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INFLATION RISK AND CAPITAL MARKET EQUILIBRIUM

Zvi Bodie

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ABSTRACT

This paper investigates the effect of inflation uncertainty on the portfolio behavior of households and the equilibrium structure of capitol market rates.

The principal findings regarding portfolio behavior are:

1. In the presence of inflation uncertainty, households will have an inflation-hedging demand for assets other than riskless nominal bonds, which will be directly proportional to the covariance between the rate of inflation and the nominal rates of return on these other assets.

2. An asset is a perfect inflation hedge if and only if its nominal return is perfectly correlated with the rate of inflation.

The principal findings regarding capital market rates are:

1. The equilibrium real yield spread between any risky security and riskless nominal bonds is directly proportional to the difference between the covariance of the security's nominal rate of return with the market portfolio and its covariance with the rate of inflation.

2. As long as the net supply of monetary assets in the economy is greater than zero, an increase in inflation uncertainty will lower the risk premia on all real assets.

3. A preliminary empirical test of the theory using rates of return on common stocks, long-term bonds, real estate and commodity futures contracts yields mixed results. The risk premia on long-term bonds and futures have the "wrong" signs.

Zvi Bodie
School of Management
Boston University
Boston, MA 02215
617/353-4160

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Inflation Risk and Capital Market Equilibrium

I. Introduction

The purpose of this paper is to investigate the effect of inflation risk on the portfolio behavior of households and the equilibrium structure of capital market rates. In a previous paper (Bodie [2]) I defined inflation risk as the uncertainty associated with the real return on single-period, riskless-in-terms-of-default nominal bonds and explored the factors determining the effectiveness of a security as an inflation hedge. This paper attempts to integrate that earlier analysis into a more complete model of capital market equilibrium.

The theoretical model used is the one pioneered by Merton [15] and applied by Fischer [6] in his study of a hypothetical capital market in which a perfect inflation hedge exists in the form of index bonds. I employ the same methodology here to examine asset demands and equilibrium yields when the available inflation hedges are less than perfect. Fischer's results then turn out to be a special case, in which a perfect hedge is available.

In their paper on equilibrium yield relationships under uncertain inflation, Friend, Landskroner and Losq (FLL) [7] highlighted the crucial role played by the covariances between nominal asset returns and unanticipated inflation and showed that the Security Market Line of the traditional Sharpe-Lintner-Mossin Capital Asset Pricing Model (CAPM)[1] is in effect a special case of their model, in which all of these covariances are zero. In this paper

I extend the analysis of FLL in several directions. First I explore the special role of the minimum-variance of real return portfolio in the asset demand functions of households and show that the effectiveness of an asset as an inflation hedge is determined by the degree of correlation between its nominal rate of return and the rate of inflation. This leads to a discussion of a special "inflation-hedge" portfolio of nominally risky assets which is combined with riskless nominal bonds to create the minimum-variance portfolio. Then by expressing the asset demand functions in terms of the parameters of the distribution of real rather than nominal returns, I am able to examine the effect of an increase in inflation uncertainty both on these asset demands and on the equilibrium yield relationships derived from them.

The paper is structured as follows. In the next section I develop the basic model for the case of two assets. I then generalize it in Part III to the case of many assets to discuss the role of the "inflation-hedge" portfolio and derive three alternative formulas for an inflation-adjusted Security Market Line. Part III ends with a discussion of the relationship between the model developed here and Black's [1] model of capital market equilibrium in the absence of a riskless asset. In Part IV I discuss the available empirical evidence, and in Part V I summarize my findings and indicate some directions for future research.

II. The Two-Asset Case.

A. Asset Demands.

We assume an infinitely-lived household receiving all its income as a return on two tradable assets: bonds offering a riskless nominal return and an asset, which we shall call equity, whose nominal return is risky. We assume further that there are no transactions costs or restrictions on short sales. The household makes its portfolio choices by maximizing the expected utility of its consumption stream.

There is a single consumption good whose price, P, follows a geometric Brownian motion described by the stochastic differential equation:[2]

$$(1) \quad \frac{dP}{P} = \pi dt + s_{\pi} dz_{\pi}$$

where π is the instantaneous mean rate of inflation per unit time and s_{π}^2 the instantaneous variance.

The behavior of the nominal return on equity is described by the equation:

$$(2) \quad \frac{dQ_M}{Q_M} = R_M dt + s_M dz_M$$

where R_M and s_M^2 are the instantaneous mean and variance per unit time of the nominal rate of return on equity.

By Itô's lemma the real return on equity is then given by:

$$(3) \quad \frac{d(Q_M/P)}{Q_M/P} = (R_M - \pi - s_{M\pi} + s_{\pi}^2)\, dt + s_M dz_M - s_{\pi} dz_{\pi}$$

where $s_{M\pi}$ is the instantaneous covariance per unit time between the rate of inflation and the nominal return on equity. Equivalently, we can express (3) in terms of real parameters as:

$$(3') \quad \frac{d(Q_M/P)}{Q_M/P} = r_M dt + \sigma_M dz'_M$$

The riskless nominal return on bonds is:

$$(4) \quad \frac{dQ_f}{Q_f} = R_f dt$$

and the real return is:

$$(5) \quad \frac{d(Q_f/P)}{Q_f/P} = (R_f - \pi + s_{\pi}^2)dt - s_{\pi} dz_{\pi}$$

or, in real terms:

$$(5') \quad \frac{d(Q_f/P)}{Q_f/P} = r_f dt + \sigma_f dz'_f$$

Let us consider the relationships between the parameters of the distributions of real and nominal rates of return implied by these equations. From equations (5) and (5') we see that the stochastic component of the real return on nominal bonds, $\sigma_f dz'_f$, is just the negative of unanticipated inflation, $-s_\pi dz_\pi$, and therefore the variance of the real return on nominal bonds, σ_f^2, is equal to the variance of the rate of inflation, s_π^2. Equations (3) and (3') show that the stochastic component of the real return on equity, $\sigma_M dz'_M$, is equal to $s_M dz_M - s_\pi dz_\pi$. Since the variance of the difference between two random variables is the sum of their variances less twice the covariance between them, we get for the variance of the real return on equity:

$$(6) \quad \sigma_M^2 = s_M^2 + s_\pi^2 - 2s_{M\pi}$$

The covariance between the real returns on equity and bonds, σ_{Mf}, is the covariance between $s_M dz_M - s_\pi dz_\pi$ and $-s_\pi dz_\pi$, which is given by:

$$(6') \quad \sigma_{Mf} = -s_{M\pi} + s_\pi^2 = -s_{M\pi} + \sigma_f^2$$

The covariance between the real return on equity and the rate of inflation, $\sigma_{M\pi}$, is just the negative of σ_{Mf}, or $s_{M\pi} - s_\pi^2$. Finally, note that the difference between the mean real returns on equity and bonds, $r_M - r_f$, which we shall call the real risk premium on equity, is equal to $R_M - R_f - s_{M\pi}$.

For convenience in later use we summarize these relationships in Table 1.

Table 1

Summary of Relationships Between Real and Nominal Returns Parameters

Variances:

Bonds $\sigma_f^2 = s_\pi^2$

Equity $\sigma_M^2 = s_M^2 + s_\pi^2 - 2s_{M\pi}$

Covariances:

Equity and inflation $\sigma_{M\pi} = s_{M\pi} - s_\pi^2$

Equity and bonds $\sigma_{Mf} = -\sigma_{M\pi} = -s_{M\pi} + s_\pi^2$

Risk premium on equity: $r_M - r_f = R_M - R_f - s_{M\pi}$

Let x be the proportion of its asset portfolio which the household will invest in equity. The change in the household's nominal wealth in any instant is then given by:

$$(7)\quad dW = \{[R_f + x(R_M - R_f)]\ W - PC\}\ dt + xWs_M dz_M$$

where C is the rate of consumption, and W is nominal wealth.

The household's optimal consumption and portfolio rules are derived by finding:

$$(8)\quad \max_{\{C,x\}} E_o \int_o^\infty [U\ C(t),\ t]dt$$

subject to (7) and $W(0) = W_o$ and with U ( ) strictly concave in C. One solves the problem by first defining the derived utility of wealth function:

$$(9)\quad J(W,P,t) = \max_{\{C,x\}} E_t \int_o^\infty U[C(\tau),\tau]\ d\tau$$

and then solving the two first order conditions:

$$(10)\ 0 = U_c(C,t) - PJ_w$$

$$(11)\quad 0 = J_w(R_M - R_f) + J_{ww}Wxs_M^2 + J_{wp}Ps_{M\pi}$$

Solving (11) for the optimal value of x gives:

$$(12)\quad x^* = \frac{-Jw(R_M - R_f)}{JwwW\ s_M^2} - \frac{J_{wp}Ps_{M\pi}}{JwwW\ s_M^2}$$

The term $\frac{-Jw}{JwwW}$ is just the inverse of Pratt's measure of relative risk aversion, which we will denote by A. Since consumption is in effect a function of real wealth, it can be shown that:[3]

$$(13)\quad \frac{J_{wp}P}{JwwW} = -\frac{Jw}{JwwW} - 1 = \frac{1}{A} - 1$$

Substituting into (12) and combining terms we get:

$$(14)\quad x^* = \frac{R_M - R_f - s_{M\pi}}{A\ s_M^2} + \frac{s_{M\pi}}{s_M^2}$$

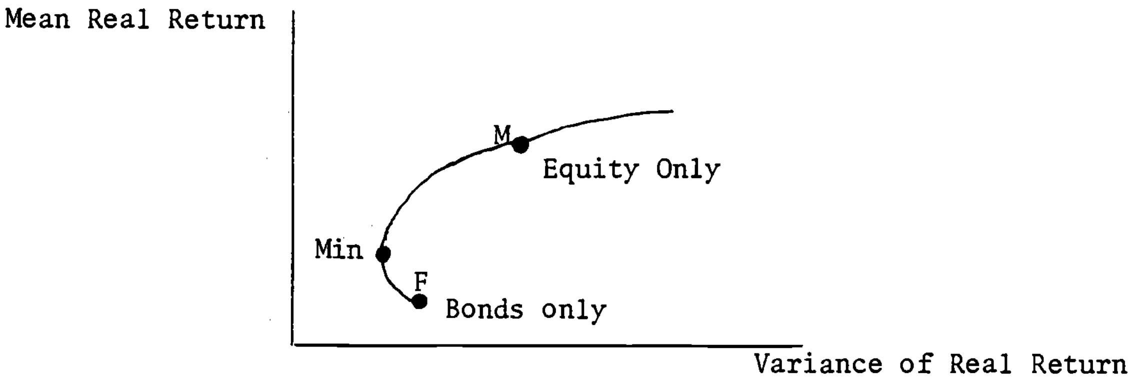
$$= \frac{r_M - r_f}{A\ s_M^2} + \frac{s_{M\pi}}{s_M^2}$$

The demand for equity is thus seen to be composed of two parts represented by the two terms on the right-hand side of (14). The first part is the "speculative" demand for equity, which is directly proportional to the real risk premium on equity (the numerator of the first term in (14)) and inversely proportional to the variance of its nominal return and to the household's degree of relative risk aversion. The second part is the "inflation-hedging" demand for equity which is equal to the covariance between inflation and the nominal return on equity, divided by the variance of the nominal return on equity. The inflation-hedging component of the demand for equity is thus independent of the investor's risk preferences, so that an extremely risk-averse investor (one whose A is very large) would still demand at least this amount of equity no matter what the risk premium on it was.

The explanation for this result is that the inflation-hedging demand is exactly the proportion of equity which must be added to riskless nominal bonds in order to create the portfolio with minimum variance of real return.[4] The inflation-hedging demand thus stems from a kind of portfolio efficiency condition. Figure 1 is a diagram showing the familiar Markowitz-Tobin portfolio frontier,[5] but with mean real returns on the vertical axis and variance of real returns on the horizontal. Points F, Min and M correspond to an all-bond portfolio (x=0), the minimum-variance portfolio, and an all-equity portfolio (x=1), respectively. Every investor, no matter what his risk preferences, would make the move from point F to point Min in order to get to the efficient part of the frontier; but how far one would go beyond point Min depends on one's tolerance for risk.

Equation (14) implies that the inflation-hedging demand for equity will be zero if and only if its nominal rate of return is uncorrelated with the rate of inflation ($s_{M\pi}=0$). In that special case the minimum-variance portfolio would simply be the all-bond portfolio. To readers familiar with mean-variance portfolio analysis this result might seem a bit puzzling because usually the minimum-variance portfolio formed between two assets whose correlation is zero will contain positive amounts of both assets. The

Figure 1 - Efficient Portfolio Frontier



puzzle is solved, however, by referring to equation (6') which reveals that a zero covariance between the nominal return on equity and the rate of inflation implies that the covariance between the real returns on equity and bonds is equal to the variance of the real return on bonds ($\sigma_{Mf} = \sigma_f^2$). Expressed in these terms it is easier to see why the minimum-variance portfolio would be all-bonds.

Returning to the general case it can be shown that the variance of the real rate of return on the minimum-variance portfolio is given by:[6]

$$\sigma^2_{min} = (1-\rho^2)\ s^2_\pi$$

where ρ is the correlation coefficient between the nominal rate of return on equity and the rate of inflation. ρ^2 is thus a natural measure of the effectiveness of equity as a hedge against unanticipated inflation. Equity would be a perfect inflation hedge if and only if its nominal return were perfectly correlated with the rate of inflation (ρ^2=1). Furthermore, as long as there are no restrictions on short sales, it does not matter whether the sign of the correlation is positive or negative.

This view of the way to measure the effectiveness of an asset as an inflation hedge differs sharply from a view advanced recently by Fama and Schwert (FS) [5]. They propose that an asset be called a "complete hedge against unexpected inflation" if and only if its <u>real</u> rate of return is <u>uncorrelated</u> with unanticipated inflation. By their definition an asset would qualify as a complete inflation hedge even if it were subject to a great deal of non-inflation risk and therefore relatively useless in reducing the risk associated with a nominal bond.

To understand better the distinction between the FS definition of an inflation hedge and ours, consider the case of a hypothetical asset, a futures contract, whose price is much more volatile than the consumer price level, yet is perfectly positively correlated with it. In a regression of the nominal return on the futures contract against unanticipated inflation, the slope coefficient, which is denoted by γ in FS' equation (4), would be much greater than 1, and therefore by their definition the futures contract would be a poor hedge against unanticipated inflation. According to our definition, however, it would be a perfect hedge, because by combining the futures contract with riskless nominal bonds an investor could create a "synthetic index bond", whose real rate of return would be completely risk-free.

Finally, note that the FS definition although different from ours, is not necessarily inconsistent with it. The only asset which would be a complete inflation-hedge under both definitions, however, is an asset with zero variance of real return, i.e., an index bond.[7]

Before concluding our discussion of the demand for equity let us consider what effect an increase in inflation uncertainty will have on it.[8] To answer this question it is best to reformulate (14) in terms of the variances and covariances of real rather than nominal rates of return. From Table 1 we know that $s_{M\pi} = \sigma_{M\pi} + s_\pi^2$ and that $\sigma_M^2 = s_M^2 + s_\pi^2 - 2s_{M\pi}$. It therefore follows that $s_M^2 = \sigma_M^2 + s_\pi^2 + 2\sigma_{M\pi}$. Substituting these expressions for $s_{M\pi}$ and s_M^2 into (14) we get:

$$(14') \quad x^* = \frac{r_M - r_f}{A(\sigma_M^2 + 2\sigma_{M\pi} + s_\pi^2)} + \frac{\sigma_{M\pi} + s_\pi^2}{\sigma_M^2 + 2\sigma_{M\pi} + s_\pi^2}$$

The effect of an increase in inflation uncertainty is found by differentiating (14') with respect to s_π^2:

$$(15)\quad \frac{\partial x^*}{\partial s_\pi^2} = \frac{-(r_M - r_f)}{A(\sigma_M^2 + 2\sigma_{M\pi} + s_\pi^2)^2} + \frac{(\sigma_M^2 + 2\sigma_{M\pi} + s_\pi^2) - (\sigma_{M\pi} + s_\pi^2)}{(\sigma_M^2 + 2\sigma_{M\pi} + s_\pi^2)^2}$$

$$= \frac{-(r_M - r_f)}{A\ s_M^4} + \frac{\sigma_M^2 + \sigma_{M\pi}}{s_M^4}$$

Inspection of the first term on the right hand side of (15) reveals that if there is a positive real risk premium on equity ($r_M > r_f$) then the speculative demand for it will fall with an increase in s_π^2; while the second term indicates that as long as $\sigma_M^2 > \sigma_{M\pi}$ the hedging demand will rise. It can be shown that the net impact on x* of the increase in s_π^2 will depend on what the initial demand for bonds was.[9] If initially the investor held a positive amount of bonds ($x^* < 1$) then the increase in s_π^2 will cause the demand for equity to rise.

B. The Equilibrium Risk Premium on Equity

Assuming exogenously given supplies of equity and "outside" (i.e., government) bonds, the equilibrium real risk premium on equity is found by aggregating the demands of all households (equation (14)) weighted by their shares of total private wealth, γ_k, and solving the resulting equation for $r_M - r_f$:

$$(16)\quad r_M - r_f = \bar{A}(\alpha s_M^2 - s_{M\pi})$$

where α is the ratio of equity to the total wealth of the private sector (i.e., equity plus government bonds), and $\bar{A}$ is $[\sum_k \gamma_k / A_k]^{-1}$, a weighted harmonic mean of the individual households' measures of relative risk aversion.

Since α is a critical parameter in (16) and in the analysis which is to follow, we should emphasize a problem which exists in assessing its value. There is considerable controversy among monetary theorists about whether all government debt should be included in the total wealth of the private sector.[10] There does, however, seem to be a consensus that at least the government's non-interest-bearing debt (i.e., the monetary base) should be included. In our analysis, the term "bonds" means all securities with a fixed nominal value and thus includes money. We shall therefore assume in general that α is less than 1, although we shall point out in every case what our results would be if α were equal to 1.

Equation (16) implies that the real risk premium on equity will be greater the higher the degree of relative risk aversion, the greater the variance of the nominal return on equity, and the higher the ratio of the outstanding supply of equity to the total wealth of the private sector. It will be smaller, however, the more positive the covariance between the nominal return on equity and the rate of inflation, i.e., the more effective equity is as an inflation hedge.

We can gain additional insight into the determinants of the real risk premium on equity by reformulating (16) in terms of the variances and covariances of real rather than nominal rates of return and dividing by the variance of the real rate of return on equity. The result is the formula for the real market price of risk (RMPR):

$$(17)\quad \frac{r_M - r_f}{\sigma_M^2} = \bar{A}\left[\alpha + (2\alpha - 1)\frac{\sigma_{M\pi}}{\sigma_M^2} - (1-\alpha)\frac{s_\pi^2}{\sigma_M^2}\right]$$

Equation (17) implies that, unless there are no government bonds (α=1), an increase in inflation risk, which leaves both the covariance structure of real returns and the average degree of relative risk aversion unchanged, will cause the RMPR to fall. The fall in the RMPR will be greater the higher the proportion of government bonds in the total wealth of the private sector. In the special case in which there are no government bonds, (17) reduces to:

$$(17')\quad \frac{r_M - r_f}{\sigma_M^2} = \bar{A}\left(1 + \frac{\sigma_{M\pi}}{\sigma_M^2}\right)$$

implying that the RMPR is independent of s_π^2. If the real return on equity is also uncorrelated with inflation then the RMPR will simply equal $\bar{A}$, as is the case in the traditional CAPM.

III. The Case of Many Assets

A. Asset Demands.

Let us assume that there are n nominally risky assets (i.e., assets with uncertain nominal rates of return) with nominal returns dynamics of:

$$(18)\quad \frac{dQ_j}{Q_j} = R_j dt + s_j dz_j \qquad j = 1, \ldots, n$$

and a riskless nominal bond as before. Let s_{ij} be the covariance between the nominal returns on assets i and j, $s_{i\pi}$ the covariance between the rate of inflation and the nominal return on asset i, and R_f the nominal return on nominal riskless bonds.

The household's portfolio demand for asset i is given by:[11]

$$(19)\quad x_i^* = \frac{1}{A}\sum_{j=1}^{n} v_{ij}(R_j - R_f - s_{j\pi}) + \sum_{j=1}^{n} v_{ij}s_{j\pi} \text{ for } i = 1, \ldots, n$$

where v_{ij} is the ij^{th} element of the inverse of the variance-covariance matrix $[s_{ij}]$. As in the previous section the demand for each nominally risky asset is seen to be composed of a speculative component, represented by the first term in (19), and an inflation-hedging component represented by the second term, which is the weight of asset i in the portfolio with minimum variance of real return:

$$(20) \quad x_i^{min} = \sum_{j=1}^{n} v_{ij}\, s_{j\pi} \quad \text{for nominally risky asset } i \qquad i=1, \ldots, n$$

$$x_i^{min} = 1 - \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} s_{j\pi} \quad \text{for nominal bonds}$$

B. The Optimal Inflation-Hedge Portfolio.[12]

In the absence of inflation uncertainty the nominal bond would be a riskless asset, and all investors regardless of their preferences would be able to satisfy their asset demands for each of the n + 1 available assets by choosing from just two assets: the nominal bond and the "optimal combination of the n nominally risky assets (OCONRA)" defined by:[13]

$$(21) \quad x_i^o = \frac{\sum_{j=1}^{n} v_{ij}\,(R_j - R_f)}{\sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}\,(R_j - R_f)} \qquad i=1, \ldots, n$$

where x_i^o is the portfolio proportion of security i.

In other words the nominal bond and the OCONRA span the nominal efficient portfolio frontier.

In the presence of inflation uncertainty, the real efficient frontier can still be spanned by two portfolios, but the nominal bond and the OCONRA will not in general have this property. However, by adding to them a third, "inflation-hedge" portfolio, all investors would again be able to satisfy their asset demands for each of the n + 1 available assets. This inflation-hedge portfolio consists of the n nominally risky assets only and is defined by:

$$(22) \quad x_i^H = \frac{\sum_{j=1}^{n} v_{ij} s_{j\pi}}{\sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} s_{j\pi}} = \frac{x_i^{min}}{\sum_{j=1}^{n} x_j^{min}} \qquad i=1, \ldots, n$$

To see how any household would be able to satisfy its asset demands under inflation uncertainty with just these three assets let us rewrite (19) for household k as:

$$(23) \quad x_{ik}^* = \frac{1}{A_k} \sum_{j=1}^{n} v_{ij} (R_j - R_f) + (1 - \frac{1}{A_k}) \sum_{j=1}^{n} v_{ij} s_{j\pi} \qquad i=1, \ldots, n$$

To satisfy (23) the household would have to invest $\frac{1}{A_k} \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} (R_j - R_f)$ of its wealth in the OCONRA portfolio, $(1 - \frac{1}{A_k}) \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} s_{j\pi}$ of it in the inflation-hedge portfolio, and the remainder in nominal bonds.

Note that the inflation-hedge portfolio is the portfolio of nominally risky assets which is combined with riskless nominal bonds to create the global minimum-variance portfolio. An alternative but equivalent way of defining the inflation-hedge portfolio is as the portfolio of risky assets whose nominal return has the highest correlation with the rate of inflation.[14]

C. Equilibrium Risk Premia.

By aggregating the individual household asset demand functions (19) and setting them equal to the exogenously given proportional supplies of assets, we obtain the following yield relationships:

$$(24) \quad R_i - R_f - s_{i\pi} = \bar{A}(\sum_{j=1}^{n} \alpha_j s_{ij} - s_{i\pi}) \qquad i=1, \ldots, n$$

where α_j is the ratio of the market value of nominally risky asset j to the total wealth of the private sector, and $\bar{A}$ is the same harmonic mean of the individual A_k defined before. Equation (24) can therefore be written as:

$$(25) \quad r_i - r_f = \bar{A}(\alpha s_{iM} - s_{i\pi})$$

where α is the ratio of the value of all nominally risky assets to the total wealth of the private sector, s_{iM} is the covariance between the nominal return on asset i and what we shall call the "market" portfolio of nominally risky assets, defined as the portfolio containing each nominally risky asset in proportion to its outstanding aggregate value.[15]

The real risk premium on each security is thus seen to be directly proportional to the difference between the covariance of its nominal return with the market portfolio of nominally risky assets and its covariance with the rate of inflation. If its covariance with inflation exceeds its covariance with the market, it will have a negative risk premium. (25) also implies that the risk premia on nominally risky securities which are positively correlated with the market portfolio will be higher the higher the ratio of nominally risky assets to total private wealth.

The impact of inflation risk on these equilibrium risk premia is best examined by reformulating (25) in terms of real rather than nominal covariances:[16]

$$(25') \quad r_i - r_f = \bar{A}[\alpha(\sigma_{iM} + \sigma_{M\pi}) - (1-\alpha)(\sigma_{i\pi} + s_{\pi}^2)]$$

Equation (25') implies that unless there are no government bonds, an increase in the variance of the rate of inflation will lower the risk premia on all nominally risky assets. This effect will be greater the larger the proportion of government bonds in total private wealth. In the special case in which there are no government bonds (α=1), the real risk premium on any security will depend only on the sum of the covariance of its real return with the market portfolio and the covariance between the real return on the market and the rate of inflation.

Applying (25) to the market portfolio of nominally risky assets, we get:

$$(26) \quad r_M - r_f = \bar{A}(\alpha s_M^2 - s_{M\pi})$$

which is exactly equivalent to the formula for the risk premium on equity derived and discussed in the two-asset model of the previous section.

Finally, applying (25) to the minimum-variance portfolio and recognizing that $s_{min,M} = s_{\pi M}$ and $s_{min,\pi} = \rho_{H\pi}^2 s_\pi^2$ we get:[17]

$$(27) \quad r_{min} - r_f = \bar{A}(\alpha s_{\pi M} - \rho_{H\pi}^2 s_\pi^2)$$

where $\rho_{H\pi}$ is the coefficient of correlation between the nominal rate of return on the inflation hedge portfolio and the rate of inflation. We thus see that the real risk premium on the minimum-variance portfolio can be either positive or negative. It will be more positive the higher the covariance between the nominal return on the market portfolio and the rate of inflation, and more negative the greater the degree of correlation between the nominal return on the inflation-hedge fund and the rate of inflation. Note that if $r_{min} - r_f$ is positive, the "cost" of hedging against inflation is negative in the sense that in moving from a portfolio of nominal bonds only to the minimum - variance portfolio an investor would experience both a reduction in variance and an increase in mean real return.

Alternatively we can express the real risk premium on the minimum-variance portfolio in terms of real covariances and variances by applying (25') and recognizing that $\sigma_{i,min} = \sigma_{min}^2$ for all i.[18] We then get:

$$(27') \quad r_{min} - r_f = \bar{A}[(2\alpha-1)\sigma_{min}^2 + \alpha\sigma_{M\pi} - (1-\alpha)s_\pi^2]$$

If α=1, then (27') reduces to:

$$(27'') \quad r_{min} - r_f = \bar{A}(\sigma_{min}^2 + \sigma_{M\pi})$$

Two special cases are of interest. The first is if the real return on nominally risky securities is uncorrelated with inflation. In that case (27") implies that the real risk premium on the minimum-variance portfolio is directly proportional to the variance of its real rate of return. The second is if there exists a perfect inflation hedge (such as an index bond), in which case the real risk premium on the minimum-variance portfolio will be proportional to the covariance between the real return on the market portfolio and the rate of inflation.[19]

D. The Inflation-Adjusted Security Market Line.

As FLL have shown, one way to adjust the traditional Security Market Line (SML) for inflation is as follows:

$$(28) \quad R_i - R_f - s_{i\pi} = (R_M - R_f - s_{M\pi}) \frac{(\alpha s_{iM} - s_{i\pi})}{(\alpha s_M^2 - s_{M\pi})}$$

Equation (28) reduces to the SML of the traditional CAPM either when there is no inflation uncertainty or when the nominal returns on all securities are uncorrelated with inflation.

But there are two other equally valid ways of adjusting the SML for inflation, each of which offers an additional perspective. The first is to reformulate (28), in real rather than nominal terms:

$$(29) \quad r_i - r_f = (r_M - r_f) \frac{\alpha(\sigma_{iM} + \sigma_{M\pi}) - (1-\alpha)(\sigma_{i\pi} + s_\pi^2)}{\alpha(\sigma_M^2 + \sigma_{M\pi}) - (1-\alpha)(\sigma_{M\pi} + s_\pi^2)}$$

If we assume further that $\alpha = 1$, we get:

$$(29') \quad r_i - r_f = (r_M - r_f) \frac{(\sigma_{iM} + \sigma_{M\pi})}{(\sigma_M^2 + \sigma_{M\pi})}$$

If the real return on the market portfolio is uncorrelated with the rate of inflation, then (29') implies that the real risk premium on a security is its real beta coefficient times the real risk premium on the market portfolio.

Although (29') would then have the same form as the traditional SML, the difference would be that all of the parameters would be in real rather than nominal terms and the nominal bond would not be riskless.

The second alternative way of adjusting the SML for inflation is in the spirit of Merton's intertemporal CAPM and is entirely in nominal terms:[20]

$$(30)\quad R_i - R_f = \beta_{iM\cdot H}(R_M - R_f) + \beta_{iH\cdot M}(R_H - R_f)$$

where $\beta_{iM\cdot H}$ and $\beta_{iH\cdot M}$ are multiple regression coefficients.

Equation (30) expresses the nominal risk premium on a security as the sum of two factors, a market factor and an inflation factor. It is a natural generalization of the traditional SML and may appropriately be called the Security Market Plane. Note that in this last formulation α does not appear. Thus the value of α does not affect the SMP relationship.

E. Connection with Other Models of Asset Market Equilibrium.

Up to this point we have been comparing and contrasting our model with the traditional form of the CAPM, in which the riskless asset is nominal bonds. Now we propose to make the connection between our model and other models which have been used to describe the equilibrium structure of asset returns in the absence of a riskless asset.

In order to make the connection we have to interpret the returns in these other models as real rates of return, and we must redefine what we mean by the market portfolio. In the previous section of this paper, the market portfolio consisted of all assets whose nominal rates of return were risky.

In this section we must redefine it to include nominal bonds as well. The two definitions of the market portfolio will be equivalent if and only if the net supply of nominal bonds is zero (i.e., if $\alpha=1$).

In a recent paper, Roll [20] summarized the state of our knowledge of efficient set yield relationships by showing that the mean real return on any asset can be expressed as:

$$(31)\quad r_i - r_A = (r_B - r_A)\frac{(\sigma_{iB} - \sigma_{AB})}{(\sigma_B^2 - \sigma_{AB})}$$

where A and B are any two "frontier" portfolios on the Markowitz efficient portfolio frontier. If the assumptions underlying the CAPM are correct then the market portfolio (inclusive of nominal bonds) will be efficient. If we choose the market portfolio as B and the minimum-variance <u>zero β</u> portfolio as A (not the global minimum-variance portfolio, but rather that frontier portfolio which is uncorrelated with the market) then we get Black's [1] generalization of the SML:

$$(32)\quad r_i - r_z = (r_M - r_z)\frac{\sigma_{iM}}{\sigma_M^2}$$

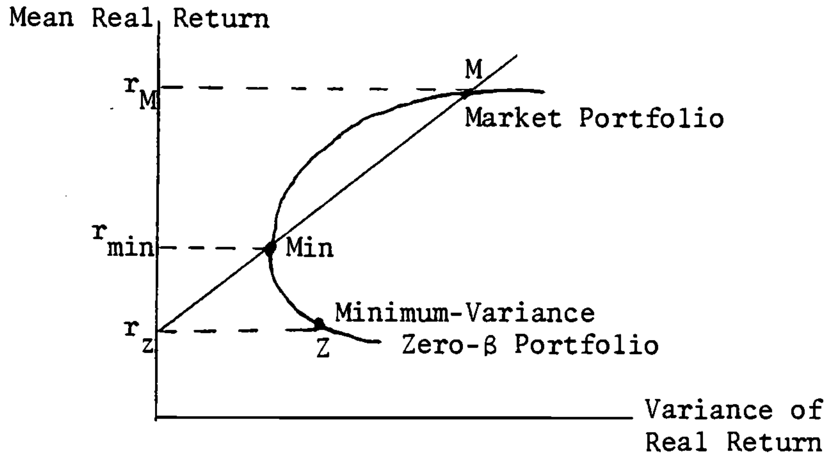
where all parameters are in real terms.

If, however, we select the global minimum-variance of real return portfolio as our second efficient portfolio and make use of the fact that the covariance between any security and the minimum-variance portfolio is equal to the variance of the minimum-variance portfolio we get:

$$(33)\quad r_i - r_{min} = (r_M - r_{min})\frac{(\sigma_{iM} - \sigma_{min}^2)}{(\sigma_M^2 - \sigma_{min}^2)}$$

Figure 2

Relationship Between Global Minimum Variance Portfolio and Minimum-Variance Zero-β Portfolio



If there exists a real riskless asset such as an index bond then both (32) and (33) converge to the same formula (the traditional SML) with the real riskless rate taking the place of r_z and r_{min}, respectively. In the absence of a perfect inflation hedge, r_{min} will exceed r_z, but the difference between them will get smaller the more effectively investors can hedge against inflation (i.e., the smaller the value of σ^2_{min}).

Figure 2 displays the relative positions of the global minimum-variance portfolio and the minimum-variance zero-β portfolio on the Markowitz frontier. The location of point Z representing the minimum-variance zero-β portfolio is found by projecting a straight line from point M (the market portfolio) through point Min to the vertical axis and then projecting a horizontal line back to the frontier. If we perform the conceptual experiment of holding r_M and r_{min} constant and allowing σ^2_{min} to go to zero, we can see that as point Min moves closer to the vertical axis, point Z converges on it from below.

IV. Empirical Evidence

The most important empirical implication of the theory developed above is the one embodied in equation (25), that the real risk premium on a security measured relative to a riskless nominal bond is a linear decreasing function of the covariance between its nominal rate of return and unanticipated inflation and a linear increasing function of its covariance with the market portfolio of nominally risky assets. In this section of the paper we briefly examine some of the available U.S. evidence on four classes of assets: common stocks, long-term government bonds, residential real estate, and commodity futures contracts, to check its consistency with this theoretical result.

Using quarterly data over the 27 year period 1950-1976, I computed the parameter estimates reported in Table 2. I used the Bureau of Labor Statistics Consumer Price Index as my measure of the price of the consumer good, the three month Treasury bill rate as the riskless nominal rate of interest,and the Standard and Poor's 500 Composite Index of Stocks as the representative common stock portfolio. The return on residential real estate was measured as the rate of increase in the Home Purchase Price component of the CPI. For U.S. Government bond returns I used Ibbotson and Sinquefield's (9) series. The commodity futures returns series represents the rate of return on a well-diversified portfolio of commodity futures contracts with an average maturity of a little over three months. The derivation of the series is explained in detail in Bodie and Rosansky (3). It is an equally weighted portfolio containing futures contracts on a number of different commodities, which grows from seven at the start of our sample period to twenty-two at the end. The quarterly series for unanticipated inflation was derived by the same procedure employed by Fama and Schwert (5), i.e. by taking the residuals from a regression of the rate of inflation on the three month T-bill rate.

Finally, as a proxy for the rate of return on the market portfolio of nominally risky assets I used an equally weighted average of the returns on three of the four representative asset categories; common stocks, long-term government bonds and residential real estate.[21] I excluded commodity futures contracts from the market proxy because their net supply is always zero, i.e., the number of short positions always equals the number of long positions.

The measure of the rate of return used in all cases was the natural logarithm of the quarterly wealth relatives $P_i(t)/P_i(t-1)$. On the assumption that the rate of return on security i follows a geometric Brownian motion:

$$\frac{dP_i}{P_i} = R_i dt + s_i dz_i$$

the log of the wealth relative over one discrete time interval will be normally distributed with mean μ_i and variance s_i^2, where

$$\mu_i = R_i - \frac{s_i^2}{2}.$$

If, in addition, we assume that these parameters were stable during our sample period, then the standard t-tests of statistical significance apply.

The last two columns in Table 2 indicate that the covariances with unanticipated inflation are negative for common stocks and long-term bonds and positive for real estate and commodity futures, while the covariances with the market portfolio of nominally risky assets are positive for stocks and bonds and negative for real estate and commodity futures. According to our theory the real risk premia should therefore be positive in the cases of stocks and bonds and negative in the cases of real estate and commodity futures.

As Table 2 shows, however, the estimates of these real risk premia derived from our data conflict with the theory in two of the four cases. Although the risk premium on common stocks is positive and on real estate it is negative as predicted by the theory, the risk premia for long-term bonds and commodity futures have the wrong signs.

Table 2

Summary Statistics of Quarterly Nominal Rates of Return: 1950-1976[a]

Asset	Mean[b] $\mu_i (x10^2)$ (t statistic)	Real Risk Premium $r_i - r_f$ $(x10^2)$	Standard Deviation s_i $(x10^2)$	Covariance with inflation $s_{i\pi}$ $(x10^4)$	Covariance with the market portfolio s_{iM} $(x10^4)$
Common stocks	2.65* (4.14)	1.93	6.65	-.498	14.82
Long-Term Bonds	0.64* (2.14)	-0.26	3.11	-.470	3.17
Real Estate	0.69* (6.55)	-0.24	1.01	.219	-.32
Commodity Futures[c]	1.66* (2.61)	1.86	6.61	1.832	-6.47
Treasury Bills	0.94		.49		
Unanticipated Inflation	0.00		.64		

*indicates statistical significance at the 5% level

Notes to Table 2

[a]The time series for real estate starts in 1954 and therefore has only 92 quarterly observations. For the other three asset categories we have 108 quarterly observations.

[b]μ_i is the arithmetic mean of the natural logarithms of the quarterly wealth relatives. An estimate of R_i can be obtained by adding $1/2\ s_i^2$ to μ_i.

[c]The nominal rate of return on a futures contract is analogous to the excess return (nominal return minus the Treasury bill rate) on other assets because no money is actually paid for such a contract at the beginning of the contract period. See Bodie and Rosansky [3] for a more complete explanation of this point.

There are several serious deficiencies in our data, which might account for this somewhat surprising result. The most important of these is the lack of a reliable measure of the return on the market portfolio. Roll [20] has recently raised serious doubts about the empirical testability of any hypothesis involving the market portfolio because of the near impossibility of measuring it or even of agreeing on its composition.

A possible theoretical explanation of our empirical results which would be consistent with the "consumer services" model employed in this paper is that there are systematic risk factors other than the two considered here which might be affecting equilibrium expected returns in offsetting ways. Merton [16], for example, has suggested several possible sources of systematic risk other than the two considered explicitly in this paper. Whether they can account for the discrepancy between theory and evidence reported above remains a subject for future research.

V. Summary and Implications for Future Research

In a capital market in which there exist only two assets, riskless nominal bonds and equity, the demand for equity will consist of two distinct components. The first is the "speculative" demand, which is directly proportional to the real risk premium on equity and inversely proportional to the variance of its nominal rate of return and to the investor's degree of relative risk aversion. The second component is the "inflation-hedging" demand which is directly proportional to the covariance between the rate of inflation and the nominal rate of return on equity. The inflation-hedging demand is exactly the amount of equity which must be added to riskless nominal bonds in order to create the portfolio with minimum variance of real return, and it is the minimum amount of equity every investor would hold regardless of his risk preferences. An increase in the variance of the rate of inflation will cause an investor's demand for equity to increase if he initially held positive amounts of both bonds and equity.

Our analysis suggests that the most natural measure of the effectiveness of an asset as an inflation hedge is the degree of correlation between its nominal rate of return and unanticipated inflation. An asset is therefore a perfect inflation hedge if and only if its nominal return is perfectly correlated with the rate of inflation. This view differs sharply from Fama and Schwert's [5] definition of a "complete hedge against unexpected inflation" as an asset whose real return is uncorrelated with unanticipated inflation.

The equilibrium real risk premium on equity measured as the difference between the mean real rates of return on equity and nominal bonds is higher the higher the variance of the rate of return on equity and the higher the ratio of equity to the total wealth of the private sector. It is lower the higher the covariance between the nominal rate of return on equity and the rate of inflation, i.e., the better equity is as an inflation hedge. Unless the supply of government bonds is zero, an increase in inflation risk will lower the real risk premium on equity.

In the many risky asset case, with no single security which is a perfect inflation hedge, there is an analogous decomposability of the demand for each asset into speculative and inflation-hedging components. We showed that there is an inflation-hedge portfolio which when combined with the riskless nominal bond and the optimal combination of nominally risky assets allows investors to span the real efficient portfolio frontier. This inflation-hedge portfolio is that portfolio composed of nominally risky assets whose nominal return has the highest correlation with unanticipated inflation.

We then showed that the equilibrium real risk premium on any security with uncertain nominal returns is directly proportional to the difference between the covariance of its nominal rate of return with the "market" portfolio of nominally risky assets and its covariance with the rate of inflation. The risk premium on any security which is positively correlated with the market portfolio will be higher the higher the ratio of nominally risky assets to total private wealth. Unless there is no government fiat money, an increase in the variance of the rate of inflation will lower the risk premia on all nominally risky assets, and this effect will be greater the larger the proportion of government bonds in total private wealth. We also showed that the "cost" of hedging against inflation might well be negative in the sense that in moving from nominal bonds to the minimum-variance portfolio an investor might be able to achieve both a reduction in the variance and an increase in the mean of his real rate of return.

FOOTNOTES

*I want to thank Robert C. Merton, Stanley Fischer, Alex Kane and an anonymous referee for helpful comments. I also benefited greatly from reading an unpublished manuscript by Stuart M. Turnbull, "Inlfation, Indexation and the Structure of Returns."

[1]See Sharpe (21), Lintner (10,11) , and Mossin (18).

[2]Fischer (6) has an appendix which presents an excellent explanation of the meaning of stochastic differential equations such as (1) and of Ito's lemma.

[3]As Fischer (6), p. 515, has shown, (13) can be derived by differentiating (10) with respect to P and then with respect to W to obtain:

$$U_{cc}\frac{\partial C}{\partial P} = J_w + PJ_{wp} \text{ and } U_{cc}\frac{\partial C}{\partial W} = PJ_{ww}$$

Then, since C is a function of W/P we have $\frac{\partial C}{\partial P} = -\frac{W}{P}\frac{\partial C}{\partial W}$

Hence, $\frac{J_{wp}P}{J_{ww}W} = \frac{-J_w}{J_{ww}W} - 1.$

[4]Let σ_p^2 be the variance of the real rate of return on any portfolio consisting of equity and riskless nominal bonds. It is given by:

$$\sigma_p^2 = x^2\sigma_M^2 + (1-x)^2\sigma_f^2 + 2x(1-x)\sigma_{Mf}$$

Using the equivalences summarized in Table 1 of the text we get by substitution:

$$\sigma_p^2 = x^2(s_M^2+s_\pi^2-2s_{M\pi}) + (1-x)^2 s_\pi^2 + 2x(1-x)(s_\pi^2-s_{M\pi})$$
$$= x^2 s_M^2 - 2xs_{M\pi} + s_\pi^2$$

The variance-minimizing proportion of equity, x_{min}, is found by setting the derivative of σ_p^2 with respect to x equal to zero:

$$\frac{d\sigma_p^2}{dx} = 2xs_M^2 - 2s_{M\pi} = 0$$

$$x_{min} = \frac{s_{M\pi}}{s^2}$$

[5]See Markowitz [13] for a definition, explanation and deriviation of the efficient portfolio frontier.

[6]Substituting $x_{min} = \frac{s_{M\pi}}{s_M^2}$ into the expression for σ_p^2 in footnote 4 we get:

$$\sigma_{min}^2 = x_{min}^2 s_M^2 - 2x_{min} s_{M\pi} + s_\pi^2$$

$$= \left(\frac{s_{M\pi}}{s_M^2}\right)^2 s_M^2 - 2\,\frac{s_{M\pi}}{s_M^2}\, s_{M\pi} + s_\pi^2$$

$$= \frac{s_{M\pi}^2}{s_M^2} - \frac{2s_{M\pi}^2}{s_M^2} + s_\pi^2$$

$$(1)\quad \sigma_{min}^2 = s_\pi^2 - \frac{s_{M\pi}^2}{s_M^2}$$

The correlation coefficient between the nominal return on bonds and inflation, ρ, is defined by:

$$\rho^2 = \frac{s_{M\pi}^2}{s_M^2 s_\pi^2}$$

so that:

$$(2)\quad \frac{s_{M\pi}^2}{s_M^2} = \rho^2 s_\pi^2$$

Substituting from (2) into (1) above we get:

$$\sigma_{min}^2 = s_\pi^2 - \rho^2 s_\pi^2 = (1-\rho^2) s_\pi^2$$

If an asset has both a nominal return which is perfectly correlated with inflation and a real return which is uncorrelated with inflation, it must have zero variance of real return.

Proof:

By definition, $\rho^2 = \dfrac{s_{M\pi}^2}{s_M^2 s_\pi^2} = \dfrac{(\sigma_{M\pi} + s_\pi^2)^2}{(\sigma_M^2 + 2\sigma_{M\pi} + s_\pi^2)s_\pi^2}$

Using the fact that $\rho^2 = 1$ and $\sigma_{M\pi} = 0$ we get

$$s_\pi^4 = (\sigma_M^2 + s_\pi^2)s_\pi^2$$

$$\sigma_M^2 s_\pi^2 = 0$$

Assuming $s_\pi^2 > 0$, σ_M^2 must be zero.

8 This question was answered by Gordon and Halpern [8] for the special case in which $\sigma_{M\pi}=0$, but as we show their result also holds for the more general case considered here. Bookstaber [4] has also examined this issue.

9 Equation (15) can be rewritten as: $\dfrac{\partial x^*}{\partial s_\pi^2} = \dfrac{A(\sigma_M^2 + \sigma_{M\pi}) - (r_M - r_f)}{A s_M^4}$

We know from (14) that $x^* = \dfrac{r_M - r_f}{A s_M^2} + \dfrac{s_{M\pi}}{s_M^2} = \dfrac{r_M - r_f}{A s_M^2} + \dfrac{\sigma_{M\pi} + s_\pi^2}{s_M^2}$

If $x^* < 1$ then:

$$A s_M^2 > r_M - r_f + A(\sigma_{M\pi} + s_\pi^2)$$

$$A(s_M^2 - \sigma_{M\pi} - s_\pi^2) - (r_M - r_f) > 0$$

$$A(\sigma_M^2 + \sigma_{M\pi}) - (r_M - r_f) > 0$$

so that:

$$\frac{\partial x^*}{\partial s_\pi^2} > 0$$

10 See for example Patinkin [19], p. 289.

[11] The n first-order conditions to be solved for the optimal portfolio weights x_i^* are:

$$0 = J_W(R_i - R_f) + J_{WW}\sum_{j=1}^{n} x_j W s_{ij} + J_{pW} s_{i\pi} \qquad (i=1,2, \ldots, n)$$

This is a set of n linear equations, which is solved by simple matrix inversion to yield (19).

[12] The material in this section is closely related to the work done by Solnik [22] and Manaster [12]. Manaster defines the hedge portfolio as the portfolio which must be added to a nominal efficient portfolio to transform it into a real efficient portfolio. But he does this in the context of a model with no nominal riskless asset, and consequently the composition and properties of his hedge portfolio are very different from mine. Solnik, however, does consider in part II of his paper what happens when a nominal riskless asset exists, thus making his analysis more comparable to mine.

[13] The proof is in Merton [15], p. 878.

[14] The proof is in Merton [17], p.91.

[15] The weight of security i in the market portfolio of nominally risky assets is:

$$\frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} = \frac{\alpha_i}{\alpha}$$

[16] The reformulation is accomplished by using the identities:

$$s_{iM} = \sigma_{iM} + \sigma_{i\pi} + \sigma_{M\pi} + s_\pi^2$$

$$s_{i\pi} = \sigma_{i\pi} + s_\pi^2$$

[17] Proof: Let Ω be the variance-covariance matrix $[s_{ij}]$, cov π be the vector $[s_{i\pi}]$ and x^{min} the vector of variance-minimizing portfolio weights defined by:

$$x^{min} = \Omega^{-1}\text{cov } \pi$$

The covariance between the nominal return on any security and the minimum-variance portfolio, $s_{i,min}$, is then given by:

$$\Omega x^{min} = \Omega\Omega^{-1}\text{cov } \pi = \text{cov } \pi$$

It follows that $s_{min,\pi} = s^2_{min}$ and that $\rho^2_{min,\pi} = \frac{s^2_{min}}{s^2_{\pi}}$.

But $\rho^2_{min,\pi} = \rho^2_{H\pi}$

Hence, $s_{min,\pi} = \rho^2_{H\pi} s^2_{\pi}$.

[18] See Merton [14], p. 1864, for the proof.

[19] This is the case discussed in Fischer [6]. As Fischer shows, however, in this special case no nominal bonds would exist.

[20] Equation (30) is derived in the appendix.

[21] The decision to use equal weights is a rough approximation based on unpublished data supplied by the U.S. Federal Reserve Board's Flow of Funds Division; however the signs of the covariances reported in the last column of Table 2 are quite robust with respect to changes in these weights.

Appendix: Derivation of Security Market Plane

From equation (25) we can get:

$$\text{(A1)} \quad R_i - R_f - s_{i\pi} = \bar{A}(\alpha s_{iM} - s_{i\pi}) \quad \text{for } i = 1, \ldots, n$$

or

$$\text{(A2)} \quad R_i - R_f = (1-\bar{A})s_{i\pi} + \bar{A}\alpha s_{iM}$$

Applying (A2) to the market portfolio of nominally risky assets and to the inflation-hedge portfolio we get:

$$\text{(A3)} \quad R_H - R_f = (1-\bar{A})s_{H\pi} + \bar{A}\alpha s_{HM}$$

$$R_M - R_f = (1-\bar{A})s_{M\pi} + \bar{A}\alpha s_M^2$$

Solving (A3) for $1-\bar{A}$ and $\bar{A}\alpha$ we get:

$$\text{(A4)} \quad (1-\bar{A}) = \frac{(R_H-R_f)s_M^2 - (R_M-R_f)s_{HM}}{s_{H\pi}s_M^2 - s_{HM}s_{M\pi}}$$

and

$$\bar{A}\alpha = \frac{(R_M - R_f)s_{H\pi} - (R_H-R_f)s_{M\pi}}{s_{H\pi}s_M^2 - s_{HM}s_{M\pi}}$$

Substituting from (A4) into (A2) we get:

$$\text{(A5)} \quad R_i-R_f = s_{i\pi}\,\frac{(R_H-R_f)s_M^2 - (R_M-R_f)s_{HM}}{s_{H\pi}s_M^2 - s_{HM}s_{M\pi}} + s_{iM}\,\frac{(R_M-R_f)s_{H\pi} - (R_H-R_f)s_{M\pi}}{s_{H\pi}s_M^2 - s_{HM}s_{M\pi}}$$

which reduces to:

$$\text{(A6)} \quad R_i-R_f = \frac{(s_{iM}s_{i\pi} - s_{i\pi}s_{HM})}{s_{H\pi}s_M^2 - s_{HM}s_{M\pi}}(R_M-R_f) + \frac{(s_{i\pi}s_M^2 - s_{iM}s_{M\pi})}{s_{H\pi}s_M^2 - s_{HM}s_{M\pi}}(R_H - R_f)$$

From footnote (17) and the fact that the minimum-variance portfolio is created by just combining the inflation-hedge portfolio with riskless nominal bonds we know that:

(A7) $s_{H\pi} = ks_H^2$ and $s_{i\pi} = ks_{iH}$

$s_{H\pi} = ks_{HM}$

where $k = \sum_i \sum_j v_{ij} s_{j\pi}$

Substitute from (A7) into (A6) the k's cancel out and we get:

$$(A8)\quad R_i - R_f = \frac{(s_{iH}s_{HM} - s_{iM}s_H^2)}{s_{HM}^2 - s_H^2 s_M^2}(R_M - R_f) + \frac{(s_{iM}s_{HM} - s_{iH}s_M^2)}{s_{HM}^2 - s_H^2 s_M^2}(R_H - R_f)$$

Since by definition:

$$\beta_{iM.H} = \frac{s_{iH}s_{HM} - s_{iM}s_H^2}{s_{HM}^2 - s_H^2 s_M^2} \quad \text{and} \quad \beta_{iH.M} = \frac{s_{iM}s_{HM} - s_{iH}s_M^2}{s_{HM}^2 - s_H^2 s_M^2}$$

we get equation (30) in the text.

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