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AN INDEX OF INEQUALITY: WITH APPLICATIONS TO HORIZONTAL EQUITY AND SOCIAL MOBILITY

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ABSTRACT

An index of inequality is constructed which decomposes into two components, corresponding to vertical and "horizontal" equity respectively. Horizontal equity is defined in terms of changes in the ordering of a distribution. The proposed index is a function to two inequality aversion parameters. One empirical application is for comparison of a pre-tax distribution with a post-tax distribution, and an example of this is given for the distribution of incomes in the UK in 1977. There is a trade-off between "horizontal" and vertical equity, and for particular combinations of the inequality aversion parameters the original distribution will be preferred to the final distribution. The paper concludes with an application of the proposed index to a model of optimal taxation.

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1. Introduction

It is conventional to assess the merits of alternative public policies in terms of a trade-off between equity and efficiency. In practice, however, a change in, say, the tax system involves three effects. First, it may have incentive or disincentive effects leading to efficiency gains or losses. Secondly, it may alter the distribution of welfare levels. Thirdly, it may alter the ranking of individuals (or households) within the distribution. These three effects correspond to efficiency, vertical equity, and (certain aspects of) "horizontal equity" respectively, and any assessment of a tax change must take into account all three. The principal assumption of this paper is that the government is concerned about the trade-off between these three effects.

The introduction of changes in the ranking into the evaluation of a distribution is an example of how non-utility considerations may enter social rankings (See Sen (1979) and Pattanaik (1980)). Abandoning "welfarism", to use Sen's terminology, enables us

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to take account of the status which policy-makers may wish to give to the pre-tax or pre-reform distribution. We shall proceed on the assumption that the relevant consideration is the effect of a policy change on the ranking of individuals in the distribution. The relationship between this concept and conventional notions of horizontal equity and social mobility is discussed below. Only certain aspects of horizontal equity as discussed in the literature, can be captured by a measure of the change in ranking of utilities, but we shall argue that in these cases an index may be constructed which has empirical relevance in describing the effects of a proposed reform to policy-makers.

Indexes of vertical equity (or "inequality") abound, whereas there is no widely used index of horizontal equity. But, as Musgrave (1959, p.160) has commented in his discussion of the two principles of vertical and horizontal equity, "an objective index of inequality is needed to translate either principle into a specific tax system". The use of an "objective" statistical index to measure the concept of vertical inequality is open to well-known objections, and has led to the adoption of normative measures (see, for example, Atkinson 1970, Cowell 1977 and Sen 1973). Similar arguments apply also to the measurement of horizontal inequity. Nevertheless, we shall show that if we stipulate certain requirements for an index of horizontal equity, there exists a unique index characterised by a parameter which describes the degree of "aversion to horizontal inequity". The index of horizontal inequity may be combined with an index of vertical inequity to give an overall measure of inequality, and in turn this can be compared with the efficiency gains or losses of a particular reform. We shall show also that the same approach yields an index of social mobility which appears to overcome many of the problems which have been encountered in previous attempts to produce an index.

In this way we hope that the proposed index will supplement the usual analysis of tax reforms in terms of their effects on the level and distribution of welfare, (as illustrated by Rosen (1976)), by allowing considerations of horizontal equity to be taken into account explicitly. The aim is to develop a theoretical measure which will provide a useful extension to the empirical analysis of tax changes. When evaluating a proposal, politicians are often interested not only in the average gain which will accrue to a particular income group, but also in the distribution of gains and losses within the group. Their concern might take the rather crude form of wishing to know simply the numbers of those who would gain and of those who would lose from the proposal in question. But, more generally, the social valuation of the <u>ex post</u> distribution will take into account the <u>ex-ante</u> distribution from which it is derived, and, in particular, any change in the ordering of the distribution which is induced by the move. The idea behind this is the following. Imagine a reform which leaves average income unchanged, but which leads to a compression of the distribution such that the Lorenz curve for the <u>ex post</u> distribution lies entirely within that for the <u>ex-ante</u> distribution. If we are concerned with vertical equity alone then there has been an unambiguous reduction in inequality, and the reform would be favoured.[1] But suppose that the reform involves also a permutation of the order of individuals in the distribution giving rise to "horizontal inequities". The evaluation of the reform may no longer be so clearcut.

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[1]Given the assumption that social welfare (as a function of individual incomes) is S-concave.

The distinction between the effect of a tax on the shape of the distribution and its effect on the ranking of the distribution is illustrated by the concept of "progression". The conventional definition of a progessive tax (on income, for example) is that the proportion of an individual's pre-tax income which is paid in tax is an increasing function of his income. But we could equally well define progression with respect to the tax paid as a proportion of post-tax income. The difference between the two is simply the difference between tax-inclusive and tax-exclusive rates of tax. Provided there is no change in the ranking of the distribution the two measures will give the same answer and a tax which is progressive (regressive) in terms of the pre-tax distribution will be progressive (regressive) also if measured in terms of the post-tax distribution. But when there is a change in the ordering the two measures give conflicting answers. This is illustrated in Table 1.

Table 1

		Income		
		Before Tax	After Tax A	After Tax B
Individual	1	10	5	4
	2	6	4	5

It shows the incomes of two individuals before tax and also after tax for two taxes A and B. The effects on the distribution of income of the two taxes are identical. Tax A is clearly progressive whether we measure progression in terms of pre-tax or of post-tax income. This is not, however, true of tax B. In terms of pre-tax incomes, tax B is certainly progressive with tax rates of 16.7% on the poorer and 60% on the richer individuals. But as a proportion of disposable income the tax weighs more heavily on the individual with the smaller post-tax income, and in this sense is clearly regressive. This conflict in the measure of progression arises because the tax changes the ordering of the distribution. The ambiguity results from a tax change which reduces vertical inequity on the one hand but creates horizontal inequity on the other. The trade-off between the two will depend upon social judgements about the two types of inequity, and one of the aims of our proposed index of horizontal equity is to help make these judgements explicit. If the policy-maker is concerned not only with the post-tax distribution of incomes, but more generally with "what is, what was, and what might have been", then he will attach some significance to the initial distribution. The process by which redistribution is brought about is relevant, and I would venture the empirical statement that politicians would like their economic advisers to provide them with information about the effects of a policy on the change in ranking of households, in addition to the usual summary statistics of the ex post distribution.

Changes in ordering may occur for a variety of reasons. Some may be deliberate on the part of government. But if we assume that those characteristics which the government wishes to recognise as warranting differential treatment (such as household size or state of health) are subsumed into the utility function, then changes in the ordering of utilities will represent undesired horizontal inequities.[1]

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[1] The term "utility function" means here the social valuation of the vector of arguments of the function for a given household.

A government may wish to pay little or no attention to these changes in ordering or it may be much concerned by them; the social valuation of these changes is an ethical judgement.

One example of a tax-induced change in ordering is provided by the effects of an income tax in a world of heterogenous preferences. Individuals with a relatively strong preference for leisure will be favoured at the expense of those with a greater preference for consumption. Administrative constraints on the amount of information which can be incorporated in the design of the tax system may also lead to horizontal inequities. Regional variations in the price level are ignored by a national tax system. Housing allowances, based on rents paid are unable to discriminate between households which face high unit costs for housing and households which choose to consume a lot of housing services. This factor is likely to be taken into account by a government contemplating the introduction of a scheme of housing allowances, and will be set against any benefits in terms of efficiency or vertical equity which the scheme offers.

Attempts to remove or reduce the privileged tax treatment of some marketable assets (such as owner-occupied housing) run up against the problem that existing distortions have been capitalised. Proposals for a gradual transition reflect concern about the horizontal inequities which would result from immediate withdrawal of the tax concessions. Finally, the administration of the tax system and the use of random audits result in erors in tax payments by some individuals. Moreover, the sheer complexity of the tax system may lead taxpayers themselves to make mistakes in their labour supply and consumption decisions.

The empirical significance of an index of horizontal equity is enhanced by the growing exploitation of micro-data files with observations on individual households. With individual observations we are able to compare the ranking of the distribution both before and after any particular change in taxes or benefits. Simulation of tax reforms can encompass horizontal as well as vertical equity. The change in ranking involved in moving from pre-tax to the post-tax distribution has been discussed by Atkinson (1980) who examines the effects of re-ranking on conventional measures of inequality such as the Gini coefficient. But he does not consider the welfare significance of mobility as such. In this paper we focus on the evaluation of changes in ranking.

Section 2 discusses further the concept of horizontal equity and an index is constructed in section 3. In section 4 we show how the index may be modified to yield an index of social mobility. An illustration of the use of the index is given in section 5, and section 6 discusses theoretical applications of the index to models in which policy changes involve a trade-off between vertical and horizontal equity. A simple example of the use of the index in models of optimal taxation is examined.

2. Horizontal Equity

There is a widespread belief that a tax system should meet the double criteria of vertical and horizontal equity (see, for example, Musgrave (1959, 1976)). But the concept of horizontal equity has played little role in the literature on optimal taxation.[1] Undoubtedly, the main reason for this is the difficulty of giving precise expression to the concept of equal treatment of those equally situated. The most useful definition of horizontal equity is that of Feldstein (1976), "if two individuals would be equally well off (have the same utility level) in the absence of taxation, they should also be equally well off if there is a tax".[2] The extension of this definition to the case of tax reform (a comparison of two positive levels of taxation) is straightforward[3]. In practice, of course, no two individuals are ever identical, and to deal with this problem we are led naturally to a comparison of the ordering of utility levels before and after a tax change. Horizontal equity implies that a tax should leave unchanged the ranking of utility levels.[4]

Horizontal equity and vertical equity are distinct, though related criteria. They are related in so far as a discussion of vertical equity cannot proceed without some assumptions about which characteristics should be incorporated directly into the measure of utility - for example, if utility is a function only of income we would normally wish to compare household incomes adjusted to "adult equivalent"levels. But more relevant for our purposes is the possible clash between horizontal and vertical equity. Pursuit of one criterion may lead to violation of the other. Atkinson and

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[1]Exceptions to this are Atkinson and Stiglitz (1976) and Stern (1979)

[2]Horizontal equity may be defined also in terms of equality of budget sets (Gordon 1976), but it is not clear that this approach, in terms of opportunities rather than preferences, is sufficiently comprehensive as the concept of differing "needs" illustrates.

[3]See Feldstein op. cit. p.95.

[4]This is also stated by Feldstein (op. cit. p.83).

Stiglitz (1980) show that even where individuals possess identical preferences and endowments, maximisation of a social welfare function expressing social judgements about vertical equity does not necessarily imply equal tax rates for identical individuals. If the feasible set is non-convex equal tax rates may characterise a local minimum of welfare, and the social optimum will involve random taxes. Once differences in preferences and endowments are admitted the conflict between horizontal and vertical equity is readily apparent. An income tax discriminates between those with different tastes for leisure and between those with different skills (which, for the same money income, may entail different working conditions and job satisfaction).

But perhaps the most striking theoretical example of the conflict between vertical and horizontal equity arises in the model used by Mirrlees (1971) to examine optimal income taxation. In that model individuals had identical preferences (defined over consumption and leisure) and differed only in respect of their potential wage rates or ability levels. Clearly, in the absence of taxation individual utility is an increasing function of ability. Suppose the government uses redistributive lump-sum taxes to achieve the first-best optimum. Then, as Mirrlees (1971, 1977) shows, the first-best optimum for a utilitarian social welfare function has the property that individual utility is a decreasing function of ability (provided leisure is a normal good). In other words, the optimum from the point of view of vertical equity is characterised by a complete reversal of the ranking of utilities. Some attention has been paid to the incentive problems which result from such an outcome, because it would not be difficult for individuals to pretend to have less ability than they in fact have.[1] Where this is possible and individuals differ in respect of only one type of ability, then the optimum is equal utility for all (see Allingham (1975) and Mirrlees (1977)).[2] This solution is identical to the optimum for the case in which ability may be observed but where there is complete aversion to horizontal inequity. In general, however, the optimum tax on ability will involve a trade-off between vertical and horizontal equity. The possiblity of a trade-off of this kind has been ignored in the literature on the model of differing abilities because it has usually been assumed that informational constraints permit the use of only one policy instrument, namely an income tax. Changes in the ordering of utilities do not arise with an income tax provided a uniform schedule applies to all individuals and the marginal tax rate does not exceed 100%. But alternative policy instruments (such as the use of lump-sum taxes when we can observe a variable which is highly, but not perfectly, correlated with ability) may well involve changes in ranking, and to evaluate these requires explicit value judgments about both horizontal and vertical equity. We shall illustrate below the role which these judgements play in the construction of an index of horizontal equity (section 3) and also in a simple model of optimal taxation (section 6).

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[1] See Mirrlees (1977), Dasgupta (1979)

[2] The equal utility outcome would also result from a "maxi-min" objective function.

3. An Index of Horizontal Equity

Consider a distribution of incomes $y_1, y_2, \ldots y_N$ ranked in increasing order of income. We measure the changes in ordering between the ex ante and ex post distributions by a scaled order statistic which we shall define as [1]

$$s_i = \frac{|r_i - \bar{r}_i|}{N - 1} \qquad (1)$$

where s_i is the scaled order statistic (SOS) for household i

r_i is the rank of household i in the ex post distribution

$\bar{r}_i$ is the rank of household i in the ex ante distribution

N is the number of households in the distribution

The SOS value for each household lies between zero and unity, these two extremes representing an unchanged order and a move from one end of the distribution to the other, respectively. The statistic is defined in the same way for a distribution of utilities, but for simplicity we shall assume that we are concerned only with the distribution of incomes. In this way our results may be easily compared with existing measures of vertical inequality. The approach, however, is general.

An "objective" index of horizontal inequity would be some function of the SOS values given in equation (1). One example would be Spearman's rank correlation coefficient. The use of such an index to compare changes in rankings, however, does not necessarily

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[1] Ties in rankings can be dealt with by the usual conventions.

correspond to any social welfare function, and it seems preferable to tackle this problem directly by constructing a normative index.

The aim is to rank distributions according to the following social welfare function

$$W = W(y_1, \dots y_N; s_1, \dots s_N) \quad (2)$$

We shall normally wish to assume that W is an increasing concave function of incomes, and a decreasing function of the SOS values.

We define first the concept of the uniform reduction in all incomes which, at the original ranking, produces the same level of social welfare as that produced by the actual distribution given the changes in ranking which take place. The unchanged ordering equivalent proportion of income, μ, is defined by

$$W(\mu y_1, \dots \mu y_N; 0, \dots 0) = W(y_1, \dots y_N; s_1, \dots s_N) \quad (3)$$

In other words, $(1 - \mu)$ measures the proportion of total income which, for a given distribution, we would be prepared to sacrifice to eliminate all changes in ranking. A natural index of horizontal inequity is therefore

$$I_H = 1 - \mu \quad (4)$$

We may define also an index of vertical inequity. Consider a compression of the distribution which maintains the ordering of individuals. In the limit the ordering is maintained but all incomes tend to a common value. Define $\hat{y}$ as the equally distributed level of income which is equivalent to the actual distribution where both are evaluated at the initial ranking

$$W(\hat{y}, \ldots \hat{y}; 0, \ldots 0) = W(y_1, \ldots y_N; 0, \ldots 0) \quad (5)$$

which gives the same level of welfare as the actual distribution at the initial ranking. Hence we may define the index of vertical inequity as

$$I_v = 1 - \frac{\hat{y}}{\bar{y}} \quad (6)$$

where $\bar{y}$ is the mean of the distribution

I_v is the proportion of total income which, for a given ranking, we would be prepared to sacrifice to eliminate all vertical inequality.

The index of overall inequality is defined in terms of the equally distributed originally ranked equivalent level of income, denoted by y^*, and defined by

$$W(y^*, \ldots y^*; 0, \ldots 0) = W(y_1, \ldots y_N; s_1, \ldots s_N) \quad (7)$$

The proportion of total income which we would be prepared to sacrifice to eliminate all vertical and horizontal inequities is given by the index of overall inequality.

$$I = 1 - \frac{y^*}{\bar{y}} \quad (8)$$

The assumptions that W is an increasing and concave function of income and a decreasing function of the scaled order statistic guarantee that the three index numbers lie between zero and unity. The relationship between them may be seen as follows. From (3) and (7) we have

$$W(\mu y_1, \ldots \mu y_N; 0, \ldots 0) = W(y^*, \ldots y^*; 0,\ldots 0) \qquad (9)$$

If we impose the assumption that the measure of vertical inequality is independent of the meanvalue of income, then from (5) and (6) it follows that

$$W(\mu y_1, \ldots \mu y_N; 0, \ldots 0) = W(\mu\hat{y}, \ldots \mu\hat{y}; 0, \ldots 0) \qquad (10)$$

We may,therefore,deduce that

$$y^* = \mu\hat{y} \qquad (11)$$

and

$$(1 - I) = (1 - I_H)(1 - I_V) \qquad (12)$$

This result shows how the index of overall inequality may be decomposed into its two component parts, the index of horizontal inequity and the index of vertical inquality. The only condition on W which is reauired for (12) to hold is that the measure of vertical inequality is independent of the mean of the distribution; a similar decomposition holds in the case of mean-dependence with $(1 - I_V)$ replaced by $m(\bar{y})$ $(1 - I_V)$ where m is some function of the mean income.[1] Existing measures of vertical inquality are based on the assumption of

[1]The reader will note that the decomposition result holds for <u>any</u> relevant characteristic denoted by s; and not just where s represents the SOS value.

mean-independence and we shall proceed in a like manner.[1]

To convert the theoretical definitions of inequality into indexes which can be empirically estimated for particular distributions, we must place some restrictions on the form of the function W. The first is that we shall assume an additively separable social welfare function.

$$W = \sum_{i=1}^{N} F(y_i, s_i) \tag{2'}$$

This is a strong assumption. Sen (1973), in particular, has discussed the restrictions implied by additive separability - namely, that the relative social valuation of the incomes of two individuals is independent of the levels of any other incomes. Nevertheless they enable us to construct an index of horizontal inequity which is both easily interpreted and comparable with normative measures of vertical equity (such as Atkinson's (1970) index) which are also based on the assumption of additive separability.[2]

The restrictions we shall place on the function F may be summarised in the following three conditions.

<u>Condition 1</u>

Let two individuals with incomes y_i and y_j, and ranks $\bar{r}_i$ and $\bar{r}_j$, exchange places in the distribution. Clearly, $s_i = s_j = s$, say. When horizontal equity is not an issue, an anonymity condition

[1]This point is discussed further in Sen (1973) pps. 60-1

[2] For a discussion of additively separable inequality measures see Cowell (1980). He discusses the decomposition of an index of inequality into indexes for separate groups which, although related to some concepts of horizontal equity, is rather different from our treatment in terms of changes in ordering.

will ensure that the social valuation of the two income levels is unchanged[1]. A condition of this type is obviously inappropriate here, but we replace it by the condition that the relative marginal social valuation of the income levels is unchanged. Formally

$$\frac{F_1(y_i, s)}{F_1(y_j, s)} = f(y_i, y_j) \text{ independent of } s \ \forall \ s, y_i, y_j \qquad (13)$$

where F_1 denotes the social marginal utility of income (the derivative of F w.r.t.y.)

The above implies that we may write F in the multiplicative form

$$F(y, s) = V(y) R(s) \qquad (14)$$

Note that this does not imply that the relative valuation of different incomes is independent of the change of ordering. Constancy of the relative valuation holds only if the individuals at these income levels have the same value of the scaled order statistic.

Condition 2

The second condition is that the degree of aversion to horizontal inequity may be represented by a single parameter, denoted by η, which is assumed to vary between zero (no aversion to changes in ordering) to infinity (complete aversion to changes in ordering).

[1] See, for example, Sen (1973) p.10.

We may, therefore, write R as

$$R = R(s, \eta) \tag{15}$$

The following restrictions imply no loss of generality. When there is no change in the ordering the social valuation of different income levels is given by the function V(y) and we may write

$$R(0, \eta) = 1 \qquad \forall \eta \tag{16}$$

Similarly, when no weight at all is attached to horizontal inequity, the valuation of the distribution disregards any re-ordering, and

$$R(s, 0) = 1 \qquad \forall s \tag{17}$$

Consider an individual with income y and scaled order statistic s. Suppose we now increase the change in his ranking by a small amount ds. The proportionate increase in his income which is equivalent (in terms of the social valuation) to the further displacement is (from (14) and (15))[1]

$$\frac{1}{y}\,\frac{dy}{ds} = -\,\frac{1}{k(y)}\cdot\frac{1}{R}\,\frac{dR}{ds} = f(s, y, \eta) \tag{18}$$

where k(y) is the elasticity of V with respect to y.

Integrating (18) we obtain

$$\log R = c - k(y)\, g(s, y, \eta) \tag{19}$$

[1] Strictly speaking, we may differentiate only when there is a continuum of individuals.

where c is the constant of integration, and g is the integral of f with respect to s. It is easy to show that (16) and (17) imply that c = 0. If we assume that k is a constant, which is equivalent to the property that our measure of vertical equity is mean-independent (Atkinson 1970), then since R is not a function of y (by condition 1), we have

$$R = e^{-kg(s,\eta)} \tag{20}$$

Consider now two individuals with incomes y_1 and y_2 and initial rankings $\bar{r}_1$ and $\bar{r}_2$. Imagine a reform which results in their exchanging positions. The cost of this reform may be measured in terms of the proportionate increase in the incomes of both individuals which would be necessary to compensate for the change in ordering. Denote the required increase by $(\lambda-1)$ and the SOS value, which is the same for both, by s. Then λ is defined by

$$V(\lambda y_1)R(s) + V(\lambda y_2)R(s) = V(y_1) + V(y_2) \tag{21}$$

With the constant elasticity form for V we have

$$\lambda^k R(s) = 1 \tag{22}$$

Imagine now that the reform had instead been implemented in two partial steps, the first of which implied an SOS value of s_1 for both individuals and the second further changed the ordering (in the same direction) with an SOS value of s_2. By construction [1]

$$s_1 + s_2 = s \tag{23}$$

The cost of each step may be evaluated as above which gives

$$\lambda_1^k R(s_1) = 1 \tag{24}$$

$$\lambda_2^k R(s_2) = 1 \tag{25}$$

Using (20) and taking logs we have

$$\begin{aligned} \log \lambda &= g(s,\eta) \\ \log \lambda_1 &= g(s_1, \eta) \\ \log \lambda_2 &= g(s_2, \eta) \end{aligned} \tag{26}$$

The final condition we shall impose is that the cost of the reform is independent of the number of stages in which it is implemented, provided that the steps converge monotonically to the final ordering.

<u>Condition 3</u>

$$\text{If } s = s_1 + s_2 \text{ then } \lambda = \lambda_1\lambda_2 \tag{27}$$

This means that when $s = s_1 + s_2$, $g(s,\eta) = g(s_1,\eta) + g(s_2,\eta)$ i.e. g is linear in s.[2] Since η represents social preferences and is constrained only to be positive we may normalise such that[3]

$$g(s,\eta) = \eta s \tag{28}$$

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[1] Note that (23) holds only when the various steps of a reform lead monotonically to the final change in ordering; when there is "overshooting" the equation does not hold.

[2] It is straightforward to generalise the proof to the case of many steps of arbitrary size provided always that the steps form a monotonic sequence for the ranking.

[3] Given that the normalisation must satisfy (17).

Hence

$$R(s, \eta) = e^{-k\eta s} \tag{29}$$

From (18) and (29) we see that this permits a natural interpretation of the parameter η measuring the degree of aversion to horizontal inequity as the proportionate increase in income required to compensate for a unit change in the ordering (measured by the SOS value).

The above results enable us to state the following theorem.

Theorem 1

Given conditions 1-3, there is a unique social valuation function F given by

$$F(y, s) = \frac{1}{k}\left(ye^{-\eta s}\right)^k \qquad \eta \geq 0, \; k \leq 1 \tag{30}$$

$$= \log y - \eta s, \text{when } k = 0$$

where k is the constant value of the elasticity of V w.r.t.y.

Atkinson (1970) has shown that if the index of vertical inequality is independent of the mean then V has the constant elasticity form (corresponding to constant relative risk aversion)

$$V(y) = \frac{y^k}{k} \qquad k \leq 1 \tag{31}$$

$$= \log y \qquad k = 0$$

Combining this with the results derived above gives the form (30). The economic explanation of the functional form (30) is the following. There is a close analogy between the standard analysis of risk aversion in consumer theory, and the aversion to vertical and horizontal inequality as expressed in the social valuation function.

Constant relative and absolute risk aversion imply special and well-known forms for individual utility functions, and this is true also for the social valuation function. The assumption of mean-independence for the index of vertical inequality is equivalent to the property of constant relative inequality aversion, and hence the social valuation function is of the constant elasticity form in income. The assumption about aversion to changes in ordering embodied in condition 3 is equivalent to the property of constant absolute re-ordering aversion, and hence the change in ordering enters the social valuation function in the exponential form. Although these are strong assumptions, they provide a useful starting point and benchmark for empirical work.

An implication of the result is that complete aversion to horizontal equity is equivalent to placing no value on the incomes of those individuals whose ordering in the distribution has changed. From (30) we see that

$$\lim_{\eta \to \infty} F(y, s) = F(0, s) \qquad \forall\ s > 0 \tag{32}$$

As the degree of aversion to horizontal inequity increases the social valuation of the incomes of all individuals whose ordering has changed decreases. In the limit a change in ordering is equivalent to reducing the incomes of those affected to zero. Complete aversion to horizontal inequity implies that a reform which alters the position in the ordering of every individual has the same effect on social welfare as wiping out the whole of national income.

Substituting (30) into equations (3)-(8) yields

Theorem 2

Given conditions 1-3, there are unique indexes of horizontal, vertical, and overall inequality (as a function of the two inequality aversion parameters) given by the following expressions. In addition, the index of overall inequality may be decomposed according to (12)

$$I_H = 1 - \left[\frac{\sum_i (y_i e^{-\eta s_i})^k}{\sum_i y_i^k} \right]^{\frac{1}{k}} \qquad k \neq 0 \qquad (33)$$

$$= 1 - \exp\left(\frac{-\eta}{N} \sum_i s_i \right) \qquad k = 0 \qquad (34)$$

$$I_v = 1 - \left[\frac{1}{N} \sum_i \left(\frac{y_i}{\bar{y}} \right)^k \right]^{\frac{1}{k}} \qquad k \neq 0 \qquad (35)$$

$$= 1 - \exp\left[\frac{1}{N} \sum_i \log \left(\frac{y_i}{\bar{y}} \right) \right] \qquad k = 0 \qquad (36)$$

$$I = 1 - \left[\frac{1}{N} \sum_i \left(\frac{y_i}{\bar{y}} \cdot e^{-\eta s_i} \right)^k \right]^{\frac{1}{k}} \qquad k \neq 0 \qquad (37)$$

$$= 1 - \exp\left(\frac{1}{N} \sum_i \left(\log \frac{y_i}{\bar{y}} - \eta s_i \right) \right) \qquad k = 0 \qquad (38)$$

It can be checked by inspection that these expressions satisfy (12). Not surprisingly, the index I is identical to the Atkinson index of inequality. Define $\varepsilon = 1-k$. Then the index of overall inequality is a function of two parameters, η and ε, both of which vary from zero to infinity, denoting the degrees of aversion to horizontal and vertical inequity respectively. When horizontal equity is a matter of no concern ($\eta = 0$) the index of overall inequality reduces to I_V; similarly, when distributional considerations (in the conventional sense of vertical equity) are disregarded ($\varepsilon = 0$) the index of overall inequality is simply I_H. In general, however, the index of inequality depends upon both horizontal and vertical equity. Given values of the parameters η and ε, and the ex ante and ex post distributions, it is a straightforward matter to compute the different measures of inequality.[1]

To calibrate the index of horizontal equity it may be helpful to take an example. Consider the case when $\varepsilon = 0$ (the utility function is logarithmic in income) and imagine a reform which completely reverses the original ranking. This corresponds exactly to the first-best lump-sum tax system in the Mirrlees (1971) model discussed above. A complete reversal of ranking of a continuum of individuals implies an average SOS value of one-half, and from (34) we see that the percentage of total income we are prepared to sacrifice to eliminate the reversal of the rankings is $1 - \exp\left(\frac{-\eta}{2}\right)$. This is equal to 4.8% when $\eta = 0.1$, 22.1% when $\eta = 0.5$, 39.3% when $\eta = 1$, and 63.2% when $\eta = 2$.

[1]The definitions of the indexes in the text are for discrete distributions; the conversion to continuous distributions is straightforward.

The value for η chosen to represent preferences about horizontal equity will depend upon the degree of attachment to the ranking of the initial distribution. For this reason it may well be that the value of η chosen will depend upon the reform being analysed.[1] In section 5 we shall illustrate the use of the equity indexes by reference to the post-tax income distribution in the UK, and we shall examine the sensitivity of the results to different assumptions about the value of η.

We may compare our index of horizontal inequity with alternative measures which have been suggested. These tend to concentrate on the unequal treatment of people assumed to be equal initially, and thus have the weakness that they are not directly concerned with changes in ordering as such. For example, Feldstein (1976) considers, and rejects, the use of the variance of post-reform utilities for a group of individuals with equal utility before the reform. Applied to a world in which everyone starts with a different level of utility this measure is completely uninformative.

Rosen (1978) has suggested measuring horizontal inequity in terms of increases in the differences between successive utility levels in the ordering brought about by tax reform. His concept of horizontal inequity is not confined to changes in ordering, which has some advantages, but it is then impossible to decompose the effects of a given tax reform into changes in vertical and horizontal equity (as in (12) above). Moreover, his index is not uniquely defined because it depends on an arbitrary randomisation

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[1] Similar considerations may apply to the value of ε. Feldstein (1976) has suggested the use of different values of the vertical inequality aversion parameter for different types of consumption.

of utility levels. Nevertheless, there may be some aspects of horizontal equity which are not captured simply by the change in ordering, in which case our index could not be equated with horizontal equity.

Johnson and Mayer (1962) suggest looking at the number of "inequalities" which could be interpreted here as the number of changes in ordering. The objection to this is the same as that to the use of other "objective" measures (e.g. the rank correlation coefficient), namely that it gives no indication of the magnitude of the change in terms of a measuring rod such as the fraction of national income we are prepared to sacrifice to eliminate the "inequities". This is precisely why we need a normative index based on an explicit statement of the preferences about horizontal equity. Only in this way is it possible to construct an index for empirical analysis of tax and other reforms which involve a trade-off between efficiency, vertical equity, and horizontal equity.

4. An Index of Social Mobility

In this section we shall try to show that the approach used above may help to overcome some of the problems which have been encountered in the literature on the measurement of social mobility. Shorrocks (1978) has shown very clearly that an index of mobility which satisfies the relatively innocuous conditions that it lie between zero and unity and be monotonic in the scaled order statistic is incompatible with an objective notion of perfect mobility.[1] The

[1] This is usually defined in terms of a transition matrix in which the probability of ending up in any income group is the same for all groups.

attempt to relax conditions on the index in order to reconcile the measure with some objective concept of mobility does not seem to have proved fruitful. Shorrocks concludes that "we may finally have to admit that no single mobility statistic has the minimum requirements regarded as essential". (*op.cit.* p.1023). An alternative approach is to construct a normative index along the lines pursued above. The only difference is that it is usual in the context of social mobility to favour changes in the ordering of the distribution. As before, we shall assume that distributions may be ranked according to the social welfare function (2) where we assume that F is now an increasing function of s. We further define the concept of the uniform increase in all incomes which, with zero mobility, produces the same level of social welfare as that produced by the actual distribution given the mobility which exists. The zero mobility equivalent proportion of income, ρ, is defined by

$$\sum_i F(\rho y_i, 0) = \sum_i F(y_i, s_i) \tag{39}$$

Given our assumptions $\rho \geqslant 1$. The proportion of total income which, from a position of zero mobility, we would be prepared to sacrifice in order to achieve the degree of mobility we observe is $\left(1 - \frac{1}{\rho}\right)$, and so a natural index of mobility is

$$I_M = 1 - \frac{1}{\rho} \tag{40}$$

We may write down an exact form for the index if we are prepared to impose further conditions on F. Parallel to conditions 1-3 above, we may define

(1') If two individuals in the distribution exchange places the relative marginal social valuation of the two income levels is unchanged.

(2') The desire for mobility may be represented by a single parameter, denoted by γ, which varies between zero (no preference for mobility) to infinity (complete aversion to immobility).

(3') The benefits of a reform which increases mobility are independent of the number of stages in which the reform is implemented, provided that no stage reverses a change in ordering brought about by a previous stage. (Note that this condition is not concerned with the speed at which a reform is implemented which, in general, would alter our valuation of changes in ordering. The social valuation function takes no account of the time dimension of reform). Since F is an increasing function of s, it is easy to show that conditions 1'-3' imply that (29) becomes

$$R(s, \gamma) = e^{k\gamma s} \tag{41}$$

We may now state

Theorem 3

Given conditions 1'-3' there is a unique index of mobility (as a function of the mobility preference parameter) given by

$$I_M = \begin{cases} 1 - \left[\dfrac{\sum\limits_i (y_i e^{\gamma s_i})^k}{\sum\limits_i y_i^k}\right]^{-\frac{1}{k}} & k \neq 0 \\[2ex] 1 - \exp\left(\dfrac{-\gamma}{N} \sum\limits_i s_i\right) & k = 0 \end{cases} \tag{42}$$

(42) shows that the index of mobility depends upon both the degree of desire for mobility and the degree of inequality aversion. Consequently, any index of mobility is based on some preferences (usually implicit) about vertical inequality. For example, Bartholomew's (1976) index of social mobility is proportional to the average change in SOS values

$$B = \frac{1}{N} \sum_i s_i \tag{43}$$

Hence when V(y) takes the logarithmic form (k = 0)

$$I_M = 1 - e^{-\gamma B} \tag{44}$$

and I_M is an increasing function of B.

In other words, the use of Bartholomew's index to rank distributions according to their degree of mobility, is equivalent to assuming a value for the inequality aversion parameter ε of unity. It is interesting to note that the extent to which Bartholomew's index is useful as a measure of mobility depends upon our attitudes towards vertical inequality.

The mobility index may be used to evaluate reforms which increase mobility at the expense of an increase in inequality. If we consider reforms which preserve the mean of the distribution then, by arguments parallel to those embodied in equations (9)-(12), it may be shown that social welfare is an increasing function of $(1 - I_V)/(1 - I_M)$. Consider a class of mean-preserving reforms characterised by a policy variable x such that the indexes are concave differentiable functions of x. The optimal policy is given by

$$E(M) = E(V) \tag{45}$$

where E(M) and E(V) are the elasticities of $(1 - I_M)$ and $(1 - I_V)$ respectively with respect to x.

5. An Empirical Application

The principal use of the index of horizontal equity is to examine the impact of tax reforms using large data files in which it is possible to compute the change in ranking for each of a large sample of individuals or households. To demonstrate the use of the index, however, we present some illustrative calculations for the distribution of household incomes in the UK. Table 2 shows the distribution of incomes by deciles for a sample of 7,198 households taken from the 1977 Family Expenditure Survey. Col. (1) shows the distribution of "original" income (defined as factor incomes plus cash transfers). Col. (2) shows the distribution of income after taxes and benefits (except for public expenditure on defence, law and order and public investment)[1].

From the available data it is impossible to calculate the changes in ranking of each household in the sample. But the Royal Commission on the Distribution of Income and Wealth had access to the same data and found that 41.5% of households in the distribution of original income had moved up or down one decile in the distribution of final income, and that 27.5% had moved by more than one decile.

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1 Details of the assumptions about incidence used to compute the distributions are given in Economic Trends January 1979.

Clearly, many of these changes were deliberate and represented the use of taxes and benefits to take account of differing "needs" such as household size and age. But others were not, and in Col. (3) we show a plausible representation of the change in ordering induced by the process of redistribution. Six of the decile groups have an unchanged order, and four move up or down one decile. Table 3 shows the index of horizontal inequality (defined by (33)) for the change in the distributions for different values of the two inequality aversion parameters. The parameters may be interpreted as follows. If we attach the same social value to a marginal dollar in the hands of someone with income y as to x dollars for someone with income λy, then $x = \lambda^{\varepsilon}$. For example, when $\varepsilon = 0.5$, one dollar taken from someone on twice average earnings has the same social value as 50 cents given to a person on one-half average earnings. The social value attached to the income, y, of a person who has changed position in the distribution is equal to the social value of an income $ye^{-\eta s}$. When $\eta = 0.5$ this is equivalent to a reduction in income of about 5% for a change in ordering of one decile, and when $\eta = 5$ the equivalent reduction is about 40%.

Table 4 shows the index of overall inequality for both the original and final distributions. The first two columns correspond also to the index of vertical inequality. It can be checked that the Lorenz curve for the final distribution lies entirely within that for the original distribution, and, therefore, as far as vertical inequality is concerned, the redistribution is unambiguously desirable. But for some values of the horizontal inequity aversion parameters the loss of horizontal equity offsets the gain in vertical equality, and the index of overall inequality is greater for the final distribution

TABLE 2 DISTRIBUTION OF HOUSEHOLD INCOMES, UK 1977

(£1 /annum)

DECILE	Original Income (including cash benefits)	After Taxes and Benefits	
		Income	Ranking in Initial Distribution
Bottom	1090	1141	2
2	1658	1674	1
3	2431	2166	3
4	3267	2668	4
5	4014	3198	5
6	4740	3709	6
7	5476	4253	8
8	6357	4914	7
9	7640	5867	9
10	11,343	8491	10
Mean	4802	3808	

Source Economic Trends January 1979, Tables 9 and 12.

TABLE 3 INDEX OF HORIZONTAL INEQUITY

		η		
		0.5	1.0	5.0
ε	0	.017	.033	.134
	0.5	.019	.038	.164
	1.0	.022	.044	.200
	2.0	.030	.059	.282

TABLE 4 INDEX OF OVERALL INEQUALITY

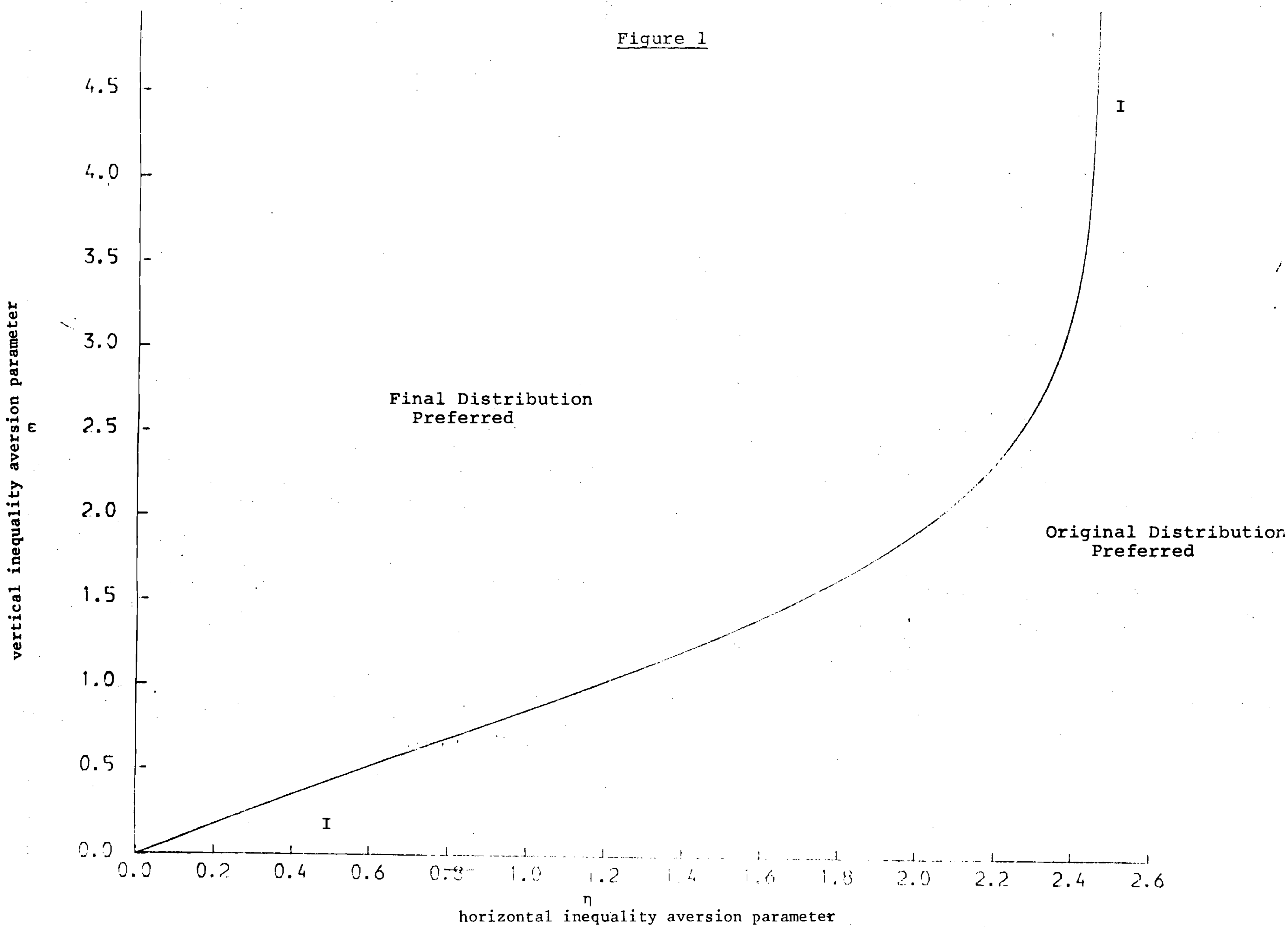
		Original Distribution	Final Distribution			
			η			
			0	0.5	1.0	5.0
ε	0	0	0	.017	.033	.134
	0.5	.093	.072	.090	.107	.224
	1.0	.187	.143	.162	.181	.314
	2.0	.361	.274	.296	.317	.479

than for the original distribution. This occurs, for example, when $\eta = 5$ for the values of ε shown in Table 4. If we ignore the efficiency costs of the process of redistribution, we may rank distributions according to the value of the index of overall inequality. From (37) we see that this means there is a trade-off between vertical and horizontal equity. The trade-off for the distribution of incomes in the UK is shown in Figure 1. For combinations of the inequity aversion parameters to the north-west of the indifference line II, the final distribution is preferred to the original distribution, whereas for combinations to the south-east the status quo is preferred. Diagrams such as figure 1 may be a useful way of presenting information about a proposed reform to policy-makers.

6. An Application to Optimal Taxation

Administrative errors are one source of horizontal inequities and we present below a highly simplified model to illustrate the application of the approach developed above. Stern (1979) has explored optimal lump-sum taxes in a model in which there was a fixed probability that indviduals (either skilled or unskilled) were incorrectly classified. He computed the optimal level of taxes for a utilitarian social welfare function. We shall consider a simple version of his model and examine explicitly the trade-off between horizontal and vertical equity. To do this we allow the probability of error to be endogenous.

Figure 1


vertical inequality aversion parameter
ε
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Final Distribution Preferred
Original Distribution Preferred
I
I
0.0
0.2
0.4
0.6
1.8
2.0
2.2
2.4
2.6
η
horizontal inequality aversion parameter

Consider a centrally planned economy in which there are two groups with fixed endowments y_1 and y_2 ($y_2 > y_1$; mean income = $\bar{y}$). social welfare is a concave function of individual consumption levels. The planner allocates consumption bundles directly (using local administrators) which can be seen as equivalent to imposing lump-sum taxes such that

$$\begin{aligned} c_1 &= y_1 + a \\ c_2 &= y_2 - a \end{aligned} \qquad a \geq 0 \tag{46}$$

When the planner can be sure that the local administrators make no mistakes he will choose taxes to give equal consumption levels. We now introduce administrative errors. Suppose that the planner knows the relative numbers of the two groups in each region and allocates consumption accordingly. The local administrators are told to allocate consumption c_i to people with income y_i. But they sometimes confuse the groups and with probability p make an error. We shall assume that p is an increasing function of a, perhaps because when the consumption bundles are very different the local administrators try hard not to make mistakes but they become careless when distributing roughly similar bundles. In general, the optimal tax will depend on the trade-off between vertical and horizontal equity.[1] There are certain special cases. When $\eta = 0$, the optimal outcome is complete equality for any positive ε, and when $\varepsilon = 0$ the solution is to impose no taxes if η is positive.

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[1]Because of the very simple structure of the model examined here, it seems less plausible that the planner would be concerned about changes in ordering as the consumption levels tend to equality, but the results are intended to be illustrative only.

Consider the case when $\varepsilon = 1$. The expected value of social welfare is

$$EW = (1 - p(a))[\log(y_1+a) + \log(y_2-a)] + p(a)[\log(y_1+a) + \log(y_2+a) - 2\eta]$$

$$= \log(y_1+a) + \log(y_2-a) - 2\eta p(a) \qquad (47)$$

The solution depends on the function p(a). A sufficient, though not necessary, condition for the first-order condition to yield a local maximum is that p be convex. Consider the simplest linear form where $p = 0$ when $a = 0$ and $p = \frac{1}{2}$ (i.e. random allocation) when $a = \bar{y} - y_1$.

$$p = \frac{a}{2(\bar{y}-y_1)} \qquad (48)$$

It is easy to show that the solution is the following

(i) if $\eta < \eta^*$, there is an interior solution given by

$$\eta(y_1+a)(y_2-a) = (y_2-y_1-2a)(\bar{y}-y_1) \qquad (49)$$

$$\eta^* = \frac{(\lambda-1)^2}{2\lambda} \quad ,\text{where } \lambda = y_2/y_1$$

(ii) if $\eta \geq \eta^*$, $a = 0$ and the optimal solution involves no taxes.

7. Conclusions

We have shown that if we are prepared to define vertical and horizontal inequity as a mean-preserving spread and a change in ranking respectively, it is possible to construct an index of overall inequality which decomposes into two measures of vertical and horizontal inequity. Provided certain conditions are accepted, these indexes are uniquely defined as a function of two inequity aversion parameters. It is hoped that the indexes will prove useful in the evaluation of tax proposals (to examine the trade-off between vertical and horizontal equity), and also as a measure of social mobility.

Horizontal equity is an elusive concept, and it is most unlikely that a single index will capture all the different interpretations which have been given to it. For example, it has been argued that little importance should be attached to a change in ordering per se, but that concern for horizontal equity restricts the use of certain tax instruments.[1] Our index is limited to those cases where changes in ordering do affect social valuations. It is based on strong assumptions, but the aim has been to construct an index which may be of use in empirical research, always bearing in mind Sen's (1973, p.76) stricture that "the glib man who can make inequality comparisons perfectly between every pair of distributions and the wise guy who finds all such comparisons 'arbitrary' both seem to miss essential aspects of the concept of inequality".

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[1]See Atkinson (1980) for a discussion of alternative views.

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