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THE OPTIMAL WEIGHTING OF INDICATORS FOR A CRAWLING PEG

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ABSTRACT

This paper derives optimal weights for current-account and reserve indicators for adjusting the exchange rate (a "crawling peg"). Keven (1975) showed that use of a current account indicator alone would not stabilize reserves, while a reserve indicator results in unstable fluctuations in the exchange rate. This paper begins by analyzing the problem in the framework of Phillips (1954), in which the current account indicator is "proportional" and the reserve indicator is "integral." We then analyze the problem in a deterministic optimal control framework, and finally as a problem in stochastic control. In all cases the optimal combination is a weighted average, which we call the Keven-Phillips formula. With a fairly low variance of the current account, its weight falls in the range 0.47 - 0.65. Rising variance reduces its weight in the optimal formula.

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I. Introduction

The problem of choosing indicators for exchange-rate adjustment will be relevant for some time to come. Most countries do not permit their exchange rates to float freely, and others must choose some rule, implicit or explicit, for adjusting the rate. In any program of exchange-rate surveillance, the IMF staff will have to take a view on the appropriate indicators. Indeed, in the early seventies, some work was done at the Fund along these lines, namely by Underwood (1973) and Williamson (1937b) and the problem was discussed by the Committee of Twenty (1974).

To our knowledge, the definitive study of the problem to date is Kenen's (1975). He did an extensive simulation study of numerous alternative schemes and found that a reserve change or basic balance indicator would not stabilize reserves, whilst a reserve indicator resulted in large fluctuations in the current account. He also noted that this reflected the problem of simple proportional or integral stabilization rules analyzed by Phillips (1954), but did not pursue a Phillips' analysis.

Even though the importance of Phillips's early contribution has been more widely noted in the macro literature, perhaps because the title of his paper refers to a "closed economy", the open economy implications are pointed out. Thus, it is said that "a country which attempts to regulate its current balance of payments, whether by means of internal credit policy or quantitative import restrictions, and in doing so responds mainly to the size of its foreign reserves (i.e., to the time integral of its current balance of payments), is applying an integral correction policy which is likely to cause cyclical fluctuations similar to those illustrated (in this paper).

The short cycles that have occurred in the balance of payments of a number of countries since the war may be in part the result of such action". Phillips also notes that "The general principles of stabilization (...) could equally well be used, for example, in investigating the stability of adjustment in international trade or the problems involved in commodity price stabilization schemes."

In this paper we generalize Kenen's results and relate them to an optimal control approach to the problem. Kenen studied arbitrarily specified adjustment mechanisms; we wish to derive an optimal specification explicitly. In section II we generalize Kenen's results analytically, and derive weights for the current account and reserve target that yield monotonically stable adjustment. We see that a current account (or, in general, flow) indicator is stable but randomizes reserves, while a reserve indicator yields a limit cycle. Conditions for a Kenen-Phillips formula weighting the two to give stable results are presented and illustrated.

In section III we derive an optimal control solution for adjustment to a given current account disturbance. Since this formula is a linear control rule, it can be derived as the solution of the minimization of a quadratic minimum energy loss function subject to a linear equation of motion. Thus optimal weights for the current account and reserve target are derived for various values of the derivative of the current account with respect to the exchange rate (B_e), and the weights of the output variables in the social loss function α.

Finally in section IV we derive the adjustment equation in a

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[1] See Phillips (1954), p. 298 footnote 1 and p. 305 footnote 1 respectively. The importance of this work is emphasized in the first paragraph of Turnovsky (1973) and in the preface of Aoki (1976), for example.

stochastic framework with continuous current account and exchange rate multiplier shocks. The separation theorem of stochastic control, known in economics as the principle of certainty equivalence, implies that the linear control rule remains applicable, given the expectation on the state variables conditional on the path of the output variables. If the two are uncorrelated, we therefore have an expression in the Kenen-Phillips form of section II.

The values of the optimal weights for various values of the parameters of this problem are compared with those obtained for the deterministic case. It is found that the weight of the current account in the optimal control rule is generally higher than the welfare weight, and also higher than the lower bound of Section II. When the variance of the effect of the exchange rate on the current account increases, however, the optimal weight approaches the lower bound and becomes smaller than the welfare weight, as was to be expected from the static analysis in Brainard (1967). The results are summarized and conclusions are drawn in Section V, which includes the summary Tables 6 and 7.

II. Flow vs. Stock Indicators

The purpose of this section of the paper is to expose the analytical problem of the choice of indicators as clearly as possible, setting the stage for the optimizing approaches of the following sections. Therefore we begin with the simplest model that illustrates the problem. Assume that the monetary authority in a given small open economy has already decided not to permit its exchange rate to float freely. This is necessary for the question of choice of indicators to arise. Further, assume zero capital mobility so that the current account balance B in foreign currency is the rate of accumulation of reserves,

(1) $\dot{R} = B$.

These two assumptions are consistent; with no stabilizing speculation on capital account, a foreign exchange market based on trade flows alone might well be unstable. [1]

The current account balance is an increasing function of the exchange rate e (units of home currency per unit of foreign exchange):

$$(2)\ B = B(e)\ ;\quad B_e > 0.$$

The sign of the derivative B_e reflects the Marshall-Lerner condition. [2] By appropriate choice of units equilibrium imports and the equilibrium exchange rate are set to one, so that $B_e = d_x + d_m - 1$, where d_x and d_m are the absolute values of the export and import price elasticities of demand. We assume that domestic absorption is manipulated by aggregate demand policy so that we can focus on the dependence of B on e alone.

The model could be amended to include capital flows as a function of uncovered interest rate differentials. In that case, given interest rates,

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[1] See Branson and Katseli-Papaefstratiou (1978) for the fully-developed argument.

[2] Writing the current account more explicitly as $B(p) = X(p)/p - M(p)$ where $p = eP^*/P$ is the real exchange rate and P/P^* an appropriate relative price index, and differentiating around equilibrium, we have

$$dB = \frac{dp}{p} M^* (d_x + d_m - 1)$$

$$\text{where } d_i = \frac{\varepsilon_i(1+\eta_i)}{\eta_i+\varepsilon_i} \qquad i = x,m$$

is a combination of supply (η) and demand (ε) elasticities of exports and imports. We set $P = P^* = 1$ in the analysis so as to preserve the simplicity of equation (1) in the text.

exchange rate expectations would have to depend on the current level of the exchange rate, and be such that $B_e > 0$, including capital flows in B. This is achieved by regressive or weakly extrapolative expectations. Adding capital movements in this "old" way would reduce analytical clarity without adding anything to the results.

A position of external balance is defined by attainment of a given target level of reserves R^*, with a zero balance on current account.[1] The latter condition defines a target value for the exchange rate:

(3) $B(e^*) = 0$,

and $R = R^*$, $e^* = e$ defines external balance. The problem of choice of objective indicators is to choose a rule for adjusting e, following observations on B or R, that converges to external balance.

One candidate suggested by Cooper (1970) would be to key adjustment of e to reserve changes, which, given (1), amount to the current account balance:

(4) $\dot{e} = -\lambda B(e)$.

This is a proportional stabilizer, in Phillips' terms. As Kenen says, "It matches a flow control to a flow target."[2] Given our assumption that $B_e > 0$, it is stable around e^*. Linearizing, we have

(5) $\dot{e} = -\lambda B_e (e - e^*)$,

and $d\dot{e}/de = -\lambda B_e < 0$. But there is no mechanism to move R to R* with this rule. The time path of R will resemble a random walk. A current

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[1] The extensive literature on optimal reserves was surveyed in Williamson (1973a). See also Bilson-Frenkel (1979).

[2] See Kenen (1975) p. 128. Cooper's proposal is analyzed on p. 118 and "given good marks" in the conclusion on p. 147.

account disturbance moving e^* will be eliminated gradually as the adjustment rule (4) moves e to the new e^*. During the adjustment period R will change. When e reaches e^* and B is again zero, there will be no further change in R.

Another candidate, proposed in 1972 by the U.S. Secretary of the Treasury would be to key adjustment of e to deviations of reserves from the target: [1]

(6) $\dot{e} = -\lambda\ (R - R^*)$.

This is an integral stabilizer in Phillips' terms. As Kenen says, "The rule marries a flow control to a stock target, a union that is always apt to be unstable." [2]

Indeed, it leads to a limit cycle in e (t). To see this, differentiate the rule in (6) with respect to time, and linearize around e^*.

(7) $\ddot{e} = -\lambda\ \dot{R} = -\lambda\ B(e) = -\lambda B_e(e-e^*)$.

This is a second-order differential equation without a term in $\dot{e}$:

$$\ddot{e} + \lambda B_e e = 0\ .$$

The roots of the equation are purely imaginary and equal to $\pm\ i\ (\lambda\ B_e)^{1/2}$. If the system were to begin at

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[1] The outline of the proposal presented at the Annual Meeting of the IMF is in IMF (1972), p. 34-44, esp. p. 39-40. The full document is in CEA (1973), p. 160-174. Alternative proposals in the C-20 are reproduced in Committee of Twenty (1974) and discussed in Williamson (1977). See Underwood (1973) for a listing of similar proposals.

[2] Kenen (1975) p. 128. The conditions for the instability of the reserve target indicator, as well as the change in reserves indicator when there are capital flows, were derived in a complete model of the "small open inflationary economy" by Martirena-Mantel (1976), who concludes that her results "seem to agree" with Kenen's. Recently, particularly in connection with the "Southern Cone problem" discussed by Diaz-Alejandro (1979), this instability was obtained in a variety of portfolio balance models: see Rodriguez (1978), Kouri (1979), Dornbusch (1979), Calvo (1979) and Krugman (1980).

$R = R^*$, $e = e^*$, a current account disturbance would set off an infinite cycle in e, B, and R.

Phillips' prescription was to combine the two rules in (4) and (6). The essential idea is to add a bit of the integral stabilizer as in (6) to the proportional rule of (4) in order to keep a stable adjustment system moving toward the reserve target. We can express this by weighting the two rules:

(8) $\dot{e} = -\beta[\lambda B(e) + (1-\lambda)(R-R^*)]; \; 0 \leq \lambda \leq 1.$

Here β gives the sensitivity or speed of adjustment of e with respect to the weighted average of off-target values of B and R. By appropriate choice of units, we can scale β to unity.

We can find a range of values for λ which will yield monotonically stable adjustment of e as follows. Differentiate (8) with respect to time and linearize B around e^* to obtain the second-order differential equation.

(9) $\ddot{e} + \lambda B_e \dot{e} + (1-\lambda) B_e e = 0.$

The roots are given by

$$r_1, r_2 = (-\lambda B_e \pm (\lambda^2 B_e^2 - 4(1-\lambda) B_e)^{1/2})/2.$$

For monotonic stability, λ should be chosen such that both roots are real and negative, which requires that the square root term be positive, or that

$$\frac{\lambda^2}{4} > \frac{1-\lambda}{B_e}$$

This, in turn, gives us a quadratic in λ with roots given by

$$r_1, r_2 = \frac{2}{B_e} \; [-1 \pm (1 + B_e)^{1/2}].$$

Since λ is positive, we discard the negative root. This yields the expression for the permissible range of λ, depending on B_e.

(10) $1 > \lambda > \frac{2}{B_e} (- 1 \pm (1 + B_e)^{1/2}) > 0.$

To obtain an intuitive understanding of the result, recall that $B_e = d_x + d_m - 1$. If both demand elasticities are unity (in absolute value) so that $B_e = 1$, we have $\lambda > 0.83$. As B_e gets smaller, the bound for λ approaches unity; as B_e gets larger, it approaches zero. Some values for B_e and λ are given in Table 1.

Table 1: Lower Bound for λ, Depending on B_e

B_e	0	.25	.50	1.0	1.5	2.0	10.0	100.0
λ	1	.94	.90	.83	.77	.49	.46	.18

As B_e increases, less weight must be given to the current account balance to get a given adjustment of the current account. A "reasonable" weighting of the two targets with B_e around unity would be 0.8 for the for the current account and 0.2 for the reserve deviation.

III. Optimal Adjustment in a Deterministic Framework

Equation (10) and Table 1 give the permissible range of weights for the stock and flow targets that yields monotonically stable adjustment to a current account disturbance in the Kenen-Phillips framework. Still considering adjustment to a one-shot current account disturbance, we now turn to an optimal control analysis from the viewpoint of external balance. In this context, the problem involves minimizing the square of the difference between actual and desired current account and level of reserves, with minimum exchange rate changes. The desired level of the current account and the deviation of reserves from some given level R* is taken to be zero. The quadratic minimum-energy loss function is then

$$L = \frac{1}{2}\left[\alpha B(e)^2 + (1-\alpha)(R-R^*)^2 + (\dot{e})^2\right].$$

Here α is the welfare weight for current account imbalance and $1-\alpha$ weights the distance from the reserve target, both being measured relative to the unit cost of exchange rate variability.

As in section II the model of the economy is given by

$$\dot{R} = B(e) = B_e(e-e^*), \tag{11}$$

where $B(e^*) = 0$ defines e^*. In this simple case, the output vector is just

$$w = \begin{bmatrix} R - R^* \\ B \end{bmatrix},$$

and we write

$$w = Cz, \tag{12}$$

where $C = \begin{bmatrix} 1 & 0 \\ 0 & B_e \end{bmatrix}$

and $z = \begin{bmatrix} R - R^* \\ e - e^* \end{bmatrix}$ is the state vector.

The state vector is also expressed as a difference of actual and given desired values, so that we can treat the problem as a simple time invariant output regulator problem. Strictly speaking, when the desired values are not zero we cannot assume that the minimum loss of the time invariant problem is finite, or that a control exists. By writing the state variables in deviation form, however, we are able to ignore the forcing function and therefore work with infinite horizon. [1] We write our objective function as

$$J = \frac{1}{2}\int_0^\infty (w'Dw + u'u) \quad dt$$

where $u = \begin{bmatrix} 0 \\ \dot{e} \end{bmatrix}$ is the control vector

and $D = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}$.

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[1] The homogeneous system we work with has of course the same eigenvalues as the inhomogeneous one. The consequences of discounting are analyzed below.

Given (12), we have a convenient state space representation of our problem. [1/]

(13) $$\text{Min } \frac{1}{2}\int_o^{\infty} (z'Qz + u'u)\ dt$$

subject to $\dot{z} = Az + Bu$

$z(o) = z_o$,

where $Q = C'DC = \begin{bmatrix} \alpha & o \\ 0 & (1-\alpha)\ B_e^{\ 2} \end{bmatrix}$;

$A = \begin{bmatrix} 0 & B_e \\ 0 & 0 \end{bmatrix}$;

$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Defining the vector of costate variables $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ we have the Hamiltonian

(14) $$H = \frac{1}{2}(z'Qz + u'u) + y'(Az + Bu),$$

which is minimized at each instant of time. The marginal conditions then are

(15) $$\frac{\partial H}{\partial u} = u' + y'B = 0,$$

since $\dfrac{\partial^2 H}{\partial u^2} = I$ is positive definite.

By the minimum principle, we have

(16) $$\frac{\partial H}{\partial y'} = \dot{z} = Az + Bu;$$

(17) $$\frac{-\partial H}{\partial z'} = \dot{y} = -Qz - A'y.$$

Using (15) transposed we write the canonical equations

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[1] Note that the system (12), (13) is both controllable and observable, since the rank of $(B \mid AB)$ and $(C' \mid A'C)$ is two.

(18) $$\dot{z} = Az - BB'y;$$

$$\dot{y} = -Qz - A'y.$$

For time invariant A, B, Q and an infinite horizon problem, the costate variable is given by

(19) $$y = Kz,$$

where $K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} = \lim_{t\to\infty} K(t)$

is a positive definite matrix given by the Riccati equation:

(20) $$-KA - A'K + KBB'K - Q = 0 .$$

By substitution we derive from (20) equations for the elements of K:

$$k_{12}^2 = \alpha ;$$

(21) $$k_{12}k_{22} = k_{11}B_e ;$$

$$k_{22}^2 - 2_e k_{12} = (1-\alpha)B_e^{\,2}.$$

Given the positive definiteness of K, k_{11}, k_{12}, $k_{22} > 0$, so that the solution is

(22) $$k_{12} = \sqrt{\alpha} ;$$

$$k_{22} = B_e \sqrt{2\sqrt{\alpha}/B_e + (1-\alpha)} ;$$

$$k_{11} = \sqrt{2\alpha/B_e + (1-\alpha)\alpha} .$$

The optimal control $\tilde{u}$ is linear in the state vector and, from (15) and (19) satisfies the equation

(23) $$\tilde{u} = -B'Kz$$

so that it can be written as a function of k_{12} and k_{22}:

$$\tilde{u}_t = -\sqrt{\alpha}(R_t - R^*) - \sqrt{2\sqrt{\alpha}/B_e + (1-\alpha)}\; B_e(e_t - e^*) .$$

Defining $\Theta = \sqrt{\alpha} + \sqrt{2\sqrt{\alpha}/B_e + (1-\alpha)}$

and $1 - \gamma = \sqrt{\alpha}/\Theta$

this yields a familiar expression for the optimal rate of crawl

(24) $$\dot{e} = -\Theta\,[\gamma B(e) + (1-\gamma)(R - R^*)] .$$

Some values of θ and γ for values of B_e and α are shown in Table 2.

Table 2

Alternate values of Θ and γ

Values of α

	0		.5		1	
Value of B_e	Θ	γ	Θ	γ	Θ	γ
0.1	1	1	4.54	.84	5.47	.82
1.0	1	1	2.09	.66	2.41	.59
10.0	1	1	1.51	.53	1.45	.31
100.0	1	1	1.43	.50	1.14	.12

The table is consistent with the result of the previous section that the weight on the current account increases as B_e decreases, but by less. In the case of $B_e = 1$, weighting equally the two targets implies that the optimal current account weight is .66, rather than .8 from Table 1. Note, however, that the change in the exchange rate is 2.09 as large as was the case before. For the same change, the lower bound at the current account weight is .38 (see Table 6 below). The optimal value is thus substantially higher than the lower bound, confirming the need to supplement the instability analysis of section II by an optimizing approach.

Solving (18) explicitly we find that the system is monotonically stable with eigenvalues given by

$$(25)\quad r_1, r_2 = -\frac{B_e}{2}\left[\sqrt{1-\alpha + \frac{2\sqrt{\alpha}}{B_e}} \pm \sqrt{1-\alpha - \frac{2\sqrt{\alpha}}{B_e}}\right]$$

The absolute values of r_1 and r_2 are displayed in Table 3 for the same values of α and B_e as Table 2.

Table 3 Eigenvalues of (6) for values of α and B_e

Values of B_e	Values of α					
	0		.5		1.0	
	r_1	r_2	r_1	r_2	r_1	r_2
.1	-.1	0	-.192(1+i)	-.192(1-i)	-.223(1+i)	-.223(1-i)
1.0	-1.0	0	-.69(1+i)	-.69(1-i)	-.707(1+i)	-.707(1-i)
10	-10	0	-7	-1	-2.23(1+i)	-2.23(1-i)
100	-100	0	-70.7	-1	-7.07(1+i)	-7.70(1-i)

The solution is of the form

$$(26)\quad \begin{bmatrix} R_t - R^* \\ e_t - e^* \end{bmatrix} = \exp(Gt)\begin{pmatrix} R_o - R^* \\ e_o - e^* \end{pmatrix}$$

where $G = A - BB'K$, and

$$\exp(Gt) = \frac{B_e}{r_2 - r_1}\begin{bmatrix} 1 & 1 \\ r_1/B_e & r_2/B_e \end{bmatrix}\begin{bmatrix} \exp(r_1 t) & 0 \\ 0 & \exp(r_2 t) \end{bmatrix}\begin{bmatrix} r_2/B_e & -1 \\ -r_1/B_e + 1 & \end{bmatrix}$$

In the case where reserves are initially at their desired level, $R_o = R^*$, the deviation of the exchange rate from its long run equilibrium value is given by

$$(27)\quad e_t - e^* = \frac{1}{2}(e_o - e^*)\left[(\theta_2-\theta_1)/\Psi \quad -(\theta_2+\theta_1)\right]$$

$$\text{where } \Psi = \sqrt{\frac{B_e(1-\alpha)-2\sqrt{\alpha}}{B_e(1-\alpha)+2\sqrt{\alpha}}},$$

and $\theta_i = \exp(r_i t); \quad i = 1, 2.$

In the case of a pure current account target, $\Psi = 1$ and the exchange rate is a weighted average of the initial and long run levels, with the former weight declining over time, as in the analysis of Kouri (1978):

$$(28)\quad e_t = e_o\theta_1 + e^*(1-\theta_1).$$

If $\alpha = 1$, however, $\Psi = i$ and the exchange rate is given by the limit cycle

$$(29)\quad e_t = \frac{1}{2}(e_o - e^*)\left[(\theta_2-\theta_1) - (\theta_2+\theta_1)i\right] + e^*i$$

Associated with the rule in (23) is a minimum value for the loss function in terms of the deviations of the state variables from their given equilibrium levels and the Riccati matrix, so that loss is zero in the steady-state. This can be written as

$$(30)\quad \tilde{L}_t = \frac{1}{2}z'_t K z_t = \frac{1}{2}k_{11}(R_t-R^*)^2 + \frac{1}{2}k_{22}(e_t-e^*)^2 + k_{12}(R_t-R^*)(e_t-e^*)$$

Let us now consider the case in which future utility is discounted at a rate ρ, so that the minimand in (13) becomes

$$(31)\quad \frac{1}{2}\int_0^\infty (z'Qz + u'u)\, e^{-\rho t}\, dt.$$

Then the Riccati equation in (20) becomes

(32) $$-KA - A'K + \rho K + KBB'K - Q = 0$$

and equations (21) for its elements become

$$k_{12}^2 - \alpha + \rho k_{11} = 0$$

(33) $$k_{12}k_{22} - k_{11}B_e + \rho k_{12} = 0$$

$$k_{22}^2 - 2k_{12}B_e - (1 - \alpha) B_e^2 + \rho k_{22} = 0$$

Solving out for k_{11}, which does not enter the optimal control solution in (23), we obtain two second-order equations which can be represented in k_{12}, k_{22} space. This is done in Figure 1, for $\rho = .2$, $\alpha = .5$ and $B_e = .1$. The intersection of the two curves at E_ρ is clearly to the southwest of R, where the rate of discount is zero and the coefficient on reserves is independent of the coefficient on the current account. Note that larger values of B_e bring E closer to R, as shown in the θ column of Table 4, which

Table 4

Alternate values of

θ and γ

for $\rho = .2$

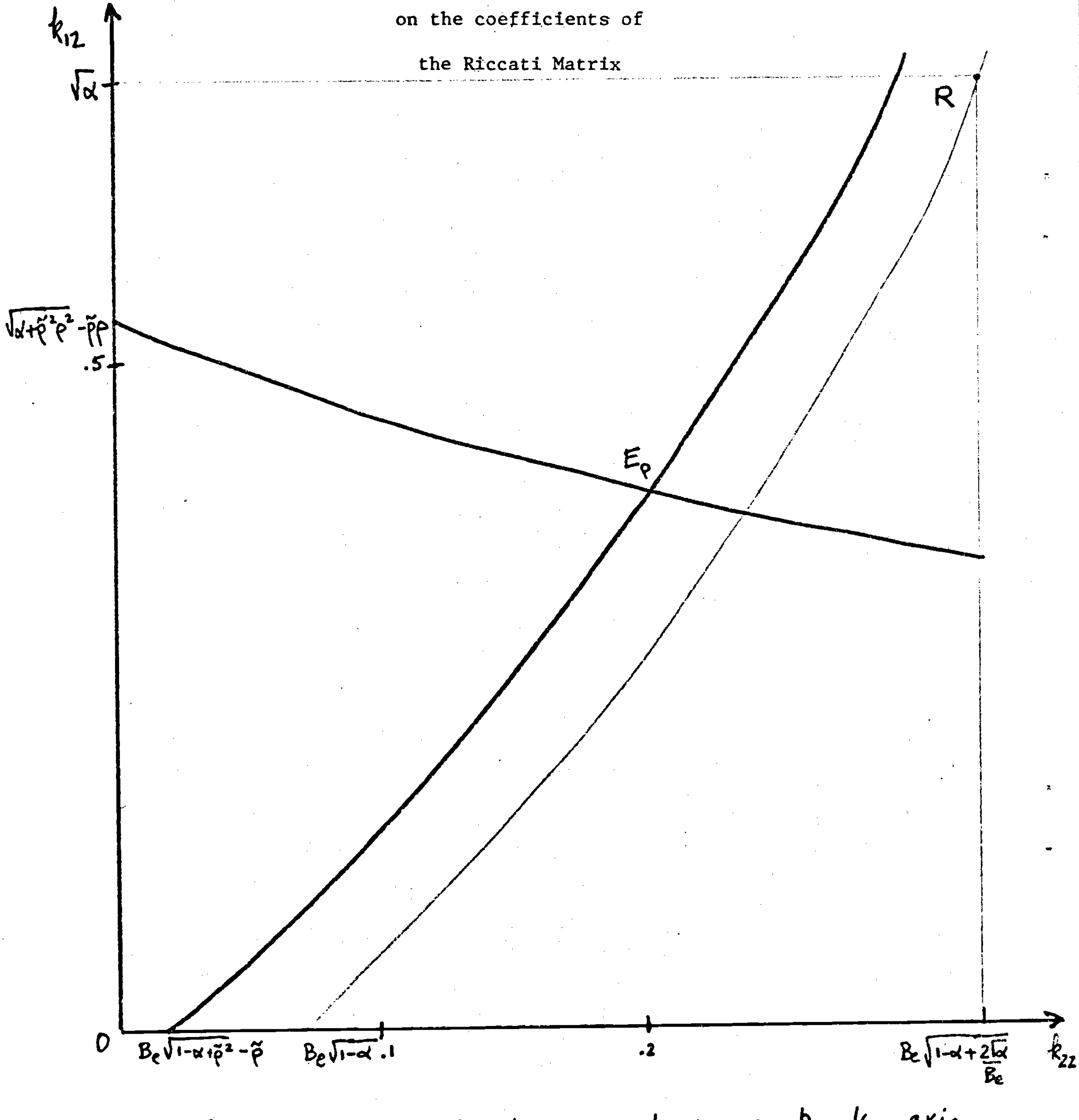
B_e \ α	0		.5		1	
	θ	γ	θ	γ	θ	γ
.1	.41	1	2.51	.84	3.34	.81
1	.91	1	1.78	.67	2.07	.58
10	.99	1	1.35	.47	1.38	.31
100	.99	1	1.35	.47	1.13	.12

Figure 1

The effect of discounting

on the coefficients of

the Riccati Matrix



Notes :- the k_{22} axis is a scale twice as large as the k_{12} axis

- $\tilde{\rho} = \rho/2B_e$

is comparable to the θ column of Table 2 above. The weights γ are however closer to those of Table 2 for lower values of B_e. As expected, discounting reduces the sensitivity of the rate of crawl to the indicators but this reduction is more than offset by a large value of the Marshall-Lerner condition so that when $B_e = 100$ (and $\alpha = .5$), $\gamma = .47$ rather than .50 as in Table 2.

IV Optimal Adjustment in a Stochastic Framework

To analyze the problem within a stochastic framework, we will modify the model given by equations (1) and (2) above. While we still assume that the exchange rate can be controlled exactly, we introduce an additive current account disturbance ω_2. We also introduce uncertainty regarding the effect of the exchange rate on the current account. This "state-dependent noise" is modelled as an additive disturbance ω_1 to B_e, possibly correlated with ω_2. If we assume further that ω_1 and ω_2 are Brownian motion with $\sigma_1^2\,dt$ and $\sigma_2^2\,dt$ as variances of their respective increments, the change in reserves can be approximated around equilitrium by a linear stochastic differential equation of the form

(34) $$dR = (B_e dt + du_1)\,(e - e^*) + du_2 .$$

The state representation of our system becomes

(35) $$dz = (Az + Bu) + S_1 z dt + S_2 \underset{\sim}{1}\, dt,$$

where $$S_i = \begin{bmatrix} \sigma_i & o \\ o & o \end{bmatrix}; \quad i = 1, 2 ;$$

$$\underset{\sim}{1}' = [1 \quad 1] .$$

We now wish to minimize

$$(36) \qquad E_o \int_o^T \frac{1}{2} (z'\theta z + u'u)\, dt ,$$

where the expectation E_o is taken conditional on the steady state of the system, z = 0,

subject to (35) and to

$$z(0) = z_o.$$

Define 1/

$$(37) \qquad J(z,t) = \min_u E_t \int_t^T \frac{1}{2} (z'\theta z + u'u)\, ds ,$$

and

$$(38) \qquad \Phi (u,z,t) = \frac{1}{2}(z'\theta z + u'u) + \mathcal{L}(J) ,$$

where $\mathcal{L}$ is the Dynkin operator.

By Ito's Lemma we find

$$dJ = J_t + \sum_{i=1}^{2} J_{z_i} dz_i + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} J_{z_i z_j} dz_i\, dz_j .$$

By definition

$$(39) \qquad \mathcal{L}(J) = \frac{1}{dt} E_t (dJ)$$

so that

$$(40) \qquad \mathcal{L}(J) = J_t + J_z'(Az+Bu) + \frac{1}{2} (z'S_1' + 1'S_2') J (S_1 z + S_2 1)$$

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[1] See a similar derivation in Macedo (1979), Appendix 1. Also Chow (1979).

By Bellman's theorem we know that there exists a control rule $\tilde{u}$ such that , from (38),

(41) $$\Phi\ (\tilde{u},\tilde{z},t) = 0.$$

Given (41), we minimize (38) with respect to u and obtain the control rule

(42) $$\tilde{u} = -\ B'J_z$$

Substituting (42) into (40), we can write the optimal value of Φ as

(43) $$\frac{1}{2}z'\theta z - \frac{1}{2}J_z'\ BB'J_z + J'_z Az + \frac{1}{2}z'S_1'J_{zz}S_1 z + \underset{\sim}{1}'S_2'J_{zz}S_1 z$$

$$+ \frac{1}{2}\underset{\sim}{1}'S_2'J_{zz}S_2\underset{\sim}{1} + J_t = 0$$

To evaluate the partial derivatives of J consider terminal loss from time zero to time zero + Δ.[1] This can be written as

(44) $$J_T(z_o,t_\theta) = J_z(z_o,t_o+\Delta) + E^{\min}_{u} \int_{T-\Delta}^{T} \frac{1}{2}(z'Qz + u'u)dt$$

Make T very large so that the expected value of the integral approaches the steady state value L. Then, dropping T subscripts, we obtain

(45) $$J(z,t) = J(z,t+\Delta) + \Delta L\ ,$$

so that

(46) $$J_t = -L\ .$$

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[1] See Chang-Sketler (1976) for a similar derivation.

Consider now J as a polynominal in z. such as

(47) $$J = z'Kz + kz + c,$$

where $k = (k_1\ k_2)$,

(48a) so that $J_z = Kz + k'$, and

(48b) $$J_{zz} = K$$

Substituting into (43) and collecting terms we have

(49) $$0 = \frac{1}{2} z' [KA + A'K - KBB'K + S_1'KS_1 + Q] z$$

$$+ [k'A - k'BB'k + \underset{\sim}{1}'S_2'KS_1] z$$

$$-L - \frac{1}{2} kBB'k + \frac{1}{2} \underset{\sim}{1}'S_2'KS_2\underset{\sim}{1}$$

The terms in brackets are equations for $\frac{dK}{dt}$, $\frac{dk}{dt}$ and $\frac{dc}{dt}$, which for sufficiently large T have approximate solutions

(50) $$KA + A'K - KBB'K + S_1'KS_1 + Q = 0 ;$$

(51) $$k'A - k'BB'k + \underset{\sim}{1}' S_2'S_1 = 0 ;$$

(52) $$\frac{1}{2} \underset{\sim}{1}'S_2'KS_2\underset{\sim}{1} - \frac{1}{2} kBB'k' = L .$$

From (50) we find the equations for the elements of the Riccati matrix as

$$k_{12}^2 - k_{11}\sigma_1^2 = \alpha ;$$

$$k_{12}\, k_{22} = k_{11}\, B_e ;$$

$$k_{22}^2 - 2 \cdot k_{12} B_e = (1-\alpha) B_e^2 .$$

Eliminating k_{11}, we find that the first and third equations define a hyperbola and parabola respectively in k_{12}, k_{22} space, just as in equations (33) above. [1] Now the parabola is the same as the third equation in (21) above whereas the hyperbola is upward sloping; their intersection E_σ is to the north east of point R, as shown in Figure 2 for $\alpha = .5$ and $B_e = \sigma_1 = 1$. The larger σ_1, the further away will E_σ be from R, in the same way that a larger ρ brought E_ρ closer to the origin and away from R. [2]

Given the elements of K, we solve for k in (51) to find

$$k_1 = \frac{k_{22}^2}{B_e^2} \sigma_{12} ;$$

(53)

$$k_2 = \frac{k_{22}}{B_e} \sigma_{12} .$$

Even though we minimized loss conditional on z = 0, the variance terms make it non zero in the steady-state, as can be seen by solving for the value of the loss function in (52):

(54) $$L = \frac{1}{2} \sigma_2^2 \frac{k_{12} k_{22}}{B_e} - \frac{1}{2} \frac{k_{22}^2}{B_e^2} \sigma_{12}^2$$

Note that if the two disturbances are uncorrelated the optimal control cannot reduce the loss and that the zero loss optimal weights are independent of σ_2.

Now using (42) and (48a) we find the optimal control rule to be

(55) $$\tilde{u} = - B'(Kz + k')$$

---

[1] If we were discounting and $\sigma_1^2 > \rho$ the interaction would be on the hyperbola to the left of E, (on the parabola cutting the k_{22} axis at $B_e(\sqrt{1-\alpha + \tilde{\rho}^2} - \tilde{\rho})$.

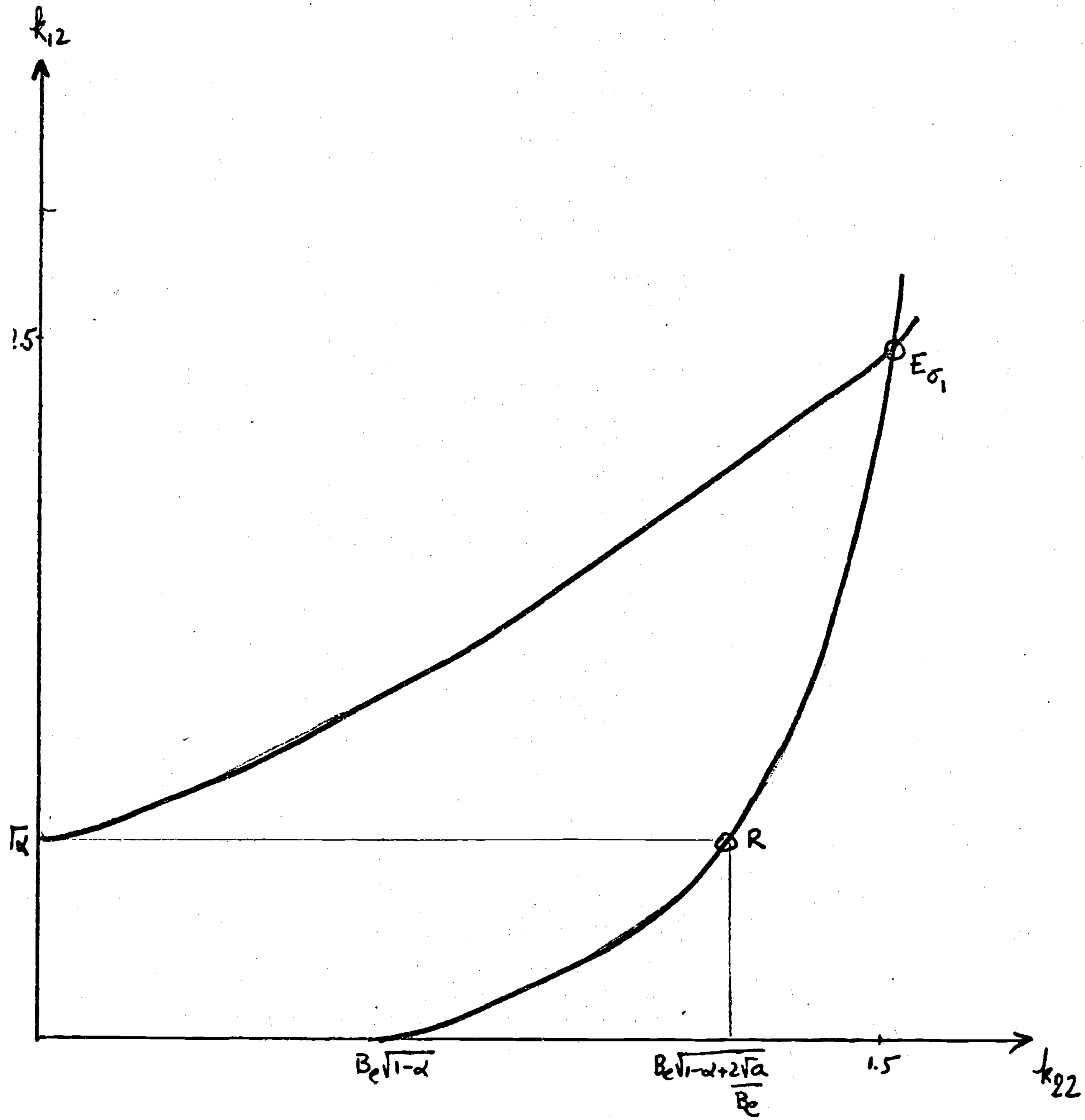
[2] The similarity between state dependent noise and a "negative discount" has been pointed out by Turnovsky (1973). Note however the difference in this model between (32) and (50) and the difference in the magnitudes of the parameters discussed below in the text.

Figure 2

The effect of state-dependent

noise on the coefficients

of the Riccati matrix



Note: scale proportions as in Figure 1

Thus, as expected from the separation theorem, the rule is the same as in the deterministic case when the two disturbances are uncorrelated, so that the forcing term k' is zero. Using (53) we can express the rule in terms of the k_{12} and k_{22} coefficients, or in the θ, γ notation of (24) as:

$$(56) \qquad \dot{e} = -\theta \ [\ \gamma\ (B(e) + \sigma_{12}) + (1-\gamma)\ (R-R^*]\ .$$

where $\theta = k_{12} + k_{22}/B_e$;

$1-\gamma = k_{12}/\theta.$

Some values of θ and γ for the usual values of B_e and α and three values of σ_1, are shown in Table 5. The optimal rate of crawl depends, in addition, on the covariance term σ_{12}. If the additive and state dependent disturbances are negatively correlated, the optimal rate of crawl is less than if they are uncorrelated. However, in (54), loss only depends on σ_1^2 and the square of the correlation coefficient, so that the sign of the covariance has no effect on loss.

Aside from the effect of the covariance, which is not included in Table 5, the interpretation is similar to Tables 2 and 4 above. In fact, we notice that the offsetting of the overall sensitivity θ by the size of B_e, which was pointed out in connection with Table 4, holds for the stochastic case. The strong effect of the variance term of the state noise is also evident from the table. Indeed, when the standard deviation of the state-dependent noise is 2, the optimum weight on the current account is 0.2 for virtually all of the values of α and B_e that have been used. The exception is the combination of a pure reserve target and "infinite" elasticities ($\alpha = 1$ and $B_e = 100$). In that case the table shows a drop in the weight on the current account from 0.2 to 0.11. In the case of a pure flow target ($\alpha = 0$) we find that, just as in Tables 2 and 4, the optimum current account weight does not depend on B_e. The weight is not unity, however, as in the deterministic case. Also, the sensitivity parameter θ declines with increases in B_e whereas it increased with B_e in the discount case of Table 4. Thus, the current account weight can be as low as 0.2 when variance is large, and is only 0.91 when variance is 0.1.

In the equal weight case (α = .5), low variance yields optimal weights that are very close to the ones obtained in the deterministic analysis. For large B_e, in fact, these weights are close to the no discount case of Table 2. For example, when B_e = 10, γ = .53 in Table 2 and γ = .47 in Table 4. When B_e = 100 γ remains unchanged in the discount case and it is equal to 0.5 in the no discount and low variance stochastic cases.

When variance is unity, however, the current account weight drops substantially and the range is reduced from .82 - .50 to .51 - .39. As mentioned, the uniform value for a variance of 4 is 0.2. As γ varies less, the range of the sensitivity parameter θ increases substantially with the variance. Thus, when σ_1^2 = .1 θ has a range of the same order of magnitude as in Table 2. (5.35 to 1.45 vs. 4.54 to 1.43) whilst in the high variance case the range is 400 to 4. It should be pointed out that the effects of state-dependent noise on the Kenen-Phillips formula might be less drastic when control-dependent noise is incorporated in the analysis, particularly if it is inversely correlated with state-dependent noise.

V. Summary and Conclusions

The numerical findings are from section II - IV are summarized in tables 6 and 7 where the values of the current account weight γ, the sensitivity coefficient θ, and the coefficient of the current account in the optimal rule (=$\gamma\theta$) are listed α = .5 and for B_e = 1 and B_e = 100 respectively. The implied lower bound is obtained by dividing .8 by θ and permits comparison of the weights for a given change in the exchange rate, this is subject to the proviso

Table 5: Alternate Values of θ and γ for

Different Values of σ_1.

	α = 0		α = .5		α = 1	
	θ	γ	θ	γ	θ	γ
$\sigma_1^2 = .1$						
$B_e = .1$	2.66	.91	5.35	.82	6.30	.79
1	1.22	.91	2.23	.65	2.50	.58
10	1.11	.91	.56	.52	1.51	.31
100	1.10	.91	1.45	.50	1.20	.12
$\sigma_1^2 = 1$						
$B_e = .1$	40.10	.50	40.55	.51	40.60	.51
1	4.82	.50	4.80	.48	5.02	.46
10	2.20	.50	2.14	.41	1.75	.29
100	2.02	.50	1.87	.39	1.20	.12
$\sigma_1^2 = 4$						
$B_e = .1$	400.06	.20	400.00	.20	400.00	.20
1	40.61	.20	40.35	.20	40.07	.20
10	7.39	.20	6.23	.20	4.56	.19
100	5.20	.20	4.00	.19	1.38	.11

that, in the stochastic case, the change in the exchange rate would be larger or smaller depending on whether the state dependent noise is positively or negatively correlated with the additive disturbance. If the variances are equal, in fact, this might mean a difference as high as ± 4 multiplying the coefficient on B in the tables.

The tables bring out the results that have been emphasized earlier. They can be summarized as four points.

First, low variance (.1 in the tables) and discounting bracket rather tightly the deterministic no discount case. For $B_e = 1$ the range is 0.65 to 0.67 (no discount $\gamma = .66$) and for $B_e = 100$ the range is 0.47 to 0.50 (no discount $\gamma = .50$). The ranges of the implied lower bound are, respectively, 0.36 to 0.38 and 0.124 to 0.133. Second, as mentioned above, the effect of discounting in reducing the sensitivity of the rate of crawl to the indicators is more than offset by large values of B_e, so that in Table 6 γ with discount is higher than γ no discount and γ low variance and in Table 7 γ with discount is lower than the other two. Third, the finding discussed above that the implied lower bound in the deterministic as discount case of section III is less than half the lower bound of section II, the same holding for the low variance and discounting cases, as shown in Table 6, is eliminated by a large B_e, as evident from Table 7. There the values are 0.18 for the lower bound of section II and .13 for the deterministic no discount case. Fourth, and perhaps most important, both tables show again that the effect of a large variance – $\sigma_1^2 = 1$ or larger – are quite strong. When $\sigma_1^2 = 4$ the lower bound goes to .02 in Table 6 and to .05 in Table 7, while θ increases to 40 and 4 respectively.

This paper has shown that the Kenen-Phillips formula for the optimal weighting of indicators in a crawling peg is obtained in the various situations of sections II – IV, given the simplest model of the economy and a drastic

Table 6

Values of Adjustment Parameters for $\alpha = .5$; $B_e = 1$

Section	Description	γ	θ	coefficient of B	implied lower bound
II	Lower bound	.83	1		
III	Deterministic no discount	.66	2.09	1.38	.38
III	Deterministic discount $\rho = .2$	.67	1.78	1.19	.45
IV	Stochastic				
	$\sigma_1^2 = .1$	.65	2.23	1.44	.36
	$\sigma_1^2 = 1$	.48	4.80	2.30	.17
	$\sigma_1^2 = 4$	.20	40.35	8.07	.02

Table 7

Values of Adjustment Parameters

for $\alpha = .5$; $B_e = 100$.

Section	Description	γ	θ	coefficient of B	implied lower bound
II	Lower bound	.18	1		
III	Deterministic $\rho = 0$ no discount	.50	1.43	.715	.126
III	Deterministic discount $\rho = .2$	.47	1.35	.635	.133
IV	Stochastic $\sigma_1^2 = .1$	.50	1.45	.725	.124
	$\sigma_1^2 = 1$	.39	1.87	.729	.096
	$\sigma_1^2 = 4$	.19	4.00	.760	.045

definition of the loss function in terms of external balance only.

In all cases the optimal formula is a weighted combination of targets with an additional parameter for the desired speed of adjustment. With a low variance a reasonable range for the current account weight seems to be 0.65 to 0.67 when $B_e = 1$ and 0.47 to 0.50 when $B_e = 100$. The optimal speed of adjustment θ is very sensitive to the estimated value of B_e and σ_1. Thus while we have shown that the optimal indicator is in general a weighted combination of the flow and stock targets, the numerical results suggest that quantitative choice of a formula will require careful econometric estimation in each particular case.

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