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DISEQUILIBRIUM GROWTH THEORY: THE KALDOR MODEL

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Abstract

Disequilibrium macroeconomic theory [e.g. Clower, and Barro and Grossman] is extended to deal with capital accumulation in the long run. A growth model a la Kaldor is chosen for a framework. The real wage is supposed to be adjusted slowly, therefore there may be excess demand or supply in the labor market. The transaction takes place at the minimum of supply and demand. Since income shares of workers and capitalists depend on which regime the labor market is in, different equations are associated to different regimes. Local stability of the steady state by the disequilibrium dynamics is demonstrated.

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Disequilibrium Growth Theory

1. Introduction

Disequilibrium macroeconomics has been one of the active areas of research for the last ten years. After Clower (1965) and Leijonhufvud (1968) proposed a new definition of the effective demand, Barro and Grossman (1971, 1976), Malinvard (1977), Hildenbrand and Hildenbrand (1978), and Muellbauer and Portes (1978) showed a static quantity-constrained equilibria for simple macroeconomic models. In those models, the price and the wage are rigid, therefore the aggregate demand is not necessarily equal to the aggregate supply. Depending on the direction and size of disequilibrium of both markets, one can trace a dynamic path of static quantity-constrained equilibria, such as Bohm (1978) and Honkapohja (1978). However, those models do not incorporate endogenous capital accumulations. In this paper, we will introduce capital accumulation into a disequilibirum macroeconomic model, where capital accumulation is determined by the saving decision of households, like neoclassical growth theory.

We develop disequilibrium growth theory as follows: we take a one-sector neoclassical growth model,[1] but do not assume perfect flexibility of the real wage. If the wage is fixed at the moment (or for a certain period) by a lazy invisible auctioneer or by the visible government, then the labor market is under a regime of excess supply (unemployment), excess demand (overemployment), or (exact) full employment. The actual transactions are assumed to take place at the minimum of demand and supply of labor force. In the regime of unemployment, workers cannot satisfy their notional wage income, while in the overemployment regime, owners of firms cannot fulfill the notional plan of production and profit. Therefore, the amount of saving depends not only on the wage rate but also on which regime the labor market is in. The amount of saving determines a path of capital accumulation, while the wage rate may be partially adjusted according to a regime of the labor market. Movements of capital per capita and the wage rate will trace the disequilibrium path of a growing economy.

We will face a question of the stability for a differential equation systems with switching regimes, each regime being associated with a different set of differential equations. This kind of "patched-up system of differential equations has not been investigated much. We will give a simple conditions for local stability of a two-dimentional patched-up system.

To sum up, a framework to deal with capital accumulation is inherited from neoclassical grwoth theory, while the short-run rigidiry of prices and the minimum transaction rule of demand and supply in non-market-clearing prices are adopted from disequilibrium macroeconomics. Therefore the present model is differenciated from the earlier models of "disequilibrium growth" or "unemployment in a theory of growth."[2]

I demonstrated the above idea in a special example in Ito (1978), where the Diamond model (1965) of overlapping generations was chosen for a framework. Since I employed in the note the Cobb-Douglas functions for utility and production and optimistic expectations about future, the model implied that workers (i.e., the young generation) save a constant fraction of realized income. Moreover, the two-period overlapping generations model without bequests implies that capitalists (i.e., the older generation) never save. It would be most desirable to generalize the Diamond model to the n-period overlapping generations model with a general utility and production function. However, it would be extremely difficult. Instead, I adopt a neoclassical model where the saving rates from the wage income and from the profit income are different.[3]

In the next section, we describe the model. In Section 3, we will examine the stability of the model over time. It will be shown that a disequilibrium dynamic path fluctuates between the unemployment and overemployment regimes, exploding away or converging to the neoclassical long-run steady state. We will prove a new theorem on stability of a dynamic path with switching regimes. An alternative wage adjustment scheme is examined in Section 4. Section 5 will be devoted to the stucy of the implicaitons of the results obtained in the earlier sections, and concluding remarks are given in Section 6.

2. The Model

We assume a neoclassical well-behaved production function. The (flow of) output at time t, Y_t, is determined by the (flow of) labor force at time t, L_t, and the (stock of) capital, K_t. The production function, F, is assumed to be twice differentiable and homogeneous of degree one, i.e.,

$$Y_t = F(K_t, L_t) \qquad \text{for } K_t \geq 0,\ L_t \geq 0,$$

and

$$\lambda F(K_t, L_t) = F(\lambda K_t, \lambda L_t) \qquad \text{for } \lambda > 0.$$

We can write in the intensive form due to the homogeneity:

$$Y_t/L_t = f(K_t/L_t) \quad . \tag{2.1}$$

Assume that the production function is "well-behaved":

$$\begin{aligned} & f(0) = 0 \\ & f'(\cdot) > 0 \text{ and } f''(\cdot) < 0 \\ & \lim_{K/L \to 0} f'(\cdot) = \infty \quad \text{and} \quad \lim_{K/L \to \infty} f'(\cdot) = 0 \end{aligned} \tag{2.2}$$

At each moment of time, the capital stock is historically given and the wage rate is also fixed. The (representative) firm maximizes the (flow of) profit, Π_t, with respect to the labor input.

$$\max_{L_t} \Pi_t \equiv Y_t - w_t L_t \quad .$$

The labor demand, L^d, is the level of labor input which satisfies

$$(2.3) \qquad w_t = F_L(K_t, L_t)$$

where $F_L \equiv \partial F/\partial L$. Since F is homogeneous of degree one, F_L is homogeneous of degree zero. Therefore we have a separable form:

$$(2.4) \qquad L_t^d = \Omega(w_t) \cdot K_t \qquad\qquad \Omega' < 0$$

We assume, for the sake of simplicity, that the labor supply per capita, ℓ, is inelastic.[4] Therefore, the aggregate labor supply, L^s, is

$$(2.5) \qquad L_t^s = \ell N_t \quad .$$

The transaction rule in disequilibrium is a usual minimum of demand and supply.

$$(2.6) \qquad L_t = \min[L_t^d, L_t^s]$$

We here introduce several notations for convenience. First, we define capital per capita by $k_t \equiv K_t/N_t$. Note that this variable does not depend on the current wage rate or the actual level of employment. Secondly, we set $\ell = 1$ by choosing an appropriate unit of measurement. Thirdly, we introduce the desired capital/labor ratio, $k_t^d \equiv K_t/L_t^d = 1/\Omega(w_t)$. This variable is a function of the current wage rate, although capital per capita is not. Using (2.3) and the Euler equation for a homogeneous function, we have

(2.7) $$w_t = f(k_t^d) - k_t^d f'(k_t^d) \quad .$$

We are going to describe our economy by two state variables, k_t and w_t. But keep in mind in the following discussion that k_t^d does not depend on k_t but on w_t.

There are three possible regimes in the labor market. We say that the labor market is under full-employment, unemployment and overemployment, if $L^d = L^s$, $L^d < L^s$ and $L^s < L^d$, respectively.

It is easy to see the following relations:

(2.8) $$\frac{K_t}{L_t} = k^d(w_t)$$ in the full employment and unemployment regimes; and

(2.9) $$\frac{K_t}{L_t} = k_t$$ in the full employment and overemployment regimes.

Hereafter we omit a subscript t, when it is possible. Capital accumulation is solely determined by the savings decision as in a neoclassical growth model. In the Kaldor model, the increase (flow) of capital is the sum of workers' saving and capitalists' saving:

(2.10) $$\begin{aligned}\dot{K} &= s_w\, wL + s_K(Y - wL) \\ &= s_K F(K,L) + (s_w - s_K)wL \\ &= s_K Lf(K/L) + (s_w - s_K)wL\end{aligned}$$

where $\dot{K} = dK/dt$, $0 < s_w < 1$, and $0 < s_K < 1$. We take those saving rates as constants.

One of the characteristics of disequilibrium growth model is that the capital accumulation equation, (2.10), is different for each regime. This is clear from the relations (2.8) and (2.9).

The population is assumed to grow at a constant rate, n, i.e., $\dot{N}/N = n$. Since there are only two commodities in an economy, we can take the price of output as numeraire. Moreover, the demand and supply for output is always balanced, because output which is not consumed becomes saving which is equivalent to investment. We now consider the nominal wage (which is also the real wage) adjustment equation.

First, we take a simple scheme of wage adjustment. Assume the law of supply and demand, i.e., the wage increases in the overemployment regime and goes down in the unemployment regime. Moreover, we assume that the wage adjustment is proportional to the rate of unemployment or overemployment , but the proportion may be different in the positive direction or the negative direction.

$$\dot{w} \begin{cases} = \xi_1 \dfrac{L^d - L^s}{L^s} & \text{if } L^d \geq L^s, \quad \xi_1 > 0, \\ = \xi_2 \dfrac{L^d - L^s}{L^s} & \text{if } L^d < L^s, \quad \xi_2 > 0 \end{cases}$$

or rewriting using the established notation

$$\text{(2.12)} \qquad \dot{w} = \xi\left(\frac{k}{k^d(w)} - 1\right) \qquad \begin{array}{ll} \xi = \xi_1 & \text{if } L^s \leq L^d \\ \xi = \xi_2 & \text{if } L^d > L^s . \end{array}$$

We will see that the above equation does not keep an economy on the neoclassical capital deepening with full employment since it requires the wage rate go up as capital deepens. We may want to use a wage adjustment scheme which would keep an economy on the neoclassical path once it is on it. Therefore the wage adjustment consists of the effect of change in productiviey and the effect of disequilibrium:

$$(2.12) \qquad \dot{w} = - f''(k)\, k\dot{k} + \xi(k/k^d(w) - 1), \qquad \begin{array}{l} \xi = \xi_1 \text{ if } L^s \leq L^d \\ \xi = \xi_2 \text{ if } L^s > L^d. \end{array}$$

A contrast of (2.11) and (2.12) will be seen in the following subsections.

Full Employment Regime

First, we start by examining a regime where the current combination of the capital per capita and the wage rate gives a state of full employment. That is, from (2.4), (2.5) and $\ell \equiv 1$,

$$\Omega(w_t)K_t = N_t \qquad \forall t \; .$$

Solving this equation for w_t, we have "the full employment wage rate" denoted by w^*, depending on the capital per capita:

$$w_t^* = \Phi(k_t)$$

Since $k_t = k^d(w_t)$ in the full employment regime, we know by (2.7),

$$(2.13) \qquad w_t^* = \Phi(k_t) = f(k_t) - k_t f'(k_t) .$$

In other words, a full employment regime is defined as a set $R_f \subset \mathbb{R}^2_+$.

$$R_f = \{(k,w) \varepsilon \mathbb{R}^2_+ \mid w = f(k) - kf'(k)\}.$$

We know that the full employment wage rate is an increasing function with respect to the capital per capita, because of assumption (2.2).

$$(2.14) \qquad \frac{dw^*}{dk} = \Phi'(k) = -kf''(k) > 0.$$

Since the full employment wage rate is a function of the capital per capita, we have a dynamic equation of capital accumulation per capita, substituting (2.9) and (2.13) into (2.10):

$$(2.15) \qquad \dot{k} \equiv k(\dot{K}/K - \dot{N}/N)$$

$$= s_w f(k) - (s_w - s_K) kf'(k) - nk$$

Note that the increase in capital per capita does not depend on the wage rate any more. Since the labor market is in equilibrium, the wage rate does not change at the moment

$$(F) \qquad \begin{cases} \dot{k}_t = s_w f(k_t) - (s_w - s_K) k_t f'(k_t) - nk_t, \\ \dot{w}_t = 0, \end{cases} \quad \text{for } (k_t, w_t) \varepsilon R_f$$

The dynamic equations of (F) implies that (k_t, w_t) will not be in (F) at the next moment unless k_t is a special value so that $\dot{k} = 0$. It is heuristically interesting to consider the neoclassical model in our framework. Full employment is assumed over time in the neoclassical models by instantaneous adjustment of the wage rate. Therefore the dynamic equation of the neoclassical model is equation (2.15) only.

$$\text{(N)} \qquad \dot{k}_t = s_w f(k_t) - (s_w - s_K) k_t f'(k_t) - nk_t, \quad \forall k_t > 0$$

while the wage rate is adjusted to the labor productivity:

$$w_t = f(k_t) - k_t f'(k_t)$$

This wage rate is guaranteed to be achieved if the wage is adjusted according to (2.12) and the initial state is full employment.

In the next two sections, we assume that the wage adjustment follows (2.11). In section 4, we will come back to the implicaitons of the wage adjustment of (2.12).

We introduce a notion of "neoclassical steady states,":

output and inputs are growing at the same rate, or the natural rate, n. Let us denote the steady state capital per capita by $\hat{k}$ and the associated wage rate by $\hat{w}$.

(2.16) $$\hat{k} = \{k \mid s_w f(k) - (s_w - s_K) k f'(k) - nk = 0\}$$

(2.17) $$\hat{w} = f(\hat{k}) - \hat{k} f'(\hat{k})$$

Note that $\hat{k}$ gives the steady state to system (F), too.

We know that a neoclassical growth model is globally stable under plausible assumptions. Our concern here is how the conclusion may change when we allow short-run disequilibria. Therefore a strategy of research is that we take for granted the stability of the neoclassical model. It is well known that the Kaldor hypothesis, i.e., $s_K \geq s_w$, is a sufficient condition of stability in the two-class model. We, however, are also interested in a case of $s_w > s_K$ which is implied by the life-cycle hypothesis. We introduce a concept of the elasticity of substitution, σ, of a production function:

$$\sigma \equiv d\log(K/L)/d\log(F_L/F_K).$$

It is easy to verify that σ is described in terms of the intensive forms:[5]

(2.18) $$\sigma = -\frac{f'(k)\{f(k) - kf'(k)\}}{kf(k)f''(k)} > 0 .$$

The sign comes from assumption (2.2).

Theorem 2.1

The neoclassical model is defined by (2.1), (2.2), and (N). There exists a unique equilibrium (steady state) value of the capital per capita, and it is globally asymptotically stable, if

$$\text{(2.19)} \qquad \sigma > \left(1 - \frac{s_K}{s_w}\right) \frac{kf'(k)}{f(k)} \qquad \forall k > 0 .$$

A proof is given in Appendix 1.

Remark 1

(2.19) is satisfied if $s_K \geq s_w$, that is the original Kaldor hypothesis, since the RHS ≤ 0 and $\sigma > 0$.

Remark 2

(2.19) is satisfied if a production function is of the Cobb-Douglas type, $f(k) = Ak^{\alpha}$, even if $s_w > s_K$. Since $kf'(k)/f(k) = \alpha < 1$ and $\{1 - (s_K/s_w)\} < 1$, the RHS < 1. It is easy to see that $\sigma = 1$ for a Cobb-Douglas production function, therefore the LHS $= 1$.

Remark 3

With the same logic, (2.19) is satisfied when we have any production function with the more-than-one elasticity of substitution for $k > 0$, in addition to assumption (2.2).

Remark 4

The above remarks imply that the only chance that the neoclassical model is unstable is in a case where σ is sufficiently less than 1 and $s_w > s_K$. It is a well known proposition that $\sigma < 1$ implies the wage share $\{f(k)-f'(k)k\}/f(k)$ increases as k increases. Since the saving rate from wage income is larger, it is plausible that $\dot{k}$ increases as k increases. This means the steady state is unstable.

In the following we assume (2.19). Now let us go back to the full employment regime of the disequilibrium model. From (2.15) and (2.16), we have the following sign condition

$$\text{(FS)} \quad \begin{cases} \dot{k} < 0, & \dot{w} = 0 \quad \text{if } \hat{k} < k, \ (k,w) \varepsilon R_f \\ \dot{k} = 0, & \dot{w} = 0 \quad \text{if } k = \hat{k}, \ (k,w) \varepsilon R_f \\ \dot{k} > 0, & \dot{w} = 0 \quad \text{if } k < \hat{k}, \ (k,w) \varepsilon R_f \end{cases} .$$

Unemployment Regime

We now turn to the unemployment regime. The current wage rate is higher than the full employment wage rate, $w^* = \Phi(k_t)$, therefore $L^d < L^s$. The unemployment regime set on the (k,w) plane is denoted as R_u,

$$R_u = \{(k,w) \varepsilon \mathbb{R}^2_+ | w > f(k) - kf'(k)\}.$$

Since the firm's demand for labor is satisfied, the marginal condition (2.3) is attained. In other words, the capital-employment ratio is determined by (2.8), and equal to the demand capital-labor ratio, $k^d(w)$:

$$(2.20) \qquad w_t = f(k^d(w_t)) - k^d(w_t)f'(k^d(w_t)).$$

Using (2.20), the rate of capital accumulation is simplified as follows:

$$(2.21) \qquad \frac{\dot{K}}{K} = s_K \frac{f(k^d)}{k^d} + (s_w - s_K)\frac{w_t}{k^d}$$

$$= s_w f(k^d)/k^d + (s_K - s_w)f'(k^d).$$

Therefore

$$(2.22) \qquad \dot{k}_t = k_t[s_w f(k^d(w_t))/k^d(w_t) + (s_K - s_w)f'(k^d(w_t)) - n]$$

Let us denote [·] in (2.22) by $h(w_t)$. Note that $\dot{k} \gtrless 0$ as $h(w_t) \gtrless 0$, respectively. However, $h(w_t)$ is independent of k_t. This is resulted from a homogeneous production function. Since the labor demand is multiplicatively separable with respect to the wage rate and the total capital stock, the capital per the employed is independent of the total capital stock, but only the ratio between the capital stock and employment matters.

For the dynamic equation for the wage rate, assume (2.11). Therefore (2.11) and (2.22) give a system of equations which describes the motion in the R_u and its boundary R_f.

$$(U) \qquad \begin{cases} \dot{k}_t = k_t[s_w f(k^d(w_t))/k^d(w_t) + (s_K - s_w)f'(k^d(w_t)) - n] \\ \dot{w}_t = \xi_2\{k_t/k^d(w_t) - 1\}, \qquad \text{for } (k_t, w_t) \in R_u \cup R_f . \end{cases}$$

Note that (U) coincides with (F) at $(k,w)\varepsilon R_f$. Now we check the sign of $\dot{k}$ in the unemployment regime. First note that $\dot{k}_t = 0$ at $(\hat{k},\hat{w})$. Secondly,

$$\left.\frac{dw}{dk}\right|_{\substack{(k,w)\varepsilon R_u \\ \dot{k}=0}} = -\left.\frac{\partial\dot{k}/\partial k}{\partial\dot{k}/\partial w}\right|_{\substack{(k,w)\varepsilon R_u \\ \dot{k}=0}} = 0 ,$$

since $k(w_t)$ in equation (2.22) is independent of k_t. Thirdly

$$\frac{\partial\dot{k}}{\partial w} = k \frac{dh}{dk^d}\frac{dk^d}{dw}$$

$$= \frac{-k_t}{k^d f''(k^d)}\left[\frac{s_w}{(k^d)^2}\{k^d f'(k^d) - f(k^d) - (k^d)^2 f''(k^d)\} + s_K f''(k^d)\right]$$

where $[\cdot] < 0$ if condition (2.19) is satisfied, by the same argument in the proof of Theorem 2.1. Therefore

$$\text{(US)}\quad\begin{cases} \dot{k}_t < 0, \quad \dot{w}_t < 0 & \text{if } \hat{w} < w_t,\ (k_t,w_t)\varepsilon R_u \\ \dot{k}_t = 0, \quad \dot{w}_t < 0 & \text{if } w_t = \hat{w},\ (k_t,w_t)\varepsilon R_u \\ \dot{k}_t > 0, \quad \dot{w}_t < 0 & \text{if } w_t < \hat{w},\ (k_t,w_t)\varepsilon R_u \end{cases}$$

gives the directions of state variables.

Overemployment Regime

Lastly, we examine a case of the overemployment regime, where $L^s < L^d$, or equivalently $w_t < w^* = \Phi(k)$. A set of combinations, (k,w), which gives the overemployment regime is denoted by R_o,

$$R_o = \{(k,w) \varepsilon \mathbb{R}^2_+ \mid w < f(k) - kf'(k)\}.$$

Since the actual employment is determined by the supply side, substitute (2.9) into (2.10) to obtain the following equation:

$$\frac{\dot{K}}{K} = \frac{s_K f(k_t)}{k_t} + (s_w - s_K)\frac{w_t}{k_t} ,$$

or in terms of the intensive form

$$\text{(2.23)} \qquad \dot{k} = s_K f(k_t) + (s_w - s_K)w_t - nk_t.$$

Since the demand for labor is quantity-constrained, the marginal condition of (2.3) is not satisfied. That is $w_t \neq f(k_t) - kf'(k_t)$. Take equation (2.11) as the wage adjustment in the overemployment regime. Therefore (2.11) and (2.23) give a system of equations which describes the motion in the R_o and its boundary R_f.

$$\text{(O)} \quad \begin{cases} \dot{k}_t = s_K f(k_t) + (s_w - s_K)w_t - nk_t \\ \dot{w}_t = \xi_1\{k_t/k^d(w_t) - 1\} , \quad \text{for } (k_t,w_t)\varepsilon R_o \subset R_f. \end{cases}$$

Note that (O) coincides with (F) at $(k,w)\varepsilon R_f$. Next we check the signs of time derivatives. First, from (2.23) we know that

$$\dot{k} \gtreqless 0 \quad \text{iff} \quad s_K f(k) + (s_K - s_w)w - nk \gtreqless 0, \text{ respectively.}$$

Define the wage rate which gives the stationary movement of k in the overemployment regime,

(2.24)
$$\begin{cases} \phi(k) = \{nk - s_K f(k)\}/(s_w - s_K) , & s_w \neq s_K . \\ \phi(k) = \hat{k} , & s_w = s_K . \end{cases}$$

Then

(OS)
$$\begin{cases} \dot{k}_t = 0 , \dot{w}_t > 0 & \text{iff } w_t = \phi(k_t); \\ \dot{k}_t > 0 , \dot{w}_t > 0 & \text{if } \{w_t - \phi(k_t)\}\{s_w - s_K\} > 0, \ s_w \neq s_K \\ & \text{or if } k_t < \hat{k} , s_w = s_K; \\ \dot{k}_t < 0 , \dot{w}_t > 0 & \text{if } \{w_t - \phi(k_t)\}\{s_w - s_K\} < 0 , \ s_w \neq s_K \\ & \text{or if } k_t > \hat{k} , s_w = s_K \\ \text{for } (k_t, w_t) \in R_O \end{cases}$$

A careful examination of relative positions of $\Phi(k)$, the full employment wage rate, and $\phi(k)$ shows that

(2.25) if $s_K < s_w$, then $\phi(k) \lesseqgtr \Phi(k)$ for $k \lesseqgtr \hat{k}$, respectively and $\phi'(k) > 0$;

(2.26) if $s_w < s_K$, then $\phi(k) \lesseqgtr \Phi(k)$ for $\hat{k} \lesseqgtr k$, respectively. and $\phi'(k) < 0$.

(See the Appendix 2 for the derivation). Now we are ready to draw a phase diagram for a disequilibrium system, which consists of (U) (F) and (O) for $(k,w) \in \mathbb{R}^2_+$. By combining (FS), (US), (OS), (2.13), (2.16), (2.17), (2.24), (2.25) and (2.26), we have Figures 1 and 2 for the cases of $s_K < s_w$ and $s_w < s_K$, respectively. If $s_K = s_w$, then $\phi(k)$ is vertical at $k = \hat{k}$.

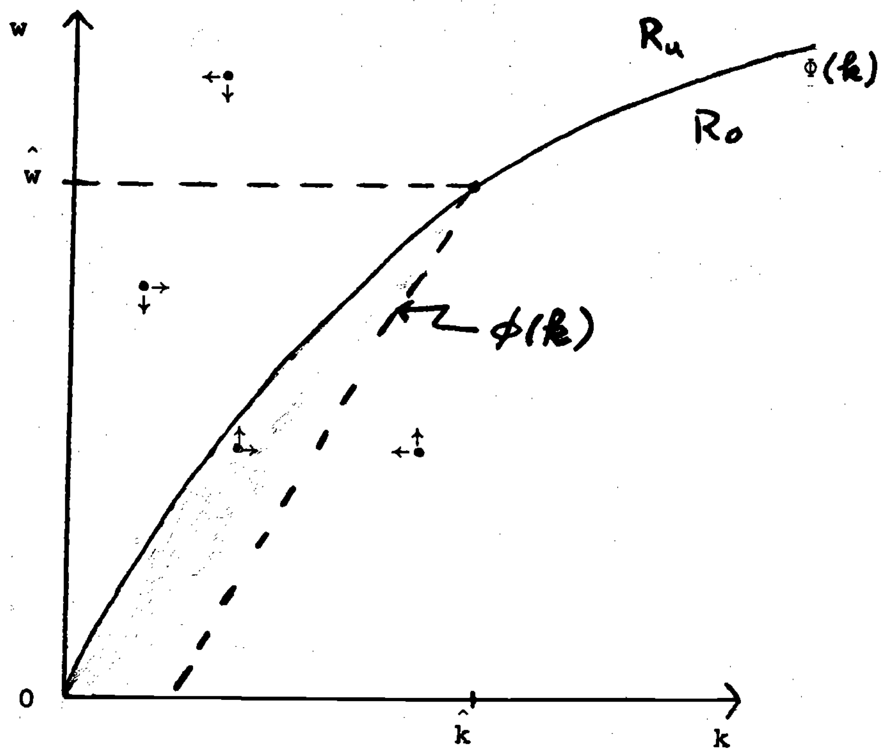

Figure 1 $s_w > s_K$

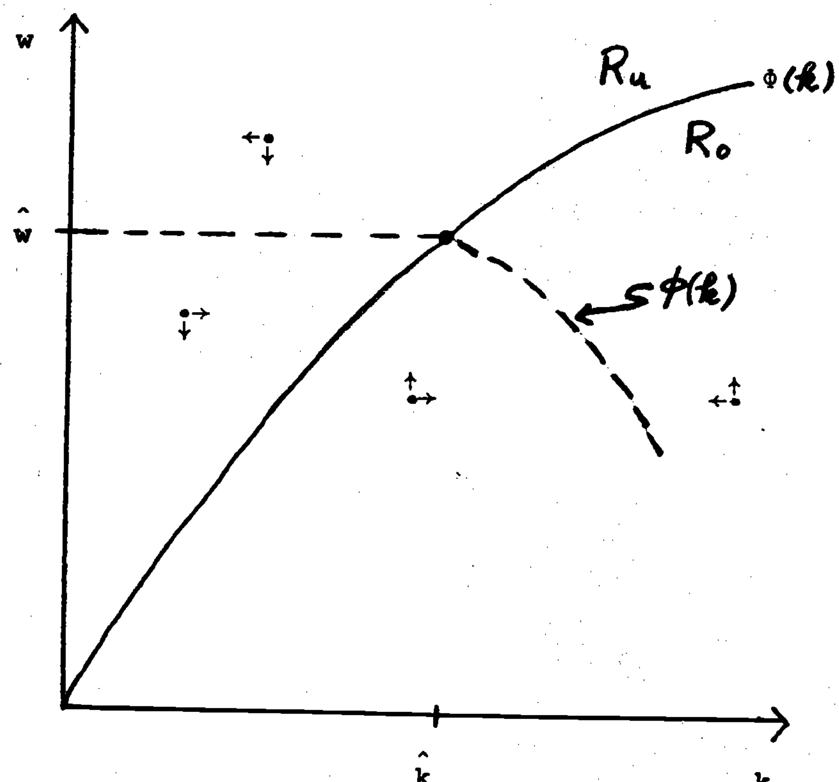

Figure 2 $s_w < s_K$

The full employment regime is described by a curve $\Phi(k)$ in Figures 1 and 2. The neoclassical growth model, (N), is a special case here in that it is restricted to a curve $\Phi(k)$. The unemployment regime is anywhere above the curve, and the overemployment regime is anywhere below the curve. The dotted curves are a combination of (k,w) which gives $\dot{k} = 0$.

In the following sections, we will examine the stability of the disequilibrium system of (U)(F) and (O), and discuss the implications of the model.

3. Stability

We have started with a question of how a model changes when we allow disequilibria while the equilibrium (neoclassical) path shows the global stability to the long-run steady state, $\hat{k}$. We are now ready to answer the above question. Our disequilibrium system consists of two different sets of differential equations, (U) and (O). They give the same values at their common boundary. (F). Since the domain is divided into two different regions, the usual theorems of the local or global stability cannot be applied without modification and restriction. Especially note that the global stability in each set of differential equations (assuming it is defined for $\mathbb{R}^2_+$) does not guarantee the global stability of a "patched-up" system. However, we can assert the global, therefore local, stability in the case of $s_K < s_W$, by just looking at the Figure 1. Notice that the region between $\Phi(k)$ and $\phi(k)$; $k < \hat{k}$ has a property to "lock in" the solution path, once it comes in the region. Since $\dot{k} \to 0$ as $k \to 0$, a solution path never hits the vertical axis even if $w > \hat{w}$. It is clear from the diagrams that a solution path does not approach to the horizontal axis. Therefore a solution path has to converge to the steady state eventually. The case can be extended to a special case $s_K = s_W$.

Therefore we have the following theorem.

Theorem 3.1

In the case of $s_K \leq s_W$, a disequilibrium system of (U) and (O) has the unique steady state $(\hat{k},\hat{w})$ and that is globally asymptotically stable.

In the case of $s_W < s_K$, it is a complicated matter to establish the local stability, and the global stability depends on the wage adjustment speeds. First, we prove that for the local stability of a patched-up system, (U) and (O), it is sufficient to prove the local stability for each (U) and (O) around the steady state point, given the boundary can be approximated by a linear line.[7]

Theorem 3.2

In the case of $s_W < s_K$, a disequilibrium system defined by (U) and (O) is locally stable at $(\hat{k},\hat{w})$, in the sense that $\exists \delta > 0$, such that

$$\lim_{t\to\infty} (k_t, w_t \mid (k_o, w_o)) = (\hat{k},\hat{w}), \text{ if } \|(k_o, w_o) - (\hat{k},\hat{w})\| < \delta$$

A proof is given in Appendix 3.

It is clear from Figure 2 that the disequilibrium system may oscillate to the limit cycle, or explode cyclically. Since the stable neoclassical growth model is a special case that $\xi_1 = \xi_2 = \infty$, one may conjecture that if ξ_1 and ξ_2 are sufficiently large, then the disequilibrium system (U) and (O) with $s_W < s_K$ is also globally asymptotically stable.

Theorem 3.3

If ξ_1 and ξ_2 are sufficiently large, a disequilibrium system defined by (U) and (O) with $s_w < s_K$ is globally asymptotically stable.

$$\exists\ \xi_1 < \infty \text{ and } \xi_2 < \infty \text{ such that } \lim_{t\to\infty} (k_t, w_t \,|\, (k_o, w_o) \in \mathbb{R}^2_+) \to (\hat{k}, \hat{w}).$$

Proof

It is a well-known theorem that if there is a Lyapunov function, $V \geq 0$, such that $V = 0$ at the unique equilibrium point and $\dot{V} < 0$ anywhere else where it is defined, then the equilibrium point is globally asymptotically stable.

Let us define a function $V(k,w)$ as follows

$$V(k_t, w_t) = \tfrac{1}{2}\{(w_t - \Phi(k_t))^2 + (k_t - \hat{k})^2\}$$

$$V(\hat{k}, \hat{w}) = 0, \text{ and } V(k_t, w_t) > 0 \quad \text{if } (k,w) \neq (\hat{k}, \hat{w}).$$

$$\dot{V} = \dot{w}(w_t - \Phi(k)) - \Phi'(k)\dot{k}_t(w_t - \Phi(k)) + (k - \hat{k})\dot{k}\ .$$

It is immediately resulted from (FS) (US) and (OS) that

$$\dot{V}(k_t w_t) < 0 \qquad \text{if (i) } w_t \leq \hat{w}$$
$$\text{and } (k_t, w_t) \in R_u \cup R_f / (\hat{k}, \hat{w})$$
$$\text{if (ii) } \phi(k_t) \leq w_t \leq \Phi(k_t)$$
$$\hat{k} \leq k_t, \text{ and } (k_t, w_t) \in R_o \cup R_f / (\hat{k}, \hat{w}).$$

Therefore we can choose ξ_2 sufficiently large so that

$$(3.1)\qquad \dot{V}(k_t,w_t) < 0 \qquad \text{if } (k_t,w_t)\varepsilon R_u \text{ and } \hat{w} < w_t$$

and we can also choose ξ_1 sufficiently large so that

$$(3.2)\qquad \dot{V}(k_t,w_t) < 0 \qquad \text{if } (k_t,w_t)\varepsilon R_o \text{ and } w_t < \phi(k_t),\ \hat{k} < k_t \text{ or } w_t < \Phi(k_t),\ \hat{k}_t < k\ .$$ [8]

4. Alternative Wage Adjustment

We showed an alternative wage adjustment ahcme, (2.12), which consists of the productivity effect and the disequilibrium effect. Then the wage adjustment equations for the regimes (F), (U), and (O) are given by (4.1), (4.2), and (4.3), respectively.

(4.1) $\dot{w} = -f''k\dot{k}$

(4.2) $\dot{w} = -f''k^2[s_w f(k^d)/k^d + (s_k - s_w)f'(k^d) - n] + \xi_2(k/k^d - 1)$

(4.3) $\dot{w} = -f''k(s_k f(k) + (s_w - s_k)w - nk) + \xi_1(k/k^d - 1)$

It is easy to see that the phase diagram now looks like Figures 3 and 4.

Unlike an economy with (2.11) illustrated in Figure 1 and 2, and economy with (2.12) shown in Figures 3 and 4 never switch regimes. Therefore an question of stability becomes simple, while the model loses its endogenous force for a business cycle. Noting that a trajectory never crosses the full-employment regime, the following theorem is obvious from examining Figures 3 and 4.

Theorem 4.1

A disequilibrium path described by (U), (F), and (O) with (4.1) (4.2) and (4.3) replacing the wage adjustment equations, is globally stable if and only if the neoclassical path is stable, i.e., (2.19) is satisfied.

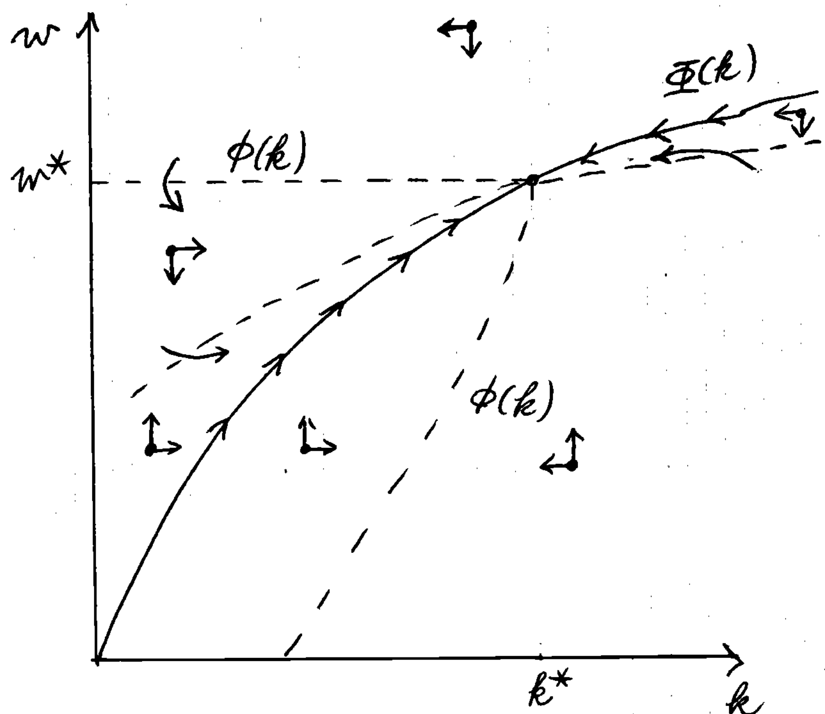

Figure 3 :
$s_w > s_k$

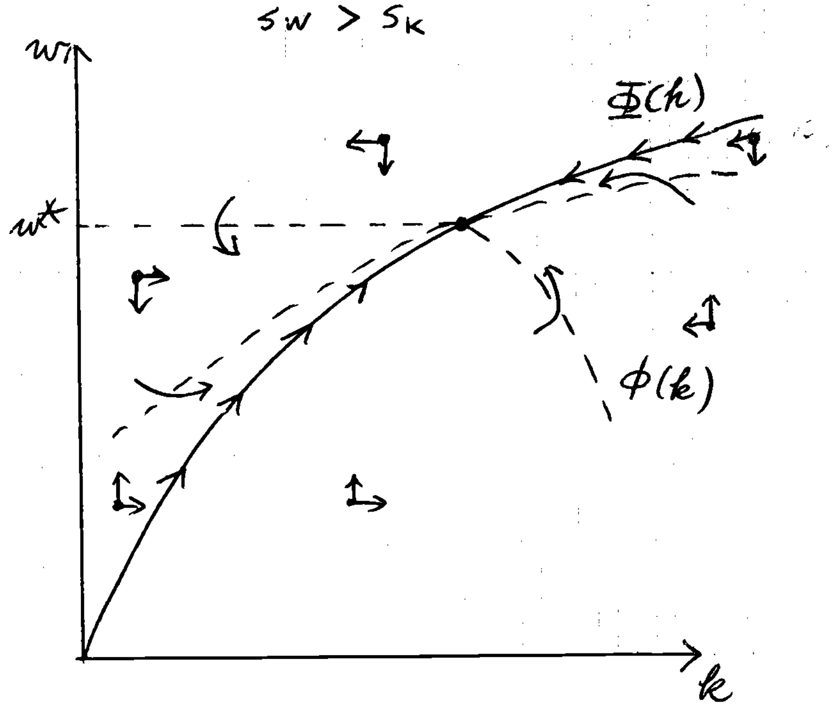

Figure 4 :
$s_w < s_k$

5. Disequilibrium Dynamics and Comparative Statics

In this section, we study the implications of the results we had in the previous sections.

Short-run and Long-run Stability

A disequilibrium growth path in a stable case can be viewed as "a sequence of short-run quantity-constrained equilibria toward the long-run market-clearing equilibrium." In a general (dis)equilibrium framework, a quantity-constraint in a market (the labor market in our model) forces constrained agents to spill-over unsatisfied demand or supply into other markets (the goods market in our model) to form the "effective" demand (a Keynesian consumption function in our model). If the number of markets in an economy is more than two, the existence of a quantity-constrained (fixed-price) equilibrium is not a trivial question, but a proposition to be proved.[9] However, our model is so simple that a spill-over from the labor market is always completely absorbed in the goods market. Therefore there is always a unique quantity-constrained equilibrium for the arbitrary wage rate and the arbitrary capital per capita. The stability of a sequence consisting of (short-run) quantity-constrained equilibria is a long-run phenomenon in the sense that the model incorporates the capital accumulation.

Business Cycles and a Stable Corridor

In the case that $s_W < s_K$ with the simple wage adjustment of (2.11) we showed that a disequilibrium path may oscillate around the equilibriu before the path, if ever, converges to the equilibrium. This is observed as business cycles over unemployment periods (i.e., depressions) and overemployment periods (i.e., booms). In each type of period, there are two sub-periods; one is the "capital-deepening" period and the other is the "capital-shallowing" (or slower-capital-deepening in the case of Harrod neutral technological progress) period. Stability theorems in the preceding section suggest that a small displacement of the real wage rate or the capital per capita from the long-run steady state results in short-run disequilibrium in the labor market; but eventually the long-run equilibrium will be resumed. Suppose an imaginary experiment of letting the displacement larger. Then it may be the case that there is a bound, depending on ξ's, beyond which a path may diverge away. Then the bounded neighborhood of the long run equilibrium may be regarded as the stable neighborhood or "a stable corridor"[10] if we recall that the long-run equilibrium in which the capital stock and the production are growing at the natural rate.

Long-run Fixed Price Equilibria

We explained that any point in Figures 1 and 2 is viewed as a quantity constrained equilibrium given the fixed real wage in the short-run. It is of interest to study a case that the real wage is fixed in the long-run as well as in the short-run. We may assume that the government or the powerful trade union impose the fixed real wage for a certain period. Suppose that the real wage is fixed less than the long-run steady state value, i.e., $\bar{w} < \hat{w}$, where $\bar{w}$ is the fixed real wage. Holding $w_t = \bar{w}$, for $t > 0$, the economy will converge to some k which induces no change in the capital per capita, i.e., $\dot{k} = 0$. Such k is calculated as $\bar{k} = \phi^{-1}(\bar{w})$. We find the long-run fixed wage equilibrium, $(\bar{k},\bar{w})$, in the overemployment regime. In the case of $s_K < s_W$, we learn, from Figure 1, that $\bar{k} < \hat{k}$, i.e., the capital accumulation is short of the neoclassical long-run capital per capita. In the case of $s_W < s_K$, Figure 2 tells us that $\hat{k} < \bar{k}$, i.e., the capital is over-accumulated relative to the neoclassical long-run capital per capita. By a similar exercise, we find that if $\bar{w} > \hat{w}$, then the capital is shallowing forever. The most interesting case occurs when the real wage rate is set to the long-run equilibrium rate "prematurely" in the sense that the capital per capita is still below its long-run steady state value, i.e., $\bar{w} = \hat{w}$, $k_t < \hat{k}$. Then the economy is stuck at the current capital per capita, i.e., $\dot{k}_t = 0$ for all $k_t < \hat{k}$ if $\bar{w} = \hat{w}$. Since it is under the unemployment regime, we observe that the steady state price signal, $\hat{w}$, may cause the quantity-constrained (underemployed) equilibria, $(\hat{w},\bar{k})$, $\bar{k} < \hat{k}$, as well as the full employment equilibrium.[10] Let us state this proposition formally.

Proposition 4.1

There are unemployment equilibria, if the real wage rate is fixed at the steady state rate in the long-run and if the capital per capita is below its steady state level.

Comparative Statics

We examine shifts of the steady states responding to changes in parameters. First, suppose that the saving rates from wage income and/or from profit increased. The full employment wage rate curve, $\Phi(k_t)$, does not change, although the steady state $(\hat{k},\hat{w})$ shifts to the north-east on the $\Phi(k_t)$ curve. Therefore an economy will be in the overemployment capital-deepening regime in the transition period from the old to new steady states. A decrease in either one or both of the saving rates causes a symmetric change in the position of the steady state. However, the transition period should be in the unemployment capital shallowing regime. Cyclic behaviors may occur in both directions of changes in the savings rate, if $s_w < s_K$ and the adjustment speeds of the wage rate is sufficiently small.

Secondly, suppose that there is a once-and-for-all shift in the production function. Assume that sudden manna is given in a multiplicatively separable form, i.e., f(k) becomes Af(k), A>1. Then the full employment wage rate curve shifts upward, while the steady state shifts to the north-east. The transition period should be in the overemployment regime.

5. Concluding Remarks

We have demonstrated an idea of disequilibrium growth theory in a simple one-sector model. It has several attractive features: explicit analysis of spill-over effects in the general (dis)equilibrium framework; simultaneous adjustment of prices and quantities in the long-run as a sequence of short-run fixed price equilibria. However, the power of analysis is severely limited by the fact that there is only one malleable commodity. The model is philosophically classical (or anti-Keynesian) in the sense that the supply side determines output. The disequilibrium labor market determines the actual employment at the minimum of demand and supply. This in turn determines actual output, which is exactly absorbed either as consumption or capital accumulation. This story is clearly what Keynes and his "faithful" followers attack. This may seem paradoxical because disequilibrium macroeconomic models are praised mainly because they claim their restoration of "Keynes's economics" as opposed to "Keynesian economics."

The present disequilibrium growth model is a hybrid of a neoclassical growth theory and a disequilibrium macroeconomic model, and the anti-Keynesian nature is inherited from a neoclassical growth theory, i.e., a one-sector model without money.

[1]Growth theory is a much explored field which studies an economy with accumulating capital, increasing population, technological progress, and changes in factor shares. For best surveys, Hahn and Matthews (1964) and Burmeister and Dobell (1970) should be consulted. We have two different breeds of growth theory: First, Keynesian growth theory, started by Harrod (1943) and Domar (1957), emphasizes the instability of the warranted growth rate. Secondly, neoclassical growth theory, initiated by Solow (1956) and Swan (1956), shows the stability of the steady state. Keynesian instability essentially comes from non-substitutability, i.e., fixed coefficients, of factors for production. The Keynesian growing economy will find itself in either chronic inflation or persistent unemployment, while factors are always fully employed by the assumption of perfect price flexibility and smooth factor substitutions in the neoclassical growing economy. Since substitutions between labor and capital seem plausible in the long run, the neoclassical models outnumbered the Keynesian models during the most fruitful period of research. But neoclassical models could not account for unemployment or inflation.

[2]There have been several works on "disequilibrium growth" or "unemployment in a theory of growth." Tobin (1955) showed that when the nominal wage is inflexible downward, an economy may experience cyclic fluctuations or stagnation depending upon whether the short-run adjustment is unstable or stable, respectively. Although the ideas are similar, our model benefits from several features derived from the modern disequilibrium macromodels, such as ability to trace disequilibrium dynamic paths of the capital per capita. Rose (1966, 1967) constructed a model to incorporate unemployment. However, he merely replaces some equalities of the equilibrium conditions by lagged adjustment equations. Hadjimichalakis' model (1971) also adopts lagged adjustment equations instead of instantaneous equalities of Tobin's monetary growth model. Hahn (1960) studied seemingly disequilibrium models where lagged adjustments are introduced both in prices and the capital-labor ratio. In his model A, the real rental rate is not always equal to the marginal productivity of capital while the instantaneous adjustment of the nominal wage makes the real wage equal to the marginal productivity of labor. Model B is a case where both marginal productivities may deviate from the real input price. However, if both real input prices are higher than their corresponding productivities, then firms are making negative profit. Our approach will be different from the above works by two features: We employ the minimum transaction rule in the disequilibrium market; and unsatisfied demand or supply spills over to the other markets. These features are inherited from disequilibrium macroeconomic models.

[3]This is similar to Kaldor's (1960) hypothesis. However, the original Kaldor hypothesis was that the profit saving rate is not only different but higher than the wage saving rate, although the Diamond model, or the life-cycle hypothesis in general, implies the opposite. Modigliani (1975) discussed theoretical and empirical comparisons between the Kaldor hypothesis and the life-cycle hypothesis. Since we take the Kaldor model where the saving rate of workers is higher than that of capitalists as an approximation of an overlapping generations model, we do not worry about the Pasinetti paradox.

[4] The qualitative nature of the model would not change even if the labor supply per capita depends on the wage rate. Assume that $\ell(0) = 0$, $\lim_{w \to \infty} \ell(w) < \infty$, and $\ell'(w) > 0$ for $0 \leq w < \infty$. Then almost all the analysis except in the case of overemployment with $s_K > s_W$ goes through, by replacing k by $k/\ell(w)$ in the following.

[5] For the derivation of (2.18), see Allen (1967; Section 3.6).

[6] Suppose that we have two systems of differential equations. Both have the stable elliptic solution paths around the equilibrium. However, one has a major axis horizontally, while the other vertically. Patch these systems in such a way that the solution path always goes from a point on a minor axis to that of a major axis, then the patched-up system is globally unstable.

7 Veendorp (1975) faced a similar problem of stability in switching regimes. The sufficient condition he used cannot be applied in our problem, since the $\dot{k}=0$ line in the unemployment regime is horizontal. Our condition for local stability cannot be applied to Veendorp's problem, either, because he has four regimes instead of two.

8 (3.1) gives the following:

$$\xi_2 > k[s_w f(k^d(w))/k^d(w) + (s_K - s_w)f'(k^d(w)) - n]$$

$$\cdot [\Phi'(k)(w-\Phi(k)) - (k-\hat{k})]/[(k/k^d(w) - 1)(w-\Phi(k))]$$

(3.2) gives the following:

$$\xi_1 > [s_K f(k) + (s_w - s_K)w - nk][\Phi'(k)(w-\Phi(k)) - (k-\hat{k})]$$

$$\div [(k/k^d(w) - 1)(w-\Phi(k))]$$

Note that $k \neq k^d(w)$ and $w \neq \Phi(k)$ by definition of regimes. Moreover if $w \to \Phi(k)$, which may occur when $\hat{k} < k$ in the unemployment capital-shallowing regime and when $k < \hat{k}$ in the over-employment capital-deepening regime, $\dot{V} < 0$ independent of the magnitude of a positive speed of adjustment.

9 See Benassy (1975) and Drèze (1975).

10 Leijonhufvud (1973) proposed an idea of a corridor in which "the market system tends to move 'automatically' towards a state where all market excess demands and supplies are eliminated." Outside the corridor, counter-acting tendencies become "weaker as the system becomes increasingly subject to 'effective demand failures'."

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Appendix 1

A proof of Theorem 2.1 is routine in a neoclassical growth model. Note that

$$\dot{k} = 0 \qquad \text{iff} \qquad \dot{K}/K - \dot{N}/N = 0 \ .$$

Since $\dot{N}/N = n$, a constant with a finite value, the following condition is enough to assert the existence, uniqueness, and globally asymptotic stability of the steady state:

(i) $\lim_{k \to 0} (\dot{K}/K) = \infty$

(ii) $\lim_{k \to \infty} (\dot{K}/K) < n$

(iii) $\frac{d}{dk}(\frac{\dot{K}}{K})$ exists and is strictly negative for $k > 0$.

Now recall that $\dot{K}/K = s_w f(k)/k + (s_K - s_w) f'(k)$

$$\lim_{k \to \infty} (\dot{K}/K) = s_w \lim_{k \to \infty} \frac{f(k)}{k}$$

$$= s_w \frac{\lim_{k \to \infty} f'(k)}{\lim_{k \to \infty} 1}$$

$$= 0, \qquad \text{(ii) is proved.}$$

To prove (i), we rewrite $\dot{K}/K$ as follows:

$$\frac{\dot{K}}{K} = s_w \{f(k) - kf'(k)\}/k + s_K f'(k)$$

$$\lim_{k\to 0} (\dot{K}/K) = s_w \frac{\lim_{k\to 0} - f''(k)\cdot k}{\lim_{k\to 0} \quad 1} + \infty = \infty \text{ , (i) is proved.}$$

It is obvious that $\dot{K}/K$ is differentiable by an assumption on F.

$$\frac{d}{dk}\left(\frac{\dot{K}}{K}\right) = s_w \left[-k^2 f''(k) - \{f(k) - kf'(k)\}\right]/k^2 + s_K f''(k).$$

$$\frac{d}{dk}\left(\frac{\dot{K}}{K}\right) < 0 \quad \text{iff} \quad \frac{-\{f(k) - kf'(k)\}}{k^2 f''(k)} - 1 > - \frac{s_K}{s_w}$$

$$\text{or} \quad - \frac{f'(k)\{f(k) - kf'(k)\}}{kf(k)f''(k)} > \left(1 - \frac{s_K}{s_w}\right) \frac{kf'(k)}{f(k)}$$

Now the LHS of the last inequality is the definition of the elasticity, (2.18). Therefore if $\sigma > (1 - \frac{s_K}{s_w})\frac{kf'(k)}{f(k)}$, which is exactly a stability condition, (2.19), then (iii) is true.

Q.E.D.

Appendix 2

$$\phi(k) - \Phi(k) = \{nk - s_K f(k)\}/(s_w - s_K) - [f(k) - kf'(k)]$$

$$= [nk - s_w f(k) + (s_w - s_K)f'(k)k]/(s_w - s_K)$$

$$= [s_w f(k) - (s_w - s_K)f'(k) \cdot k - nk]/(s_K - s_w)$$

Note that [·] in the last line is the definition of $\dot{k}$ in the full employment regime, i.e., equation (2.15). Therefore

(i) $\phi(\hat{k}) = \Phi(\hat{k})$,

and $\phi(k) \neq \Phi(k)$ for $k \neq \hat{k}$,

(ii) if $s_K < s_w$ and $\hat{k} \lesseqgtr k$, i.e., $\dot{k}\Big|_{\dot{k} = (2.15)} \lesseqgtr 0$,

then $\phi(k) \gtreqless \Phi(k)$, respectively.

(iii) if $s_w < s_K$ and $\hat{k} \lesseqgtr k$, i.e., $\dot{k}\Big|_{\dot{k} = (2.15)} \lesseqgtr 0$,

then $\phi(k) \lesseqgtr \Phi(k)$, respectively.

Next, we check the sign of ϕ' at $k = \hat{k}$.

$$\phi'(k) = \frac{dw}{dk}\Bigg|_{\substack{(k,w)\varepsilon R_o \\ \dot{k}=0}} = - \frac{\partial\dot{k}/\partial k}{\partial\dot{k}/\partial w}\Bigg|_{\substack{(k,w)\varepsilon R_o \\ \dot{k}=0}}$$

$$= - \frac{s_K f'(k_t) - n}{s_w - s_K}\Bigg|_{\substack{(k,w)\varepsilon R_o \\ \dot{k}=0}}$$

$$\phi'(\hat{k}) = \left.\frac{dw}{dk}\right|_{\substack{(k,w)\varepsilon R_o \\ \dot{k}=0 \\ k\to+\hat{k}}} = \frac{s_w[f(\hat{k}) - \hat{k}f'(\hat{k})]}{\hat{k}\ (s_w - s_K)} \begin{array}{ll} < 0 & \text{if } s_w < s_K \\ > 0 & \text{if } s_K < s_w \end{array}$$

Appendix 3

First, we establish a theorem on the (global/local) stability of linear differential equations on a two dimensional Euclidean space.

Suppose that two sets of linear differential equations are defined on $\mathbb{R}^2$:

(*1) $$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad (x,y)\in\mathbb{R}^2$$

(*2) $$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad (x,y)\in\mathbb{R}^2$$

A "patched-up" system with a linear boundary is defined by the following:

(*) $$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad \begin{array}{l} i=1 \text{ if } (x,y)\in R_I \\ i=2 \text{ if } (x,y)\in R_{II}\end{array}$$

$$R_I = \{x\in\mathbb{R}^2 \mid hx + ky \geq 0\}$$
$$R_{II} = \{x\in\mathbb{R}^2 \mid hx + ky < 0\}$$

There exists at least one solution path for (*), if solution paths are connected properly over the boundary, $hx + ky = 0$: a solution path at a point on the boundary coming from one region has to go out to the other region. In other words, the direction of solution curves on the linearized boundary relative to the direction of the boundary itself should agree for both systems of differential equations. Existence of such a path is guaranteed by Henry (1972).

In mathematical notation

$$\text{(E)} \qquad [(h\ k)\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} k \\ -h \end{pmatrix}][(h\ k)\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} k \\ -h \end{pmatrix}] > 0 .$$

Assume that (*1) and (*2) have a unique (common) equilibrium on the boundary; $\exists !\ (\hat{x},\hat{y})$, such that $h\hat{x} + k\hat{y} = 0$ and

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad i = 1,2 .$$

<u>Theorem A</u>

<u>Suppose a piecewise linear differential equation system defined by (*) satisfies condition (E). If the unique equilibrium is stable with (*1) and (*2)</u>

$$\text{(S)} \qquad \begin{cases} a_i + d_i < 0 , \\ a_i d_i - b_i c_i > 0 , \qquad i=1,2 \end{cases}$$

<u>then the "patched-up" system with a linear boundary, (*), has a stable solution path, i.e.,</u>

$$\lim_{t\to\infty} x(t|(x_o,y_o)) = \hat{x}$$

$$\lim_{t\to\infty} y(t|(x_o,y_o)) = \hat{y} \qquad \text{for } (x_o,y_o)\in\mathbb{R}^2$$

Let us assume without loss of generality that the origin is the equilibrium point, $(\hat{x},\hat{y}) = (0,0)$.

Proof of Theorem A

First, note that the i-th system of differential equations is symmetric about the origin:

$$\text{(Y)}\qquad \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} -x \\ -y \end{pmatrix} = -\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad i=1,2 .$$

Secondly, a system is homogeneous:

$$\text{(H)}\qquad \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} m\,x \\ m\,y \end{pmatrix} = m\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{matrix} m>0 \\ i=1,2 . \end{matrix}$$

Now consider a solution path starting from (x_o,y_o), a point on the boundary. Let us define a solution path of (*) as

$$\{(x_t,y_t)\,|\,(x_o,y_o)\} \text{ where } \begin{aligned} x_t &= x(t|(x_o,y_o)) \\ y_t &= y(t|(x_o,y_o)) \end{aligned}$$

$$\text{such that } hx_o + ky_o = 0.$$

Since a solution path of either (*1) or (*2) alone must converge to the origin, the solution path $\{(x_t,y_t)\,|\,(x_o,y_o)\}$ must either converge to the origin without switching regimes, or intersect the boundary to switch regimes. We claim that <u>(x_t,y_t) (x_o,y_o) must intersect somewhere in the interval, $((-x_o,-y_o),\ (0,0))$ before the first switching, if any, of regimes</u>.

Once the above claim is proved, we know that every time regimes switch one from another, the distance of switching point from the origin shrinks. Moreover, by the homogeneity the ratio of shrinking (of every two switchings) stays constant, so that $\{(x_t,y_t)\,|\,(x_o,y_o)\}$ does converge to the origin.

The solution path $\{(x_t,y_t)|(x_o,y_o)\}$ is directed (i) toward the origin staying on the linearized boundary, (ii) away from the origin staying on the linearized boundary, or (iii) toward the interior of one of the two regions, say, R_I without loss of generality. In a case of (ii), it violates the stability of each system itself, (S). In case of (i), the solution path converges to the origin whenever it starts at the linearized boundary. It implies that a solution path starting at an arbitrary initial point stays on the same region forever (because solution paths cannot "meet" or "cross"). Therefore (S) is enough to assert that (*) is locally asymptotically stable. In a case of (iii), we need a careful examination. By homogeneity, (H), directions of solution paths starting at all points on a linearized boundary on the side of (x_o, y_0) are proportional. Therefore a solution path starting from (x_0, y_0) cannot intersect that side of the linearized boundary as the first switching point. Next suppose that the solution path overshoots the origin and intersects the linearized boundary further than the symmetric initial point.

If the I-st system of differential equations is defined for the entire plane, the solution path starting at $(-x_o,-y_o)$ according to (*1) must have a symmetric path to the one starting from (x_o,y_o) by (Y), and must intersect the linearized boundary beyond (x_o,y_o). This implies the I-st system of differential equations has instability, and contradicts assumption (S). The above examination leaves the possibilities that the solution path must intersect somewhere in $[(0,0), (-x_o,-y_o))$ after travelling in R_I. In a case of arriving at the origin, other solution paths starting anywhere in $((0,0), (x_o,y_o))$ should arrive at the origin by homogeneity. Now suppose that the solution path arrives at $(-x_1,-y_1) \varepsilon ((-x_o,-y_o), (0,0))$, where the system switches to II-nd set of differential equations. However, the parallel argument to the above applies to a solution path starting at $(-x_1,-y_1)$, so that it must intersect somewhere in $[(0,0), (x_1,y_1))$. We repeat this process and have the converging path to the origin obeying the "patched-up" system of differential equations. It is obvious, by homogeneity, that any solution path starting from a point in $((0,0), (x_o,y_o))$ should proportionately shrink its distance from the origin everytime it crosses the boundary of (x_o, y_o) side. Finally, we note that any path starting at an interior point of a region should converge to the origin travelling only within the region or intersect the boundary to go into the other region, since otherwise it contradicts (S).

For the latter case, examine the behavior after the intersecting point at boundary and it has been proved to converge to the origin.

Q.E.D.

Next we show the local stability of a "patched-up" system of non-linear differential equations.

Suppose that $\mathbb{R}^2$ is partitioned into two regions, R_I and R_{II} in such a way that $R_I \cup R_{II} = \mathbb{R}^2$ and $S = R_I \cup R_{II}$ is a connected line. In each phase a system of differential equations which satisfy the Lipschitz condition is defined:

$$(N^*) \quad (\dot{x},\dot{y}) = (f_i(x,y),\ g_i(x,y)), \qquad \begin{array}{l} i = I \text{ if } (x,y) \in \tilde{R}_I \\ i = II \text{ if } (x,y) \in \tilde{R}_{II} \end{array}$$

where $\tilde{R}_I$ and $\tilde{R}_{II}$ are open neighborhoods of R_I and R_{II}, respectively, and f_i and g_i are of C^1 class.

Note that at any point on the boundary which separates two regions, a solution path traveling over the two regions is connected smoothly, i.e., f_i and g_i are continuous over the two regions:

$$(f_I(x,y),\ g_I(x,y) = (f_{II}(x,y),\ g_{II}(x,y)) \qquad \forall (x,y) \in S$$

Suppose also that both systems of differential equations have a unique common equilibrium point;