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THE (INTERESTING) DYNAMIC PROPERTIES OF THE NEOCLASSICAL GROWTH MODEL WITH CES PRODUCTION

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Part of this paper was written while I was the Kaiser Visiting Professor at the Department of Economics, Stanford University. This paper was, in part, written in response to a challenge given by Robert Barro to his graduate macroeconomics class, of which I was very fortunate to be a part of, to see if it was possible to generalize the analytical results presented in Barro and Sala-i-Martin (1995) to a more flexible specification of technology. I have benefitted from comments by Robert Dennis, Paul Evans, Doug Hamilton, Angelo Mascaro, John Sabelhaus, Jan Walliser. The Editor (Gary Hansen) helped improve the paper's exposition and an anonymous referee provided several valuable suggestions for making the proofs more rigorous. The views expressed in this paper are those of the authors and not necessarily those of the National Bureau of Economic Research.

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ABSTRACT

Despite being the standard growth model for several decades, little is actually known analytically about the dynamic properties of the neoclassical Ramsey-Cass-Koopmans growth model. This paper derives analytically the properties of the endogenous saving rate when technology takes the Constant Elasticity of Substitution (CES) form. For a factor substitution elasticity between capital and labor less than unity, the saving rate decreases along the transition path after the capital stock reaches a critical value identified analytically herein. But before reaching this critical value, the saving rate might increase and so, taken as a whole, the saving rate path might manifest "overshooting." Similarly, for a factor substitution elasticity greater than unity, the saving rate increases along the transition path after the capital stock reaches a critical value. Before reaching this critical value, the saving rate might decrease and so the saving rate path might manifest "undershooting." A simulation illustrating these interesting dynamics is presented.

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I. Introduction

Cass (1965) and Koopmans (1965) completed the standard growth model by merging Ramsey's (1928) theory of consumer optimization with neoclassical growth, allowing for an endogenous saving rate. Since that time, economists have wanted to understand the properties of the time path for the saving rate. Intriligator (1971, p. 438) noted that, barring specific assumptions on the form of the utility function and technology, the saving rate cannot necessarily be shown to be always increasing or decreasing along the optimal path. Intuitively, competing income and substitution effects produce ambiguous results in general. Years later, Robert Barro and Xavier Sala-i-Martin (1995) provided an analytical breakthrough by proving that the endogenous saving rate is either monotonically increasing, decreasing or constant throughout the entire transition path for the case of isoelastic utility (assumed to obtain a steady state) and Cobb-Douglas (CD) technology. Most other analysis has been numerical, prohibiting general statements.

This short paper extends the analysis of Barro and Sala-i-Martin to Constant Elasticity of Substitution (CES) technology that nests CD. For a factor substitution elasticity between capital and labor less than unity, the saving rate decreases along the transition path after the capital stock reaches a critical value identified analytically herein. But the path need not be monotonic. Before the capital stock reaches this critical value, the saving rate might increase and so the entire saving rate path manifests "overshooting." Similarly, for a factor substitution elasticity greater than unity, the saving rate increases along the transition path after the capital stock reaches a critical value. Before reaching this value, the saving rate might decrease and so the saving rate path "undershoots."

II. The Neoclassical Growth Model with CES Production

The following variable and parameter table is provided for easy reference.

[Place Table 1 here]

An agent's utility is defined over his/her consumption per labor unit at time t, c_t. Utility is isoelastic and equal to $u(c_t) = c_t^{1-\theta}/(1-\theta)$, where $\theta > 0$ is the constant relative risk aversion parameter. Households face the following familiar problem first proposed by Frank Ramsey in 1928 that can be interpreted á la Robert Barro (1974) as implying operative intergenerational linkages. In particular, households maximize,

(1) $$\int_0^\infty u(c_t)e^{-(\rho-n)t}dt,$$

where ρ is the rate of time preference and n is the population growth rate (with $\rho > n$). The dynamic budget constraint equals

(2) $$\frac{da_t}{dt} = w_t + r_t a_t - c_t - na_t \quad .$$

w_t is the wage, r_t is the interest rate, and a_t are assets per labor unit. The transversality condition is

(3) $$\lim_{t\to\infty} a_t \nu_t$$

where ν_t is the present-value shadow price of income from the following present-value Hamiltonian,

(4) $$H = u(c_t)e^{-(\rho-n)t} + \nu_t\left[w_t + r_t a_t - c_t - na_t\right]$$

Production takes the CES form,

(5) $$f(\hat{k}_t) = \left[\alpha \hat{k}_t^{\,1-\frac{1}{\sigma_{KL}}} + (1-\alpha)\right]^{\frac{1}{1-\frac{1}{\sigma_{KL}}}}$$

where $f(\hat{k}_t)$ is output per *effective* labor unit before depreciation at time t. σ_{KL} is the elasticity of substitution between capital and labor while α is the capital weight in production, $0 < \alpha < 1$. Cobb-

Douglas, $f(\hat{k}_t) = \hat{k}_t^{\alpha}$, is a special case ($\sigma_{KL} = 1$) and α is then equal to the capital share. $\hat{k}_t = k_t e^{-xt}$ is the capital stock per effective labor input at time t where k_t is capital stock per labor input and x is the rate of labor-augmenting technological change. General equilibrium requires $k_t = a_t$.

Combining the household and firm problems, we get the following equations of motion:

$$\text{(6)} \qquad \frac{d\hat{c}_t/dt}{\hat{c}_t} = \frac{1}{\theta}[r_t - \rho - \theta x]$$

$$\text{(7)} \qquad d\hat{k}_t/dt = f(\hat{k}_t) - \hat{c}_t - (x + n + \delta)\hat{k}_t$$

$r_t = f'(\hat{k}_t) - \delta$ with a depreciation rate of δ, and $\hat{c}_t = c_t e^{-xt}$ is consumption per effective labor unit.

Denoting the limiting values of variables with asterisks, the limiting values for the interest rate and the saving rate on the balanced growth path with CES production are as follows:

$$\text{(8)} \qquad r^* = \rho + \theta x = f'(\hat{k}^*) - \delta$$

$$\text{(9)} \qquad s^* = (x + n + \delta)\left(\frac{\alpha}{r^* + \delta}\right)^{\sigma_{KL}} .$$

III. Transitional Dynamics with CES Production

Let $z_t = 1 - s_t = \hat{c}_t / f(\hat{k}_t)$ equal the share of income consumed at time t and let $\gamma_{z(t)}$ equal the growth rate in the share of income consumed at time t. Then

$$\text{(10)} \qquad \gamma_{z(t)} \equiv \frac{dz_t/dt}{z_t} = \frac{d\hat{c}_t/dt}{\hat{c}} - \frac{f'(\hat{k}_t) d\hat{k}_t/dt}{f(\hat{k}_t)}$$

$$= \frac{1}{\theta}\left[f'(\hat{k}_t) - \delta - \rho - \theta x\right] - (1 - z_t) f'(\hat{k}_t) + \alpha^{\sigma_{KL}} f'(\hat{k}_t)^{1-\sigma_{KL}} (x + n + \delta)$$

$$= f'(\hat{k}_t)\left[\frac{1}{\theta} - (1 - z_t)\right] + \left\{ s^* \left[\frac{f'(\hat{k}_t)}{\rho + \theta x + \delta}\right]^{1-\sigma_{KL}} - \frac{1}{\theta}\right\} (\rho + \theta x + \delta)$$

where we have used equations (6) through (9) and the following relationship:

(11) $$\left(\frac{\alpha}{r_t+\delta}\right)^{\sigma_{KL}} = \frac{\hat{k}_t}{f(\hat{k}_t)} \quad .$$

The next proposition, proven in the Appendix, characterizes the endogenous transitional saving rate for the Ramsey economy with CES technology.

PROPOSITION 1 *Assume that $f(\hat{k})$ is the CES production function and that $\hat{k}_1 < \hat{k}^*$. Then*

(A) If $0 < \sigma_{KL} < 1$ and the value of θ is such that $s^*\left(\frac{f'(\hat{k}_1)}{f'(\hat{k}^*)}\right)^{1-\sigma_{KL}} < \frac{1}{\theta}$, then the saving rate is decreasing along the transition path from $\hat{k}_1$ to $\hat{k}^*$. Moreover, there exists a value of $\hat{k}_0 < \hat{k}_1$ such that $s^*\left(\frac{f'(\hat{k}_0)}{f'(\hat{k}^*)}\right)^{1-\sigma_{KL}} > \frac{1}{\theta}$, and the saving rate at $\hat{k}_0$ is increasing $\Leftrightarrow$

$$z_0 < 1 + \frac{1}{\theta}\left[\frac{f'(\hat{k}^*)}{f'(\hat{k}_0)} - 1\right] - s^*\left[\frac{f'(\hat{k}^*)}{f'(\hat{k}_0)}\right]^{\sigma_{KL}} .$$

(B) If $\sigma_{KL} > 1$ and the value of θ is such that $s^*\left(\frac{f'(\hat{k}_1)}{f'(\hat{k}^*)}\right)^{1-\sigma_{KL}} > \frac{1}{\theta}$, then the saving rate is increasing along the transition path from $\hat{k}_1$ to $\hat{k}^*$. Moreover, there exists a value of $\hat{k}_0 < \hat{k}_1$ such that $s^*\left(\frac{f'(\hat{k}_0)}{f'(\hat{k}^*)}\right)^{1-\sigma_{KL}} < \frac{1}{\theta}$, and the saving rate at $\hat{k}_0$ is decreasing $\Leftrightarrow$

$$z_0 > 1 + \frac{1}{\theta}\left[\frac{f'(\hat{k}^*)}{f'(\hat{k}_0)} - 1\right] - s^*\left[\frac{f'(\hat{k}^*)}{f'(\hat{k}_0)}\right]^{\sigma_{KL}} .$$

(C) If $\sigma_{KL} = 1$ and $s^* = 1/\theta$, then the saving rate is constant along the transition path from $\hat{k}_0$ to $\hat{k}^*$. (Barro and Sala-i-Martin [1995])

Case (A) considers a substitution elasticity between capital and labor less than unity. In this case, the saving rate will forever decrease along the transition path after the capital-labor ratio $\hat{k}_1$, defined implicitly by the expression $s^* \left(f'(\hat{k}_1)/f'(\hat{k}^*)\right)^{1-\sigma_{KL}} \leq \frac{1}{\theta}$, is reached. The values of $f'(\hat{k}^*)$ and s^* are given by equations (8) and (9), respectively. Before the capital-labor ratio $\hat{k}_1$ is reached, however, there also exists a value of the capital-labor ratio $\hat{k}_0$ such that the saving rate is increasing if and only if the second condition shown in Part (A) of the proposition holds. If this second condition is satisfied, the path of the saving rate, taken as a whole, overshoots its final steady state. Whether a simulation manifests overshooting depends, in part, on whether the initial value of the capital stock is sufficiently smaller than $\hat{k}_1$. If, for example, the initial value is chosen between $\hat{k}_0$ and $\hat{k}_1$ then overshooting may not occur. Simulation evidence below suggests that overshooting can exist with a reasonably parameterized economy.

Case (B) considers a substitution elasticity between capital and labor greater than unity. This case demonstrates that the saving rate will forever increase along the transition path after the capital-labor ratio $\hat{k}_1$, defined implicitly by the expression $s^* \left(f'(\hat{k}_1)/f'(\hat{k}^*)\right)^{1-\sigma_{KL}} \geq \frac{1}{\theta}$, is reached. In this case, the economy might also manifest "undershooting" where the saving rate first decreases over the transition path and then increases.

Case (C), given for completeness, shows the conditions when the saving rate is constant along the transition path. This case was considered already by Barro and Sala-i-Martin.

Cases (A) - (C) nest the possible equilibrium paths corresponding to the Cobb-Douglas production function ($\sigma_{KL} = 1$) considered by Barro and Sala-i-Martin. In the Cobb-Douglas case, the first inequalities presented in cases (A), (B) and (C) reduce to the simple mathematical formulations: (A) $s^* < 1/\theta$, (B) $s^* > 1/\theta$ and (C) $s^* = 1/\theta$, respectively. The absence of the capital-labor ratio in these formulations reflect the fact that the transitional endogenous saving rate is monotonically (*A*) decreasing, (*B*) increasing or (*C*) unchanging, respectively, throughout the entire transition path. But the subsequent conditions necessary for generating non-monotonic paths cannot hold. As shown above, however, things become more complicated with CES production.

IV. Illustrative Example: Overshooting

The policy function corresponding to the overshooting case ($\sigma_{KL} < 1.0$) is drawn in the phase diagram in Figure 1. Also drawn in Figure 1 is the policy function corresponding to the constant saving rate case ($\sigma_{KL} = 1.0$ and $s_t = s^* = 1/\theta$): $\hat{c}(\hat{k}) = (1-s^*)f(\hat{k})$. Both policy functions share the same value of s^*. For Cobb-Douglas production (not drawn), the policy function would never cross the constant saving rate policy function; this non-crossing property reflects the monotonic nature of the optimal transitional saving rate in the CD economy. For CES production however the policy function can cross the constant saving rate policy function as shown in Figure 1, resulting in an overshooting of the saving rate.

Figure 2 reports simulation evidence of overshooting. The policy function corresponding to the transition path can be calculated numerically using either the "shooting" method or the "time elimination" method described in the Appendix. I assume that per-capita income increased seven fold during the past 100 years. Let the index variables, Λ_T and Λ_X, describe the fraction of growth in per-capita output between period 0 (100 years ago) and time t that is attributed to transitional dynamics and technological change, respectively. In particular, $\Lambda_T = \log[f(\hat{k}_t)/f(\hat{k}_0)]/\log[f(k_t)/f(k_0)]$ and $\Lambda_X = x{\cdot}t/\log[y_t/y_0] = 1-\Lambda_T$. The implied rate of technological change equals

$$(12) \qquad x = \frac{\Lambda_X \log 7}{100}$$

where Λ_X is set equal to one half. The value of $\hat{k}_0$ (the capital-labor ratio 100 years ago) equals

$$f(\hat{k}_0) = \frac{f(\hat{k}^*_{100})}{e^{\Lambda_T \cdot \log 7}} \quad . \tag{13}$$

To reduce the amount of notation, I assume, without any impact to the numerical calculations, that year 100 (today) begins (i.e., is close enough to) the new steady state, i.e., $f(\hat{k}_{100}) = f(\hat{k}^*)$. A value of $\sigma_{KL} = 0.85$ is chosen. The steady-state capital-output ratio is equal to 4.5 which, as Barro and Sala-i-Martin explain, can be interpreted as reflecting a broader measure of the capital stock including at least some human capital. Figure 2 reports several variables including the gross saving rate at time t, $s_t = 1 - \hat{c}_t / f(\hat{k}_t)$, the net saving, $s_t^n = s_t - \delta\, \hat{k}_t / f(\hat{k}_t)$, and the μ_t gross capital share at time t. Saving rates and capital shares include human capital investment. Notice that the capital-output ratio increases from 2.5 to 4.5 which is reasonable, especially under a broad interpretation of capital that includes human capital, which has increased significantly during the past century. The value of r_0 is 0.19 and the half-life for income convergence is a respectable 22 years. There is little movement in the capital share. A simulation with "undershooting" is available from the author.

V. Conclusion

This paper derived the analytical properties of the optimal endogenous saving rate along the transition path for the standard neoclassical (Ramsey) growth model with CES technology. The saving rate decreases [increases] monotonically when the capital-labor factor substitution elasticity is below [above] unity, after the capital stock reaches a critical value derived herein. Before reaching this critical value, however, the saving rate might increase [decrease] and the saving path, taken as a whole, might manifest "overshooting" [undershooting]. In contrast, in the case of Cobb-Douglas production, convergence is monotonic throughout the entire transition path.

Appendix

Proof of Proposition 1

Consider case (*A*). Notice $\sigma_{KL} < 1$ implies $\left(\dfrac{f'(\hat{k}_1)}{f'(\hat{k}^*)}\right)^{1-\sigma_{KL}} > 1$ since $\hat{k}_1 < \hat{k}^*$ and $f''(k) < 0$. Hence, $s^* < 1/\theta$. For the economy to approach a steady state, $\gamma_Z^* \equiv \lim_{t\to\infty} \gamma_{Z(t)} = 0$. Equations (8) and (10) imply $\gamma_Z^* = 0 = f'(k^*)\left[\frac{1}{\theta} - (1-z^*)\right] + \left[s^* - \frac{1}{\theta}\right](\rho + \theta x + \delta)$. Since the second term on the RHS of the equality is negative, the first term is positive. Since $f' > 0$, we have $\frac{1}{\theta} - (1-z^*) > 0$, or $z^* > \frac{\theta-1}{\theta}$. Now suppose that there is a value of *t* such that $z_t \leq \frac{\theta-1}{\theta}$. By equation (10), $\gamma_{Z(t)} < 0$ and so $\dot{z}_s < 0$ and $\gamma_{Z(s)} < 0 \;\forall\, s > t$. Hence, $z^* < \frac{\theta-1}{\theta}$, which is a contradiction. It follows that $z_t > \frac{\theta-1}{\theta} \;\forall\, t$. Now differentiate (10) with respect to *t*:

$$\text{(A.1)}\qquad \begin{aligned} \dot{\gamma}_{z(t)} &= f''(\hat{k}_t)\frac{d\hat{k}_t}{dt}\left[z_t - \frac{\theta-1}{\theta}\right] + f'(\hat{k}_t)\gamma_{z(t)}z_t \\ &\quad + s^*(r^*+\delta)(1-\sigma_{KL})\left(\frac{f'(\hat{k}_t)}{f'(\hat{k}^*)}\right)^{-\sigma_{KL}}\frac{f''(\hat{k}_t)}{f'(\hat{k}^*)}\frac{d\hat{k}_t}{dt} \end{aligned}$$

Equation (A.1) implies that $\gamma_{Z(t)} > 0 \;\forall\, t$. To prove this claim, note that the first and third terms on the RHS of equation (A.1) are both strictly negative. Now suppose that there is a value of *t* such that $\gamma_{Z(t)} \leq 0$. Then the second term is weakly negative, which implies that $\dot{\gamma}_{z(t)} < 0 \Rightarrow \gamma_{Z(s)} < 0$ and $\dot{\gamma}_{z(s)} < 0 \;\forall\, s > t$. Hence, $\gamma_Z^* < 0$, which again is inconsistent with the economy approaching a steady state. It follows that $\gamma_{Z(t)} > 0 \;\forall\, t$ and therefore $\dot{s} < 0 \;\forall\, t$, which completes the first part of the proof. The proof that there exists a value of $\hat{k}_0 < \hat{k}_1$ such that $s^*\left(f'(\hat{k}_0)\big/ f'(\hat{k}^*)\right)^{1-\sigma_{KL}} > 1/\theta$ follows immediately from the properties of the function, *f*, and the assumption $\sigma_{KL} < 1$. In particular, for CES production, $\lim_{\hat{k}\to 0} f'(\hat{k}) = \infty$ and so $\lim_{\hat{k}_0\to 0}\left(f'(\hat{k}_0)\big/ f'(\hat{k}^*)\right)^{1-\sigma_{KL}} = \infty$ since $\sigma_{KL} < 1$. By the general equilibrium condition, $k_t = a_t$, we know that $s^* > 0$; otherwise, the marginal product of capital is infinite. Hence, for any value of $\theta > 0$, there exists a $\hat{k}_0 > 0$ *s.t.* $s^*\left(f'(\hat{k}_0) \Big/ f'(\hat{k}^*) \right)^{1-\sigma_{KL}} > 1/\theta$. The necessary and sufficient condition for saving to be increasing at $\hat{k}_0$ shown in the Proposition then follows from setting $\gamma_{z(0)} < 0$ in equation (10) and then reducing algebraically.

Case B can be proven in a similar fashion. In particular, $\sigma_{KL} > 1$ implies $\left(\dfrac{f'(\hat{k}_1)}{f'(\hat{k}^*)} \right)^{1-\sigma_{KL}} < 1$ since $\hat{k}_1 < \hat{k}^*$ and $f''(k) < 0$. Hence, $s^* > 1/\theta$. Now in order to satisfy the steady-state equation, $\gamma_Z^* = 0 = f'(k^*)\left[\frac{1}{\theta} - (1 - z^*)\right] + \left[s^* - \frac{1}{\theta}\right](\rho + \theta x + \delta)$, the inequality, $z^* < \dfrac{\theta - 1}{\theta}$, must hold. The same type of proof by contradiction used above can be used to show that $z_t < \dfrac{\theta - 1}{\theta}\ \forall t$. Now notice that the first and third terms on the RHS of equation (A.1) are both strictly positive. It immediately follows, along the same line of argument above, that $\gamma_{Z(t)} < 0\ \forall t$ and therefore $\dot{s} > 0$ $\forall t$, which completes the first part of the proof. To establish the remainder of the Proposition for part B, notice $\lim_{\hat{k}\to 0} f'(\hat{k}) = \infty$ now implies $\lim_{\hat{k}_0\to 0}\left(f'(\hat{k}_0) \Big/ f'(\hat{k}^*) \right)^{1-\sigma_{KL}} = 0$ since $\sigma_{KL} > 1$. Since s^* is finite (otherwise, the general-equilibrium condition implies there is no marginal return to saving), there exists a $\hat{k}_0 > 0$ *s.t.* $s^*\left(\dfrac{f'(\hat{k}_0)}{f'(\hat{k}^*)} \right)^{1-\sigma_{KL}} < \dfrac{1}{\theta}$. The necessary and sufficient condition for saving to be increasing at $\hat{k}_0$ follows from setting $\gamma_{z(0)} > 0$ in equation (10) and then reducing algebraically.

Case C was proven already in Barro and Sala-i-Martin (1995).■

Solving the Policy Function

The policy function for the model herein can be described implicitly as the solution to the following differential equation:

(A.2) $$\frac{d\hat{c}(\hat{k})}{d\hat{k}} = \frac{\hat{c}(\hat{k})\left[f'(\hat{k}) - \delta - \rho - \theta x\right]}{f(\hat{k}) - \hat{c}(\hat{k}) - (x + n + \delta)\hat{k}},$$

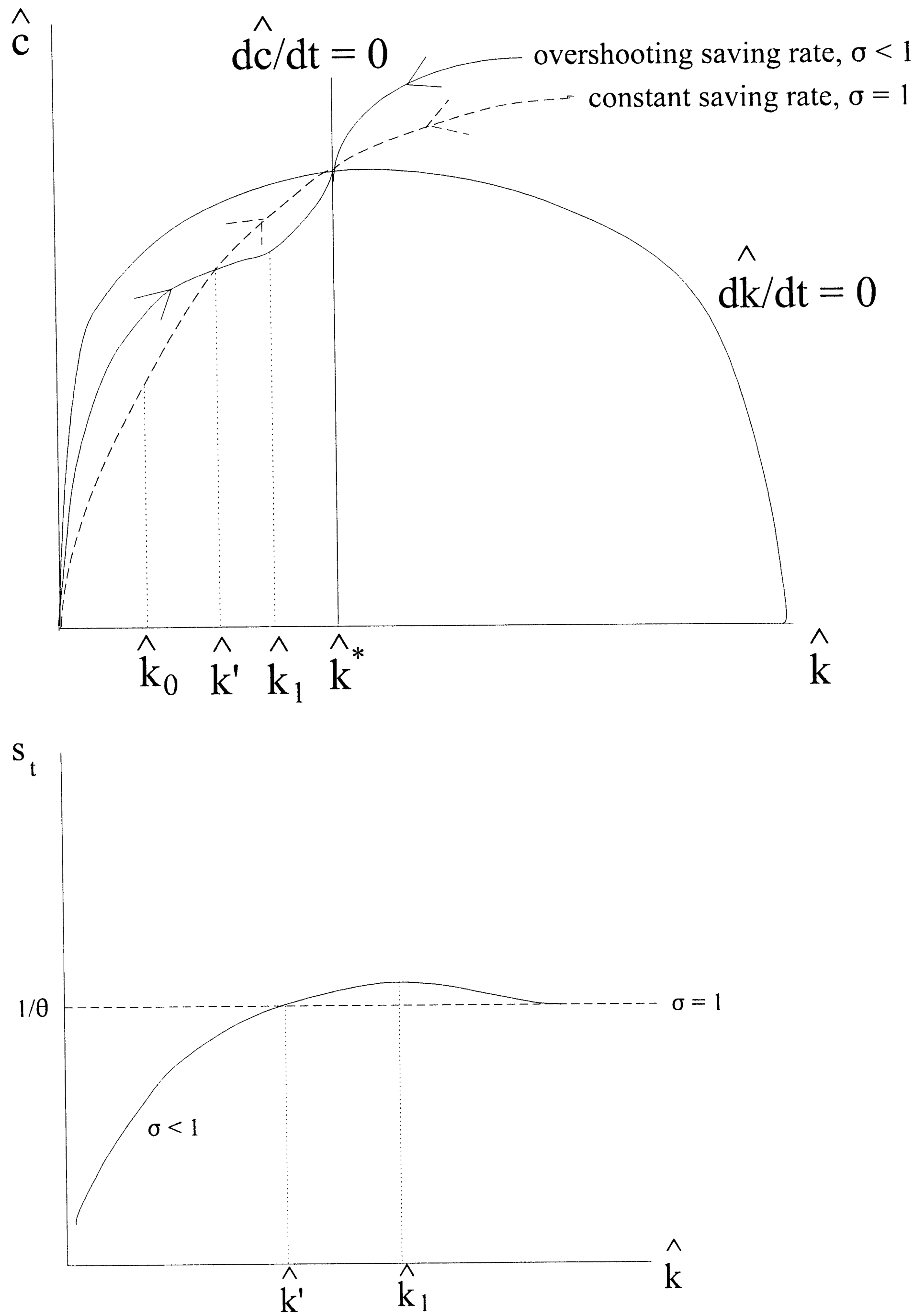
with the boundary (or initial) values given by $(\hat{c},\hat{k}) = (\hat{c}^*,\hat{k}^*)$. To see this, note that $\frac{d\hat{c}}{d\hat{k}} = \frac{d\hat{c}(\hat{k}_t)/dt}{d\hat{k}_t/dt}$ where the numerator and the denominator are given by equations (6) and (7), respectively. Equation (A.2) is solved numerically for the policy function using the *shooting method* or the *time-elimination method.* The standard *shooting method* employs the bisection updating technique to solve for the correct value of $\hat{c}_0$ such that the generated policy function satisfies the transversality condition. The *time-elimination method* is outlined by Casey Mulligan and Xavier Sala-i-Martin (1993). This novel approach transforms the standard boundary-value problem into an easier initial-value problem by employing L'Hôpital's rule at the limit point $(\hat{c},\hat{k}) = (\hat{c}^*,\hat{k}^*)$ (the slope of the policy function, equation (A.2), is indeterminant at the limit point). Applying L'Hôpital's rule to equation (A.2) at the limit point renders a quadratic equation in $\partial\hat{c}(\hat{k}^*)/d\hat{k}$. The positive root of this quadratic equation is the correct solution since $d\hat{c}(\hat{k}^*)/d\hat{k} > 0$ (Figure 1). After a little algebra, it can be shown that the positive root in the model herein is given by

(A.3) $$\frac{\partial\hat{c}(\hat{k}^*)}{d\hat{k}} = \frac{\left[f'(\hat{k}^*) - (x+n+\delta)\right] + \left\{\left[f'(\hat{k}^*) - (x+n+\delta)\right]^2 - 4\frac{\hat{c}^*}{\theta}f''(\hat{k}^*)\right\}^{1/2}}{2}.$$

Table 1
Variable Table

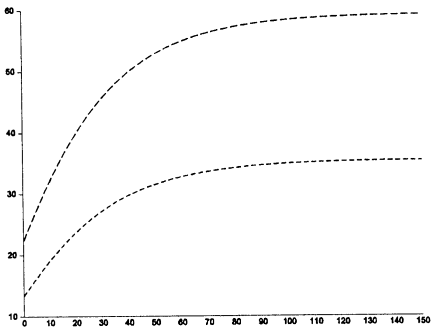
a_t	=	assets per labor input at time t.
c_t	=	consumption per labor input at time t.
$\hat{c}_t$	=	consumption per *effective* labor input at time $t = c_t e^{-xt}$.
$f(\hat{k}_t)$	=	output per *effective* labor unit gross of depreciation at time t.
k_t	=	capital stock per labor input at time t.
$\hat{k}_t$	=	capital stock per *effective* labor input at time $t = k_t e^{-xt}$.
n	=	population growth rate.
r_t	=	interest rate at time t.
s_t	=	gross saving rate at time $t = 1 - \hat{c}_t/f(\hat{k}_t)$.
s_t^n	=	net saving rate at time $t = s_t - \delta \cdot \hat{k}_t/f(\hat{k}_t)$
$u(\cdot)$	=	isoelastic utility (felicity) function, $u(c) = (c^{1-\theta}-1)/(1-\theta)$.
w_t	=	wage rate at time t
x	=	rate of labor-augmenting technological change.
z_t	=	share of income consumed at time $t = 1-s_t = \hat{c}_t/f(\hat{k}_t)$.
α	=	capital weight in production, $0 < \alpha < 1$.
δ	=	constant rate at which the capital stock depreciates.
$\gamma_{Z(t)}$	=	growth rate in the share of income consumed at time $t = (\partial z_t/\partial t)/z_t$.
μ_t	=	gross capital share at time $t = \alpha$ for Cobb-Douglas production (σ_{KL}=1).
ρ	=	rate of time preference.
Λ_T	=	$\log[f(\hat{k}_t)/f(\hat{k}_0)]/\log[f(k_t)/f(k_0)]$ = fraction of cumulative growth in per-capita output between period 0 and t due to transitional dynamics.
Λ_X	=	$x \cdot t/\log[y_t/y_0]$ = fraction of growth due to technological change = 1-Ψ_T.
σ_{KL}	=	elasticity of substitution between capital and labor.
θ	=	constant relative risk aversion
ν_t	=	present-value shadow price of income

Phase Diagram for the Neoclassical Production Function with CES Production and σ_{KL} < 1.0 and the Implied Saving Rate: The Case of an Overshooting Saving Rate
(Figure 1)

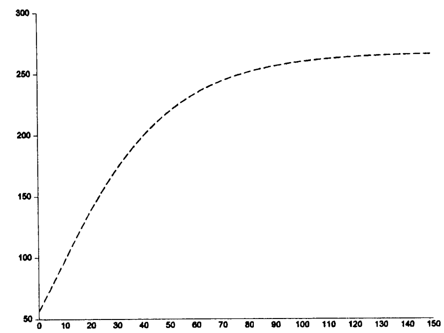


The Transition Path for Parameterization Noted Below
(Figures 2 [*a* - *f*])

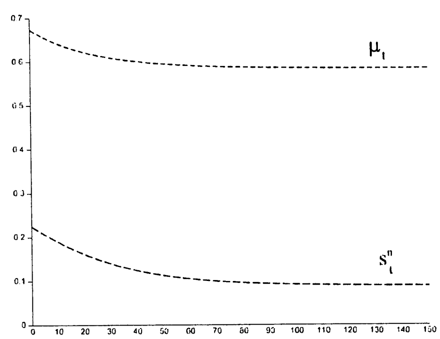
(*a*) $f(\hat{k}_t)$, $\hat{c}_t$



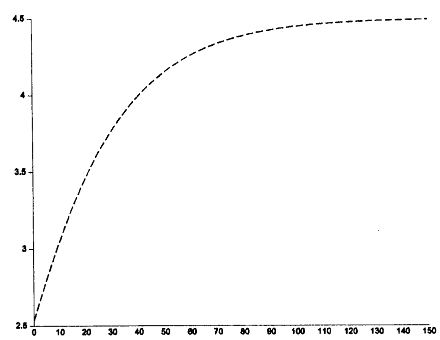
(*b*) $\hat{k}_t$



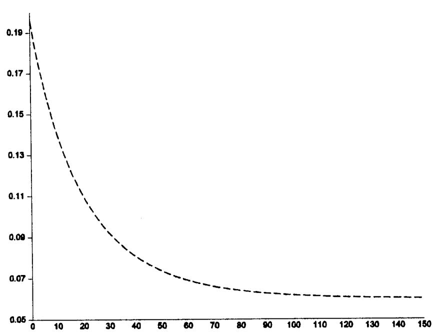
(*c*) μ_t , s_t^n



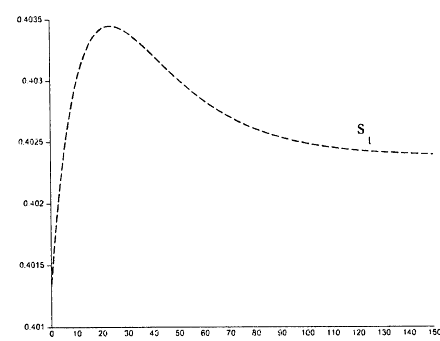
(*d*) $\hat{k}_t / f(\hat{k}_t)$



(*e*) r_t



(*f*) s_t



Parameter Vector: r^*=0.06, $\hat{k}^* / f(\hat{k}^*)$=4.5, σ_{KL}=0.85, Λ_T=0.50, α = 0.75, θ = 3.0, n = 0.01
Implied Values: δ = 0.07, ρ = 0.031, x = 0.01, half-life of output = 22 years.

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