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A MODEL OF STOCHASTIC PROCESS SWITCHING

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ABSTRACT

In this paper we develop a rational expectations exchange rate model which is capable of confronting explicitly agents' beliefs about a future switch in exogenous driving processes. In our set-up the agents know with certainty both the initial exogenous process and the new process to be adopted when the switch occurs. However, they do not know with certainty the timing of future switch as it depends on the path followed by the (stochastic) exchange rate.

The model is discussed in terms of the British return to pre-war parity, in 1925. However, our results are applicable to a variety of situations where process switching depends on the motion of a key endogenous variable.

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Often, a policy authority such as a central bank operates by establishing a policy rule to set the variables under its control. Such a rule is allowed to operate freely as long as certain endogenous variables of interest to the authority remain within particular bounds; however, when these endogenous variables cross their bounds, the authority switches to a new policy rule which it had prepared to meet this contingency. Since variables such as prices are determined partly by agents' beliefs about future events, agents' behavior injects the probabilities that policy switches will occur at particular future times into current price determination.

In this paper we explore in a formal model the determination of current prices when future policy regime switches are possible. In order to do this we develop a new aspect of an otherwise standard exchange rate model; this key component is the probability density function (p.d.f) for the first passage through a barrier of the endogenous variable (the exchange rate) which interests the policy authority. Since analytical solutions for first passage p.d.f.'s are available for only a limited number of stochastic processes, we are restricted to these processes in formulating our example. However, within this class of processes, our results are generally applicable to many different kinds of macroeconomic problems.

We present our ideas in the context of a model of exchange rate determination. Our choice of a specific example is intended to add concreteness to the analysis but should not be interpreted as setting limits on the applicability of the analysis. Indeed, the structure of the problem at hand virtually duplicates the structures which would be appropriate for studying problems such as a monetary authority's possible return to an interest rate rule, the possible introduction of wage and price controls, possible tax reform or virtually any other uncertain future policy switch.

In the specific example we study, when agents know that at a given future time the exchange rate will be fixed at a known level, then in a rational expectations world the solution for the current exchange rate assumes a form reflecting such knowledge. If the level is known while the timing is uncertain, then, though the solution technique is analogous, the actual solution for the current exchange rate will be a more complex form. As an example of the latter case, our results are particularly applicable to explaining the movements of the French and British exchange rates in the early 1920's.

In the main body of the paper we set up the model of the exchange rate when future fixing is possible and solve for the exchange rate. Most mathematical derivations are left to the appendix.

I) Determining the Current Exchange Rate When Future Fixing is Likely

That the exchange rate between two currencies is allowed to float freely means that governments do not currently intervene in exchange markets to set the rate. However, it is possible that under some future contigencies a government may intervene and establish a fixed rate system; this possibility will partly determine the current floating rate through its effect on expectations.

The specific example that we have in mind is that of Britain in the 1920's. The British decision to return to the gold standard at the pre-war parity of $4.86 was announced in the Budget Speech of April 28, 1925, and effective in the exchange market the next day (Moggeridge 1969, p. 9). However, as early as 1918 the Treasury and Ministry of Reconstruction appointed a Committee on Currency under Lord Cunliffe, which reported in 1919 "in our opinion it is imperative that after the war the conditions necessary to the maintenance of an effective gold standard should be restored without delay" (Moggeridge 1969, p. 12). Since the dollar was fixed to gold at that time, the British government was indicating that in the future it would fix the dollar-pound exchange rate at its pre-World War I level; the timing depended on achieving purchasing power parity at the pre-war exchange rate. Adopting such a policy affects the current exchange rate. Here we present a model in which this result is explicit.

The basis of our analysis is the standard model for the monetary approach to the exchange rate [see e.g. Frenkel (1978), Mussa (1978), Bilson (1978)]; together with a semi-log form for money demand in two countries, we include assumptions of interest parity and purchasing power parity.

Money demand in each country can be described by

$$m^{d}(t) - p(t) = \alpha_0 + \alpha_1 y(t) - \alpha_2 i(t) + v(t) \tag{1a}$$

$$\overset{*}{m}{}^{d}(t) - \overset{*}{p}(t) = \alpha_0^{*} + \alpha_1 \overset{*}{y}(t) - \alpha_2^{*} \overset{*}{i}(t) + \overset{*}{v}(t) \tag{1b}$$

where $i(t)$, $m^d(t)$, $p(t)$ and $y(t)$ are the nominal interest rate and the logarithms of the nominal money stock demanded, the nominal price level, and real income, respectively, in the home country. An asterisk represents corresponding variables in the foreign country. The α's are fixed parameters, all of which are greater than zero; and the $v(t)$'s are stochastic disturbances whose description we defer until later.

Purchasing power parity requires that

$$x(t) = p(t) - \overset{*}{p}(t) \tag{2}$$

where $x(t)$ is the logarithm of the exchange rate, measured as the number of units of domestic currency per unit of foreign currency. Recent work by Frenkel (1980a, b) indicates that equation (2) holds up reasonably well in the 1920's but fails in the 1970's.[1/] Thus we feel comfortable with (2) for studying a 1920's episode.

To form our exchange-rate equation we subtract (1b) from (1a) and substitute from (2) to derive (3)

$$\begin{aligned} x(t) = \gamma_0 &+ (m(t) - \overset{*}{m}(t)) + \alpha_1(\overset{*}{y}(t) - y(t)) \\ &+ \alpha_2(i(t) - \overset{*}{i}(t)) + \overset{*}{v}(t) - v(t) \end{aligned} \tag{3}$$

where $\gamma_0 = \alpha_0^* - \alpha_0$.

Define $K(t) \equiv \gamma_0 + (m(t) - \overset{*}{m}(t)) + \alpha_1(\overset{*}{y}(t) - y(t)) + \overset{*}{v}(t) - v(t)$, so that (3) can be rewritten as

$$x(t) = K(t) + \alpha_2(i(t) - \overset{*}{i}(t)). \tag{4}$$

Finally, we assume open interest parity. The anticipated rate of exchange rate depreciation, $E(\dot{x}(t) \mid I_t)$, is thus

$$E(\dot{x}(t) \mid I_t) = i(t) - i^*(t) \tag{5}$$

where $E(\mid I_t)$ is the conditional mathematical expectation operator with conditioning information set I_t, which includes all elements of the model's structure and all relevant variables dated t or earlier. Substituting in (4) yields

$$x(t) = K(t) + \alpha_2 E(\dot{x}(t) \mid I_t) \tag{6}$$

Equation (6) is the standard sort of equation that monetary approach models have produced and is a semi-reduced form which is consistent with a wide variety of underlying model specifications.[2/] In order to address its application to the problem of the future fixing of an exchange rate we must specify both the stochastic nature of the exogenous forcing function K(t) and the nature of the policy rule whereby the monetary authority decides the time for fixing the exchange rate. With rational expectations, the decision to fix the exchange rate implies a decision to change the stochastic nature of K(t). This follows from equation (6); when x(t) is fixed, with rational expectations, $E(\dot{x}(t) \mid I_t)$ must be zero, hence K(t) must be fixed. For the purposes of this example we will assume that, as long as the monetary authority does not actively fix the exchange rate, K(t) is a random walk with drift, i.e. K(t) can be written as

$$K(t) = K(0) + \eta t + e(t) \tag{7}$$

where η is the drift rate and e(s) is a Wiener process, i.e. $e(s) \sim N(0, \sigma^2 s)$.

In order to specify a policy rule for when the exchange rate will be fixed, we suppose that the monetary authorities in the foreign country will fix the exchange rate when purchasing power parity holds at some particular $\bar{x}$, i.e. when $\bar{x} = p(t) - \overset{*}{p}(t)$. Since (by assumption) the domestic price level minus the foreign price level is too low for this to obtain currently, we expect $p(t) - \overset{*}{p}(t) = x(t)$ to make a first passage through $\bar{x}$ from below at the time of the exchange rate's fixing.[3/] At any time t, the moment T in the future at which this first passage occurs is random with a p.d.f. $f(T|\bar{x}, x(t), K(t))$, which is conditional on $x(t)$, $\bar{x}$, and $K(t)$.

Taking expectations of both sides of (6) conditional on the information set I_t available to agents at time t, we find

$$E(x(t)|I_t) = E(K(t)|I_t) \quad \alpha_2 E(\dot{x}(t)|I_t). \tag{8}$$

This is a differential equation in the expected exchange rate conditional on I_t; rearranging, we have

$$E(\dot{x}(t)|I_t) = -\frac{1}{\alpha_2} E(K(t)|I_t) + \frac{1}{\alpha_2} E(x(t)|I_t) \tag{9}$$

Given a terminal condition we can solve (10) for the expected (and therefore acutal) exchange rate at time t.

Suppose first that purchasing power parity at the exchange rate $\bar{x}$ occurs at time T; then the exchange rate is fixed at $\bar{x}$ for $t > T$ and $x(T) = \bar{x}$. Since $x(T)$ is fixed at T its expected rate of change conditional on fixing at T is zero at T and hence, from (6), $\bar{x} = K(T)$. That $x(t)$ makes a first passage through $\bar{x}$ at T is equivalent to $K(t)$ making a first passage through $\bar{x}$ at T.

Conditional on first passage at T, the current exchange rate (and its current expectations) can be determined as

$$E(x(t)|I_t, T) = \bar{x}\exp\{\frac{t-T}{\alpha_2}\} \qquad (10)$$

$$\frac{1}{\alpha_2}\exp\{\frac{t}{\alpha_2}\}\int_t^T E(K(\tau)|I_t, T)\exp\{-\frac{\tau}{\alpha_2}\}d\tau$$

where $E(K(\tau)|I_t, T)$ indicates the expected path of $K(\tau)$, $t \le \tau \le T$, given I_t and $K(T) = \bar{x}$ for the first time. The unconditional exchange rate is then the integral of (11) weighted by the first passage p.d.f.

$$x(t) = \int_t^\infty E(x(t)|I_t, T)f(T - t|\bar{x}, x(t), K(t))dT \qquad (11)$$

Equation (11) is of the form of a typical solution to a rational expectations model. The problem which remains is to express the right hand side of (11) in terms of a finite number of (in principle) observable variables. In linear rational expectations models this final step is often accomplished by conjecturing that the solution is a linear function of the state variables and then requiring the unknown coefficients in the conjectured solution to obey the model at hand. This is the method of undetermined coefficients recently popularized by Lucas (1972). Our problem, however, is substantially more difficult because the (as yet) unknown non-linear form of the solution must be constructed from first principles.

To obtain the reduced form exchange-rate equation we proceed in two steps, first finding the density function $f(T-t|\bar{x}, x(t), K(t))$ and second finding $E(K(\tau)|I_t, T)$. Analytical expressions for these two magnitudes may then be substituted into (10) and (11), yielding the reduced form we seek.

The solution for the first passage p.d.f. of a Wiener process with drift is available in standard texts (see Karlin and Taylor, p. 363). The p.d.f. over the first passage of $K(t)$ through $\bar{x}$, given $K(t) < \bar{x}$ is

$$f(T - t \mid \bar{x}, x(t), K(t)) = \frac{\bar{x} - K(t)}{\sigma\sqrt{2\pi}(T - t)^{3/2}}\exp\{-\frac{1}{2}\left[\frac{\bar{x} - K(t) - \eta(T - t)}{\sigma^2(T - t)}\right]^2 \quad (12)$$

The derivation of $E(K(\tau) \mid I_t, T, t \leq \tau \leq T)$ is an exercise in stochastic processes. Although there is an analytical solution for this expectation, it is fairly complicated so we write it and its derivation in the appendix. Although analytical expressions for $f(T \mid I_t)$ and $E(K(\tau) \mid I_t, T)$ exist, the integrations in (11) and (12) appear difficult, so after substitution of these formulas into (12), the solution of $x(t)$ is still expressed as a double integral.

Applications

The reduced form solution for $x(t)$ is not displayed here but is the nonlinear equation which results from substituting the appendix expression (A1) into (10) and substituting the result into (11) with $f(T - t \mid \bar{x}, x(t), K(t))$ being replaced by the right hand side of (12). The nonlinear exchange rate equation resulting from the above can in principle be estimated using a combination of standard nonlinear techniques and numerical integration sub-routines.

The unfortunate feature of our result is that it implies that an exchange-rate equation estimated by typical linear methods during a period when agents are anticipating stochastic process switching will be subject to parameter drift. For example Frenkel and Clements (1980) estimate a US/UK exchange rate equation similar to our equation (3) over the period February 1921 to May 1925, which encompasses a large part of the period when agents may have been anticipating process switching. To allow for the endogenity of interest rate differentials Frenkel and Clements used a linear two stage least squares procedure. According to our results the first stage of their procedure should have been specified in accord with our non-linear exchange rate equation.

It seems to us that the problem encountered in Frenkel and Clements may be quite widespread. Indeed, whenever policy makers deliberate, they inject into agents' forecasting problems an element of stochastic process switching. However, it is atypical of such deliberations that they result in a stochastic process switching problem as clearly defined as the British return to pre-war parity.

Appendix

The Conditional Expectation of $K(\tau)$, $E(K(\tau)|I_t, T)$

The derivation of $E(K(\tau)|I_t, T)$ is an exercise in stochastic processes. In this appendix we first write out the explicit formula for this expectation and explain its components. Then we explain the determination of $E(K(\tau)|I_t, T)$. We are extremely grateful to J. H. Kemperman for showing us how to derive the conditional expectation of $K(\tau)$.

Recall that $E(K(\tau)|I_t, T, t \leq \tau \leq T)$ is the expectation of $K(\tau)$ given $K(t)$ and given that at time $T, K(T) = \bar{x}$ for the first time. Let $T_1 \equiv T - t$, $\tau_1 \equiv \tau - t$, $Z \equiv \bar{x} - K(t)$ and $\overset{*}{Z} \equiv \frac{Z}{\sigma}\sqrt{\frac{1 - \tau_1/T_1}{\tau_1}}$. Then the explicit formula for the conditional expectation is

$$E(K(\tau)|I_t, T) = C_2/C_1 \qquad \text{(A1)}$$

where

$$C_2 \equiv [1 - \frac{\tau_1}{T_1}]\{[(1 - \frac{\tau_1}{T_1})Z^2 + \sigma^2\tau_1]\Phi(\overset{*}{Z}) + \sigma Z\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(-\overset{*}{Z}) \qquad \text{(A2)}$$
$$- \exp\{(1 - \frac{\tau_1}{T_1})\frac{2\eta Z}{\sigma^2}\}[(1 - \frac{\tau_1}{T_1})Z^2 + \sigma^2\tau_1^2]\Phi(-\overset{*}{Z}) - Z\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(-\overset{*}{Z})]\}$$

and

$$C_1 \equiv [\sigma\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(\overset{*}{Z}) + Z(1 - \frac{\tau_1}{T_1})\Phi(\overset{*}{Z})]$$
$$- \exp\{(1 - \frac{\tau_1}{T_1})\frac{2\eta Z}{\sigma^2}\}[\sigma\tau_1^{\frac{1}{2}}(1 - \frac{\tau_1}{T_1})^{\frac{1}{2}}\phi(\overset{*}{Z}) - Z(1 - \frac{\tau_1}{T_1})\Phi(-\overset{*}{Z})]$$

In these formulas, $\phi(x) \equiv \frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}$ and $\Phi(x) \equiv \int_{\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{y^2}{2}\}dy$.

To derive formulas (A1) - (A3), we must find the conditional density of $K(t)$, given T, the time of first passage through $\bar{x}$, where $T > t$. Call this density function $h(K(t)|T)$. Then we need only multiply by $K(t)$ and integrate to determine the first moment. We can find this density function by first determining the joint density over $(K(t), T)$, which we denote by $g(K(t), T)$. For simplicity, let us assume that we are looking forward from time $t = 0$ and that $K(0) = 0$.

The joint density function equals the conditional density function over $K(t)$ multiplied by the marginal density function over T, $f(T)$, i.e.

$$g(K(t), T) = h(K(t)|T)f(T). \tag{A4}$$

The joint density also equals the conditional density over T, given $K(t)$, which we denote by $F(T|K(t))$, multiplied by the marginal density over $k(t)$, $H(K(t))$, i.e.

$$g(K(t), T) = F(T|K(t)H(K(t)) \tag{A5}$$

$H(K(t))$ does not depend on the time $T > t$ at which first passage occurs, but it is conditional on first passage not having occurred prior to t.

Since they are relatively easy to derive, we will use the density functions in (A5) to construct the joint density $g(K(t), T)$; then we can determine the conditional density $h(K(t)|T)$ from (A4). First we develop $H(K(t))$.

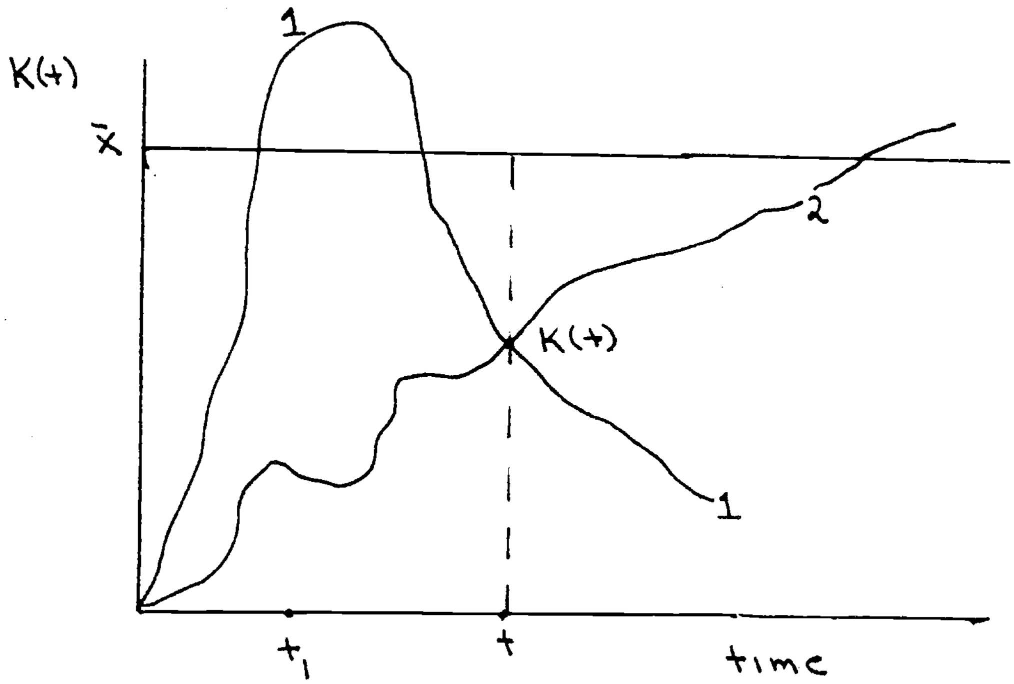

Figure 1

Since K(t) is a Wiener process with drift and with a starting value K(0) = 0, K(t) has an unconditional p.d.f.

$$G(K(t)) = \frac{1}{\sqrt{2\pi t}\sigma} \exp\{-\tfrac{1}{2}\frac{(K(t) - \eta t)^2}{\sigma^2 t}\}$$

$$= \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{K(t) - \eta t}{\sigma\sqrt{t}}\right) \tag{A6}$$

where $\phi(x) \equiv \frac{1}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2}x^2\}$ (See Karlin and Taylor, p. 356). H(K(t)) is the density of K(t) conditional on K(t)'s having remained below $\bar{x}$ prior to t. Thus, from the probability weight given to K(t) by G(K(t)), we must subtract the weight associated with all paths (like 1 in Figure 1) which pass through K(t) at t but which also have passed through $\bar{x}$ prior to t.

There are an infinity of paths which, like path 1, pass through $\bar{x}$ for the first time at t_1 and through K(t) at time t. Given that a path starts at $\bar{x}$ at t_1, the unconditional probability weight associated with its passing through K(t) at time t is again given by (A6):

$$G(K(t)|K(t_1) = \bar{x}) = \frac{1}{\sigma\sqrt{t - t_1}}\phi\left(\frac{K(t) - \bar{x} - \eta(t - t_1)}{\sigma\sqrt{t - t_1}}\right) \quad \text{(A7)}$$

The probability weight associated with a path's passing through $\bar{x}$ for the first time at t_1 is (from equation 13).

$$f(t_1|K(0) = 0) = \frac{\bar{x}}{\sigma\sqrt{2\pi}(t_1^{3/2}} \exp\{-\frac{(\bar{x} - \eta t_1)^2}{2\sigma^2 t_1}\}$$

$$= \frac{\bar{x}}{\sigma t_1^{3/2}}\phi\left(\frac{\bar{x} - t_1}{\sigma\sqrt{t_1}}\right) \quad \text{(A8)}$$

Then the probability weight associated with the set of all paths which both pass through $\bar{x}$ for the first time at t_1 and pass through K(t) at t is

$$G(K(t)|K(t_1) = \bar{x})f(t_1|K(0) = 0) =$$

$$\frac{\bar{x}}{\sigma t_1^{3/2}}\phi\left(\frac{\bar{x} - \eta t_1}{\sigma\sqrt{t_1}}\right)\frac{1}{\sigma\sqrt{t - t_1}}\phi\left(\frac{K(t) - \bar{x} - (t - t_1)}{\sigma\sqrt{t - t_1}}\right) \quad \text{(A8)}$$

Therefore, the probability weight associated with <u>all</u> paths which both pass through $\bar{x}$ at some time prior to t and equal K(t) at t is the integral over t_1 of (A8):

$$\int_0^t \frac{\bar{x}}{\sigma t_1^{3/2}}\phi\left(\frac{\bar{x} - \eta t_1}{\sigma\sqrt{t_1}}\right)\frac{1}{\sigma\sqrt{t - t_1}}\phi\left(\frac{K(t) - \bar{x} - \eta(t - t_1)}{\sigma\sqrt{t - t_1}}\right)dt_1 \quad \text{(A9)}$$

To determine H(K(t)) up to a normalizing constant we need only subtract (A9) from (A6)

$$H(K(t)) = \frac{1}{\sigma\sqrt{t}}\phi\left(\frac{K(t) - t}{\sigma\sqrt{t}}\right) - \int_0^t \frac{\bar{x}}{\sigma t_1^{3/2}}\phi\left(\frac{\bar{x} - \eta t}{\sigma\sqrt{t_1}}\right)\frac{1}{\sigma\sqrt{t - t_1}}\phi\left(\frac{K(t) - \bar{x} - \eta(t - t_1)}{\sigma\sqrt{t - t_1}}\right)dt_1 \qquad \text{(A10)}$$

To determine the joint p.d.f. g(K(t), T) we must multiply H(K(t)) by the conditional p.d.f. F(T|K(t)). But again from equation (13),

$$F(T|K(t)) = \frac{\bar{x} - K(t)}{\sigma\sqrt{2\pi}(T - t)^{3/2}} \exp\{-\frac{(\bar{x} - K(t) - \eta(T - t))^2}{2\sigma(T - t)} \qquad \text{(A11)}$$

$$= \frac{\bar{x} - K(t)}{\sigma(T - t)^{3/2}}\phi\left(\frac{\bar{x} - K(t) - (T - t)}{\sigma\sqrt{T - t}}\right).$$

Finally,

$$g(K(t), T) = CH(K(t))F(T|K(t)) \qquad \text{(A12)}$$

where C is a normalizing constant. h(K(t)|T) is simply (A12) divided by f(T) and evaluated at a particular value of T. In the formula (A1), C_1 is simply the inverse of the normalizing constant, i.e.

$$C_1 = \int_{\infty}^{\bar{x}} H(K(t))F(T|K(t))dK(t).$$

C_2 is the unnormalized first moment of (A12) evaluated at a particular T, i.e.

$$C_2 = \int_{\infty}^{\bar{x}} K(t)H(K(t))F(T|K(t))dK(t).$$

The mean of K(t) conditional on K(0) = 0 and K(T) = $\bar{x}$ for the first time is then C_2/C_1. Deriving the actual formulas (A2) - (A3) requires the cranking out of some horrendous integrals, which we avoid here.

Footnotes

1/ Frenkel's results (1980a, p. 238) provide tests of purchasing power parity for traded goods during the period Feb. 1921 - May 1925. Presently we are assuming all goods to be traded and recognize that the presence of a large nontraded goods sector would slightly modify our model.

2/ Equation (6) is of a form which is relevant to many macro-models. In a closed economy setting, price would replace the exchange rate.

3/ In our example we are treating the U.S. as the home country and the U.K. as the foreign country so that $\overline{x} = \ln(\$4.86/£)$.

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