Solutions to Linear Rational Expectations Models: A Compact Exposition

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Abstract: An elementary exposition is presented of a convenient and practical solution procedure for a broad class of linear rational expectations models. The undetermined-coefficient approach utilized keeps the mathematics very simple and permits consideration of alternative solution criteria.

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I. Introduction

The purpose of this note is to present a compact and easily understood exposition of a convenient and practical procedure for solving linear rational expectations (RE) models. The procedure, which is applicable to a class of models that is broad enough to include most cases of practical interest, can be implemented by means of a MATLAB routine provided by Paul Klein (1997).[1] The present exposition departs from Klein's, however, by relying upon the elementary undetermined-coefficients (UC) approach discussed in McCallum (1983). Indeed, the current exposition can be viewed as only an extension of the appendix to this last paper. It is, however, an extension that is nontrivial and essential for practical purposes. Here it is accomplished by use of the generalized Schur decomposition theorem discussed by Klein. The UC reasoning utilized in the present paper is much more elementary mathematically than Klein's and in addition is useful for consideration of alternative criteria for the selection of a single RE solution.

2. Undetermined Coefficient Setup

Let y_t be a $M\times 1$ vector of non-predetermined endogenous variables, k_t be a $K\times 1$ vector of predetermined variables, and u_t be a $N\times 1$ vector of exogenous variables. The model can then be written as

(1) $A_{11} E_t y_{t+1} = B_{11} y_t + B_{12} k_t + C_1 u_t$

(2) $u_t = R u_{t-1} + \varepsilon_t$

[1] Klein's (1997) approach builds upon earlier contributions of King and Watson (1995) and Sims (1996). Other significant recent contributions are Uhlig (1997) and Binder and Pesaran (1995), which use UC analysis. The Uhlig paper also features a useful procedure for linearizing models that include nonlinear relationships.

where A_{11} and B_{11} are square matrices while ε_t is a N×1 white noise vector.[2] Thus u_t is formally a first-order autoregressive process, which can of course be defined so as represent AR processes of higher orders for the basic exogenous variables. Also, for the predetermined variables we assume

(3) $$k_{t+1} = B_{21}y_t + B_{22}k_t + C_2u_t$$

If only once-lagged values of y_t were included in k_t, then we would have $B_{21} = I$, $B_{22} = 0$, and $C_2 = 0$, but the present setup is much more general. Crucially, the matrices A_{11}, B_{21}, and B_{22} may be singular.

In this setting a UC solution will be of the form

(4) $$y_t = \Omega k_t + \Gamma u_t$$

(5) $$k_{t+1} = \Pi_1 k_t + \Pi_2 u_t,$$

where the Ω, Γ, Π_1, and Π_2 matrices are real. Therefore, $E_t y_{t+1} = \Omega E_t k_{t+1} + \Gamma E_t u_{t+1} = \Omega(\Pi_1 k_t + \Pi_2 u_t) + \Gamma R u_t$. Substitution into (1) and (3) then yields

[2] Here $E_t y_{t+1}$ is the expectation of y_{t+1} conditional upon an information set that includes all of the model's variables dated t and earlier.

(6) $A_{11}[\Omega(\Pi_1 k_t+\Pi_2 u_t) + \Gamma R u_t] = B_{11}[\Omega k_t+\Gamma u_t] + B_{12}k_t + C_1 u_t$

and

(7) $(\Pi_1 k_t+\Pi_2 u_t) = B_{21}(\Omega k_t+\Gamma u_t) + B_{22}k_t + C_2 u_t$.

Collecting terms in k_t, it is implied by UC reasoning that

(8) $$\begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \Omega\Pi_1 \\ \Pi_1 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} \Omega \\ I \end{bmatrix}$$

whereas the terms in u_t imply

(9) $A_{11}\Omega\,\Pi_2 + A_{11}\Gamma R = B_{11}\Gamma + C_1$

(10) $\Pi_2 = B_{21}\Gamma + C_2$.

3. Solution

Let A and B denote the two square matrices in (8), and assume that $|B-\lambda A|$ is nonzero for some complex number λ. This last condition will not hold if the model is poorly formulated (i.e., fails to place any restriction on some endogenous variable); otherwise it will be satisfied even with singular A_{11}, B_{21}, B_{22}.[3] Then the generalized Schur decomposition theorem guarantees the existence of unitary (therefore invertible) matrices Q and Z such that QAZ = S and QBZ = T,

[3] See King and Watson (1995) or Klein (1997).

where S and T are triangular.[4] The ratios t_{ii}/s_{ii} are generalized eigenvalues of the matrix pencil B - λA;[5] they can be rearranged without contradicting the foregoing theorem. Such rearrangements correspond to selection of different UC solutions as discussed in McCallum (1983, pp. 145-147 and 165-166). We shall return to this topic in Section 4; for the moment let us assume that the eigenvalues t_{ii}/s_{ii} (and associated columns of Q and Z) are arranged in order of their moduli with the largest values first.

Now premultiply (8) by Q. Since QA = SH and QB = TH, where $H \equiv Z^{-1}$, the resulting equation is

$$(11) \quad \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega\Pi_1 \\ \Pi_1 \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}$$

and its first row can be written as

$$(12) \quad S_{11}(H_{11}\Omega+H_{12})\Pi_1 = T_{11}(H_{11}\Omega+H_{12}).$$

The latter will be satisfied for Ω such that

$$(13) \quad \Omega = -H_{11}^{-1}H_{12} = -H_{11}^{-1}\left(-H_{11}Z_{12}Z_{22}^{-1}\right) = Z_{12}Z_{22}^{-1},$$

[4] See Golub and Van Loan (1996, p. 377).

[5] Or, in the terminology used by Uhlig (1995), are eigenvalues of B with respect to A.

where the second equality results because HZ = I. Thus we have a solution for Ω, provided that Z_{22}^{-1} exists.[6]

Next, writing out the second row of (11) we get

(14) $$S_{21}(H_{11}\Omega+H_{12})\Pi_1 + S_{22}(H_{21}\Omega+H_{22})\,\Pi_1 = T_{21}(H_{11}\Omega+H_{12}) + T_{22}(H_{21}\Omega+H_{22}).$$

Then using (13) and HZ = I we can simplify this to

(15) $$S_{22}\, Z_{22}^{-1}\, \Pi_1 = T_{22}\, Z_{22}^{-1}$$

so since S_{22}^{-1} exists by construction [7] we have

(16) $$\Pi_1 = Z_{22}\, S_{22}^{-1}\, T_{22}\, Z_{22}^{-1}.$$

To find Γ and Π_2 we return to (9) and (10). Combining them we have

(17) $$G\Gamma + A_{11}\Gamma R = F$$

[6] This is the same condition as that required by Klein (1997, p. 13) and King and Watson (1995, pp. 9-11). It appears to provide no difficulties in practice. The King and Watson example of a system in which the condition does not hold is one in which $B_{12} = 0$ in my notation so the MSV solution has $\Omega = 0$ and the other solution matrices follow easily.

[7] By the arrangement of generalized eigenvalues, S_{22} has no zero elements on the diagonal (and is triangular).

where $G \equiv A_{11}\Omega B_{21} - B_{11}$ and $F \equiv C_1 - A_{11}\Omega C_2$. If G^{-1} exists, which it typically will with nonsingular B_{11}, the latter becomes

$$(18) \quad \Gamma + G^{-1}A_{11}\Gamma R = G^{-1}F.$$

This can be solved for Γ by the steps given in McCallum (1983, p. 163) or can be obtained as

$$(19) \quad \text{vec}(\Gamma) = [I + R'\otimes G^{-1}]^{-1}\, \text{vec}(G^{-1}F),$$

as in Klein (1997, p. 28).[8] Finally, Π_2 is obtained from (10). In sum, the UC solution for a given ordering of the eigenvalues is given sequentially by equations (13), (16), (19) and (10).

4. Solution Criteria

Different values of Ω, and thus different solutions, will be obtained for different orderings of the generalized eigenvalues t_{ii}/s_{ii}. What ordering should be used to obtain the economically relevant solution? Many writers, following Blanchard and Kahn (1980), arrange them in order of decreasing modulus and conclude that a unique solution obtains if and only if the number with modulus less than 1.0 ("stable roots") equals K, the number of predetermined variables. The minimal-state-variable (MSV) procedure of McCallum (1983), by contrast, is to choose the arrangement that would yield $\Omega = 0$ if it were the case that $B_{12} = 0$—this step relying

[8] This uses the identity that if A, B, C are real conformable matrices, $\text{vec}(ABC) = (C'\otimes A)\,\text{vec}(B)$. See Golub and Van Loan (1996, p. 180).

upon the continuity of eigenvalues with respect to parameters.[9] Uhlig (1997, p. 17) correctly notes that this procedure is difficult to implement and also that in most cases it will lead to the same solution as the Blanchard-Kahn stability criterion. Adoption of the decreasing-value arrangement will therefore usually be attractive, even for MSV adherents. In such cases it seems unnecessary, however, to limit one's attention to problems in which there are exactly K stable roots. If there are fewer than K stable roots, the MSV criterion would produce a single explosive solution whereas if there are more than K stable roots, it would yield the single stable solution that is bubble-free—both of these being solutions that may be of particular scientific interest. In those exceptional cases in which an MSV analyst suspects that the Blanchard-Kahn and MSV criteria would call for different solutions, he/she could plot eigenvalues for various values of B_{12} and then adjust the ordering if necessary. But usually the decreasing-value arrangement will be appropriate.

[9] With $B_{12} = 0$, k_t does not appear in the system (1) (2), so k_t represents extraneous variables of a bootstrap, bubble, or sunspot nature.

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