2} t \right) \ dt \ du. \qquad (A34)$$

From Equation (3.471.9) of Gradshteyn and Ryzhik (2000), this expression becomes,

$$\int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi\sigma^2}} \frac{1}{1+qe^u} \exp\left(\frac{\mu u}{\sigma^2}\right) \left(\frac{u^2}{\mu^2+2c\sigma^2}\right)^{1/4} K_{1/2}\left(2\sqrt{\frac{u^2(\mu^2+2c\sigma^2)}{4\sigma^4}}\right) du, \qquad (A35)$$

where $K_{1/2}(\cdot)$ is the modified Bessel function of order $1/2$ (see Abramowitz and Stegum (1970) Chapter 9). From the identity relations for Bessel functions of order equal to an integer plus one half given in Gradshteyn and Ryzhik Equation (8.469.3), however, the above expression can be expressed as,

$$\frac{1}{\psi} \int_{-\infty}^{\infty} \frac{1}{1+qe^u} \exp\left(\frac{\mu u}{\sigma^2}\right) \exp\left(-\frac{\psi}{\sigma^2}|u|\right) du, \qquad (A36)$$

where

$$\psi = \sqrt{\mu^2 + 2c\sigma^2}.$$

In turn, Equation (A5) can be written

$$\frac{1}{\psi} \int_0^{\infty} \frac{1}{1+qe^u} \exp\left(\gamma u\right) du \; + \; \frac{1}{\psi} \int_{-\infty}^0 \frac{1}{1+qe^u} \exp\left(\theta u\right) du, \qquad (A37)$$

where

$$\gamma = \frac{\mu - \psi}{\sigma^2}, \qquad \theta = \frac{\mu + \psi}{\sigma^2}.$$

Define $y = e^{-u}$. By a change of variables Equation (A37) can be written

$$\frac{1}{q\psi} \int_0^1 \frac{1}{1+y/q} y^{-\gamma} dy \; + \; \frac{1}{\psi} \int_0^1 \frac{1}{1+qy} y^{\theta-1} dy. \qquad (A38)$$

From Abramowitz and Stegum Equation (15.3.1), this expression becomes

$$\frac{1}{q\psi(1-\gamma)} F(1,\; 1-\gamma;\; 2-\gamma;\; -1/q) \; + \; \frac{1}{\psi\theta} F(1,\; \theta;\; 1+\theta;\; -q). \qquad (A39)$$

Substituting in for q and dividing by a gives the expression for $A(a,b,c)$ in Equation (11)

$$A(a,b,c,) = k_1 \; (X/Y) \; F\left(1,\; 1-\gamma;\; 2-\gamma;\; -\frac{aX}{bY}\right) \; + \; k_2 \; F\left(1,\; \theta;\; 1+\theta;\; -\frac{bY}{aX}\right), \qquad (A40)$$

where

$$k_1 = \frac{1}{b\psi(1-\gamma)}, \qquad k_2 = \frac{1}{a\psi\theta}.$$

The function $A(a,b,c;t)$ is identical to $A(a,b,c)$ with the exception that X_t and Y_t appear in the expression rather than X and Y. The function $B(a,b,c)$ is defined as the expectation,

$$B(a,b,c) = E\left[\int_0^{\infty} e^{-ct} \left(\frac{Y_t}{aX_t + bY_t}\right) dt\right]. \qquad (A41)$$

The evaluation of this expectation is omitted since it is virtually the same at that for $A(a, b, c)$ above. The resulting expression for $B(a, b, c)$ is

$$B(a, b, c) = k_3 \ (Y/X) \ F\left(1, \ 1 + \theta; \ 2 + \theta; \ -\frac{bY}{aX}\right) \ - \ k_4 \ F\left(1, \ -\gamma; \ 1 - \gamma; \ -\frac{aX}{bY}\right), \qquad (A42)$$

where

$$k_3 = \frac{1}{a\psi(1 + \theta)}, \qquad k_4 = \frac{1}{b\psi\gamma}.$$

Substituting these solutions for $A(a, b, c)$ and $B(a, b, c)$ into Equations (A6) and (A7) gives the expressions for P and Q in Equations (16) and (17). Similarly, substituting in these solutions into the first-order conditions in Equations (20) through (23) leads immediately to Equations (A18) through (A21).

5. Expected Utility.

For brevity, we present the closed-form expressions for the agents' expected utilities in the liquid-market and illiquid-market cases without detailed derivations. These closed-form expressions, however, are obtained by straightforward applications of the results in the previous section. In solving for the expected utility of the agents, we make repeated use of the decomposition of $\ln(X_t + Y_t)$ into $\ln(X_t) + \ln(1 + X_t/Y_t)$.

To obtain the expected utility for the first agent in the liquid-market case, we substitute the expression for the optimal C_t into Equation (3) and use the above decomposition. This allows the first agent's expected utility to be written as

$$\ln C + E\left[\int_0^\infty e^{-\beta t} \ln X_t \ dt\right] + E\left[\int_0^\infty e^{-\beta t} \ln(1 + Y_t/X_t) \ dt\right]$$

$$- \int_0^\infty e^{-\beta t} \ln\left(1 + \frac{X + Y - C}{C} e^{(\beta - \delta)t}\right) \ dt. \qquad (A43)$$

The first term is given from Equation (9). The second is determined by substituting in the expectation $E[\ln X_t] = \ln X + (\mu_X - \sigma_X^2/2)t$ and evaluating the resulting integral. The third term can be reduced to an integral of the form evaluated in the previous section using integration by parts. Similarly for the fourth term. Collectively, these results imply that the first agent's expected utility is

$$\ln C + \frac{\ln X}{\beta} + \frac{\mu_X - \sigma_X^2/2}{\beta^2} + \left(\frac{\gamma - \theta}{\psi\gamma\theta}\right)\ln(1 + Y/X)$$

$$+ \frac{1}{\psi\gamma^2}F(1, -\gamma; 1 - \gamma; -X/Y) - \frac{Y}{X\psi\theta(1 + \theta)}F(1, 1 + \theta; 2 + \theta; -Y/X) - \frac{1}{\beta}\ln\left(\frac{X + Y}{C}\right)$$

$$- \frac{\beta - \delta}{\beta^2}F(1, \beta/(\beta - \delta); \beta/(\beta - \delta) + 1; -C/(X + Y - C)), \qquad (A44)$$

where

$$\psi = \sqrt{\mu^2 + 2\beta\sigma^2}, \qquad \gamma = \frac{\mu - \psi}{\sigma^2}, \qquad \theta = \frac{\mu + \psi}{\sigma^2}.$$

A similar approach can be used to obtain the expected utility of the second agent in the liquid-market case,

$$\ln(X+Y-C) + \frac{\ln X}{\delta} + \frac{\mu_X - \sigma_X^2/2}{\delta^2} + \left(\frac{\bar{\gamma}-\bar{\theta}}{\bar{\psi}\bar{\gamma}\bar{\theta}}\right)\ln(1+Y/X) + \frac{1}{\bar{\psi}\bar{\gamma}^2}F(1,-\bar{\gamma};1-\bar{\gamma};-X/Y)$$

$$- \frac{Y}{X\bar{\psi}\bar{\theta}(1+\bar{\theta})}F(1,1+\bar{\theta};2+\bar{\theta};-Y/X) + \frac{1}{\delta}\ln\left(\frac{X+Y-C}{C}\right) + \frac{\beta-\delta}{\delta^2} - \frac{1}{\delta}\ln\left(\frac{X+Y}{C}\right)$$

$$- \frac{\beta-\delta}{\delta^2}F(1,\delta/(\beta-\delta);\delta/(\beta-\delta)+1;-C/(X+Y-C)), \tag{A45}$$

where

$$\bar{\psi} = \sqrt{\mu^2 + 2\delta\sigma^2}, \qquad \bar{\gamma} = \frac{\mu-\bar{\psi}}{\sigma^2}, \qquad \bar{\theta} = \frac{\mu+\bar{\psi}}{\sigma^2}.$$

In the illiquid-market case, the expected utility of the first agent is given by

$$\ln C + \frac{\ln(NX)}{\beta} + \frac{\mu_X - \sigma_X^2/2}{\beta^2} + \left(\frac{\gamma-\theta}{\psi\gamma\theta}\right)\ln(1+\tfrac{MY}{NX})$$

$$+ \frac{1}{\psi\gamma^2}F(1,-\gamma;1-\gamma;-\tfrac{NX}{MY}) - \frac{MY}{NX\psi\theta(1+\theta)}F(1,1+\theta;2+\theta;-\tfrac{MY}{NX}). \tag{A46}$$

Similarly, the expected utility of the second agent in the illiquid-market case is

$$\ln(X+Y-C) + \frac{\ln((1-N)X)}{\delta} + \frac{\mu_X - \sigma_X^2/2}{\delta^2} + \left(\frac{\bar{\gamma}-\bar{\theta}}{\bar{\psi}\bar{\gamma}\bar{\theta}}\right)\ln(1+\tfrac{(1-M)Y}{(1-N)X})$$

$$+ \frac{1}{\bar{\psi}\bar{\gamma}^2}F(1,-\bar{\gamma};1-\bar{\gamma};-\tfrac{(1-N)X}{(1-M)Y}) - \frac{(1-M)Y}{(1-N)X\bar{\psi}\bar{\theta}(1+\bar{\theta})}F(1,1+\bar{\theta};2+\bar{\theta};-\tfrac{(1-M)Y}{(1-N)X}). \tag{A47}$$

These expressions hold when there is an unconstrained equilibrium. When the equilibrium is constrained, similar (but simpler) expressions for the expected utilities of the two agents can be obtained.

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Table 1

Optimal Portfolios. This table reports the optimal portfolio for the first agent. Div. denotes the initial dividend for the first asset, where the total initial dividends are normalized to one. The terms w and v denote the number of shares of the two assets with which the first agent is endowed. By the market clearing condition, the second agent is endowed with $1 - w$ and $1 - v$ shares of the two assets. N and M are the number of shares of the two assets chosen by the first agent at time zero. By the market clearing condition, the second agent chooses $1 - N$ and $1 - M$ shares of the two assets. The column denoted Liquid reports portfolio weights when there is no illiquidity period (the liquid-market case). The remaining columns report portfolio weights for illiquidity horizons ranging from 1 week to infinity (the illiquid-market case).

Div.	w	v	Shares	Liquid	1 Wk.	3 Mo.	1 Yr.	2 Yr.	5 Yr.	10 Yr.	∞
0.25	1.00	0.00	N	0.071	0.826	0.825	0.823	0.821	0.814	0.805	0.759
			M	0.071	0.006	0.006	0.006	0.007	0.007	0.009	0.016
	0.50	0.50	N	0.457	1.000	1.000	1.000	1.000	1.000	1.000	0.980
			M	0.457	0.370	0.368	0.362	0.356	0.343	0.335	0.335
	0.00	1.00	N	0.777	1.000	1.000	1.000	1.000	1.000	1.000	0.994
			M	0.777	0.726	0.724	0.719	0.713	0.701	0.693	0.691
0.50	1.00	0.00	N	0.151	0.860	0.861	0.864	0.867	0.873	0.874	0.852
			M	0.151	0.014	0.014	0.014	0.013	0.013	0.013	0.022
	0.50	0.50	N	0.457	1.000	1.000	1.000	1.000	1.000	1.000	0.973
			M	0.457	0.274	0.271	0.261	0.250	0.224	0.203	0.200
	0.00	1.00	N	0.634	1.000	1.000	1.000	1.000	1.000	1.000	0.986
			M	0.634	0.475	0.470	0.454	0.436	0.397	0.366	0.355
0.75	1.00	0.00	N	0.305	0.898	0.899	0.903	0.907	0.918	0.926	0.919
			M	0.305	0.032	0.032	0.031	0.029	0.026	0.023	0.032
	0.50	0.50	N	0.457	0.985	0.986	0.986	0.986	0.982	0.974	0.942
			M	0.457	0.129	0.127	0.120	0.111	0.088	0.065	0.053
	0.00	1.00	N	0.515	1.000	1.000	1.000	1.000	0.993	0.981	0.946
			M	0.515	0.180	0.177	0.166	0.151	0.116	0.081	0.060

Table 2

Equilibrium Asset Prices. This table reports the price-dividend ratios for the benchmark liquid-market case and the percentage differences between the prices in the illiquid cases and the benchmark liquid-market case. Div. denotes the initial dividend for the first asset, where the total initial dividends are normalized to one. The terms w and v denote the number of shares of the two assets with which the first agent is endowed. By the market clearing condition, the second agent is endowed with $1-w$ and $1-v$ shares of the two assets. P and Q are the prices of the two assets. The column denoted Liquid Price-Dividend reports the equilibrium price-dividend ratios when there is no illiquidity period (the liquid-market case). The remaining columns report the percentage differences in prices for illiquidity horizons ranging from 1 week to infinity (the illiquid-market case).

Div.	w	v	Price	Liquid Price-Dividend	Percentage Differences						
					1 Wk.	3 Mo.	1 Yr.	2 Yr.	5 Yr.	10 Yr.	∞
0.25	1.00	0.00	P	13.47	0.02	0.25	0.98	1.93	4.44	7.57	14.58
			Q	52.45	-0.03	-0.36	-1.42	-2.78	-6.41	-10.91	-21.02
	0.50	0.50	P	5.67	0.17	2.20	8.04	14.35	26.25	34.09	38.15
			Q	11.88	-0.03	-0.35	-1.28	-2.28	-4.18	-5.42	-6.07
	0.00	1.00	P	4.71	0.07	0.89	3.29	5.94	11.06	14.50	16.14
			Q	6.89	-0.00	-0.01	-0.04	-0.08	-0.15	-0.20	-0.22
0.50	1.00	0.00	P	8.36	0.02	0.20	0.78	1.51	3.33	5.40	9.17
			Q	43.42	-0.05	-0.65	-2.54	-4.88	-10.79	-17.49	-29.72
	0.50	0.50	P	5.19	0.11	1.39	5.26	9.72	19.19	26.63	31.96
			Q	15.46	-0.04	-0.47	-1.77	-3.27	-6.44	-8.95	-10.74
	0.00	1.00	P	4.65	0.10	1.23	4.68	8.69	17.38	24.27	28.52
			Q	10.66	-0.00	-0.03	-0.12	-0.23	-0.45	-0.63	-0.74
0.75	1.00	0.00	P	6.18	0.01	0.10	0.40	0.77	1.71	2.80	4.55
			Q	40.36	-0.06	-0.80	-3.11	-5.97	-13.25	-21.67	-35.19
	0.50	0.50	P	5.27	0.03	0.42	1.67	3.32	7.77	12.74	18.76
			Q	25.50	-0.02	-0.26	-1.03	-2.06	-4.82	-7.90	-11.63
	0.00	1.00	P	5.05	0.04	0.51	2.05	4.05	9.59	15.76	22.90
			Q	21.92	-0.00	-0.02	-0.08	-0.17	-0.39	-0.65	-0.94

Table 3

Market Capitalization. This table reports the market price-dividend ratio for the benchmark liquid-market case and the percentage differences between the value of the market in the illiquid cases and the benchmark liquid-market case. Div. denotes the initial dividend for the first asset, where the total initial dividends are normalized to one. The terms w and v denote the number of shares of the two assets with which the first agent is endowed. By the market clearing condition, the second agent is endowed with $1 - w$ and $1 - v$ shares of the two assets. The column denoted Liquid Price-Dividend reports the market price-dividend ratio when there is no illiquidity period (the liquid-market case). The remaining columns report the percentage differences in market capitalization for illiquidity horizons ranging from 1 week to infinity (the illiquid-market case).

Div.	w	v	Liquid Price-Dividend	Percentage Differences						
				1 Wk.	1 Qtr.	1 Yr.	2 Yr.	5 Yr.	10 Yr.	∞
0.25	1.00	0.00	42.70	-0.02	-0.31	-1.23	-2.40	-5.55	-9.45	-18.20
	0.50	0.50	10.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.00	1.00	6.34	0.01	0.16	0.58	1.04	1.93	2.53	2.82
0.50	1.00	0.00	25.89	-0.04	-0.52	-2.00	-3.85	-8.51	-13.79	-23.44
	0.50	0.50	10.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.00	1.00	7.66	0.03	0.35	1.34	2.48	4.97	6.94	8.15
0.75	1.00	0.00	14.73	-0.04	-0.52	-2.00	-3.85	-8.54	-13.97	-22.68
	0.50	0.50	10.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.00	1.00	9.27	0.02	0.20	0.79	1.56	3.68	6.06	8.80

Table 4

Optimal Consumption. This table reports the optimal level of consumption C for the first agent at time zero as well as the optimal level of consumption C^+ immediately thereafter. The corresponding optimal levels of consumption for the second agent are equal to one minus the level of consumption for the first agent. Div. denotes the initial dividend for the first asset, where the total initial dividends are normalized to one. The terms w and v denote the number of shares of the two assets with which the first agent is endowed. By the market clearing condition, the second agent is endowed with $1 - w$ and $1 - v$ shares of the two assets. The column denoted Liquid reports consumption when there is no illiquidity period (the liquid-market case). The remaining columns report consumption for illiquidity horizons ranging from 1 week to infinity (the illiquid-market case).

Div.	w	v	Cons.	Liquid	1 Wk.	1 Qtr.	1 Yr.	2 Yr.	5 Yr.	10 Yr.	∞
0.25	1.00	0.00	C	0.603	0.603	0.605	0.609	0.614	0.628	0.646	0.685
			C^+	0.603	0.211	0.211	0.211	0.210	0.209	0.208	0.202
	0.50	0.50	C	0.944	0.944	0.944	0.944	0.944	0.944	0.944	0.944
			C^+	0.944	0.528	0.526	0.522	0.517	0.507	0.501	0.496
	0.00	1.00	C	0.986	0.986	0.986	0.985	0.985	0.985	0.984	0.984
			C^+	0.986	0.795	0.793	0.789	0.785	0.776	0.770	0.767
0.50	1.00	0.00	C	0.780	0.780	0.782	0.786	0.791	0.803	0.818	0.844
			C^+	0.780	0.437	0.438	0.439	0.440	0.443	0.444	0.437
	0.50	0.50	C	0.944	0.944	0.944	0.944	0.944	0.944	0.944	0.944
			C^+	0.944	0.637	0.636	0.631	0.625	0.612	0.602	0.587
	0.00	1.00	C	0.972	0.972	0.972	0.971	0.970	0.968	0.966	0.965
			C^+	0.972	0.737	0.735	0.727	0.718	0.699	0.683	0.670
0.75	1.00	0.00	C	0.898	0.898	0.898	0.901	0.904	0.911	0.919	0.933
			C^+	0.898	0.682	0.682	0.685	0.688	0.695	0.701	0.697
	0.50	0.50	C	0.944	0.944	0.944	0.944	0.944	0.944	0.944	0.944
			C^+	0.944	0.771	0.771	0.770	0.767	0.759	0.747	0.720
	0.00	1.00	C	0.955	0.955	0.955	0.954	0.954	0.951	0.949	0.946
			C^+	0.955	0.795	0.794	0.791	0.788	0.774	0.756	0.725

Table 5

Welfare Losses from Market Illiquidity. This table reports the welfare losses for the agents for the indicated parameter values. The welfare loss is the percentage decrease in the wealth of an agent facing liquid markets that would equate his welfare to the level of welfare he would have in the illiquid-market case with his entire wealth. Div. denotes the initial dividend for the first asset, where the total initial dividends are normalized to one. The terms w and v denote the number of shares of the two assets with which the first agent is endowed. By the market clearing condition, the second agent is endowed with $1 - w$ and $1 - v$ shares of the two assets.

Div.	w	v	Welfare Loss of First Agent	Welfare Loss of Second Agent
0.25	1.00	0.00	51.63	2.59
	0.50	0.50	36.80	20.77
	0.00	1.00	16.21	53.12
0.50	1.00	0.00	36.11	5.90
	0.50	0.50	34.22	12.81
	0.00	1.00	27.97	21.66
0.75	1.00	0.00	21.34	11.74
	0.50	0.50	27.08	7.60
	0.00	1.00	28.52	5.38