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HORIZON LENGTH AND PORTFOLIO RISK

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ABSTRACT

In this paper, we compare the attitude towards current risk of two expected-utility-maximizing investors that are identical except that the first investor will live longer than the second one. In one of the models under consideration, there are two assets at every period. The first asset has a zero sure return, whereas the second asset is risky without serial correlation of yields. It is often suggested that the young investor should purchase more of the risky asset than the old investor in such circumstances. We show that a necessary and sufficient condition to get this property is that the Arrow-Pratt index of absolute tolerance (T_u) be convex. If we allow for a positive risk-free rate, the necessary and sufficient condition is T_u convex, plus $T_u(0) = 0$. It extends the well-known result that investors are myopic in this model if and only if the utility function exhibits constant relative risk aversion.

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1 Introduction

How should the length of one's investment horizon affect the riskiness of his portfolio? This question confronts startup companies choosing which ventures to pursue before they go public, ordinary investors building a nest egg, investment managers concerned with contract renewals and executives seeking strong performance before their stock options come due, among others.

Portfolio decisions are the focus of this analysis. However, the horizon-risk relationship extends beyond finance. Thus, students may vary their strategy on grades – say how venturesome a paper to write – over the course of the grading period; presidents may adjust the risk of their political strategies from early through mid-term and then as they approach an election.

Popular treatments suggest that short horizons often lead to excessively conservative strategies. Thus, the decisions of corporate managers, graded on their quarterly earnings, are said to focus too much on safe, short-term strategies, with underinvestment say in risky R & D projects. Privately-held firms, it is widely believed, secure substantial benefit from their ability to focus on longer-term projects. Mutual fund managers, who get graded regularly, are also alleged to focus on strategies that will assure a satisfactory short-term return, with long-term expectations sacrificed.[1]

Economists and decision theorists, speculators and bettors, have long been fascinated by the problem of repeated investment. Thus, long ago Bernoulli provided the first motivation of utility theory when he confronted the St. Petersburg Paradox, whose components can be reformulated as payoffs from an infinite series of actuarially fair, double-till-you-lose bets. Successful speculators must manage their money effectively when making a series of bets that are actuarially favorable. They must determine how much to allocate to each gamble given its odds, future prospects, and the time horizon.[2]

In recent years, this class of problems has been pursued in two different

[1]The experience of the U.S. mutual fund Twentieth Century Giftrust is instructive. It requires monies to be left with it for 10 years at least. Managers of the fund suggest that thanks to this 10-year no-withdrawal rule the fund was able to return 24 percent annually since 1985, nearly 10 points better than the S & P500.(See Newsweek 6/19/95, page 60)

[2]With n periods, utility function $u(w_n)$ on terminal wealth and initial wealth w_0, he must lay out a contingent strategy of how much to put at risk each period. This dynamic programming problem becomes tractable with an analytical solution only for HARA utility functions.

literatures. one on utility theory the other on dynamic investment strategies. A recurring theme in both is that the opportunity to make further investments affects how one should invest today. These two literatures have not merged. in part because a key question has gone unresolved. How should the length of the investment horizon affect the riskiness of one's investments? Some special cases have yielded results, as in the case of logaritmic and power utility functions. And with any particular utility function and set of investment opportunities actual calculations, perhaps using a simulation, could answer that question within a dynamic programming framework. But the central theoretical question of the link between the structure of the utility function and the horizon-riskiness relationship remained unresolved. This paper attempts to resolve that question.

In the formal literature, the horizon-riskiness issue has received the greatest attention addressing portfolios appropriate to age. Samuelson [1989a] and several others have asked: "As you grow older and your investment horizon shortens, should you cut down your exposure to lucrative but risky equities?" Conventional wisdom answers affirmatively, stating that long-horizon investors can tolerate more risk because they have more time to recoup transient losses. This dictum has not received the imprimatur of science, however. As Samuelson [1963, 1989a] in particular points out, this "time-diversification" argument relies on a fallacious interpretation of the Law of Large Numbers: repeating an investment pattern over many periods does not cause risk to wash out in the long run.

Early models of dynamic risk-taking, such as those developed by Samuelson [1969] and Merton [1969] find no relationship between age and risk-taking. This is hardly surprising, since the models, like most in continuous time finance, employ utility functions exhibiting harmonic absolute risk aversion (HARA). This choice is hardly innocuous: If the risk-free rate is zero, myopic investment strategies are rational iff the utility function is HARA (Mossin, 1968). Given positive risk-free rates, empirically the normal case, only constant relative risk aversion (CRRA) utility functions allow for myopia. A myopic investor bases each period's decision on that perio'd initial wealth and investment opportunities, and maximizes the expected utility of wealth at the end of the period. The value function on wealth exhibits the same risk aversion as the utility function on consumption; the future is disregarded.

Judging by empirical evidence, constant relative risk aversion does not seem to be a reasonable assumption. First, given CRRA, the optimal share of wealth invested in various assets would be independent of wealth level. However, as Kessler and Wolff [1991] show, the portfolios of households with low wealth contain a disproportionately large share of risk free assets. Measuring by wealth, over 80 % of the lowest quintile's portfolio was in liquid assets, whereas the highest quintile held less than 15 % in such assets.[3]

This work is also linked to the literature on compound risks. Samuelson [1963], looking at how many replications of an actuarially favorable gamble to take, opened the utility literature in this area, which considers more generally the effects of taking one gamble on the willingness to take other independent gambles. The multiple-period portfolio problem, in essence, considers successive sets of independent gambles. In each period the investor can shift his portfolio, depending on past results and the new time horizon.

A number of recent articles examine portfolio choice given independent risks. Pratt and Zeckhauser [1987] investigate the interaction between independent risks in a static model. They introduce the concept of proper risk aversion. Said crudely, risk aversion is proper if adding an undesirable risk to wealth has a negative impact on the attitude towards other risks. This concept is clearly related to our problem, with the existence of future risks making current risks more or less desirable. However, this concept is not directly useful to our problem, since future risks that will be undertaken are inherently desirable (otherwise they would not be undertaken). Moreover, Pratt and Zeckhauser do not consider problems with dynamic structures. The new concepts of standard risk aversion (Kimball, 1993) and risk vulnerability (Gollier and Pratt, 1995) both directly address portfolio composition with independent risks. These approaches all relate to required powers on the third and fourth derivatives of the utility function.

Our analysis focuses solely on investment risks and returns. Thus, it does not consider a range of background risks that are likely to prove more significant for a young than an old individual. For example, if both work, wages when old are more distant for the young individual, hence likely to be less certain. If work is only conducted when young, then only a young person faces undesirable background risk for wages. If utility is vulnerable to risk (Gollier and Pratt [1995]). the risk on human capital increases aversion to

[3]If individuals have CRRA utility functions but differ in risk aversion, given the extraordinary relative performance of equities in the postwar world, empirically we will find that those with lesser risk aversion will hold more stocks and hold greater wealth.

other independent risks, this will inhibit risk taking while young.

Guiso, Jappelli, and Terlizzese [1996] tested the relation between age and risk-taking in a cross-section of Italian households. Their empirical results show that young people, presumably facing greater income risk than old, actually hold the smallest proportion of risky assets in their portfolios. The share of risky assets increases by 20% to reach its maximum at age 61 [Guiso et al. p. 165].

Bodie, Merton and Samuelson [1992] consider the advantage of young people, with more periods to live and work, who can adjust their labor supply in response to uncertain investment returns. This ability, they find, induces younger individuals with a CRRA utility function to take on greater investment risk than older ones. King and Leape [1987] find young people hold less diversified portfolios than old; they identify the accretion of financial information over the lifetime as a possible explanation.

This paper consider properties of utility functions. It examines their implications for the horizon-risk relationship, where investors seek to maximize the expected utility of terminal wealth. Our holy grail is the set of necessary and sufficient conditions on u that guarantee that an investor with a longer horizon will assume more risk in his portfolio. Given such conditions, we say that *duration enhances risk*, abbreviated DER.

To forewarn the reader of our results, if the risk-free rate is zero, DER will be satisfied iff the Arrow-Pratt measure of absolute risk tolerance $T_u(z) = -u'(z)/u''(z)$ is convex (this measure is the inverse of the measure of absolute risk aversion). If the risk-free rate is positive, DER holds iff T_u is convex and $T_u(0) = 0$. The latter condition we label superhomogeneity; it is equivalent to infinite risk aversion at 0. Convex absolute risk tolerance is a condition involving the second, third and fourth derivatives of the utility function.

Through Section 4, our analysis considers a two-period model with a zero risk-free rate, and consumption only at the end of the second period. The paper is organized as follows: In Section 2, we address a world with complete markets at every period. The proof of the necessity and sufficiency of convex risk tolerance is straightforward in that case. The standard portfolio problem with two aseets and two periods is considered in Section 3. In Section 4, we analyze the concept of convex risk tolerance and link it to other concepts. Section 5 introduces three areas of extension for our analysis: a positive risk-free rate, the possibility of intermediate consumption and production, and multiple-period models. Section 6 concludes.

2 A simple model with complete markets

We consider a two-period model in which investors maximize the expected utility of their terminal wealth. Young investors invest and live for two periods, old investors merely for one. Each investor is endowed with wealth w at the beginning of period 1, and has utility function u, which is assumed to be twice differentiable, increasing and concave. The risk free rate is zero. At the end of any period t, $t = 1, 2$, the realization of random variable $\tilde{s}_t$ is revealed. In order to eliminate intertemporal hedging strategies that are well-known, we assume that $\tilde{s}_1$ and $\tilde{s}_2$ are independent, but not necessarily identically distributed. F_t denotes the cumulative distribution function of $\tilde{s}_t$.

At the beginning of every period t, each surviving investor has to take a decision $\theta_t \in \Theta_t$ that yields a payoff $\phi(\tilde{s}_t, \theta_t)$ at the end of the period. The problem of the old investor in period $t = 1$ is written as follows:

$$\max_{\theta \in \Theta_1} Eu(w + \phi(\tilde{s}_1, \theta)), \tag{1}$$

where E is the expectation operator. By backward induction, the problem of the young investor in period $t = 1$ is

$$\max_{\theta \in \Theta_1} Ev(w + \phi(\tilde{s}_1, \theta)), \tag{2}$$

where the value function v is defined as

$$v(z) = \max_{\theta \in \Theta_2} Eu(z + \phi(\tilde{s}_2, \theta)). \tag{3}$$

Variable z hereafter denotes the wealth available at the beginning of the second period, i.e. $z = w + \phi(s_1, \theta_1)$. The young investor is less risk-averse than the old one if v is less concave than u in the sense of Arrow-Pratt. The concept of decreasing risk aversion is useful for many comparative statics problems. For example, Pratt [1964] and Dybvig and Lippman [1983] showed that agent A will always accept more lotteries than agent B if and only if agent A is less risk-averse than agent B. Determining whether v is less concave than u only requires considering the second period problem. Since the comparative statics of an decrease in risk aversion (in period 1) is well-known, we limit the analysis to the second period problem here and in the remainder of the paper. This allows us to remove unnecessary time-indexes.

In this section, we assume a complete set of contingent claims markets. The decision problem is to determine the optimal bundle $\theta = c(\tilde{s})$ of contingent claims to purchase. The exogenous (probability-adjusted) contingent price is $\pi(s) \geq 0$. Since the risk free rate is zero, the price schedule must have an expectation equaling unity: $E\pi(\tilde{s}) = 1$. Let $c(s)$ be the demand for the contingent claim associated with event s. Problem (2) can now be rewritten as

$$v(z) = \max Eu(c(\tilde{s})) \tag{4}$$

$$\text{s.t. } E\pi(\tilde{s})c(\tilde{s}) = z. \tag{5}$$

Namely, the problem of the individual endowed in period 2 with wealth z is to maximize the expected utility of his final consumption under budget constraint (5). The standard first-order condition for (4) is

$$u'(c(s)) = \lambda\pi(s) \tag{6}$$

for any s. Parameter λ is the Lagrangian multiplier associated to the budget constraint. It is a function of z. Let $T_u(c) = -u'(c)/u''(c)$ and $T_v(z) = -v'(z)/v''(z)$ denote the index of absolute risk tolerance for respectively function u and function v. The following Lemma is instrumental for our result.

Lemma 1 $T_v(z) = E\pi(\tilde{s})T_u(c(\tilde{s}))$, *where function* $c(s)$ *is the solution of program (4).*

Proof: See the Appendix.

The absolute risk tolerance of an individual is a martingale with respect to risk-adjusted probabilities $dG(s) = \pi(s)dF(s)$. Cox and Leland [1982] and He and Huang [1994] obtain the same property in a continuous-time framework. G is a well-defined distribution function with $dG \geq 0$ and $\int dG(s) = \int \pi(s)dF(s) = 1$. The assumption that the risk-free rate is zero is central in this interpretation of T_v. If $\hat{E}$ denotes the expectation operator with respect to distribution G of $\tilde{s}$, we get

$$T_v(z) = \hat{E}T_u(c(\tilde{s})),$$

whereas the budget constraint is rewritten as $\hat{E}c(\tilde{s}) = z$. By Jensen's inequality, $T_v(z)$ is larger than $T_u(z) = T_u(\hat{E}c(\tilde{s}))$ if and only if T_u is convex. This proves the following result.

Proposition 1 *Consider the two-period investment problem with a complete set of markets at every period and a zero risk-free rate. Young investors are less risk-averse than old investors if the absolute risk tolerance of final wealth is convex. If absolute risk tolerance is not convex, one can find a distribution of state prices such that young investors are not less risk-averse than old investors.*

This means that investors with a longer horizon purchase a riskier portfolio of contingent claims, or duration enhances risk (DER).[4] Of course, the converse property also holds: young investors are more risk-averse than old ones, independent of the state price distribution if and only if absolute risk tolerance is concave.

3 The standard portfolio problem

In most investment contexts, complete contingent claims markets are not available. With the "standard portfolio problem", there are two assets available in each period. The first offers a zero sure return. The return $\tilde{s}$ on the second asset is random; first-period decisions and outcomes are independent of second-period returns.

With such an investment program, the existence of the second-period investment opportunity exerts two effects on first-period risk taking, which we label the flexibility and background risk effects. In general, the investor will adjust optimal risk exposure in the second period to the outcome in the first. With decreasing absolute risk aversion, for example, the better the first-period outcome the more of the risky asset is purchased in the second period.

[4]As shown by Gollier [1995], for example, this formally means that the contingent consumption of the young is more sensitive than the consumption of the old to changes in the contingent price.

The opportunity to adjust one's portfolio is an advantage; this flexibility effect always reduces aversion to current risks.

Beyond this, the presence of a second-period risk can be analyzed as a "background risk" with respect to the first-period choice problem. With a zero risk-free rate available, the opportunity to make a second-period investment is weakly advantageous, and is only neutral with a CARA utility function or when none of the risky asset is ever purchased, i.e.. when $E\tilde{s} = 0$. There is a lively literature on the effect of a background risk on attitudes towards another risk, but it does not address cases of favorable risks.[5]

The critical question is when the flexibility effect can be assured to overcome (be weaker than) a potential negative background risk effect. The required condition is that absolute risk tolerance be convex (concave). We shall now conduct the optimization, identify our two effects, and then proceed to the proposition.

Let $\theta = \alpha$ denote the demand for the risky asset. It depends upon the level z of wealth available at the beginning of the second period. The investor selects $\alpha(z)$ which solves program (7).

$$v(z) \;==\; \max_a Eu((z-a) + a(1+\tilde{s})) \;=\; \max_a \; Eu(z + a\tilde{s}) \qquad (7)$$

The problem is to establish the relationship between the degree of concavity of u and the degree of concavity of v. If $\tilde{s}$ is a two-state random variable, markets are complete, and we know from Proposition 1 that the convexity of T_u is necessary and sufficient for a negative effect of age on risk-taking. We herafter shows that this result is robust to the introduction of additional states. The first-order condition on $\alpha(z)$ is written as

$$E\tilde{s}u'(z + \alpha(z)\tilde{s}) = 0. \qquad (8)$$

Fully differentiating this condition yields

$$\frac{d\alpha}{dz} = -\frac{E\tilde{s}u''(z+\alpha\tilde{s})}{E\tilde{s}^2u''(z+\alpha\tilde{s})}. \qquad (9)$$

[5]Pratt and Zeckhauser (1987) introduce the concept of properness for situations where background risk is undesirable. Kimball (1993) develops standardness, when background risk increases expected marginal utility. Gollier and Pratt (1995) consider background risks with a non-positive mean. None of these restrictions is satisfied by background risk $\alpha\tilde{s}$ under consideration here.

As is well-known, $\alpha(z)$ is increasing, constant or decreasing depending upon whether absolute risk aversion is decreasing, constant or increasing. The envelope theorem yields

$$v'(z) = Eu'(z + \alpha\tilde{s}). \tag{10}$$

Fully differentiating again this equality yields

$$v''(z) = Eu''(z + \alpha\tilde{s}) + \frac{d\alpha}{dz} E\tilde{s}u''(z + \alpha\tilde{s}). \tag{11}$$

Combining conditions (9), (10) and (11) allows us to write

$$\begin{aligned} -\frac{v''(z)}{v'(z)} &= -\frac{Eu''(z+\alpha\tilde{s})}{Eu'(z+\alpha\tilde{s})} + \frac{d\alpha}{dz}\frac{-E\tilde{s}u''(z+\alpha\tilde{s})}{Eu'(z+\alpha\tilde{s})} \\ &= -\frac{Eu''(z+\alpha\tilde{s})}{Eu'(z+\alpha\tilde{s})} + \frac{[E\tilde{s}u''(z+\alpha\tilde{s})]^2}{E\tilde{s}^2u''(z+\alpha\tilde{s})Eu'(z+\alpha\tilde{s})}. \end{aligned} \tag{12}$$

Notice that by the Cauchy-Schwarz inequality, $[E\tilde{s}u''(z+\alpha\tilde{s})]^2$ is smaller than $E\tilde{s}^2u''(z+\alpha\tilde{s})Eu''(z+\alpha\tilde{s})$. Combining this property with equation (12) implies that $-v''/v'$ is positive: a longer time horizon length never transforms risk-averse investors into risk-lovers.

Our aim in this paper is to compare $-v''(z)/v'(z)$ to $-u''(z)/u'(z)$. Our two effects emerge from condition (12). The flexibility effect is expressed by the second term in the right-hand side of (12), which is negative.[6] The background risk effect corresponds to the first term in the right-hand side of (12). Future risk $\tilde{y} =_p \alpha\tilde{s}$ can be interpreted as a background risk with respect to the independent current risk. If α would be fixed and independent of the realization of the first-period risk (i.e. $d\alpha/dz = 0$), the degree of risk aversion in period 1 would equal $-Eu''(z+\tilde{y})/Eu'(z+\tilde{y})$. This can be either larger or smaller than $-u''(z)/u'(z)$. The next Proposition details the resolution of these two effects.

Proposition 2 *Consider the two-period investment problem with a zero yield risk-free asset and another risky asset. Young investors are less risk-averse*

[6] Gollier, Lindsey and Zeckhauser [1996] examine the effect of flexibility on the acceptance of risk.

than old investors if the absolute risk tolerance of final wealth is convex. If absolute risk tolerance is not convex, one can find a distribution of returns of the risky asset such that young investors are not less risk-averse than old investors.

Proof: See the Appendix.

The result is reversed if T_u is concave, which relates to the Mossin [1968] result that rational agents will be myopic, i.e. age has no effect on risk taking, iff the utility function exhibits harmonic absolute risk aversion (linear absolute risk tolerance).

No information is required whether absolute risk tolerance is increasing or decreasing – only the second derivative matters – which is surprising since the second-period investment opportunity raises both average wealth and utility. Interestingly, though we do not have a condition that an imposed favorable gamble increases the riskiness of accompanying preferred gamble, we can deduce a simple formula for examining how the availability of a favorable second-period gamble affects first-period behavior.

Proposition 2, like Proposition 1, provides a qualitative result. We would like to know the quantitative magnitude of the duration or age effect on risk taking, and could determine that if we knew the first four derivatives of the utility function. For now, consider an illustration for the case of $u(z) = z + \ln z$, with $\tilde{s} = (-1, 2; 1/2, 1/2)$. After some tedious computations. we get $T_u(5) = 30.0$, whereas $T_v(5) = 64.1$: the young investor is more than twice as risk-tolerant than the old investor. This implies that if the expected excess return of the risky asset is small, the young will invest twice as much in it as willthe old!

4 Convex risk tolerance

Most of the utility functions used in economics, as already noted. belong to the HARA class, i.e., they have linear absolute risk tolerance. Such functions, including exponentials (constant T_u), power and logarithmic functions ($T_u(z) = kz$), and quadratics ($T_u(z) = c - z$). Given such linearity, the length of the investment horizon does not affect the riskiness of the optimal portfolio.

The popularity of HARA functions, we believe, relates to their tractability, not their realism. If we think that longer horizons could lead to greater risk taking, then we must consider utility functions that exhibits convex absolute risk tolerance. The following functions do so:

(i) $u(z) = az - be^{-cz}$ with $a > 0,\ b > 0,\ c > 0$. It yields $T_u(z) = \frac{a}{bc^2}e^{cz} + \frac{1}{c}$.

(ii) $u(z) = az + b\frac{z^{1-\gamma}}{1-\gamma}$ with $a > 0,\ b > 0,\ \gamma > 0$. It yields $T_u(z) = \frac{z}{\gamma}\left[1 + \frac{az^{\gamma}}{b}\right]$.

(iii) $u(z) = az + b\ \ln(z)$ with $a > 0,\ b > 0$. It yields $T_u(z) = z\left[1 + \frac{az}{b}\right]$.

In short, if a function exhibits linear increasing absolute risk tolerance, then combining it with a linear function of z makes absolute risk tolerance convex. Finding a simple utility functions with concave risk tolerance is more difficult. An example of such a function is $u(z) = \int_0^z e^{-2\sqrt{w}}dw$, for which $T_u(z) = \sqrt{z}$.

Is convex absolute risk tolerance a reasonable assumption? To address this question, we consider two other economic problems in which convex risk tolerance plays a role: the static portfolio problem (7), and the group risk-sharing problem. In the portfolio problem, one can easily verify that the optimal demand for the risky asset is convex in initial wealth iff absolute risk aversion is convex, at least for small risks. This corresponds to the observation that the share of wealth invested in risky assets should increase with wealth, which seems reasonable.

Consider now the risk-sharing problem.[7] A group of n identical individuals with utility u have to share aggregate wealth $\tilde{z}$ which is risky. A sharing rule is a vector of payoff functions $(w_1(z), ..., w_n(z))$ that for every value of z expresses the share that goes to each member of the pool. If we assume that the risk-sharing rule that is adopted by the group is Pareto-efficient, it maximizes $Eh(\tilde{z}) = \sum_i \lambda_i Eu(w_i(\tilde{z}))$ for some vector of positive scalars $(\lambda_1, ...\lambda_n)$, under the feasibility constraint $\sum_i w_i(\tilde{z}) = \tilde{z}$. If the λ_is are not identical, the efficient rule will not be fair. In order to determine

[7]See Gollier [1993] for a survey.

the attitude towards risk of the group. we need to know the degree of risk tolerance of the group's value function h. From the theory of efficient risk-sharing, we know that the absolute risk tolerance $T_h(z)$ for the group equals $\sum_i T_u(w_i(z)) = n[\sum_i \frac{1}{n} T_u(w_i(z))]$. By Jensen's inequality, observe that $T_h(z)$ is larger than $nT_u(\sum_i w_i(z)/n) = nT_u(z/n)$ if and only if T_u is convex. In short, adopting a Pareto-efficient risk-sharing rule that is not fair increases the risk tolerance of the group if and only if the individual's absolute risk tolerance is convex. The intuition is that by adopting an unfair sharing rule, more risk can be allocated to those whose allocated wealth positions make them less risk-averse.[8]

If $A_u(z) = T_u^{-1}(z)$ denotes absolute risk aversion, convex risk tolerance is equivalent to

$$A_u''(z) \leq 2\frac{[A_u'(z)]^2}{A_u(z)}. \tag{13}$$

If absolute risk aversion is concave, absolute risk tolerance is automatically convex. But under the familiar condition of decreasing A_u, the concavity of A_u is not plausible, since a function cannot be positive, decreasing and concave everywhere. Condition (13) means in fact that absolute risk aversion may not be too convex. There is no obvious relationship between the necessary and sufficient condition (13) and the properness necessary condition $A_u'' \geq A_u' A_u$.[9] In order to relate our condition to standardness, one can easily verify that convex risk tolerance is equivalent to the condition that function

$$\phi(z) \equiv \frac{P_u(z)}{A_u(z)}, \tag{14}$$

be decreasing in z, where $P_u(z) = -u'''(z)/u''(z)$ denotes absolute prudence. This condition should be related to standardness which is characterized by the condition that both A_u and P_u be decreasing.

We conclude that none of the recent concepts mentioned above is sufficient to sign the effect of investment horizon on risk-taking. To simplify the analysis thus far, we assumed that the risk-free rate was zero, that terminal

[8] With an unfair allocation, those with a large λ will be wealthier on average. If absolute risk aversion is not constant, this generates a wealth effect on the members' attitude towards risk.

[9] The same remark can be done for the risk vulnerability necessary condition $A_u'' \geq 2A_u' A_u$.

wealth was the only concern, that there were but two periods, and that the utility function was twice differentiable. The next section examines whether our results can withstand relaxing those assumptions.

5 Extensions

To extend our analysis, we consider in turn a positive risk-free rate, intermediate consumption and production, multiple periods and nondifferentiable marginal utility.

5.1 Positive risk-free rate

A zero risk-free rate was invoked in sections 2 and 3 to eliminate a wealth effect that introduces a noise with respect to our main message. Over period 1926-1987, the arithmetic mean annual return of inflation-adjusted U.S. Treasury Bills is 0.5% (geometric mean is 0.4%), whereas it is 1.9% (geometric mean is 1.7%) for inflation-adjusted intermediate-term U.S. Government Bonds.[10] This suggests that a positive risk-free rate should be considered. Let $R \geq 1$ denote one plus the risk-free rate. In the standard portfolio problem discussed in section 3, the problem is rewritten as

$$v_R(z) = \max_a Eu((z-a)R + a(1+\tilde{r})) = \max_a Eu(zR + a\tilde{s}), \qquad (15)$$

where $\tilde{r}$ is the return of the risky asset and $\tilde{s} = 1 + \tilde{r} - R$ is its excess return over the risk-free asset. Function v_R is the value function of the young investor under a positive risk-free rate R. It is thus seen that

$$v_R(z) = v(Rz), \qquad (16)$$

where v is the value function under $R = 1$ that has been examined in section 3. The reader can easily verify that the same equivalence (16) holds for the complete markets model analyzed in section 2, with $E\pi(\tilde{s}) = R > 1$. From condition (16), one can state that a positive interest rate introduces two

[10] See Ibbotsen and Sinquefield [1989], page 74.

changes in the attitude towards current risks (that makes z risky). First. as mentioned above, there is a wealth effect expressed by the fact that $ER\tilde{z} > E\tilde{z}$. Second, there is a magnification effect. The risk borne today will be magnified by investing the part of the payoff from the current risk in the risk-free asset during the next period.

Differentiating twice condition (16) yields

$$T_{v_R}(z) = \frac{1}{R} T_v(Rz), \tag{17}$$

where T_{v_R} is the absolute risk tolerance of the young investor. The two effects of a positive risk-free rate are apparent in equation (17). The wealth effect takes the form of absolute risk tolerance measured at $Rz > z$. The magnification effect is equivalent to absolute risk tolerance divided by factor k.

Proposition 3 *In the complete markets model and in the standard portfolio model, if the risk-free rate can take any nonnegative value, the young investor is always less risk-averse than the old investor if and only if the absolute risk tolerance of the utility function u is convex and superhomogeneous.*[11]

Proof: The proof of sufficiency proceeds as follows: Propositions 1, 2 and the convexity of T_u yields

$$T_{v_R}(z) = \frac{1}{R} T_v(Rz) \geq \frac{1}{R} T_u(Rz).$$

Using the superhomogeneity of T_u yields in turn

$$T_{v_R}(z) \geq T_u(z).$$

For the necessity of convex absolute tolerance, take $R = 1$ and apply Propositions 1 and 2. For an outline of the proof of the necessity of superhomogeneity, suppose by contradiction that T_u is not superhomogeneous. Then, there exists z and $R > 1$ such that $T_u(z) > \frac{1}{R} T_u(Rz)$. Consider a distribution of $\tilde{s}$ with an arbitrary small expectation. Then, $\alpha(Rz) \cong 0$ and $\alpha'(Rz) \cong 0$, so

[11] A function g is superhomogeneous (resp. subhomogeneous) if and only if $g(Rz) \geq$ (resp $\leq$) $Rg(z)$ for all z and all $R \geq 1$.

that $T_v(Rz) \cong T_u(Rz)$. Thus, $T_u(z) > \frac{1}{R}T_v(Rz) = T_{v_R}(z)$, a contradiction. ∎

Superhomogeneity of T_u guarantees that a young investor will put his end-of-period-1 wealth in the risk-free asset is less risk-averse in period 1 than the old investor who simply consumes his end-of-period-1 wealth. If the young investor does not invest all his wealth in the risk-free asset in period 2, the convexity of T_u reinforces the magnification effect. If only negative risk-free rates are considered, then the condition of superhomogeneity must be replaced by subhomogeneity, with current risks being reduced through time. Examples of a concave utility function with convex and superhomogeneous risk tolerance are provided by cases (ii) and (iii) in the previous section, i.e. a sum of a linear function and a CRRA function. Samuelson [1989b] showed that the same age-phased property holds in the case $u(z) = (z-S)^{1-\gamma}/(1-\gamma)$, $1 \neq \gamma > 0$, or $u(z) = \ln(z-S)$, for any (minimum-consumption) constant $S > 0$. Since absolute risk tolerance is linear and superhomogenous in these cases, age-phasing occurs only due to the risk-free rate effect. A counterexample is obtained with the exponential utility function which yields $T_{v_R}(z) = \frac{1}{R}T_u(z)$. In that case, only the magnification effect is at work, with young investors *reducing* their risky investments by factor R in relation to the strategy of old investors.

Notice that a positive function which is superhomogenous must be increasing. Therefore, contrary to the results presented in sections 2 and 3 in which the convexity of T_u alone was relevant, we need here the absolute risk tolerance to be convex and increasing. If we limit the analysis to utility functions that are defined over R^+ (Remark: this excludes the cited Samuelson example), then the superhomogeneity condition can be simplified. Indeed, if a function T_u defined over R^+ is positive and convex, superhomogeneity is equivalent to the condition that this function evaluated at 0 is zero.[12] We can thus rewrite the above result as follows.

Corollary 1 *Consider the complete markets model or the standard portfolio problem with a utility function defined over R^+. In the absence of intermedi-*

[12] A formal proof of this result is available upon request.

ate consumption and with a nonnegative risk-free rate, the young investor is always less risk-averse than the old investor if and only if T_u is convex and $T_u(0) = 0$.

Mossin [1968] showed that introducing a positive risk-free rate in the standard portfolio problem reduces the set of utility function yielding rational myopia. Namely, with $R > 1$, myopia is rational if $T_u(z)$ is not only linear in z. but also *proportional* in z, i.e., $T_u(0) = 0$. The same restriction applies here.

A side result of this analysis is obtained by examining the problem of access to markets. Consider two young investors who live for two periods. In the second period there are two markets, one for a risk-free asset. and one for a risky asset. The two investors have access to the risk-free asset market, but only one of the two has access to the risky asset market. Should he be more or less risk-averse than the other in period 1? It depends upon whether $v_R(z) = v(Rz)$ is more or less concave than $u_R(z) = u(Rz)$. or, in other words, whether $\frac{1}{R}T_v(Rz)$ is larger or smaller than $\frac{1}{R}T_u(Rz)$. From Proposition 2, we directly get the result that *the investor who will have access to the risky asset market in the future takes more risk today if and only if T_u is convex.*

5.2 Intermediate consumption

Thus far we have assumed that the investor's utility function applies solely to terminal wealth; he has a pure investment problem. In a real world contexts, investors consume a portion of their lifetime wealth each period. Introducing intermediate consumption raises several interesting questions. For example, Kimball [1990] and others addressed the problem of optimal saving given exogenous future uncertainty. In this paper, we extend this approach by considering future risks that are endogenous. Another important point is that allowing for intermediate consumption makes young people potentially more willing to take risks than in the pure investment problem because current risks can be attenuated by spreading consumption over time. For example, in the case of the absence of future risk opportunity, if complete consumption-smoothing is optimal, a \$1 loss on current investment will be split into a fifty cent reduction in current consumption and a fifty cent reduction in future consumption. Under a concave utility on consumption, that has a smaller effect on total utility than a straight $1 reduction in final consumption in the investment problem.

We consider a model in which consumption is chosen at the end of each period, after having observed the realization of the random variable characterizing that period's risk.[13] For tractability and to allow for time-consistency of decision, we consider an expected utility model with a time-separable utility function u on consumption. If parameter β is the discount factor, the dynamic structure of the problem is described by the value function v_R that is defined as follows:

$$v_R(z) \;=\; \max_{c,\theta\in\Theta} u(c) + \beta E u(R(z-c) + \phi(\tilde{s},\theta)). \tag{18}$$

Using the definition of function v as in (3), this can be rewritten as:

$$\begin{aligned} v_R(z) &= \max_c \; u(c) + \beta \; \max_{\theta\in\Theta} \; Eu(R(z-c) + \phi(\tilde{s},\theta)) \\ &= \max_c \; u(c) + \beta v(R(z-c)). \end{aligned} \tag{19}$$

The first-order condition on c obtains:

$$u'(c) = \beta R v'(R(z-c)). \tag{20}$$

It yields $c'(z) = \beta R^2 v''(R(z-c))/[u''(c) + \beta R^2 v''(R(z-c))]$. The envelope theorem gives $v_R'(z) = \beta R v'(R(z-c)) = u'(c)$. Differentiating this equality and rearranging terms yields

$$T_{v_R}(z) = T_u(c(z)) + \frac{1}{R} T_v(R(z - c(z))). \tag{21}$$

If T_u is convex, one can apply Propositions 1 and 2 to write:

$$T_{v_R}(z) \geq T_u(c(z)) + \frac{1}{R} T_u(R(z - c(z))).$$

Using again the convexity of T_u together with Jensen's inequality yields

[13] See Bodie, Merton and Samuelson [1992] where labor supply, not consumption, adjusts to counterbalance poor investment outcomes.

$$\begin{aligned} T_{v_R}(z) &\geq \frac{R+1}{R}\left[\frac{R}{R+1}T_u(c(z)) + \frac{1}{R+1}T_u(R(z-c(z)))\right] \\ &\geq \frac{R+1}{R}T_u(\frac{R}{R+1}z). \end{aligned}$$

If we assume that the return on the risk-free asset is larger than -1, i.e. $R \geq 0$, the left-hand side of the last inequality is larger than $T_u(\frac{R+1}{R}\frac{R}{R+1}z) = T_u(z)$ if and only if T_u is *sub*homogeneous. This proves the sufficiency of the following Proposition. The proof of its necessity follows the same line as the one for Proposition 3 and is left to the reader.

Proposition 4 *In the complete markets model and in the standard portfolio model, if intermediate consumption is allowed, the young investor is always less risk-averse than the old investor if and only if the absolute risk tolerance of the utility function u is convex and subhomogeneous.*

When intermediate consumption is introduced, the subhomogeneity of T_u must be added to its convexity in order to get an unambiguous comparative static property. This reverses the superhomogeneity condition that proves necessary in the investment problem with a positive risk-free rate. There is a simple intuition for this reversal. As an illustration, consider the above model with $\beta = R = 1$ and $\tilde{s} = 0$ with probability 1. Then $v_R(z) = 2u(z/2)$: consumption is perfectly smoothed in the two periods, with no precautionary saving in the absence of any future risk. Value function $v_R(z)$ thus has the same degree of concavity than $u(z/2)$. There is a 50% *reduction* effect on the risk borne in period 1. Namely, by splitting the current risk makes the young investor more willing to accept risk than the old investor. This is a correct interpretation of the well-known concept of "time-diversification". As superhomogeneity was necessary to take care of the magnification effect of a positive risk-free rate in the investment problem, the condition of subhomogeneity is necessary to take care of the reduction effect of time-diversification with intermediate consumption. Convexity of T_u assures that the existence of the future risk does not reverse this effect. This reversal has already been observed by Samuelson [1989b] in the case of $u(z) = (z-S)^{1-\gamma)}/(1-\gamma)$, $S > 0$, $\gamma > 0$, in which only the magnification/reduction effect exists.

Of course, one can easily verify that T_u concave and superhomogeneous is necessary and sufficient for young people to be more risk-averse than old people, in a model with intermediate consumption. The limit case is obtained with T_u being linear and homogeneous, i.e. the CRRA case. Not suprisingly, this is the standard assumption in dynamic risk-taking models.

5.3 Multiperiod models

In this section, we would like to know under which conditions there will be a monotone effect of age on risk aversion. In our basic models of sections 2 and 3, it would require T_v to be also convex. If the convexity of T_u implies the convexity of T_v, then it would be clear that an investor who still has three periods to live is less averse to current risks than an investor with only two periods left. In the Appendix, we prove that such a property holds when there is a complete set of contingent markets at every period. This is not a surprise. One can consider the two last periods to be a single "mega-period". As is well-known, because there are complete markets in both of these periods, there are implicitly complete markets contingent to the history realized over the mega-period. One can then apply Proposition 1 over the mega-period to get the result.

Unfortunately, this is *not* true in the case of the standard portfolio problem with more than two states in every period. In its simplest form, the problem can be written as:

$$v_t(z) = \max_{\alpha} \; Ev_{t+1}(Rz + \alpha\tilde{s}), \tag{22}$$

with $v_T(z) = u(z)$ for some horizon $T > 2$.[14] However, T_{v_t} may not be convex, even if T_u is convex. This implies that under a convex T_u, a younger investor may not be less risk-averse.[15] With three age groups — young, middle-aged. and old — the convexity of absolute risk tolerance only guarantees that the

[14] As noticed elsewhere, the only fixed points for this problem are functions with constant relative risk aversion. Among others, Huberman and Ross [1983] show that under some conditions investors behave as if v_t has constant relative risk aversion when $T-t$ tends to infinity. What these authors basically show is that if the risk-free rate is positive, the optimal portfolio of a "very" young investor is the optimal portfolio of a myopic agent with an infinite wealth; i.e. the magnification effect predominates.

[15] A counterexample developed with Mathematica is available upon request.

middle-aged is less risk-averse than the old. In short, the convexity of T_u is necessary but not sufficient for risk-taking to increase with age. In order to get sufficiency, we must find a condition for T_{v_t} to be convex. We have little hope that any tractable and interpretable condition will be found in the near future. To get it, the inverse of the left-hand side of (12) must be differentiated twice and the result must be signed.

5.4 Non-differentiable u'

Throughout this analysis we have assume that utility was twice differentiable. If the utility function does not have this property, absolute risk tolerance may not be defined correctly and our theory is no longer relevant.[16] We now show an example of a utility function that is not twice differentiable and that yields the property that young people are less risk-averse than older ones. Take

$$u(z) = \min(z, (1-t)(z-D) + D), \tag{23}$$

with $t \in [0,1]$ and any scalar D. This function is continuous, piecewise linear and concave. It is drawn in Figure 1. Consider the standard portfolio problem (7) with no intermediate consumption, $R = 1$ together with $\tilde{x} = (x_-, x_+; 1/2, 1/2)$, $x_- < 0 < x_+$ and $0.5(x_- + x_+) > 0$. It can be shown that a bounded solution exists for this problem with utility function (23) if and only if

$$t > \frac{x_- + x_+}{x_+}.$$

Under this condition, the optimal demand for the risky asset equals

$$\alpha(z) = \begin{cases} \dfrac{D-z}{x_+} & \text{if } z < D, \\ \dfrac{z-D}{\mid x_- \mid} & \text{if } z \geq D. \end{cases}$$

The value function is then written as

[16] Notice that we do not need any restriction on the differentiability of T_u itself to get our result, because the Jensen's inequality does not require such a condition. However, if T_u is convex, it must be continuous.

$$v(z) = \min \left[\frac{x_+ - x_-}{2x_+} z + D \frac{x_+ + x_-}{2x_+}, (1-t) \frac{x_+ - x_-}{2 \mid x_- \mid} (z - D) + D \right],$$

whose graph is depicted in Figure 1.

The value function is obviously less concave than the utility function in the usual sense: v is a convex transformation of u. Young investors purchase more of the risky asset than old investors. However, the absolute risk tolerance is not convex in the proper sense. Notice also that v is piecewise linear as is u. Applying the same technique as above yields that investors who have three periods to live are less risk-averse than those who have only two periods to live and similarly for three and four periods, etc. Piecewise linear utility functions are examples in which there is a monotone relationship between age and risk aversion.

6 Conclusion

Time and uncertainty are inextricably entwined in investment decisions. Economists have developed useful instruments to treat either uncertainty or time in decision processes. When models entail both time *and* uncertainty, policy recommendations have been provided only under very specific conditions on preferences. Namely, analytical results are available only when absolute risk tolerance is linear in wealth. In such models, no relationship emerges between age and the attitude towards risk. In this paper, we have extended these analyses to a range of other cases. In particular, we have determined the qualitative effect of age on risk-taking in relation to the properties of the utility function for consumption or wealth.

The findings presented here illuminate the essential link between time horizon length and risk aversion. Some of their most important applications, however, may lie somewhat afield. For example, they may help us to revisit the role of liquidity constraints in dynamic models in finance, to consider how the frequency of market openings affects optimal risk-taking, or to better understand the advantages for accurate description of adding a second period, with rebalancing, to otherwise static investment models.

References

Bell, D.E., [1988], One-switch utility functions and a measure of risk, *Management Science*, 34, pp. 1416-1424.

Bodie, Z., R.C. Merton and W.F. Samuelson. [1992], Labor supply flexibility and portfolio choice in a life cycle model, *Journal of Economic Dynamics and Control*, 16, 427-449.

Cox. J. and H. Leland, [1982], Notes on dynamic investment strategies, Proceedings of Seminar on the Analysis of Security Prices, Center for Research in Security Prices (CRSP), Graduate School of Business, University of Chicago.

Dybvig, P.H., and S.A. Lippman, [1983], An alternative characterization of decreasing absolute risk aversion, *Econometrica*, 51, 223-224.

Gollier, C., [1993], Economic Theory of Risk Exchanges : A Review, in *Contributions to Insurance Economics,* G. Dionne editor, Kluwer Academic Press, 3-23.

Gollier, C., [1995], Decreasing absolute prudence: characterization and applications to second-best risk-sharing, *European Economic Review*, forthcoming.

Gollier, C. and J.W. Pratt, [1995], Risk vulnerability and the tempering effect of background risk, *Econometrica*, forthcoming.

Gollier, C., J. Lindsey and R.J. Zeckhauser, [1996], Investment Flexibility and the Acceptance of Risk, mimeo.

Guiso, L., T. Jappelli and D. Terlizzese, [1996], Income Risk, Borrowing Constraints, and Portfolio Choice, *American Economic Review,* 86, 158-172.

He, H. and C.-F. Huang, [1994], Consumption-portfolio policies: An inverse optimal problem, *Journal of Economic Theory*, 62, 257-293.

Huberman, G. and S. Ross, [1983], Portfolio turnpike theorems, risk aversion, and regularly varying utility functions, *Econometrica,* 51, 1345-1361.

Ibbotson, R.G., and R.A. Sinquefield, [1989], *Stocks, Bonds, Bills, and Inflation: Historical Returns (1926-1987)*. Dow Jones-Irwin, New York.

Kessler, D. and E.N. Wolff, [1991], A comparative analysis of household wealth patterns in France and in the United States, *Review of Income and Wealth*, 37, 249-266.

Kimball, M.S., [1990], Precautionary Saving in the Small and in the Large, *Econometrica*, 58, 53-73.

Kimball, M.S., [1993], Standard Risk Aversion, *Econometrica*, 61, 589-611.

King, M.A. and J.I. Leape, [1987], Asset Accumulation, Information, and the Life Cycle, NBER (Cambridge, MA) Working Paper 2392.

Merton, R.C., [1969], Lifetime Portfolio Selection under Uncertainty: The Continuous- Time Case, *Review of Economics and Statistics*, 51, pp. 247-257.

Mossin, J., [1968], Optimal Multiperiod Portfolio Policies, *Journal of Business*, 215-229.

Pratt, J.W., and R. Zeckhauser, [1987], Proper risk aversion, *Econometrica*, 55, 143-154.

Samuelson, P.A., [1963], Risk and uncertainty: the fallacy of the Law of Large Numbers, *Scientia*, 98, 108-113.

Samuelson, P.A., [1989a], The judgement of economic science on rationale portfolio management: indexing, timing, and long-horizon effects, *Journal of Portfolio Management*, Fall 1989, 3-12.

Samuelson, P.A., [1989b], A case at last for age-phased reduction in equity, *Proceedings of the National Academy of Sciences*, 86, 9048-9051.

Appendix

Proof of Lemma 1

As is well-known, we have that

$$v'(z) = \lambda.$$

Fully differentiating the above equality, inverting and multiplying by $-v'(z)$ yields

$$T_v(z) = -\frac{v'(z)}{v''(z)} = -\frac{\lambda}{\frac{d\lambda}{dz}}. \tag{24}$$

The value of the right-hand side of (24) is obtained by fully differentiating condition (6). It yields $u''(c(s))\frac{dc(s)}{dz} = \pi(s)\frac{d\lambda}{dz}$. Replacing $\pi(s)$ by $u'(c(s))/\lambda$ yields in turn

$$\frac{dc(s)}{dz} = -T_u(c(s))\frac{1}{\lambda}\frac{d\lambda}{dz}. \tag{25}$$

Now, by the budget constraint (5), we obtain

$$1 = E\pi(\tilde{s})\frac{dc(\tilde{s}))}{dz} = -\frac{1}{\lambda}\frac{d\lambda}{dz}E\pi(\tilde{s})T_u(c(\tilde{s})).$$

Combining the above condition with condition (24) concludes the proof. ■

Proof of Proposition 2

Consider any specific z. Without loss of generality, we assume $\alpha(z) = 1$. i.e. $E\tilde{s}u'(z+\tilde{s}) = 0$. We have to determine under which condition we have

$$-\frac{u''(z)}{u'(z)} \geq -\frac{Eu''(z+\tilde{s})}{Eu'(z+\tilde{s})} + \frac{[E\tilde{s}u''(z+\tilde{s})]^2}{E\tilde{s}^2u''(z+\tilde{s})Eu'(z+\tilde{s})} \tag{26}$$

for all z and $\tilde{s}$ satisfying $E\tilde{s}u'(z+\tilde{s}) = 0$. Let F denote the cumulative distribution function of $\tilde{s}$. Let also define $\tilde{x}$ as the random variable with cumulative distribution function G, with

$$dG(x) = \frac{u''(z+x)dF(x)}{\int u''(z+s)dF(s)}.$$

We verify that dG is positive under risk aversion, ad that $\int dG(x) = 1$. Using this change of variable, the problem is rewritten as

$$E\tilde{x}T(z+\tilde{x}) = 0 \implies \frac{1}{T(z)} \geq \frac{1}{ET(z+\tilde{x})} - \frac{(E\tilde{x})^2}{E\tilde{x}^2 ET(z+\tilde{x})},$$

or, equivalently,

$$E\tilde{x}T(z+\tilde{x}) = 0 \implies T(z)\left[1 - \frac{(E\tilde{x})^2}{E\tilde{x}^2}\right] \leq ET(z+\tilde{x}). \qquad (27)$$

Lemma 2 *Condition (27) holds for any $\tilde{x}$ if it holds for any $\tilde{x}$ with a three-point support.*

Proof: We look for the characteristics of the $\tilde{x}$ which is the most likely to violate condition (27). To do so, we solve the following problem for any scalar λ:

$$\begin{array}{rl} \max_{dG\geq 0} & T(z)(1-\lambda\int x dG(x)) - \int T(z+x)dG(x) \\ \text{subject to} & \int xT(z+x)dG(x) = 0; \\ & \int x(1-\lambda x)dG(x) = 0; \\ & \int dG(x) = 1. \end{array} \qquad (28)$$

If the solution of this problem for any λ is negative, condition (27) would be proven. Observe that the above problem is a standard linear programming problem. Therefore the solution contains no more than three x such that $dG(x) > 0$, since they are three equality constraints in the program. Thus, condition (27) would hold for any $\tilde{x}$ if it holds for any $\tilde{x}$ with three atoms. This concludes the proof of Lemma 2. Notice that Pratt and Zeckhauser (1987) and Kimball (1993) use the same kind of technique. ■

Lemma 3 *Condition (27) holds for any $\tilde{x}$ with a two-point support if and only if T is convex.*

Proof: This is a direct consequence of Proposition 1. Indeed, if $\tilde{x}$ (thus $\tilde{s}$) has only two points in its support, we have as many possible states of the world as the number of assets. Thus, markets are complete in each period. Notice that this Lemma implies that the convexity of T is necessary for our result. ■

Lemma 4 *If condition (27) holds for any $\tilde{x}$ with a two-point support, then condition (27) holds for any $\tilde{x}$ with a three-point support.*

Proof: Take $\tilde{x}$ that is distributed as $(x_1, \pi_1; x_2, \pi_2; x_3, \pi_3)$. Let g_i denote $x_i T(z + x_i)$. Suppose that $\sum_i \pi_i g_i = 0$. It implies that the g_i must alternate in sign. Without loss of generality, suppose that $g_1 > 0$, $g_2 < 0$ and $g_3 \geq 0$.

We first show that there exist a $\tilde{y}_1$ distributed as $(x_1, p; x_2, 1-p)$ and a $\tilde{y}_2$ distributed as $(x_2, 1-q; x_3, q)$ such that $E\tilde{y}_i T(z + \tilde{y}_i) = 0$, $i = 1, 2$, and such that $\tilde{x}$ is distributed as $(\tilde{y}_1, \lambda; \tilde{y}_2, 1-\lambda)$. To check this, we must solve the following system:

$$\lambda p = \pi_1 \tag{29}$$

$$\lambda(1-p) + (1-\lambda)(1-q) = \pi_2 \tag{30}$$

$$(1-\lambda)q = \pi_3 \tag{31}$$

$$pg_1 + (1-p)g_2 = 0 \tag{32}$$

$$(1-q)g_2 + qg_3 = 0 \tag{33}$$

with p, q and λ in $[0, 1]$. This system is solved with

$$p = \frac{-g_2}{g_1 - g_2} \tag{34}$$

$$q = \frac{-g_2}{g_3 - g_2} \tag{35}$$

$$\lambda = \pi_1 \frac{g_1 - g_2}{-g_2} \tag{36}$$

One can easily check that this solution is in $[0,1]^3$.

Suppose now that condition (27) holds for $\tilde{y}_1$ and $\tilde{y}_2$ that have a two-point support. Thus, we knwo that

$$T(z)\left[1-\frac{(E\tilde{y}_i)^2}{E\tilde{y}_i^2}\right] \leq ET(z+\tilde{y}_i), \tag{37}$$

for $i = 1, 2$. It implies that

$$T(z)\left[1-\left\{\lambda\frac{(E\tilde{y}_1)^2}{E\tilde{y}_1^2}+(1-\lambda)\frac{(E\tilde{y}_2)^2}{E\tilde{y}_2^2}\right\}\right] \leq \lambda ET(z+\tilde{y}_1)+(1-\lambda)T(z+\tilde{y}_2). \tag{38}$$

Observe that

$$\lambda ET(z+\tilde{y}_1)+(1-\lambda)T(z+\tilde{y}_2) = ET(z+\tilde{x}) \tag{39}$$

since $\tilde{x}$ is distributed as $(\tilde{y}_1, \lambda; \tilde{y}_2, 1-\lambda)$. Notice also that $(E\tilde{x})^2/E\tilde{x}^2$ is convex in the distribution of $\tilde{x}$, i.e. [17]

$$\lambda\frac{(E\tilde{y}_1)^2}{E\tilde{y}_1^2}+(1-\lambda)\frac{(E\tilde{y}_2)^2}{E\tilde{y}_2^2} \leq \frac{(E(\lambda\tilde{y}_1+(1-\lambda)\tilde{y}_2))^2}{E(\lambda\tilde{y}_1^2+(1-\lambda)\tilde{y}_2^2)} = \frac{(E\tilde{x})^2}{E\tilde{x}^2}. \tag{40}$$

Combining conditions (38), (39) and (40) yields condition (27). This construction can be done for any random variable $\tilde{x}$ with three atoms. ■

Proposition 2 is a direct consequence of the three above Lemma.

[17]The proof of this claim is obtained by easy manipulations of (40). Denoting $a_i = E\tilde{y}_i$ and $b_i = E\tilde{y}_i^2 > 0$, condition (40) is equivalent to

$$2a_1a_2 \leq a_1^2\frac{b_2}{b_1}+a_2^2\frac{b_1}{b_2}.$$

This is in turn equivalent to

$$\left[a_1\sqrt{\frac{b_2}{b_1}}+a_2\sqrt{\frac{b_1}{b_2}}\right]^2 \geq 0,$$

which is always true.

Proof that T_v is convex with complete markets

Proposition 5 *Consider the complete markets model introduced in section 2. The absolute risk tolerance of v is convex (resp. concave) if the absolute risk tolerance of u is convex (resp. concave).*

Proof: Without loss of generality, let us rank the states of the world in order to have π increasing with s. As we know from Lemma 1, $T_v(z) = E\pi(\tilde{s})T_u(c(\tilde{s}))$. Using (24) and (25), it implies

$$T_v'(z) = \frac{E\pi(\tilde{s})T_u(c(\tilde{s}))T_u'(c(\tilde{s}))}{E\pi(\tilde{s})T_u(c(\tilde{s}))}.$$

Define function η as

$$\eta(s) = \frac{\pi(s)T_u(c(s))}{E\pi(\tilde{s})T_u(c(\tilde{s}))}.$$

This allows us to rewrite the above expression as $T_v'(z) = E\eta(\tilde{s})T_u'(c(\tilde{s}))$. Notice that this directly implies that the sign of the derivative of T_u is passed on to the derivative of T_v, i.e. DARA is preserved. Differentiating again and using (24) and (25) yields

$$T_v''(z) = \frac{E\eta(\tilde{s})T_u(c(\tilde{s}))T_u''(c(\tilde{s}))}{E\pi(\tilde{s})T_u(c(\tilde{s}))} + E\frac{d\eta(\tilde{s})}{dz}T_u'(c(\tilde{s})).$$

The first term of the RHS above is positive if T_u is convex. Under the same condition, the second term is also positive for the following reason. We have

$$\frac{d\eta(s)}{dz} = \frac{\pi(s)T_u(c(s))}{(E\pi(\tilde{s})T_u(c(\tilde{s}))^3}\left[T_u'(c(s))E\pi(\tilde{s})T_u(c(\tilde{s})) - E\pi(\tilde{s})T_u(c(\tilde{s}))T_u'(c(s))\right].$$

From this equation, the convexity of T_u and the fact that c is decreasing in s, there exists a scalar s^* such that $(s-s^*)\frac{d\eta}{dz}(s) \leq 0$ for all s. Using again the convexity of T_u and $c'(s) < 0$ yields

$$E\frac{d\eta(\tilde{s})}{dz}T_u'(c(\tilde{s})) \geq T_u'(c(s^*))E\frac{d\eta(\tilde{s})}{dz} = 0.$$

This concludes the proof. ■

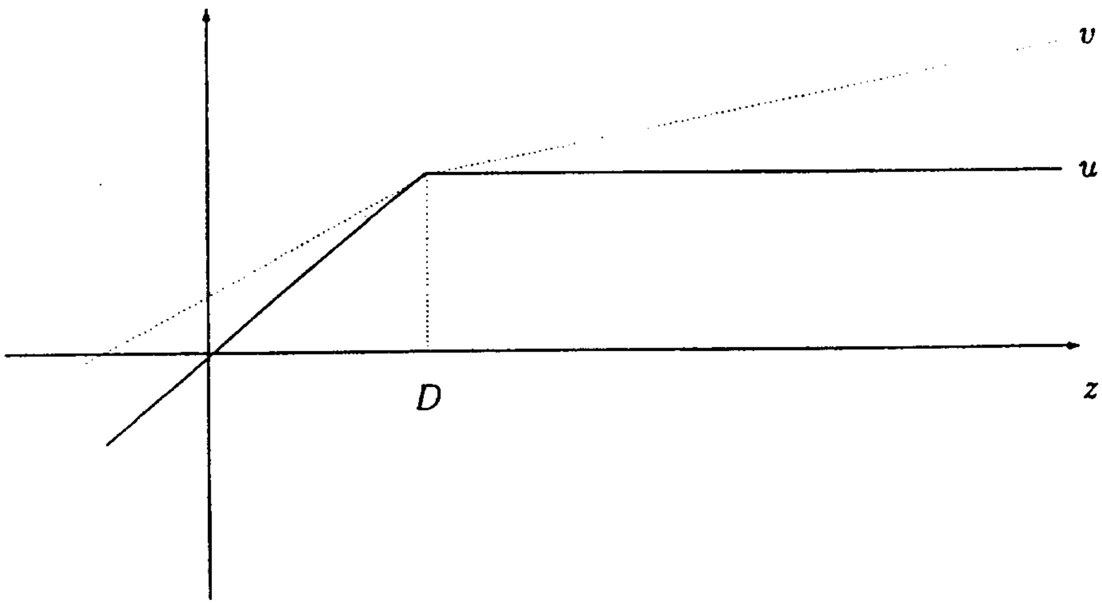

Figure 1: The case of a non-differentiable u'.