NBER WORKING PAPER SERIES

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Working Paper No. 345

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge MA 02138

May 1979

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## Abstract

Among the numerous familiar sets of specific assumptions sufficient to derive mean-variance portfolio behavior from more general expected utility maximization in continuous time, the assumptions of constant relative risk aversion and joint normally distributed asset return assessments are also jointly sufficient to derive asset demand functions with the two desirable (and frequently simply assumed) properties of wealth homogeneity and linearity in expected returns. In addition, in discrete time constant relative risk aversion and joint normally distributed asset return assessments are sufficient to yield linear homogeneous asset demands as approximations if the time unit is small.

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March, 1979

# A NOTE ON THE DERIVATION OF LINEAR HOMOGENEOUS ASSET DEMAND FUNCTIONS

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The asset demand functions used for both analytical and empirical research, especially in the monetary economics literature, are often assumed to exhibit the two convenient properties of wealth homogeneity and linearity in expected asset returns.[1] The convenience afforded by the tractability of the linear form is apparent enough, and the wealth homogeneity property in particular is often especially important in empirical applications to aggregate data.[2] Despite the frequent use of linear homogeneous asset demand functions, however, there exists (to the authors' knowledge) no readily available source setting forth the derivation of such demands from underlying principles of expected utility maximization.[3]

The object of this note is to show that, among the numerous familiar sets of specific assumptions sufficient to derive mean-variance portfolio behavior from more general expected utility maximization in continuous time, the assumptions of (a) constant relative risk aversion and (b) joint normally distributed asset return assessments are also jointly sufficient to derive asset demand functions with the two desirable (and frequently simply assumed) properties of wealth homogeneity and linearity in expected returns. In addition, in discrete time constant relative risk aversion and joint normally distributed asset return assessments are sufficient to yield linear homogeneous asset demands as approximations if the time unit is small.[4]

Analysis in Continuous Time

To begin with expected utility maximization, the investor's objective as of time t, given initial wealth $W_t$, is

$$\max_{\underline{\alpha}_t} E[U(\tilde{W}_{t+dt})] \tag{1}$$

subject to

$$\underline{\alpha}_t'\underline{1} = 1, \tag{2}$$

where $E(\cdot)$ is the expectation operator, $U(W_\tau)$ is utility as a function of wealth, and $\underline{\alpha}_t$ is a vector expressing the portfolio allocations in proportional form

$$\underline{\alpha}_t \equiv \frac{1}{W_t} \cdot \underline{A}_t \tag{3}$$

for vector $\underline{A}_t$ of asset holdings.

Assumption (a) noted above is that $U(W_\tau)$ is any power (or logarithmic) function such that the coefficient of relative risk aversion

$$\rho \equiv - W_\tau \cdot \frac{U''(W_\tau)}{U'(W_\tau)} \tag{4}$$

is constant.[5] Assumption (b) is that the investor perceives asset returns $\tilde{r}_{i\tau}$, $i = 1,\ldots,n$, to be generated as Wiener processes with respective means $r^e_{i\tau}$, standard deviations $\sigma_{i\tau}$ and correlations $\phi_{ij\tau}$, where the tilde sign indicates a random variable, and the time subscript generalizes the investor's assessments to permit variation over time. Given the assumption of Weiner processes for the asset yields, $\tilde{W}_{t+dt}$ is in turn generated by

$$\tilde{W}_{t+dt} = W_t \cdot \sum_i^n \alpha_{it}(1 + r^e_{it}dt + \sigma_{it}\tilde{z}_{it}\sqrt{dt}) \tag{5}$$

where $\tilde{z}_i$ is the unit normal random variable corresponding to each yield $\tilde{r}_i$.

Expanding $U(\tilde{W}_{t+dt})$ about $W_t$, for dt sufficiently small, and then taking the expectation yields a representation of the maximand in the form

$$E[U(\tilde{W}_{t+dt})] = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot U^{(k)}(W_t) \cdot E[\tilde{W}_{t+dt} - W_t]^k \tag{6}$$

where the notation $U^{(k)}(\cdot)$ indicates the k-th derivative of $U(\cdot)$.

Substituting from (5) and omitting terms of higher than second order in dt yields

$$\begin{aligned} E[U(\tilde{W}_{t+dt})] &= U(W_t) + U'(W_t) \cdot W_t \cdot \underline{\alpha}_t' \, \underline{r}_t^e \, dt \\ &\quad + \frac{1}{2} U''(W_t) \cdot W_t^2 \cdot \underline{\alpha}_t' \, \Omega_t \, \underline{\alpha}_t \, dt \end{aligned} \tag{7}$$

where $\Omega_t$ is a variance-covariance matrix consisting of elements $\sigma_{it}\sigma_{jt}\phi_{ijt}$. Forming the Lagrangean for the maximization of (7) subject to (2), differentiating with respect to $\underline{\alpha}_t$, and equating the derivative to zero yields the first-order condition for the solution of (1) as

$$\underline{\alpha}_t^* = B_t \, \underline{r}_t^e + \underline{\pi}_t \tag{8}$$

where the asterisk indicates the optimum allocation, and if there is no risk-free asset (because of price inflation, for example)

$$B_t = -\frac{1}{\rho} [\Omega_t^{-1} - (\underline{1}'\Omega^{-1}\underline{1})^{-1}\Omega_t^{-1}\underline{1}\,\underline{1}'\Omega_t^{-1}] \tag{9}$$

$$\underline{\pi}_t = (\underline{1}'\Omega_t^{-1}\underline{1})^{-1}\Omega_t^{-1}\underline{1}. \tag{10}$$

Alternatively, in the presence of a risk-free asset $\Omega_t$ is singular, so that it is necessary to partition the system of demands. The resulting solution, in which $\hat{\underline{\alpha}}_t$, $\hat{\underline{r}}_t^e$ and $\hat{\Omega}_t$ refer to the risky assets only, is

$$\hat{\underline{\alpha}}_t^* = \hat{B}_t \, \hat{\underline{r}}_t^e \tag{8'}$$

where

$$\hat{B}_t = -\frac{1}{\rho} \hat{\Omega}_t^{-1} \tag{9'}$$

and the optimum portfolio share for the risk-free asset is just $(1 - \hat{\underline{\alpha}}_t^{*\prime}\, \underline{1})$.[6]

It is apparent by inspection that the optimum portfolio allocations in both (8) and (8') exhibit the two properties of wealth homogeneity and linearity in expected returns. Moreover, since $\Omega_t$ (or $\hat{\Omega}_t$) is a variance-covariance matrix, the Jacobean $B_t$ (or $\hat{B}_t$) indicates symmetrical asset substitutions associated with cross-yield effects.

## Analysis in Discrete Time

In the discrete-time analog to the model developed above, the investor's single-period objective as of time t, given initial wealth $W_t$, is

$$\max_{\underline{\alpha}_t} E[U(\tilde{W}_{t+1})] \tag{11}$$

where

$$\tilde{W}_{t+1} = W_t \cdot \underline{\alpha}_t' (\underline{1} + \tilde{\underline{r}}_t) \tag{12}$$

and assessments of $\tilde{\underline{r}}_t$ (i.e., asset returns between time t and time t+1) are distributed as

$$\tilde{\underline{r}}_t \sim N(\underline{r}_t^e, \Omega_t). \tag{13}$$

Expanding $U(\tilde{W}_{t+1})$ about $E(\tilde{W}_{t+1})$ and then taking the expectation yields a representation of the maximand in the form

$$E[U(\tilde{W}_{t+1})] = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot U^{(k)}[E(\tilde{W}_{t+1})] \cdot \{E[\tilde{W}_{t+1} - E(\tilde{W}_{t+1})]^k\}. \tag{14}$$

It follows from the moment generating function of the normal distribution that the term within brackets in (14) has value

$$E[\tilde{W}_{t+1} - E(\tilde{W}_{t+1})]^k = \frac{k!}{2^{(k/2)} \left(\frac{k}{2}\right)!} [\mathrm{var}(\tilde{W}_{t+1})]^{(k/2)} \tag{15}$$

for k an even integer and

$$E[\tilde{W}_{t+1} - E(\tilde{W}_{t+1})]^k = 0 \tag{16}$$

for k an odd integer. Hence (14) simplifies to

$$E[U(\tilde{W}_{t+1})] = \sum_{m=0}^{\infty} \frac{1}{2^m m!} \cdot U^{(2m)}[E(\tilde{W}_{t+1})] \cdot [\mathrm{var}(\tilde{W}_{t+1})]^m. \tag{17}$$

Substituting from (12) and omitting terms of higher than second order yields

$$E[U(\tilde{W}_{t+1})] = U[E(\tilde{W}_{t+1})] + \frac{1}{2} U''[E(\tilde{W}_{t+1})] \cdot W_t^2 \cdot \underline{\alpha}_t' \, \Omega_t \, \underline{\alpha}_t. \tag{18}$$

Forming the Lagrangean for the maximization of (18) subject to (2), differentiating with respect to $\underline{\alpha}_t$, and equating the derivative to zero yields the first-order condition for the solution of (11) if there is no risk-free asset as (again omitting terms of higher than second order)

$$\underline{\alpha}_t^* = B_t \, (\underline{r}_t^e + \underline{1}) + \underline{\pi}_t \tag{19}$$

where

$$B_t = \left\{\frac{-U'[E(\tilde{W}_{t+1})]}{W_t \cdot U''[E(\tilde{W}_{t+1})]}\right\} [\Omega_t^{-1} - (\underline{1}'\Omega_t\underline{1})^{-1}\Omega_t^{-1}\underline{1}\,\underline{1}'\Omega_t^{-1}] \tag{20}$$

and $\underline{\pi}_t$ is again as in (10). Alternatively, in the presence of a risk-free asset the resulting solution is (for $\hat{\underline{\alpha}}_t$, $\hat{B}_t$ and $\hat{\underline{r}}_t^e$ as defined above)

$$\hat{\underline{\alpha}}_t^* = \hat{B}_t \, (\hat{\underline{r}}_t^e + \underline{1}) \tag{19'}$$

where

$$B_t = \left\{\frac{-U'[E(\tilde{W}_{t+1})]}{W_t \cdot U''[E(\tilde{W}_{t+1})]}\right\} \hat{\Omega}_t^{-1} \tag{20'}$$

and the optimum portfolio share for the risk-free asset is again just $(1 - \hat{\underline{\alpha}}_t^{*\prime} \, \underline{1})$.

If the time unit is sufficiently small to render $W_t$ a good approximation to $E(\tilde{W}_{t+1})$ for purposes of the underlying expansion, then the scalar term within brackets in (20) and (20') reduces to the constant coefficient of relative risk aversion, and the discrete-time model yields the same linear homogeneous asset demand functions developed above.

Isomorphic Assumptions

Other combinations of assumptions, if they are isomorphic to constant relative risk aversion and joint normally distributed asset return assessments, also yield linear homogeneous asset demand functions either in continuous time or as an approximation in discrete time with small time units. For example, the negative exponential utility function with coefficient of absolute risk aversion inversely dependent on initial wealth yields equivalent results.[7] Alternatively, the logarithmic utility function, in conjunction with the assumption of joint lognormally distributed returns, yields asset demand functions that are homogeneous in wealth and log-linear in expected returns, in either continuous or discrete time; but in this case yet a further (apparently reasonable) approximation is necessary, because a linear combination of lognormally distributed returns is not itself distributed lognormally.[8]

## Footnotes

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The authors are grateful to John Lintner for many helpful discussions, and to the National Science Foundation (grant APR77-14160) and the Alfred P. Sloan Foundation for research support.

1. Brainard and Tobin [1] and the voluminous work following their lead provide numerous examples in both abstract and empirical work.

2. Friedman [4] and deLeeuw [3] in particular provided useful discussions of the importance of the homogeneity property. For an alternative view, however, see Goldfeld [6,7].

3. A large literature has investigated the conditions under which, in the presence of a risk-free asset, the ex post demands for risky assets that emerge from the market clearing process are linear in expected returns and linear homogeneous with respect to the total amount invested in risky assets only; see, for example, Sharpe [16], Lintner [9], Hakansson [8], Cass and Stiglitz [2] and Merton [12]. Nevertheless, these results do not apply to the ex ante demand relations that are usually the focus of analysis in the monetary economics literature, as exemplified by Tobin [17]. Moreover, these results do not carry over in general to cases in which there is no risk-free asset; and even when there is a risk-free asset the homogeneity is not with respect to total wealth (as is usually assumed in the monetary economics literature) and does not apply to the demand for the risk-free asset.

4. The rationale for mean-variance analysis provided by Samuelson [15] and Tsiang [18] suggests that mean-variance analysis per se is only an approximation that depends on (among other factors) a small time unit.

5. Friend and Blume [5], who proceeded along the lines followed here (as did Ross [14]), offered empirical evidence supporting the assumption of constant relative risk aversion.

6. In the case including a risk-free asset, vector $\hat{r}_t^e$ expresses the mean risky returns in excess of the risk-free return. See Roley [13] for a detailed treatment of the distinctions based on the presence or absence of a risk-free asset.

7. For given initial wealth, this assumption is equivalent to expressing utility as a function of portfolio rate of return, with constant absolute risk aversion; see Melton [11].

8. See Lintner [9] for a comprehensive treatment of portfolio behavior based on the logarithmic utility function.

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