A HETEROGENEOUS-EXPECTATIONS MODEL OF
THE VALUE OF BONDS BEARING CALL OPTIONS

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Abstract

This paper develops a dynamic programming model of the optimal refunding strategy and the corresponding value of a callable bond. The model differs from previous work on this subject primarily in that it explicitly admits the possibility of differences between the issuer's expectations of future interest rates and an investor's corresponding expectations. This generalization facilitates the application of the model to determine what a specific bond (issued, for example, by a particular corporation) is worth to any given investor. Additional analytical features of the model, which differ from corresponding aspects of some previous models, include the use of a stochastic discounting rate and the use of continuous distributions to characterize the relevant interest rate expectations.

For the bond issuer, his own expectations (together with the bond's coupon and call features) suffice to indicate the critical refunding yield as well as the expected value of the bond in each time period until the bond matures. For an investor, however, the analytical solution of the model and the illustrative numerical examples presented in the paper show that the issuer's expectations and the investor's own both matter if the two differ.

A HETEROGENEOUS-EXPECTATIONS MODEL OF THE VALUE OF BONDS BEARING CALL OPTIONS

Zvi Bodie and Benjamin M. Friedman*

Long-term corporate bonds, in modern times the primary vehicle of corporate external financing in the United States, are not pure bonds. Instead, the security issued by a corporation and purchased by investors consists of a bond less an option, retained by the issuer, to call the bond at a specified price after some specified deferment period.[1] In recent years, therefore, the bond call option itself has understandably attracted substantial attention. As callable debt has constituted an ever greater component of many major companies' balance sheets, whether or not to exercise the call options on specific bonds has come to be an important element in corporate financial decision making. Similarly, as the volume of outstanding corporate bonds has grown to $350 billion -- in comparison, for example, with a total market value of $850 billion for all corporate equities -- investors have had ample incentive to develop techniques for evaluating these bonds, including their call features, as carefully as possible.[2]

Pye's [20] seminal contribution a decade ago used a dynamic programming methodology to solve a problem which simultaneously determined the bond issuer's optimal refunding strategy and the corresponding value of the bond. In order to solve this problem, Pye assumed a finite time horizon, risk neutrality, and a discrete interest rate distribution with a finite number of states. Subsequent researchers have extended Pye's analysis in a number of ways.[3] Elton and Gruber [9] and Kraus [16], for example, showed the

implications of introducing an infinite horizon. In addition, they and other

writers considered the practically relevant case which constrains the issuer

to finance a refunding operation with a new issue of long-term bonds. Most

recently, Brennan and Schwartz [6] have applied the option-pricing methodol-

ogy of Merton [18] and Black and Scholes [3] under the assumption of a

specific dynamic stochastic process generating interest rates.

The object of this paper is to develop a model of the optimal refunding

strategy and the corresponding value of a callable bond which differs from

previous work on this subject primarily in that it explicitly admits the pos-

sibility of differences between the issuer's expectations of future interest

rates and an investor's corresponding expectations. This generalization

facilitates the application of the model to determine what a specific bond

(issued by a particular corporation) is worth to any given investor. As is

now typical in the literature of bond refunding,[4] the model takes as a given

constraint that, in the event of a call, the issuer will finance the refunding

with a new issue of long-term bonds; in contrast to some familiar work on this

problem, however, the model developed here admits a stochastic discounting

rate.[5] In addition, the model assumes, analogously to some previous work,

that interest rate expectations -- including both the issuer's and the

investor's, if the two differ -- are characterized by continuous, rather than

discrete distributions; continuous distributions not only are probably more

realistic but also are especially convenient in the context of applications

of the model to market data to draw inferences about market participants'

expectations.[6]

Section I indicates the general principles and specific assumptions underlying the optimal refunding and valuation model for callable bonds. Section II develops the model in full for the general case involving no specific assumption about how the issuer finances a refunding, and Section III develops the model under the long-term refinancing constraint. Section IV illustrates several of the model's key properties, including the implications of differing issuer and investor expectations, by applying the model to several hypothetical examples based on the standard call features used in the U.S. corporate bond market. Section V briefly summarizes the paper's principal conclusions and suggests directions for further research.

I. Introductory Remarks

The value of a bond subject to a call option equals the value of an otherwise identical noncallable bond less the value of the call option.

The value of the option to the option holder -- the issuer in the case of a callable bond -- follows from a decision problem in which the option holder maximizes an objective, subject to both the specified characteristics of the option itself and his expectations of the relevant future prices (or interest rates). At any moment of time, a bond issuer with callable bonds outstanding must decide whether or not to call the bonds. In an abstraction to a discrete-time model, the issuer must decide, at the beginning of each time period, whether to call the bonds or to leave them outstanding until the beginning of the next time period. Calling a bond at the beginning of time period t implies an immediate payment of C_t the current call price,[7] while not calling the bond implies a payment of coupon k at the end of the current time period t and a new decision at the beginning of the subsequent time period t + 1.

The value of the option to the option writer -- the investor in the case of a callable bond -- is somewhat more complicated. Even given the investor's own objective and expectations of the relevant future prices (or interest rates), the value of the option to the investor depends upon actions to be taken at the sole discretion of the issuer (the option holder). To evaluate the option, therefore, the investor must second-guess the issuer's decisions, and to perform this task the investor must in turn have beliefs about the issuer's objective and expectations.[8]

Two convenient assumptions about the behavior of the firm facilitate

modeling the issuer's optimal decision. First, in order to abstract from considerations relating to the firm's production, sales, and investment decisions, it is useful to assume that the firm makes the call decision independently of the "revenue" side of its activities. Secondly, in order to have a decision criterion, it is useful to assume that the objective determining the call decision is to minimize the present discounted value of the expected stream of payments associated with the outstanding bond (or its refinancing replacement).[9] The value of a callable bond to the issuer as of the beginning of time period t is therefore

$$V_t = \min \{C_t, \frac{1}{1+\rho_t} [k + E_t(\tilde{V}_{t+1})]\} \qquad (1)$$

where ρ_t is the time rate of discount which prevails in time period t, E_t indicates an expectation as of the beginning of time period t, and the tilde symbol recalls that $\tilde{V}_{t+1}$ is a random variable as of the beginning of time period t. The appearance of $\tilde{V}_{t+1}$ in one of the two alternative expressions for V_t immediately suggests a recursion relation requiring solution by dynamic programming.

An assumption about the investor's behavior is analogously necessary in order to calculate the value of the bond to the investor. If the investor is also a present value maximizer, then the value of a callable bond to the investor as of the beginning of time period t is

$$V'_t = \min \{C_t, \frac{1}{1+\rho'_t} [k + E'_t(\tilde{V}'_{t+1})]\} \qquad (1')$$

where all variables are as in (1) and the prime symbol indicates the value, discount rate and expectation pertaining to the investor, should any or all of these differ from the issuer's.

The importance of expectations of future yields emerges clearly in the explicit evaluation of the expectation on the right-hand side of (1),

$$E_t(\tilde{V}_{t+1}) = {}_tP_{t+1} \cdot C_{t+1} + (1 - {}_tP_{t+1}) \cdot E_t [\frac{1}{1 + \tilde{\rho}_{t+1}}(k + E_{t+1}(\tilde{V}_{t+2}))] \qquad (2)$$

where ${}_tP_{t+1}$ is the probability, assessed as of the beginning of time period t, that the optimal decision at the beginning of time period t+1 will be to refund the bond. Hence the expectation $E_t(\tilde{V}_{t+1})$ equals the linear combination of the known call price C_{t+1}, premultiplied by probability ${}_tP_{t+1}$, and the conditional expectation of the value of the bond in the event that the optimal decision at the beginning of time period t+1 is not to refund it, premultiplied by probability $(1 - {}_tP_{t+1})$. One element of that conditional expectation is in turn $E_{t+1}(\tilde{V}_{t+2})$ -- the expectation, as of the beginning of time period t+1, of the value of the bond at the beginning of yet the next time period, once again conditional on the issuer's optimal decision being not to refund the bond in either of time periods t or t+1. The explicit evaluation of that conditional expectation, in turn, continues the sequence analogously. A similar sequence emerges from explicit evaluation of the expectation $E'_t(\tilde{V}'_{t+1})$ on the right-hand side of (1').

Finally, the model developed in Sections II and III below abstracts from default risk, as well as from tax considerations such as the differential tax treatment of the payment (or receipt) of interest versus call premiums.

II. The General Model

For a conventional bond, the terminal condition which anchors the dynamic programming solution to the issuer's optimal refunding problem is simply the statement of the bond's par-value maturity property,

$$V_{T+1} = 1 \qquad\qquad (3)$$

where T denotes the final time period in the bond's term to maturity. The solution follows as a series of critical discount rate values ρ_T^*, ρ_{T-1}^*,...., such that the optimal decision is to call the bond at the beginning of period t if and only if $\rho_t < \rho_t^*$.

At the beginning of time period T, the issuer's problem in the general form (1) simply reduces to

$$V_T = \min \{C_T, \frac{1}{1+\rho_T}(k + 1)\}, \qquad\qquad (4)$$

and the solution to the minimization follows from comparing the newly emerged ρ_T with the previously known coupon k and call price C_T to find the critical value

$$\rho_T^* = \frac{k+1}{C_T} - 1. \qquad\qquad (5)$$

which equates the two alternatives within the right-hand side of (4). If $C_T = 1$, as is typically the case (except for transactions costs) in U.S. corporate bond indentures,[10] the critical value ρ_T^* in (5) reduces to the bond's coupon rate k, so that the issuer's optimal refunding decision at the beginning of time period T is to call the bond or not according to $\rho_T \gtrless k$.

For each subsequent time period t, t = T-1, T-2,. . . (going backward), the general problem is simply (1) and its solution is

$$\rho^*_t = \frac{k + E_t(\tilde{V}_{t+1})}{C_t} - 1 \tag{6}$$

where, given the solution from the immediately prior (i.e., chronologically later) time period of the backward induction process,

$$E_t(\tilde{V}_{t+1}) = C_{t+1} \int_0^{\rho^*_{t+1}} f_t(\tilde{\rho}_{t+1})d\rho$$

$$+ [k + E_{t+1}(\tilde{V}_{t+2})] \int_{\rho^*_{t+1}}^{\infty} \frac{1}{1+\tilde{\rho}_{t+1}} f_t(\tilde{\rho}_{t+1})d\rho. \tag{7}$$

and $f_t(\tilde{\rho}_\tau)$, $\tau = t+1,\ldots,T$, are independent probability density functions describing the issuer's expectations, as of time period t, about the (unknown) future path of the stochastic discount rate $\tilde{\rho}_\tau$.[11]

The T-th successive application of this procedure finally yields the new-issue value of the bond as

$$V_1 = \min \left\{ C_1, \frac{1}{1+\rho_1} [k + E_1(\tilde{V}_2)] \right\} \tag{8}$$

where

$$E_1(\tilde{V}_2) = C_2 \int_0^{\rho^*_2} f_1(\tilde{\rho}_2)d\rho + [k + E_2(\tilde{V}_3)] \int_{\rho^*_2}^{\infty} \frac{1}{1+\tilde{\rho}_2} f_1(\tilde{\rho}_2)d\rho. \tag{9}$$

On the assumption that bonds are issued at par, the new-issue value V_1 in (8) must presumably satisfy $V_1 \leq 1$, or else the borrower would be unwilling to issue the new bonds in the first place.

In practice, all newly issued U.S. corporate bonds bear call options which come into effect only after some time interval, say D periods, after the issuance of the bond. In this case, the backward solution technique is

relevant only as far as the beginning of time period D+1, at which point

$$V_{D+1} = \min \{C_{D+1}, \frac{1}{1+\rho_{D+1}} [k + E_{D+1}(\tilde{V}_{D+2})]\}. \tag{10}$$

Solution of the issuer's refunding decision problem as in (1), (6) and (7) yields the previous period's expectation of the bond's value, as of the first call opportunity, as

$$E_D(\tilde{V}_{D+1}) = C_{D+1} \int_0^{\rho^*_{D+1}} f_D(\tilde{\rho}_{D+1}) d\rho$$

$$+ [k + E_{D+1}(\tilde{V}_{D+2})] \int_{\rho^*_{D+1}}^{\infty} f_D(\tilde{\rho}_{D+1}) d\rho. \tag{11}$$

Since no call option is available at the beginning of time period D, however, the value of the bond at that time is simply

$$V_D = \frac{1}{1+\rho_D} [k + E_D(\tilde{V}_{D+1})], \tag{12}$$

and the previous period's expectation of that value is simply

$$E_{D-1}(\tilde{V}_D) = [k + E_D(\tilde{V}_{D+1})] \int_0^{\infty} \frac{1}{1+\tilde{\rho}_D} f_{D-1}(\tilde{\rho}_D) d\rho. \tag{13}$$

Successive application of this simple discounting procedure during the D time periods of deferment of the call option leads to the expression of the value of the bond in any time period t = 1,..., D, as [12]

$$V_t = \frac{1}{1+\rho_t} [k + E_t(\tilde{V}_{t+1})] \tag{14}$$

where

$$E_t(\tilde{V}_{t+1}) = [k + E_{t+1}(\tilde{V}_{t+2})] \int_0^\infty \frac{1}{1+\tilde{\rho}_{t+1}} f_t(\tilde{\rho}_{t+1}) d\rho. \tag{15}$$

The D-th such successive discounting application finally yields the new-issue value of the deferred callable bond as

$$V_1 = \frac{1}{1+\rho_1} [k + E_1(\tilde{V}_2)]. \tag{16}$$

Once again, on the assumption that new bonds are issued at par, V_1 must satisfy $V_1 \leq 1$.

It remains to consider the more complicated problem of determining the value of the bond to an investor whose relevant discount rate ρ' is not necessarily identical to discount rate ρ which is relevant for the bond issuer, and whose interest rate expectations $f'(\rho')$ and $f'(\rho)$, may differ in unspecified ways from the issuer's expectations $f(\rho)$. Beginning the backward induction solution procedure in period T, as above, the value of the bond to the investor is

$$V_T' = \begin{cases} C_T & \text{if } \rho_T < \rho_T^* \\ \frac{1}{1+\rho_T'} [k+1] & \text{if } \rho_T > \rho_T^* \end{cases} \tag{4'}$$

where ρ_T^* is as in (5).

For each subsequent time period $t = T-1,\ldots,D+1$, the general procedure for solving the investor's valuation problem is first to solve the issuer's decision problem (using the issuer's expectations $f(\rho)$) for the critical values ρ_t^*, $t = D+1,\ldots,T$, and then to use these ρ^* values as limits of integration in the straightforward evaluation of the expectation embracing the two

alternative possibilities that the bond will or will not be called in any given time period $t = D+1,\ldots,T$.[13] For any time period $t = D+1,\ldots,T$, the value of the bond to the investor as in (1') is therefore

$$V'_t = \begin{cases} C_t \text{ if } \rho_t < \rho^*_t \\ \\ \frac{1}{1+\rho'_t} [k + E'_t(\tilde{V}'_{t+1})] \text{ if } \rho_t > \rho^*_t \end{cases} \tag{1''}$$

where ρ^*_t is as in (6) and

$$E'_t(\tilde{V}'_{t+1}) = C_{t+1} \int_0^{\rho^*_{t+1}} f'_t(\tilde{\rho}_{t+1})\,d\rho$$

$$+ [k + E'_{t+1}(\tilde{V}'_{t+2})] \int_{\rho^*_{t+1}}^{\infty} f'_t(\tilde{\rho}_{t+1})\,d\rho \cdot \int_0^{\infty} \frac{1}{1+\tilde{\rho}_{t+1}} f'_t(\tilde{\rho}'_{t+1})\,d\rho'. \tag{7'}$$

Finally, for time periods $t = 1,\ldots,D$, during which the call option deferment is in effect, the investor's discounting procedure is precisely analogous to that for the issuer in (14) and (15), substituting $f'(\rho')$ for $f(\rho)$. On the assumption that new bonds are issued at par, V'_1, the value to the investor at the time of issue, must satisfy $V'_1 \geq 1$.

III. The Case of Long-Term Refinancing

A major obstacle to empirically implementing a general model of the is-
suer's optimal refunding decision like that developed in Section II is iden-
tifying the appropriate discounting rate ρ to use in such calculations, and
this question has received substantial attention in the literature.[14] Some
writers, for example, have argued that bond refunding is just another capital
budgeting decision, so that ρ properly represents the firm's average cost of
capital. The consensus of the recent literature, however, is that ρ should
instead represent the currently prevailing market interest rate on the firm's
outstanding long-term debt. The rationale usually cited in support of this
choice is that the long-term debt yield properly reflects the risk of the
cash flows being discounted. In addition, a further assumption which rein-
forces this choice is that the refunding decision with respect to outstanding
long-term securities presupposes no change in the firm's liability structure
in the sense of term to maturity (apart from call features). While this as-
sumption in principle introduces some possibility for misunderstanding the
issuer's refunding decision problem, it is probably not too far off the mark
for practical purposes, since most issuers do in fact consider a refunding
situation in the somewhat limited context of replacing outstanding long-term
bonds with a new issue of long-term bonds (presumably bearing a lower coupon).[15]
In the context of the relationship between using the firm's cost of long-term
debt in the optimal refunding calculation and assuming that a new long-term
bond is the choice of refinancing vehicle, the primary question is whether this
simplifying assumption introduces more or less possibility for misunderstanding
or empirical irrelevance than does an alternative assumption identifying the

relevant discount rate ρ with some other specific observable variable.

Optimal refunding decision model (1) relies on an implicit two-fork decision tree approach in which, at the beginning of any time period t, the "terminal" (refund) fork is the immediate payment of call price C_t while the "continuing" (do not refund) fork is the payment of coupon k at the end of the time period (i.e., the beginning of the next time period) and a new decision then. If the issuer presupposes financing of the refunding by a new issue of long-term bonds, then the "terminal" fork is no longer an end point, and its value is not simply the call price but rather the stream of future payments of interest and principal on the newly issued security. It is therefore necessary to consider the coupon rate k_t which, as of the beginning of time period t, is the market yield on new issues of the firm's long-term bonds. Furthermore, since new issues of immediately callable long-term bonds do not exist in the U.S. markets, it is appropriate to consider k_t as the coupon rate on a new issue of long-term bonds bearing call options subject to deferral for some interval.[16]

In the restricted context of a predetermined liability maturity structure, then, the issuer's refunding decision involves a comparison of the present value of two alternative streams of payments. One, as in (1), is the payment of coupon k for all time periods until the bond is called, with a new refunding decision possible in each successive time period. The other is the immediate payment of call price C_t, which requires simultaneously issuing a new bond in the amount of C_t, and subsequent payment of the new coupon k_t for at least

D time periods (where D is the length of the deferment period on the new
bond's call option), with refunding decisions possible only in later time
periods. The fixed-maturity analog to the issuer's refunding decision (1)
is therefore

$$V_t = \min \{R_t, \frac{1}{1+\rho_t} [k + E_t(\tilde{V}_{t+1})]\}$$ (17)

where the value of the stream of payments associated with the refinancing
bond is[17]

$$R_t = \frac{C_t}{1+\rho_t} \{k_t[1 + \sum_{\tau=t+1}^{T} (\prod_{s=t+1}^{\tau} \int_0^\infty \frac{1}{1+\tilde{\rho}_s} f_t(\tilde{\rho}_s) d\rho)]$$

$$+ \prod_{\tau=t+1}^{T} \int_0^\infty \frac{1}{1+\tilde{\rho}_\tau} f_t(\tilde{\rho}_\tau) d\rho \}.$$ (18)

The issuer's fixed-maturity refunding decision therefore involves two
distinct sets of interest rate expectations, as is clear from considering
the conditional expectation on the right-hand side of (17),[18]

$$E_t(\tilde{V}_{t+1}) = {_tP_{t+1}} \cdot E_t(\tilde{R}_{t+1}) + \frac{1 - {_tP_{t+1}}}{1 + \rho_t} [k + E_{t+1}(\tilde{V}_{t+2})].$$ (19)

As of the beginning of time period t, k_t is known but $\tilde{k}_{t+1}$ is not. Hence
$\tilde{R}_{t+1}$ is random, as of the beginning of time period t, because of uncertainty
associated not only with future $\tilde{\rho}_\tau$, $\tau=t+1,...,T$, but also with the future

coupon rate $\tilde{k}_{t+1}$. The reason why expectations of future coupon rates enter

the problem is the "lock-in" feature associated with the noncallability (or

call option deferment) of the refinancing bonds. Refinancing with immedi-

ately callable bonds is not possible, and refinancing with noncallable bonds

requires the issuer to pay coupon k_t in every subsequent time period,

regardless of the potential emergence of even lower new-issue coupon rates

later on.[19] Even if an issuer can refinance the outstanding k-coupon bonds

by issuing new bonds with some coupon k_t which is enough lower than k to

warrant paying the current call price, therefore, he may gain even more by

waiting to refinance, say s time periods later, at the then-prevailing

coupon rate $\tilde{k}_{t+s}$ which may be even less than k_t. It is this feature of the

problem which gives the variance, or volatility, of future long-term new-

issue yields an added importance for the optimal refunding decision and the

resulting value of the callable bond. Even if the issuer expects the long-

term new-issue yield to be high in the future on average, if he expects this

yield to be volatile he may value greatly the opportunity to refund at pre-

cisely the time when the yield has fallen below his expectation of its

long-term average.

In complete form, therefore, the solution to the fixed-maturity refunding

decision problem leads to a backward succession, from time period T, of expres-

sions involving integration over two sets of density functions -- one for the

discount rate and one for the coupon rate. Expectations about these two

interest rates, however, are unlikely to be independent. A useful simplifica-

tion which avoids altogether the need for two different sets of integrals is

simply to assume that expectations about one rate are one-for-one equivalent

to expectations about the other, i.e.,

$$f_t(\tilde{\rho}_{t+s}) = f_t[g_{t+s}(\tilde{k}_{t+s})], \quad F_t(\tilde{k}_{t+s}) = F[g_{t+s}^{-1}(\tilde{\rho}_{t+s})], \qquad s > 0, \quad (19)$$

for some nonstochastic transformation $g_{t+s}(\cdot)$ for each time period t+s. Under this covariation assumption it is possible to restate the issuer's refunding decision problem in terms of expectations of future new-issue yields on long-term (noncallable) bonds, so that (18) becomes[20]

$$R_t = \frac{c_t}{1+g_t(k_t)} \left\{ k_t \left[1 + \sum_{\tau=t+1}^{T} \left(\prod_{s=t+1}^{\tau} \int_0^\infty \frac{1}{1+g_s(\tilde{k}_s)} F_t(\tilde{k}_s)dk \right] \right. \right.$$

$$\left. \left. + \prod_{\tau=t+1}^{T} \int_0^\infty \frac{1}{1+g_\tau(\tilde{k}_\tau)} F_t(\tilde{k}_\tau)dk \right\}, \qquad (21)$$

while the analogous expectation on the right-hand side of (19) becomes

$$E_t(\tilde{R}_{t+1}) = c_{t+1} \int_0^\infty \frac{1}{1+g_{t+1}(\tilde{k}_{t+1})} \left\{ \tilde{k}_{t+1} \left[1 + \sum_{\tau=t+2}^{T} \left(\prod_{s=t+2}^{\tau} \int_0^\infty \frac{1}{1+g_s(\tilde{k}_s)} F_t(\tilde{k}_s)dk \right] \right. \right.$$

$$\left. \left. + \prod_{\tau=t+2}^{T} \int_0^\infty \frac{1}{1+g_\tau(\tilde{k}_\tau)} F_t(\tilde{k}_\tau)dk \right\} F_t(\tilde{k}_{t+1})dk. \qquad (22)$$

At the beginning of time period T, the problem is simply

$$V_T = \min \left\{ R_T, \frac{1}{1+g_T(k_T)}(k+1) \right\} \qquad (23)$$

where, because k_T is known, R_T is the nonrandom

$$R_T = \frac{c_T}{1+g_T(k_T)}(k_T+1). \qquad (24)$$

The critical value k_T^* follows, analogously to ρ_T^* in (4), as the new-issue rate which equates the two alternatives within the right-hand side of (23), and the optimal refunding decision is to call the bond or not according to $k_T \lessgtr k_T^*$. If $C_T = 1$, then $k_T^* = k_T$, and the final-period refunding decision simply involves a comparison of the current (one-period) borrowing rate with the outstanding bond's coupon.[21]

For each subsequent time period t, $t = T-1,\ldots,D+1$, the problem is just (17), where R_t is as in (21),

$$E_t(\tilde{V}_{t+1}) = \int_0^{k_{t+1}^*} R_{t+1} F_t(\tilde{k}_{t+1})\,dk$$

$$+ \; [k + E_{t+1}(\tilde{V}_{t+2})] \int_{k_{t+1}^*}^{\infty} \frac{1}{1+g_{t+1}(\tilde{k}_{t+1})} F_t(\tilde{k}_{t+1})\,dk, \quad (25)$$

and the critical value k_t^* follows as the $(T-t+1)$-period new-issue rate which equates the two alternatives within the right-hand side of (17). Following this procedure backward to the solution for $E_D(\tilde{V}_{D+1})$, and then applying the simple discounting procedure of (14) and (15) for the earlier time periods during which the bond is noncallable, completes the solution of the issuer's fixed-maturity refunding problem.

Finally, as in the general model of Section II, solving for the value of the bond to the investor requires first applying the procedure developed above (using the issuer's expectations $f(\rho)$ and $F(k)$) to solve the issuer's fixed-maturity refunding problem for the critical new-issue yields k_t^*, $t = D+1,\ldots,T$. In the absence of any assumption about the investor's application

of the proceeds of a call, the value of the bond for any time period $t = D+1,\ldots,T$, then follows from re-evaluating the relevant probabilities of call on the basis of the investor's corresponding expectations $f'(\rho)$ and $F'(k)$ and the relevant discounted values on the basis of the investor's expectations $f'(\rho')$, so that the expressions analogous to (1') and (7') are

$$V'_t = \begin{cases} C_t & \text{if } k_t < k^*_t \\ \dfrac{1}{1+\rho'_t} [k_t + E'_t(\tilde{V}'_{t+1})] & \text{if } k_t > k^*_t \end{cases} \tag{17'}$$

$$E'_t(\tilde{V}'_{t+1}) = C_{t+1} \int_0^{k^*_{t+k}} F'_t(\tilde{k}_{t+1})\,dk$$

$$+ [k + E_{t+1}(\tilde{V}'_{t+2})] \int_{k^*_{t+1}}^{\infty} \frac{1}{1+g'_{t+1}(\tilde{k}_{t+1})}\,dk \tag{25'}$$

where $g'(\cdot)$ is the transformation relating the investor's discount rate to the issuer's new-issue rate. Alternatively, under the assumption that the investor will apply the proceeds of the call to re-invest in the refinancing bond, the analogous expressions are

$$V'_t = \begin{cases} R'_t & \text{if } k_t < k^*_t \\ \dfrac{1}{1+\rho'_t} [k + E'(\tilde{V}_{t+1})] & \text{if } k_t > k^*_t \end{cases} \tag{17''}$$

$$E'_t(\tilde{V}'_{t+1}) = \int_0^{k^*_{t+1}} \tilde{R}'_{t+1} F'(\tilde{k}_{t+1})\,dk$$

$$+ [k + E'_{t+1}(\tilde{V}'_{t+2})] \int_{k^*_{t+1}}^{\infty} \frac{1}{1+g'_{t+1}(\tilde{k}_{t+1})} F'_t(\tilde{k}_{t+1})\,dk \tag{25''}$$

where[22]

$$R_t' = \frac{C_t}{1+g_t'(k_t)} \left\{ k_t \left[1 + \sum_{\tau=t+1}^{T} \left(\prod_{s=t+1}^{\tau} \int_0^\infty \frac{1}{1+g_s'(\tilde{k}_s)} F_t'(\tilde{k}_s)\,dk \right] \right. $$

$$\left. + \prod_{\tau=t+1}^{T} \int_0^\infty \frac{1}{1+g_\tau'(\tilde{k}_\tau)} F_t'(\tilde{k}_\tau)\,dk \right\}. \tag{21''}$$

For time periods $t=1,\ldots,D$, the value to the investor again follows from the simple discounting procedure indicated in Section II.

IV. Some Illustrations

A few hypothetical examples may serve to illustrate some of the pertinent

properties of the solutions yielded by the bond valuation and optimal

refunding decision solution procedures developed above.

Consider, for example, a 25-year bond with a coupon of 8% per annum

payable semi-annually. In addition, suppose that the relevant discounting

rate is also 8% per annum, and that both the issuer and investors expect the

market yield to continue to be 8% per annum. If the bond is noncallable,

then the initial value is simply 100.[23] Since the expected market yield is

8% per annum for the next 25 years, the bond's expected price in any future

period until maturity is also 100.

If the same 25-year 8% bond is callable, however, and if either the

issuer or investors (or both) believe that the market yield may fall below 8%

per annum at some point during the bond's 25-year life (i.e., there is some

non-zero variance associated with the 8% expectation), then neither the

initial value nor the expected price in any future time period before maturity

equals 100. The income stream generated by this bond (and its replacement,

in the event of refunding) will under no circumstances be greater than that

of the initial 8% coupon rate. Because of the possibility of refunding,

which in turn depends upon future interest rates and expectations thereof,

this income stream may be less than that of the initial 8% coupon rate. Given

a discounting factor of 8%, therefore, the bond's price will always be

strictly less than 100, unless interest rate expectations are such as to

preclude the possibility of refunding.

The first two columns of Table 1 present the solution of the bond valuation and optimal refunding decision model of Section III for the case of the immediately callable 25-year 8% bond described immediately above. The key assumptions underlying this solution are that both the issuer's and investors' expectations of future coupon interest rates, $F(\tilde{k}_t)$ and $F'(\tilde{k}_t)$, $t=1:II,\ldots,$ $T(=25:II)$ are characterized by a truncated normal distribution with mean 8% and standard deviation 1%,[24] and that the expected relationship between the issuer's and investors' discounting rates (ρ_t and ρ_t') and coupon rate k_t as in (19) is simply $\rho_t = \rho_t' = k_t$, $t=1,\ldots,T$.

The first column of the table shows, for each half-year period during the first ten years and for alternative half-year periods thereafter, the critical refunding yield k_t^* which follows as the $(T-t+1)$-period new-issue rate which equates the two alternatives within the right-hand side of (17). The issuer's optimal decision, at the beginning of time period t, is to refund the bond if $k_t < k_t^*$ and not to refund it if $k_t > k^*$. The monotonically increasing series of k_t^* values shown in the first column of the table illustrates clearly the "lock-in" implication of a call-protected refinancing bond as emphasized in Section III. In particular, it is not optimal to refund the outstanding bond at every coupon rate which satisfies $k_t < 8\%$, even though doing so would lead to an interest saving. Unless $k_t \leq k_t^* < 8\%$ (except for the final period), the present discounted value of the interest saving achieved by refunding at $k_t < 8\%$ is smaller than the present discounted value of the expected interest saving from refunding in some later time period $t + \tau$ at the then-prevailing coupon rate $k_{t+\tau} < k_t$.[25] Given the

TABLE 1

CRITICAL REFUNDING YIELDS AND EXPECTED PRICES FOR CALLABLE 25-YEAR 8% BONDS

t	k_t^*	$E(\tilde{V}_t) = E'(\tilde{V}_t')$ Immediately Callable	Callable After 5 Years	Callable After 10 Years
1:I	–	91.25	94.26	96.68
1:II	6.57%	90.90	94.03	96.55
2:I	6.58	90.94	93.79	96.41
2:II	6.58	90.99	93.54	96.26
3:I	6.59	91.05	93.28	96.11
3:II	6.60	91.11	93.01	95.95
4:I	6.60	91.17	92.73	95.79
4:II	6.61	91.24	92.44	95.62
5:I	6.62	91.32	92.13	95.44
5:II	6.62	91.40	91.81	95.26
6:I	6.63	91.48	91.48	95.06
6:II	6.64	91.57	91.57	94.87
7:I	6.65	91.67	91.67	94.66
7:II	6.66	91.77	91.77	94.44
8:I	6.66	91.88	91.88	94.22
8:II	6.67	92.00	92.00	93.98
9:I	6.68	92.12	92.12	93.74
9:II	6.69	92.24	92.24	93.49
10:I	6.70	92.38	92.38	93.22
10:II	6.71	92.52	92.52	92.95
11:I	6.72	92.67	92.67	92.67
12:I	6.74	92.99	92.99	92.99
13:I	6.76	93.33	93.33	93.33
14:I	6.78	93.71	93.71	93.71
15:I	6.82	94.12	94.12	94.12
16:I	6.85	94.57	94.57	94.57
17:I	6.88	95.05	95.05	95.05
18:I	6.92	95.56	95.56	95.56
19:I	6.96	96.10	96.10	96.10
20:I	7.01	96.67	96.67	96.67
21:I	7.08	97.26	97.26	97.26
22:I	7.15	97.87	97.87	97.87
23:I	7.24	98.47	98.47	98.47
24:I	7.38	99.06	99.06	99.06
25:I	7.63	99.59	99.59	99.59
25:II	8.00	99.81	99.81	99.81

Assumptions: $E(\tilde{k}_t) = E'(\tilde{k}_t) = 8\%$, $\sigma(\tilde{k}_t) = \sigma'(\tilde{k}_t) = 1\%$, $\rho_t = \rho_t' = k_t$, all t.

assumed constant mean and nonzero variance of the distributions describing

expectations of future interest rates, the earlier is the time period t

the lower k_t must be to justify refunding.

The second column of the table shows, for the same half-year periods,

the price $V_t = V_t'$ expected to prevail at the beginning of the period for

this (immediately) callable 25-year 8% bond. The value of the bond's call

option is the difference between this price and 100, which would be the

expected price of a noncallable 8% bond. Given the assumed parameters of

expectations of future interest rates, the immediately callable bond will

sell for only 91.25 at its initial time of issue, so that the call option

is initially worth 8.75. After the initial half-year period, the expected

price $E(\tilde{V}_t) = E'(V_t')$ increases monotonically with time period t, at a

steadily growing rate of increase. The initial new-issue price is somewhat

greater than the expected price at the beginning of the next several time

periods because of the certainty of receiving the 8% coupon payment at the

end of the first half year; within the simplified discrete-time framework of

this paper, even the call option of an "immediately callable" bond is implic-

itly deferred for one time period.

As an illustration of more generally deferred call options, the third

and fourth columns of Table 1 present analogous solution values for the

expected prices of callable 25-year 8% bonds bearing the standard utility

and industrial call deferments of five years and ten years, respectively,

using the same assumptions about expectations of future interest rates which

underlie the solution for the immediately callable bond. Once the five- or

ten-year deferment period expires, the initially call-protected bond no longer

differs from the immediately callable bond, so that the critical refunding

yield and expected price series for time periods t > D (= 5:II or 10:II)

are identical to those shown in the first two columns of the table. For

time periods t = 1,...,D, the critical refunding yield is undefined for a

call-protected bond, and the expected price is greater than that of the

immediately callable bond, since the coupon payments are certain to be at

the 8% rate (i.e., the upper limit for the immediately callable bond). As

the call deferment interval elapses, the expected price declines monotonically

to reflect the decreasing number of certain 8% coupon payments. Figure 1,

which summarizes the expected price relationships among the immediately

callable and two call-protected 25-year 8% bonds by plotting the three

expected price series as functions of time period t, shows clearly how the

different bonds become indistinguishable upon the expiration of the relevant

call deferments.

Next, it is useful to consider explicitly the dependence of the results

presented in Table 1 on the assumed uncertainty, characterized by standard

deviation $\sigma(\tilde{k}_t) = \sigma'(\tilde{k}) = 1\%$, associated with expectations of future interest

rates. Table 2 shows the relationship between the assumed standard deviation

and the bond's new-issue price $V_1 = V_1'$ for four kinds of 8% 25-year bonds:

immediately callable bonds (i.e., bonds callable after the first half year),

bonds callable after five years, bonds callable after ten years, and non-

callable bonds.[26] For a given set of call provisions (except for the non-

callable bond[27]), the greater is the uncertainty the lower is the bond's

price. Conversely, for given uncertainty, the earlier the bond is callable
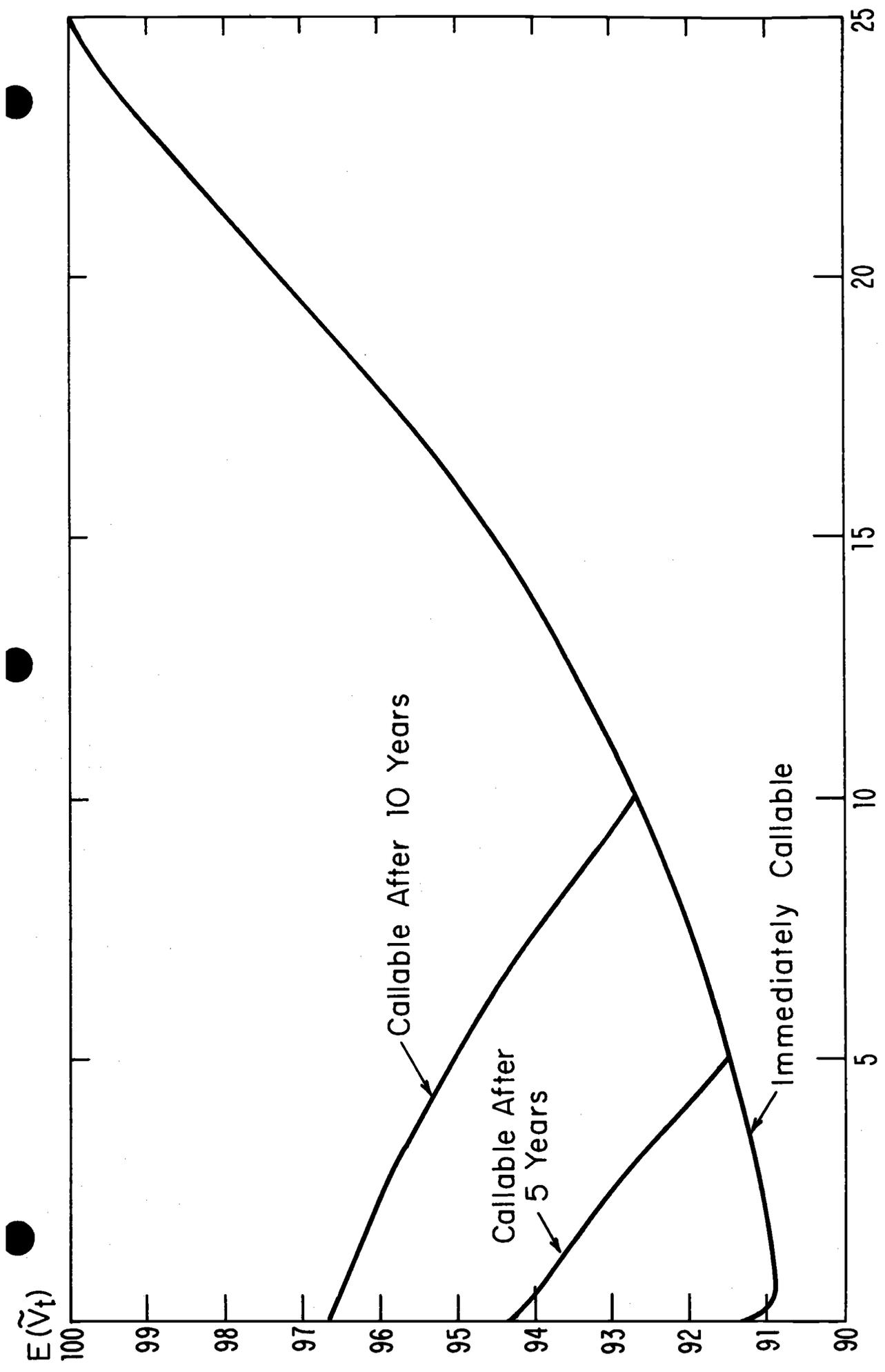
the lower is its price.

FIGURE 1

EXPECTED PRICES FOR CALLABLE 8% 25-YEAR BONDS

TABLE 2

NEW-ISSUE PRICES OF CALLABLE AND NONCALLABLE

8% 25-YEAR BONDS UNDER HOMOGENEOUS EXPECTATIONS

$\sigma(\tilde{k}_t) = \sigma'(\tilde{k}_t)$	Immediately Callable	Callable After	Callable After	Noncallable
0.2%	99.93	99.94	99.94	100.00
0.5%	97.55	98.27	98.93	100.01
1.0%	91.25	94.26	96.68	100.05
1.5%	84.59	90.09	94.38	100.11
2.0%	77.86	85.88	92.07	100.20
2.5%	71.09	81.66	89.76	100.31

Assumptions: $E(\tilde{k}_t) = E'(\tilde{k}_t) = 8\%$, $\rho_t = \rho'_t = k_t$, all t.

Table 3 presents analogous results for the general circumstance in which $\sigma(\tilde{k}_t)$, the standard deviation which the issuer associates with expectations of future interest rates, differs from the investor's corresponding standard deviation for the single case of a 25-year 8% bond callable after five years. The entries along the diagonal of the table, for which $\sigma(\tilde{k}_t) = \sigma'(\tilde{k}_t)$, are identical to those in the second column of Table 2. When the issuer and the investor disagree, however -- i.e., off the table's diagonal -- the results differ sharply. Given the issuer's expectations, the greater is the issuer's uncertainty the lower is the bond's price. By contrast, given the investor's expectations, the bond's price is lowest when the issuer agrees -- i.e., when $\sigma(\tilde{k}_t) = \sigma'(\tilde{k}_t)$; and the greater the disagreement the higher the price, regardless of the direction of the disagreement (although the rate of increase is asymmetrical).

Because the 8% coupon of all of the callable bonds considered in Tables 1-3 is identical to the assumed mean of the distribution of expected future interest rates $E(\tilde{k}_t) = E'(\tilde{k}_t)$, and because these expectations are not held with perfect certainty, the expected prices of all three of these bonds are always less than 100, even at the time of issue. By contrast, in the United States as well as in most other countries, new long-term bonds are typically issued at or near 100 even when they bear call options. Since a coupon rate equal to the mean of stationary expectations of future market discount rates results in callable bond prices less than 100, some coupon rate greater than this mean is necessary to make the callable bond worth 100 at the time of issue. For any given set of stationary expectations of future interest rates, the amount by which the coupon rate must exceed the fixed mean depends

TABLE 3

NEW-ISSUE PRICES OF A 5-YEAR NONCALLABLE

8% 25-YEAR BOND UNDER HETEROGENEOUS EXPECTATIONS

$\sigma(k_t)$	$\sigma'(k_t)$			
	0.5%	1.0%	1.5%	2.0%
0.5%	98.27	95.26	92.60	89.97
1.0%	99.34	94.26	90.85	88.01
1.5%	99.90	95.42	90.09	86.49
2.0%	99.97	97.64	90.97	85.88

not only upon the bond's call features but also, following the model developed

above, upon the higher moments of the density functions describing interest

rate expectations.

Table 4 and Figure 2 summarize the implications of the bond refunding

and valuation model for bonds having the same call provisions as those

assumed in Table 1 but different coupons, under the same assumptions about

interest rate expectations as in Table 1.[28] The first two columns of Table 4

show the critical refunding yields and the expected prices for an immediately

callable 25-year 10.60% bond. As the first entry in the price column

indicates, 10.60% is precisely the coupon rate necessary to render the bond's

new-issue price equal to 100, given the bond's call provisions and given the

issuer's and investors' expectations of future interest rates characterized

by truncated normal distributions with $E(\tilde{k}_t) = E'(\tilde{k}_t) = 8\%$ and $\sigma(\tilde{k}_t) = \sigma'(\tilde{k}_t)$

$= 1\%$. As in the example in Table 1, the critical refunding yield k_t^* (also

plotted as the solid line in panel (a) of Figure 2) in the first column of

Table 4 rises monotonically so that it equals 10.60%, the bond's coupon rate,

in the final half year of the bond's term to maturity. Since the mean

interest rate expectation is well below the 10.60% coupon rate, the critical

refunding yield remains far below 10.60% -- indeed below $E(\tilde{k}_t) = E'(\tilde{k}_t) = 8\%$

-- until only three years before the bond's maturity. The bond's expected

price (also plotted as the solid line in panel (b) of Figure 2) declines for

seven years -- with the largest decline coming after the first half year,

when the implicit one-period call deferment expires -- then rises until

shortly before the bond's maturity, and finally declines back to 100.

TABLE 4

CRITICAL REFUNDING YIELDS AND EXPECTED PRICES FOR

CALLABLE 25-YEAR BONDS ISSUED AT PAR

t	10.60% Bond Immediately Callable		8.92% Bond Callable After 5 Years		8.40% Bond Callable After 10 Years	
	k_t^*	$E(\tilde{V}_t)$	k_t^*	$E(\tilde{V}_t)$	k_t^*	$E(\tilde{V}_t)$
1:I	–	100.00	–	100.00	–	100.00
1:II	7.13%	98.74	–	99.54	–	99.82
2:I	7.14	98.69	–	99.06	–	99.60
2:II	7.14	98.65	–	98.56	–	99.38
3:I	7.15	98.61	–	98.04	–	99.15
3:II	7.16	98.57	–	97.50	–	98.91
4:I	7.17	98.54	–	96.93	–	98.65
4:II	7.17	98.51	–	96.35	–	98.39
5:I	7.18	98.49	–	95.74	–	98.12
5:II	7.19	98.47	–	95.11	–	97.84
6:I	7.20	98.45	6.88	94.45	–	97.55
6:II	7.21	98.43	6.89	94.50	–	97.24
7:I	7.22	98.42	6.90	94.55	–	96.92
7:II	7.23	98.42	6.91	94.61	–	96.59
8:I	7.24	98.42	6.92	94.68	–	96.25
8:II	7.25	98.42	6.93	94.75	–	95.89
9:I	7.26	98.43	6.93	94.82	–	95.51
9:II	7.27	98.44	6.94	94.90	–	95.13
10:I	7.28	98.45	6.95	94.99	–	94.73
10:II	7.30	98.47	6.96	95.08	–	94.31
11:I	7.31	98.50	6.97	95.18	6.84	93.88
12:I	7.33	98.56	7.00	95.40	6.87	94.15
13:I	7.36	98.65	7.03	95.64	6.89	94.45
14:I	7.40	98.75	7.06	95.91	6.92	94.77
15:I	7.44	98.88	7.09	96.20	6.95	95.13
16:I	7.48	99.02	7.12	96.52	6.98	95.52
17:I	7.53	99.18	7.16	96.87	7.02	95.93
18:I	7.58	99.35	7.21	97.24	7.06	96.38
19:I	7.65	99.54	7.26	97.64	7.11	96.85
20:I	7.73	99.72	7.33	98.05	7.17	97.35
21:I	7.83	99.91	7.41	98.47	7.23	97.86
22:I	7.97	100.07	7.51	98.90	7.32	98.38
23:I	8.15	100.20	7.64	99.30	7.44	98.89
24:I	8.45	100.25	7.84	99.66	7.60	99.36
25:I	9.09	100.14	8.23	99.90	7.92	99.75
25:II	10.60	100.00	8.92	99.95	8.40	99.89

Assumptions: $E(\tilde{k}_t) = E'(\tilde{k}_t) = 8\%$, $\sigma(\tilde{k}_t) = \sigma'(\tilde{k}_t) = 1\%$, $\rho_t = \rho_t' = k_t$, all t.
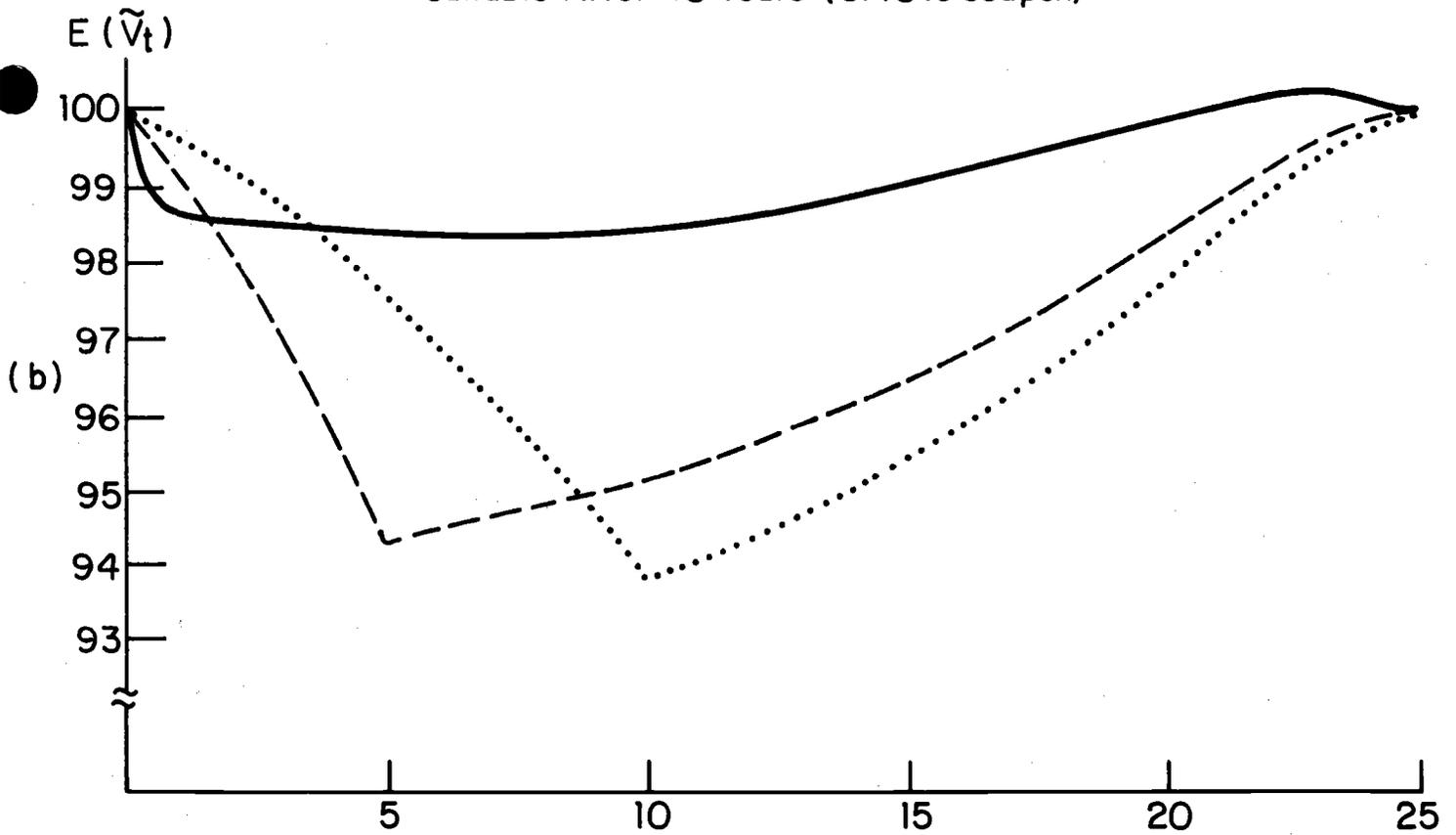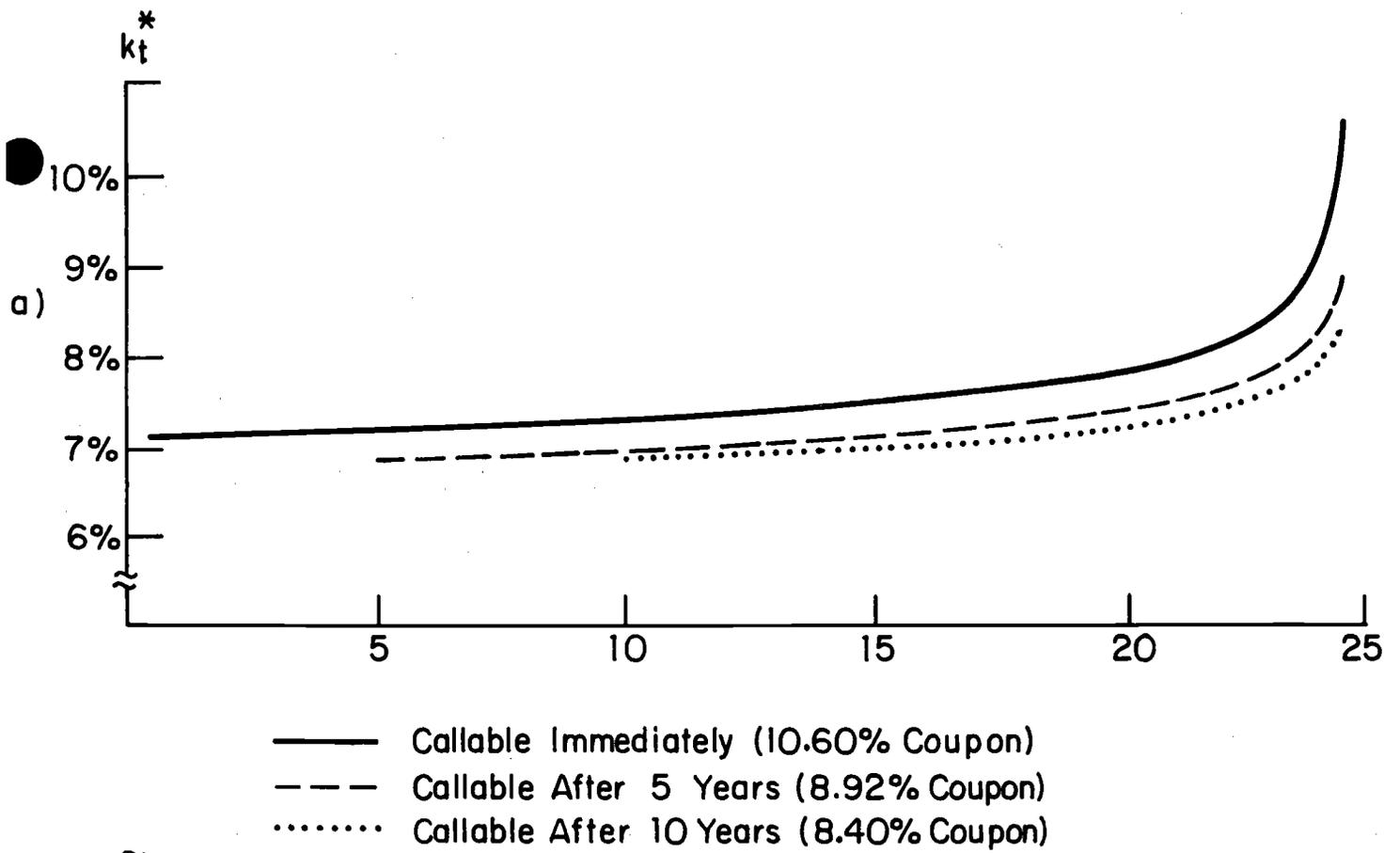
FIGURE 2

CRITICAL REFUNDING YIELDS AND EXPECTED PRICES
FOR 25-YEAR BONDS ISSUED AT PAR

The remaining columns of Table 4, and the broken and dotted lines in Figure 2, show the analogous k_t^* and $E(\tilde{V}_t)$ solution values for a 25-year 8.92% bond callable after five years and a 25-year 8.40% bond callable after 10 years. Because of the series of certain coupon payments guaranteed by their respective call deferments, the coupon rates necessary to render these bonds' new-issue prices equal to 100 are less than for the immediately callable bond, and the ten-year call-protected bond requires a lower coupon rate for this purpose than does the five-year call-protected bond. The two call-protected bonds' respective critical refunding yields, which are defined only for time periods after the expiration of their call deferments, again rise monotonically to equal the respective coupon rates in the final half year of the bonds' term to maturity. In contrast to the examples in Table 1, however, in which the three bonds depicted each have the same coupon rate and therefore have the same k_t^* series, in Table 4 the different coupon rates on the three bonds imply different k_t^* series for each. The expected price series for the five- and ten-year call-protected bonds shown in Table 4 decline monotonically until the respective bonds become callable, and then rise monotonically until the bonds' maturity, according to the pattern familiar from Table 1. Once again, however, because the three bonds illustrated in Table 4 bear different coupons, their expected price series also differ throughout. In particular, since it has the lowest coupon (8.40%), the price of the ten-year call-protected bond falls to the lowest level of any of the three bonds illustrated in Table 4.

Finally, Table 5 summarizes the implications of interest rate uncertainty for the coupon rate necessary to render a bond's new-issue price equal to 100.

TABLE 5

COUPON RATES REQUIRED FOR PAR NEW-ISSUE PRICE

FOR CALLABLE AND NONCALLABLE 25-YEAR BONDS

$\sigma(\tilde{k}_t) = \sigma'(\tilde{k}_t)$	Immediately Callable	Callable After 5 Years	Callable After 10 Years	Noncallable
0.2%	8.00%	8.00%	8.00%	8.00%
0.5%	8.48	8.25	8.12	8.00
1.0%	10.60	8.92	8.40	8.00
1.5%	13.18	9.65	8.70	7.99
2.0%	16.00	10.40	9.00	7.98
2.5%	19.03	11.14	9.30	7.97

Assumptions: $E(\tilde{k}_t) = E'(\tilde{k}_t) = 8\%$, $\rho_t = \rho'_t = k_t$, all t.

As in Table 2, the computations underlying Table 5 assume homogeneous issuer's and investors' expectations with mean 8%. In contrast to the examples in Table 2, however, in which the four bonds depicted all have 8% coupon rates so that the callable bonds have new-issue prices below 100, in Table 5 the four bonds' respective coupon rates differ according to whatever coupon is necessary for a par new-issue price. Following the same logic which indicates new-issue prices less than 100 for the three 8% callable bonds in Table 2, the callable bonds in Table 5 all have coupon rates greater than 8%. For a given set of call provisions (except for the noncallable bond[29]), the greater the uncertainty the higher is the required coupon rate. Conversely, for given expectations, the earlier the bond is callable the higher is the required coupon rate.

V. Summary of Conclusions

Expectations about future interest rates are essential to the bond issuer's optimal refunding problem and the associated problem of determining the value of a callable bond. For the issuer, his own expectations (together with the bond's coupon and call features) suffice to indicate the critical refunding yield as well as the expected value of the bond in each time period until the bond matures. For an investor, however, the issuer's expectations and his own both matter if the two differ.

In the general refunding decision and callable bond valuation model developed in Section II, in which there is no specific presumption about how the issuer finances a refunding operation, the relevant interest rate expectations refer only to the issuer's and investor's respective discounting rates. In the less general -- but probably more important for practical applications -- model developed in Section III, in which the issuer finances the refunding by issuing a new long-term (call deferred) bond, the relevant interest rate expectations refer not only to the issuer's and investor's respective discounting rates but also to the new-issue rate on the issuer's long-term bonds. Once again, if an investor's expectations about the future course of this interest rate differ from the issuer's, both sets of expectations are relevant to determining the value of the bond to an investor. The hypothetical examples presented in Section IV illustrate the relationship, within the model, of both the mean and the associated uncertainty of the issuer's and investors' interest rate expectations for a number of the model's parameters including the bond's coupon, call deferment, critical refunding yield, new-issue price, and expected price in future time periods.

A number of simplifying assumptions in the analysis of this paper probably represent interesting topics for further research. The use in the illustrations in Section IV of a constant discounting factor mean, for example, is equivalent to the assumption of a flat term structure of interest rates. To the extent that yields on debts of different term to maturity differ, the individual coupon payments should be discounted by different factors. Since the actually prevailing term structure is typically fairly flat after the first several years, the approximation involved in assuming a totally flat term structure probably leads to significant over-valuation or under-valuation only for the first few coupon payments. Nevertheless, the assumption itself is merely convenient and is in no way a necessary element of the model, so that allowing for any specified non-flat term structure would be straight-forward, as would allowing for any specified non-flat (probably increasing) standard deviation.

A second subject for further research along these lines concerns risk aversion. While the model's assessments of probabilities explicitly incorporate the full distributions of interest rate expectations, its valuations rely only on expected values and ignore preferences with respect to higher moments. Allowing for risk aversion would simply require substituting any specified utility function for the simple present discounted value function applied in Sections II and III.

Yet a third extension of the model would be to replace the normal distributions used in Section IV to characterize interest rate expectations with different (perhaps asymmetrical) distributions. Here again, the assumption of normally distributed expectations is in no way necessary for the model, and substituting an alternative distribution would be straightforward.

Finally, the model admits a variety of potentially interesting empirical applications, including not only direct use as an aid to corporations' financial decisions and/or investors' portfolio decisions but also use in reverse to infer market participants' (unobservable) expectations from observed market data.[30]

<center>Footnotes</center>

1. The security is the bond <u>less</u> the option, because in effect the purchaser of the bond is required to write an option to the issuer, permitting the issuer to call the bond, i.e., to repay the loan. Industrial bonds and utility bonds almost always bear ten years and five years, respectively, of call protection.

2. Data are from Board of Governors of the Federal Reserve System, Flow of Funds Accounts, and are as of yearend 1976.

3. In addition to the references cited in the text, see also Bowlin [5], Pye [21], Jen and Wert [13], and Ofer and Taggart [19]. For an analysis which antedated Pye's, see Crockett's Appendix to Hess and Winn [11].

4. See, for example, the treatment in Elton and Gruber [10].

5. For analysis of the bond refunding problem, assuming a nonstochastic discount rate misses the heart of the problem, and both Pye and Elton and Gruber [9] incorporated a stochastic discount rate in their respective models. By contrast, much of the literature analyzing the refunding option has been deficient in assuming a nonstochastic discount rate; see, for example, Kraus [16], Kalymon [14] and Elton and Gruber [10].

6. See Bodie and Friedman [4].

7. To the extent that the refunding operation involves transactions costs, such as investment bankers' fees or use of management time, C_t includes these costs as well as the call price per se.

8. The model developed in this paper abstracts from a number of complications which arise from the option writer's need to estimate the option holder's objective and expectations, e.g., the "gaming" implications which would result if bond issuers and investors deliberately attempted to conceal their true objective and expectations from one another; see Cyert and de Groot [7, 8] for a treatment of an analogous problem in a different context.

9. Other objectives, perhaps incorporating risk aversion, would also be suitable, both here and below for the investor. The only requirement, which follows from the use of dynamic programming to solve the model, is that the objective function exhibit the intertemporal separability (additivity) property; for reference, see Bellman [1] and Bellman and Dreyfus [2]. In conjunction with a further assumption extending the present discounted value criterion to other aspects of the issuer's behavior, this objective is consistent with maximization of stockholders' wealth.

10. The call price typically provides for some premium above the bond's principal amount, with that premium varying positively with the number of years remaining to maturity at the time of exercise of the call option. One commonly used schedule of call prices, for example, begins at 100% of principal amount plus one year's coupon interest, if the bond is called within one year of its issue, and declines linearly to 100% of principal amount some few years before maturity.

11. No general solution exists for the refunding decision problem when expectations of the stochastic discount rate are state-dependent. To solve this problem, therefore, it is necessary to specify a unique stochastic process which the issuer assumes to generate the future ρ_τ, so that the resulting solution depends on the particular stochastic process chosen. Pye [20] dealt with this problem by assuming that the discount rate followed a first-order Markov process, with a small number of possible states, and using either estimated values or arbitrarily assigned values for the elements of the transition matrix. The state-independence assumption introduced above renders the problem solvable by straightforward dynamic programming techniques while still permitting a general form for the density functions $f_t(\rho_\tau)$.

12. Successive application of the time discounting procedure in (14) and (15), for a given set of expectations about the future discount rate, of course yields a form of the familiar expectations hypothesis of the term structure of interest rates; see Hicks [12] and Lutz [17].

13. Integrals $\int_0^{\rho_t^*} f'_{t-1}(\tilde{\rho}_t)\,d\rho$ and $\int_{\rho_t^*}^{\infty} f'_{t-1}(\tilde{\rho}_t)\,d\rho$ represent the investor's perception of probabilities analogous to $_tP_{t+1}$ and $(1 - {_tP_{t+1}})$, respectively, as in (2).

14. See, for example, Bowlin [5], Ofer and Taggart [19], and the references cited therein.

15. More precisely, issuers most often use a refunding situation to extend somewhat the maturity of their outstanding debt; Elton and Gruber's [9] generalization of Pye's model focuses on this point.

16. U.S. issuers of long-term corporate bonds almost always issue bonds bearing call options deferred for either five or ten years, depending on the business of the issuer. Issuers refunding bonds which originally bore five-year call deferments will typically issue new bonds bearing five-year deferments. Issuers refunding bonds which originally bore ten-year deferments will typically issue new bonds bearing ten-year deferments.

17. This simplified expression overstates R_t in that it does not take account of the possibility of subsequently calling the refinancing bond before maturity. The correct expression is

$$R_t = \frac{C_t}{1+\rho_t} \left\{ k_t \left[1 + \sum_{=t+1}^{t+D} \left(\prod_{s=t+1}^{\tau} \int_0^{\infty} \frac{1}{1+\tilde{\rho}_s} \tilde{f}_t(\tilde{\rho}_s) ds \right) \right] \right.$$

$$\left. + E_{t+D}(\tilde{R}_{t+D+1}) \cdot \prod_{\tau=t+1}^{t+D} \int_0^{\infty} \frac{1}{1+\tilde{\rho}_s} \tilde{f}_t(\tilde{\rho}_s) ds \right\}$$

where $E_{t+D}(\tilde{R}_{t+D+1})$ follows from a further backward induction solution analogous to that shown below except using k_t in place of k, and expressions (21) and (21') below are analogously altered. Hence the model is in general able to accommodate an entire series of potential refundings, as would be the case for D small relative to T. The possibility of multiple refundings, however, means that the computational requirements of the solution no longer increase only linearly with T.

18. Once again, this expectation is conditional on the bond's not being called as of the beginning of time period t.

19. The problem is therefore similar to the optimal one-time sale of an asset in the context of an uncertain stream of bids; see, for example, Karlin's [15] highly instructive treatment of that question.

20. See again footnote 17.

21. For the later periods of the outstanding bond's term (i.e., the first periods treated in the backward induction), the term to maturity of the refinancing bond to be issued is very short under the fixed-maturity assumption. For the limiting case of the final time period T as in (22) and (23), for example, k_T is a one-period borrowing rate, so that the discounting rate is $g_T(k_T) = k_T$. For discount rates of the order of those in the numerical examples presented below, however, any term-structure implications of this aspect of the fixed-maturity assumption in far-off time periods have only very small effects on the resulting calculations for current refunding decisions and valuations.

22. See again footnote 17.

23. Footnote 27 below qualifies this statement.

24. The normal distribution is truncated so as to preclude expectations of negative nominal interest rates.

25. Note that $C_t > 100$ provides yet another reason, apart from interest rate expectations, why it may not be optimal to refund at $k_t < 10.60\%$. The effect of a declining call price, so that $C_{t+\tau} < C_t$, could be to make it optimal to wait to refund even if $k_{t+\tau} \geq k_t$.

26. The remaining assumptions underlying the solution values presented in Table 1 are the same as those used in Figure 1, including flat, normally distributed expectations with $E(\tilde{k}_t) = E'(\tilde{k}_t) = 8\%$ for all t and truncated so as to preclude expectations of negative nominal interest rates.

27. Because $E(\frac{1}{1+\tilde{k}_t}) > \frac{1}{1+E(\tilde{k}_t)}$ for $(\tilde{k}_t) > 0$ (Jensen's Inequality), there is always a small effect in the opposite direction, i.e., requiring a <u>higher</u> price as $\sigma(\tilde{k}_t)$ becomes larger. For callable bonds, the effect of $\sigma(\tilde{k}_t) > 0$ on the valuation of the call option predominates. For noncallable bonds, the smaller (opposite) effect stands alone. As the "noncallable" column of Table 2 indicates, however, this effect is quantitatively very small for normally distributed interest rate expectations.

28. In each case the schedule of call prices declines linearly from 100 plus one year's interest in the first half year to 100 in the final half year of the bond's term to maturity.

29. The reason for the coupon rates <u>less</u> than 8% for the noncallable bond is again the effect of Jensen's Inequality; see again footnote 26.

30. See Bodie and Friedman [4] for an application of the model in reverse, to market data for 1969-76, to infer the uncertainty associated with expectations of future interest rates.

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