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STATISTICAL MECHANICS APPROACHES  
TO SOCIOECONOMIC BEHAVIOR

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**ABSTRACT**

This paper provides a unified framework for interpreting a wide range of interactions models which have appeared in the economics literature. A formalization taken from the statistical mechanics literature is shown to encompass a number of socioeconomic phenomena ranging from out of wedlock births to aggregate output to crime. The framework bears a close relationship to econometric models of discrete choice and therefore holds the potential for rendering interactions models estimable. A number of new applications of statistical mechanics to socioeconomic problems are suggested.

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## 1. Introduction

This paper is designed to provide a unified discussion of the use of statistical mechanics methods<sup>1</sup> in the study of socioeconomic behavior. The use of these methods in the social sciences is still in its infancy. Nevertheless, a growing body of work has shown how statistical mechanics and related probability techniques may be used to study the evolution and steady state behavior of heterogeneous populations. Examples of the range of applications of statistical mechanics methods include social pathologies such as out of wedlock births and crime (Brock and Durlauf (1996), Glaeser, Sacerdote, Scheinkman (1996)), asset price behavior (Brock (1993,1995)), expectation formation (Brock and Hommes (1995)), business cycles (Bak *et al* (1993) and Durlauf (1991,1994)), and endogenous preferences (Bell (1995)). In addition, work such as Blume (1993,1996) has shown how these methods can provide insight into the structure of abstract game-theoretic environments.<sup>2</sup>

These disparate phenomena are linked by the possibility that each is determined at least partially by direct interactions between economic actors. Put differently, each of these phenomena is a case where the decisions of each individual are influenced by the choices of others with whom he interacts. This interdependence leads to the possibility that polarized behavior can occur at an aggregate level solely due to the collective interdependence in decisionmaking. This explanation of polarized group behavior may be contrasted with explanations which rely on the presence of highly correlated characteristics among members of a group.

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<sup>1</sup>The statistical mechanics models I employ are also referred to as interacting particle system or random fields models.

<sup>2</sup>Alternative approaches to the modelling of complex interaction environments include Arthur (1987,1989), Ioannides (1990), Kirman (1983), and Krugman (1996).

Interactions between economic actors are commonplace in economic models. However, these interactions are typically mediated through markets. What distinguishes the bulk of the interactions in the recent inequality literature is the focus on interdependencies which are direct.<sup>3</sup> Examples of such direct interactions are role model, social norm and peer group effects.

At first glance, statistical mechanics methods, which underlie the theory of condensed matter, would appear to have little to do with socioeconomic phenomena related to inequality. However, strong metaphorical similarities exist between the two fields of research. The canonical question in statistical mechanics concerns the determinants of magnetization in matter. As a magnetized piece of matter is one in which a substantial majority of the atoms share a common spin (which can either be up or down), magnetization would appear to be an extremely unlikely phenomenon, as it would require the coincidence of many atoms sharing a common property. However, if the probability that one atom has a particular spin is a function of the spins of surrounding atoms, the possibility of collective interdependence renders magnetism understandable. As techniques for the study of economic phenomena, statistical mechanics approaches have proven valuable for studying the aggregate behavior of populations facing interdependent binary choices. In particular, statistical mechanics methods hold the promise of providing a general framework for understanding how collective interdependence can lead to the emergence of interesting and rich aggregate behavior.<sup>4</sup>

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<sup>3</sup>While markets may not exist to directly mediate these interactions, this does not imply that economic actors do not alter their behavior in order to account for them. For example, as discussed in Bénabou (1993,1996a) and Durlauf (1995,1996a), the presence of within-neighborhood interactions can play a primary role in determining the composition of neighborhoods.

<sup>4</sup>See Crutchfield (1993) for a discussion of the meaning of emergent phenomena.

Of course, any metaphorical similarity between physical and social models of interdependent behavior is of little interest unless the specific substantive models underlying each can be shown to have similar structures. An important goal of this paper is to show how statistical mechanics structures naturally arise in a number of socioeconomic environments. Also, it is important to recognize that the potential for interesting aggregate behavior to emerge from individual decisions has been explored in previous economic contexts. Two prominent examples include Becker's (1962) work on aggregate demand with zero-intelligence agents and Schelling's (1971) analysis of racial segregation.<sup>5</sup> Statistical mechanics approaches should thus be regarded as complementary to disparate strands of previous work.

The rest of this paper is organized as follows. Section 2 outlines some general issues in modelling binary choices with interactions. Section 3 analyzes binary choice models with global interaction structures. Section 4 analyzes binary choice models with local interactions. Section 5 provides a discussion of limitations and outstanding questions which arise in statistical mechanics models of social behavior. Section 6 contains summary and conclusions.

## 2. General considerations

The statistical mechanics-inspired models of socioeconomic phenomena which have typically been studied focus on environments where a group of individuals each faces a binary choice. Many individual decisions which are relevant to understanding inequality are binary in nature. Standard examples include decisions to have a child out of wedlock, drop out of school, commit a crime. Not only are these decisions of interest in

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<sup>5</sup>Schelling's model possesses a structure which is quite similar to some of the statistical mechanics models which are discussed below.

their own right, but they are well known to affect a broad range of individual socioeconomic outcomes over large time horizons. While the importance of these binary decisions in cross-section and intertemporal inequality is beyond dispute, there is considerable controversy over the role of interactions and so one purpose of the particular formulation I choose is to develop interactions based models in a way which makes contact with the econometric literature on discrete choice. In addition, this approach provides a method for exploring the interconnections between different approaches to modelling interactions. Brock (1993) and Blume (1993) originally recognized the connection between discrete choice and statistical mechanics models. The current development follows Brock and Durlauf (1995).

Binary decisions of this type may be formalized as follows. Consider a population of  $I$  individuals. Individual  $i$  chooses  $\omega_i$ , whose support is  $\{-1, 1\}$ . The vector consisting of the choices of all agents in the population is  $\underline{\omega}$  and the vector of all decisions other than that of agent  $i$  is  $\underline{\omega}_{-i}$ .

Individuals are heterogeneous in three different respects. First, agents differ with respect to personal attributes which are characterized by the vector  $\underline{X}_i$ . This vector can include elements ranging from family background to community environment to past behavior. Second, individuals possess distinct expectations concerning the behavior of the population as a whole. Specifically, each agent is associated with a conditional probability measure  $\mu_i^e(\underline{\omega}_{-i})$ . Third, each agent experiences a pair of unobservable random shocks,  $\epsilon(\omega_i)$ , which influence the payoff of each of the possible choices. The shock  $\epsilon(1)$  is distinguished from  $\epsilon(-1)$ , as certain types of random shocks are only relevant for one of the choices. For example,  $\epsilon(1)$  might refer to a shock which reflects mathematical talent and so is only relevant if the person stays in school. Similarly, if  $\epsilon(-1)$  is an innovation to the sensitivity of one's fingertips, a virtue in safecracking, then the shock only affects the payoff to dropping out and becoming a

criminal.

Taken together, an individual's choice problem may be specified as

$$\max_{\omega_i \in \{-1, 1\}} V(\omega_i, \mathcal{X}_i, \mu_i^\epsilon(\omega_{-i}), \epsilon(\omega_i)) \quad (1)$$

At this level of generality, of course, virtually nothing can be said about the properties either of the individual decisions or about the behavior of the population as a whole. Two standard restrictions have been made on the general decision problem (1) to permit explicit analysis.

First, the decision problem is assumed to be additive in three components,

$$V(\omega_i, \mathcal{X}_i, \mu_i^\epsilon(\omega_{-i}), \epsilon(\omega_i)) = u(\omega_i, \mathcal{X}_i) + S(\omega_i, \mathcal{X}_i, \mu_i^\epsilon(\omega_{-i})) + \epsilon(\omega_i) \quad (2)$$

In this specification,  $u(\omega_i, \mathcal{X}_i)$  represents deterministic private utility,  $S(\omega_i, \mathcal{X}_i, \mu_i^\epsilon(\omega_{-i}))$  represents deterministic social utility, and  $\epsilon(\omega_i)$  represents random private utility. The two private utility components are standard in the economics of discrete choice. Recent work is distinguished by the introduction of social utility considerations.

Second, the form of the social utility and the form of the probability density characterizing random private utility are generally given particular functional forms. (It will become clear that restricting the form of the private utility term has no qualitative effect on the properties of the aggregate population.) The social utility component is formalized by exploiting the intuition that individuals seek to conform in some way to the behavior of others in the population. Formally, a specification which subsumes many specific models may be written as

$$S(\omega_i, \mathcal{X}_i, \mu_i^\epsilon(\omega_{-i})) = -E_i \sum_{j \neq i} \frac{J_j(\mathcal{X}_i)}{2} (\omega_i - \omega_j)^2 \quad (3)$$

$E_i(\cdot)$  represents the conditional expectation operator associated with agent  $i$ 's beliefs. The term  $\frac{J_j(\underline{X}_i)}{2}$  represents the interaction weight which relates  $i$ 's choice to  $j$ 's choice and is typically assumed to be nonnegative. Note that this specification can accommodate any assumptions concerning who interacts with whom.<sup>6</sup>

The random utility terms are assumed to be extreme-value distributed, so that

$$Prob(\epsilon(-1) - \epsilon(1) \leq z) = \frac{1}{1 + \exp(-\beta(\underline{X}_i)z)}; \beta(\underline{X}_i) \geq 0. \quad (4)$$

Anderson, Thisse and dePalma (1992) provide a nice survey of the logistic density and its relationship to interpretations of the random payoff terms. The random terms are assumed to be independent across agents.

These assumptions permit the model to be manipulated into a more transparent form. Observe first that since the support of  $\omega$  is  $\{-1, 1\}$ , the first term in the expression can be replaced with  $h(\underline{X}_i)\omega_i + k(\underline{X}_i)$  so long as the functions  $h(\underline{X}_i)$  and  $k(\underline{X}_i)$  obey

$$h(\underline{X}_i) + k(\underline{X}_i) = u(1, \underline{X}_i) \quad (5)$$

and

$$-h(\underline{X}_i) + k(\underline{X}_i) = u(-1, \underline{X}_i) \quad (6)$$

since the linear function coincides with the original function on the support of the individual choices.

Second, since  $\omega^2 = 1$ , social utility can be rewritten as

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<sup>6</sup>Of course,  $J_j(\underline{X}_i)$  could be further generalized. For example, bilateral interactions might depend on the characteristics of both agents, producing interaction weights of the form  $J_j(\underline{X}_i, \underline{X}_j)$ .

$$S(\omega_i, \underline{X}_i, \mu_i^e(\underline{\omega}_{-i})) = \sum_{j \neq i} J_j(\underline{X}_i) \cdot (\omega_i E_i(\omega_j) - 1). \quad (7)$$

which means that social utility is linear in the expected values of the choices in one's reference group.

Now, since the probability of an individual's choice conditional on his characteristics and expectations is

$$\begin{aligned} & \text{Prob}(\omega_i | \underline{X}_i, \mu_i^e(\underline{\omega}_{-i})) = \\ & \text{Prob}(V(\omega_i, \underline{X}_i, \mu_i^e(\underline{\omega}_{-i}), \epsilon(\omega_i)) > V(-\omega_i, \underline{X}_i, \mu_i^e(\underline{\omega}_{-i}), \epsilon(-\omega_i))), \end{aligned} \quad (8)$$

substituting (5), (6), and (7) into (8) implies

$$\begin{aligned} & \text{Prob}(\omega_i | \underline{X}_i, \mu_i^e(\underline{\omega}_{-i})) \sim \\ & \exp(\beta(\underline{X}_i)h(\underline{X}_i)\omega_i + \sum_{j \neq i} \beta(\underline{X}_i)J_j(\underline{X}_i) \cdot \omega_i E_i(\omega_j)). \end{aligned} \quad (9)$$

The  $k(\underline{X}_i)$  terms do not appear in this expression as they are irrelevant to the comparison of utilities which drive individual choices.

Finally, observe that each of the individual choices is independent once one has conditioned on the set of individual-specific expectations. Hence

$$\begin{aligned} & \text{Prob}(\underline{\omega} | \underline{X}_1, \dots, \underline{X}_I, \mu_1^e(\underline{\omega}_{-1}), \dots, \mu_I^e(\underline{\omega}_{-I})) = \prod_i \text{Prob}(\omega_i | \underline{X}_i, \mu_i^e(\underline{\omega}_{-i})) \sim \\ & \prod_i \exp(\beta(\underline{X}_i)h(\underline{X}_i)\omega_i + \sum_{j \neq i} \beta(\underline{X}_i)J_j(\underline{X}_i) \cdot \omega_i E_i(\omega_j)) \end{aligned} \quad (10)$$

Equation (10) provides a general form for the joint probability measure of individual choices. It has the general form of a Gibbs measure, which is not coincidental. A deep theorem in the statistical mechanics literature, due to Averintsev (1970) and Spitzer (1971a) is that models of

stochastic interactions of the type which have been outlined will generically possess probability measures with Gibbs representations.

Different specializations of the functional forms in these expressions will yield many of the particular statistical mechanics models which have been studied in the literature. These different functional forms differ substantively with respect to the nature of the interaction structure which connects individual decisions.

### 3. Global interactions

A first class of models has focused on environments in which each individual interacts symmetrically with all other members of the population. I focus on the model studied by Brock and Durlauf (1995); a related example is Aoki (1995).

In this model, each individual is assumed to derive utility from conforming with the average behavior of his reference group. Operationally, this occurs for the special case (relative to the general specification of bilateral interactions)

$$J_j(\underline{X}_i) = \frac{J(\underline{X}_i)}{I} \quad (11)$$

so that

$$S(\omega_i, \underline{X}_i, \mu_i^e(\omega_{-i})) = J(\underline{X}_i) \cdot (\omega_i m_i^e - 1) \quad (12)$$

where  $m_i^e$  is individual  $i$ 's expectation of the average population choice level.

Under these assumptions, the joint probability measure characterizing all agents choices obeys

$$\begin{aligned} & Prob(\omega \mid \underline{X}_1, \dots, \underline{X}_I, \mu_1^e(\omega_{-1}), \dots, \mu_I^e(\omega_{-I})) \sim \\ & \prod_i \exp(\beta(\underline{X}_i)h(\underline{X}_i)\omega_i + \beta(\underline{X}_i)J(\underline{X}_i) \cdot \omega_i m_i^e) \end{aligned} \quad (13)$$

The large sample behavior of the average choice level in this economy may be analyzed as follows. First, assume that all agents share a common expectation of the average choice level, *i.e.*

$$m_i^e = m^e \quad \forall i \quad (14)$$

Second, let  $dF_{\underline{X}}$  denote the limit of the sequence of empirical probability density functions associated with individual characteristics  $\underline{X}_i$ , where the limit is taken with respect to the population size  $I$ . (I assume that such a limit exists.) The strong law of large numbers implies, for any common expected mean, that the sample mean  $\bar{m}_I$  of population choices converges with a limit equal to

$$\lim_{I \rightarrow \infty} \bar{m}_I = \int \tanh(\beta(\underline{X})h(\underline{X}) + \beta(\underline{X})J(\underline{X})m^e) dF_{\underline{X}} \quad (15)$$

The model is closed by imposing self-consistency in the large economy limit, so that limit of the sample mean corresponds to the common expected average choice level. A self-consistent equilibrium mean,  $m^*$ , is any root of

$$m^* = \int \tanh(\beta(\underline{X})h(\underline{X}) + \beta(\underline{X})J(\underline{X})m^*) dF_{\underline{X}} \quad (16)$$

When all agents are identical in that they are associated with the same  $\underline{X}$ , the mean choice level in the economy will correspond to the roots of

$$m^* = \tanh(\beta h + \beta J m^*). \quad (17)$$

In this special case, the model corresponds to the mean field approximation of the Curie-Weiss model. Brock (1993) is the first instance in which the Curie-Weiss model was given an economic interpretation; the current formulation differs in emphasizing the equivalence between the mean field approximation of the model and the assumption of a noncooperative interaction environment. The following theorem, taken from Brock and Durlauf (1995), characterizes the number of self-consistent steady states.

**Theorem 1. Existence of multiple average choice levels in equilibrium**

*i.* If  $\beta J > 1$  and  $h = 0$ , there exist three roots to eq. (17). One of these roots is positive, one root is zero, and one root is negative.

*ii.* If  $\beta J > 1$  and  $h \neq 0$ , there exists a threshold  $H$ , (which depends on  $\beta$  and  $J$ ) such that

*a.* for  $|\beta h| < H$ , there exist three roots to eq. (17), one of which has the same sign as  $h$ , and the others possessing the opposite sign.

*b.* for  $|\beta h| > H$ , there exists a unique root to eq. (17) with the same sign as  $h$ .

Notice that the model exhibits nonlinear behavior with respect to both the parameters  $\beta h$  and  $\beta J$ . This makes sense intuitively. Conditional on a given private utility difference between the choices 1 and  $-1$ , which equals  $2h$ , there is a level which the conformity effect  $\beta J$  must reach in order to produce multiple self-consistent mean choice behavior. Recall that the random utility shocks are *iid*, so that in absence of the conformity effects, there would be a unique mean whose sign is the same as  $h$ .

Conditional on a conformity effect  $\beta J > 1$ , as  $\beta h$  increases in magnitude, any multiplicity will eventually be eliminated. This occurs because eventually the private utility differential between the choices will overcome any tendency for the conformity effect to produce a self-consistent mean with the opposite sign.

This type of model illustrates the complementary nature of the roles of economic fundamentals and social norms in explaining the degree of social pathologies in different neighborhoods. For example, Theorem 1 states that high degrees of conformity can lead to mean choice levels opposite to that dictated by private utility. To be concrete, even if economic fundamentals as embodied in  $h$  imply that the average teenager should stay in school, conformity effects can produce an equilibrium in which most teenagers drop out.

It is straightforward to show (Brock and Durlauf (1995)) that the equilibrium in which the sign of  $h$  is the same as the sign of the mean choice level produces higher average utility than the equilibrium in which the signs are opposite. Hence this model illustrates the potential for collectively undesirable behavior, such as high out-of-wedlock birth rates, which is individually optimal. This ranking of average utility provides the appropriate stochastic generalization to the Pareto rankings of equilibria studied in Cooper and John (1988).

#### 4. Local Interactions

An alternative approach in the study of interactions has focused on the aggregate implications of local interactions. Such models assume that each individual interacts with a strict subset of others in the population. For individual  $i$ , this subset is  $n_i$  and is referred to as the individual's neighborhood. Hence, for agent  $i$ ,

$$J_j(\underline{X}_i) = 0 \text{ if } j \notin n_i \quad (18)$$

Typically, the index  $i$  has been interpreted so that  $|i - j|$  measures distance between individuals. This allows one to construct a neighborhood for agent  $i$  by taking all agents within some fixed distance from  $i$ . (The distance can vary with direction.) This latter assumption is what renders the interactions local.

### *i. endogenous preferences*

Föllmer (1974) introduced statistical mechanics methods to economics by considering an economy with locally interdependent preferences. Specifically, he wished to understand under what circumstances randomness in individual preferences will fail to disappear in the aggregate. Agents in the model are arrayed on the two-dimensional integer lattice  $Z^2$  and possess one of two possible utility functions denoted as  $U_1$  and  $U_{-1}$  respectively; agent  $i$ 's preferences are coded by  $\omega_i$  so that the first preference type corresponds to  $\omega_i = 1$  and the second type to  $\omega_i = -1$ . The probability that an agent has one utility function is an increasing function of the number of neighbors with the same preferences. In particular, Föllmer assumes that the probability density characterizing the preferences of agents is of the form of the Ising model of statistical mechanics (see Liggett (1985) or Spitzer (1971b)), which means that

$$\begin{aligned} & \text{Prob}(\omega_i | \omega_j \forall j \neq i) = \\ & \text{Prob}(\omega_i | \omega_j \forall j \text{ such that } |i - j| = 1) \sim \exp\left(J \sum_{|i - j| = 1} \omega_i \omega_j\right) \quad (19) \end{aligned}$$

so that the joint probability measure over agent preferences is

$$Prob(\underline{\omega}) \sim \exp\left(J \sum_i \sum_{|i-j|=1} \omega_i \omega_j\right) \quad (20)$$

As is well known, there exists a critical value  $J_c$  such that if  $J < J_c$ , then the sample mean produced by a realization of this system will converge to zero (almost surely) when  $I$  is infinite, whereas if  $J > J_c$ , then the sample mean from a realization of this economy will converge (when averaged over larger and larger finite squares each of whose centers is a particular agent on the lattice) to one of two possible nonzero values. Hence, Föllmer concluded that for his preference structure, idiosyncratic shocks may have aggregate consequences.

Föllmer's model of endogenous preferences does not directly translate into the discrete choice problem formulated in Section 2. The reason for this is that he places a conditional probability structure on agent characteristics in order to study interdependence; in fact there is no choice component to his model.<sup>7</sup> However, there exist two ways in which one can modify Föllmer's model so as both to render its probability structure interpretable within a discrete choice formulation. In each case, one reinterprets the  $\omega_i$ 's as individual choices (of some commodity, for example) in which the utility of a particular choice depends on whether one's neighbors have made the same choice. Further, the model must be made explicitly intertemporal. This intertemporal interpretation of preference interdependence has the feature that it closely preserves the structure of neoclassical utility theory without losing any of Föllmer's qualitative insights; see Becker and Stigler (1977) for a defense of this approach to generalizing neoclassical preferences. For clarity of exposition, I reindex relevant variables by time where appropriate.

In one possible modification of Föllmer's model, one moves to a

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<sup>7</sup>In fact, I am unaware of any way to explicitly formulate a set of individual decision problems in a noncooperative environment such that Föllmer's conditional probability structure emerges as an equilibrium.

discrete time setting and assumes that the utility of a particular choice is a function of whether others in an agent's neighborhood have made the same choice the previous period.<sup>8</sup> To reformulate Föllmer's model as a discrete choice problem in this fashion, this means that

$$J_j(\underline{X}_{i,t}) = 0 \quad (21)$$

which eliminates any contemporaneous interactions. Second, by assuming that private utility depends on the past behavior of one's nearest neighbors in a way such that the total number of neighbors making a particular choice at  $t-1$  is all that matters with respect to utility at  $t$ , the deterministic private utility weight can be expressed as

$$h(\underline{X}_{i,t}) = h \cdot \sum_{|i-j|=1} \omega_{j,t-1}. \quad (22)$$

This embeds all local interactions in the private utility component. Third, set  $\beta(\underline{X}_{i,t}) = \beta$ , which may be done without loss of generality. Together, this implies that

$$\begin{aligned} Prob(\omega_{i,t} | \omega_{j,t-1} \forall j \text{ such that } |i-j|=1) \sim \\ exp(\beta h \sum_{|i-j|=1} \omega_{i,t} \omega_{j,t-1}) \end{aligned} \quad (23)$$

and

$$Prob(\underline{\omega}_t) \sim exp(\beta h \sum_i \sum_{|i-j|=1} \omega_{i,t} \omega_{j,t-1}) \quad (24)$$

The role of idiosyncratic preference shocks in aggregate outcomes,

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<sup>8</sup>This way of formulating interdependent preferences is strongly defended by Becker (1996).

can be reformulated for this model as follows. Do all initial configurations of choices  $\omega_0$  produce the same limiting behavior for the average choice level  $m_\infty$ ? The answer parallels Föllmer's original case. There exists an  $h_c$  such that if  $h < h_c$ , then the mean choice is zero, whereas if  $h > h_c$ , the mean choice will converge to one of two nonzero values. The probability of converging to any one of these values will depend on  $\omega_0$ .

While providing qualitatively similar features to the original Föllmer formulation, the absence of any contemporaneous interactions does create some differences. In particular, the  $h_c$  in the dynamic model does not equal the  $J_c$  in the static model. This occurs because of the absence of any contemporaneous interactions. An exact equivalence between the static and dynamic models (with respect to parameter values and invariant measures) can be achieved through the following continuous time formulation. Suppose that agents adjust asynchronously in time. In particular, each agent is associated with a separate Poisson process such that at each arrival time for his process, the agent's choice is such that he takes the choices of the nearest neighbors as given. As the probability that any two agents choose at the same time is zero even in the infinite population limit. This model will generate multiple regimes, with  $J_c = h_c$ .<sup>9</sup>

Föllmer's model illustrates that it is difficult to provide a straightforward interpretation of contemporaneous local interactions in which the interdependence occurs with respect to choices. Alternatively, one can interpret this discussion as demonstrating the sensitivity of local interactions models to the assumptions placed on expectations. To see this, consider a local interaction formulation of social utility which preserves the Ising interaction structure,

$$S(\omega_i, X_i, \mu_i^e(\omega_{-i})) = J \sum_{|i-j|=1} E_i(\omega_j). \quad (25)$$

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<sup>9</sup>See Liggett (1985) for a formal discussion.

Unlike the global interactions model, it is not clear how to formulate individual expectations. At first glance, it might appear that since the analysis is dealing with nearest neighbors, expectations should exhibit perfect foresight with probability 1, *i.e.*

$$E_i(\omega_j) = \omega_j. \quad (26)$$

However, this assumption will not lead to the conditional probability structure (10) for individual choices. The reason is simple. Under perfect foresight, the choice of individual's  $i$  and  $j$  will be interdependent whenever  $|i - j| = 1$ , and so each choice will be a function of both  $\epsilon_i(\omega_i)$  and  $\epsilon_j(\omega_j)$ . Therefore, the equilibrium probability will not be the product of a set of densities of *i.i.d.* extreme-value innovations. The Gibbs representation of the equilibrium probability measure will no longer be valid; no characterization of the equilibrium in this case is known.

An alternative assumption on expectations is that all agents assign the same expectations to each other

$$E_i(\omega_j) = E_k(\omega_l) \quad \forall i, j, k, l \quad (27)$$

In this case, the model can easily be seen to be observationally equivalent to the global interactions model that has already been examined, as (25) and (27) combine to yield

$$S(\omega_i, X_i, \mu_i^e(\omega_{-i})) = 4JE(\omega) \quad (28)$$

implying that

$$Prob(\underline{\omega}) \sim \exp(4J \sum_i \omega_i E(\omega)) \quad (29)$$

which is the same form as (13). This might appear odd, given the explicit local interaction structure of preferences. In fact, the equivalence is not surprising. When all expectations are identical, and the sample mean is required to equal the population mean, then agents globally interact through the expectations formation process.<sup>10</sup>

While much remains to be understood about these models, the import of this discussion is that intertemporal interactions are at the present time far more tractable than contemporaneous ones in general local interaction environments.

## ii. growth and economic development

A second area where statistical mechanics approaches have been applied is that of cross-country inequality. Durlauf (1993) constructs a model based on local technological interactions. A set of infinitely-lived industries is analyzed, each of which maximizes the discounted value of profits. Industries are assumed to be the aggregation of a large number of firms who act identically but noncooperatively. Letting  $Y_{i,t}$  denote industry  $i$ 's output at  $t$ ,  $K_{i,t}$  denote industry  $i$ 's capital investment at  $t$ , and  $\mathcal{F}_t$  denote information available at  $t$ , discounted expected profits will equal

$$\Pi_{i,t} = E\left(\sum_{j=0}^{\infty} \beta^{t+j}(Y_{i,t+j} - K_{i,t+j}) \mid \mathcal{F}_t\right). \quad (30)$$

Each industry can produce output using one of two production techniques. Technique choices are coded so that  $\omega_{i,t} = 1$  if technique 1 is chosen,  $-1$  if technique 2 is chosen. Capital fully depreciates after one period of use. Output is produced with a one-period lag, so that investment

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<sup>10</sup>Notice that this version of the Ising model is equivalent to the mean field version of the Curie-Weiss model, indicating how, in noncooperative environments with common expectation assumptions, very different interaction structures may be observationally equivalent.

at  $t$  produces output available at  $t+1$ . The production functions at  $t$  depend on the technique choices at  $t-1$ . Together, these assumptions may be represented by a pair of production functions of the form

$$Y_{i,t+1} = f_1(K_{i,t-F}, \zeta_{i,t}, \omega_{j,t-1} \forall j \in \Delta_{k,l}) \quad (31)$$

if technique 1 is chosen, or

$$Y_{i,t+1} = f_{-1}(K_{i,t}, \eta_{i,t}, \omega_{j,t-1} \forall j \in \Delta_{k,l}) \quad (32)$$

if technique 2 is chosen.  $F$  is a fixed capital cost which must be paid to employ technique 1 at  $t$ .  $\zeta_{i,t}$  and  $\eta_{i,t}$  are industry-specific productivity shocks, and are assumed to be *i.i.d.* across industries and time. The term  $\Delta_{k,l}$  refers to the interaction range of an industry. For each industry  $i$ ,  $\Delta_{k,l} = \{i-j, \dots, i+k\}$  so that the spillovers are all taken to be local: past technique choices influence the relative productivity of current techniques. The relative positions of industries with respect to the  $i$  index is interpreted as measuring technological similarity.

Finally, the relative productivity of technique 1 at  $t$  is enhanced by choices of technique 1 at  $t-1$ . If  $\underline{\omega}'$  and  $\underline{\omega}''$  denote two realizations of  $\underline{\omega}_{t-1}$  such that  $\omega'_j \geq \omega''_j \forall j \in \Delta_{k,l}$ , then

$$\begin{aligned} & f_1(K, \zeta_{i,t}, \omega_{j,t-1} = \omega'_j \forall j \in \Delta_{k,l}) - \\ & f_{-1}(K, \eta_{i,t}, \omega_{j,t-1} = \omega'_j \forall j \in \Delta_{k,l}) \geq \\ & f_1(K, \zeta_{i,t}, \omega_{j,t-1} = \omega''_j \forall j \in \Delta_{k,l}) - \\ & f_{-1}(K, \eta_{i,t}, \omega_{j,t-1} = \omega''_j \forall j \in \Delta_{k,l}). \end{aligned} \quad (33)$$

The long run output dynamics of the model will thus depend on the evolution of technique choices.

The assumptions outlined are sufficient to show that the technique choices in the economy will obey the structure

$$Prob(\omega_{i,t} | \mathcal{F}_{t-1}) = Prob(\omega_{i,t} | \omega_{j,t-1} \forall j \in \Delta_{k,l}) \quad (34)$$

which is similar to the dynamic variant of the Ising model discussed above.

Durlauf (1993) examines the stability of the choice of technique 1 by assuming that

$$Prob(\omega_{i,t} = 1 | \omega_{j,t-1} = 1 \forall j \in \Delta_{k,l}) = 1 \quad (35)$$

and studying the conditions under which  $\omega_\infty = \underline{1}$  is the unique invariant measure of the system. Using the following bounds on the system of conditional probabilities (34),

$$\Theta_{k,l}^{\min} \leq Prob(\omega_{i,t} = 1 | \omega_{j,t-1} = -1 \text{ for some } j \in \Delta_{k,l}) \leq \Theta_{k,l}^{\max} \quad (36)$$

the following result is proven.

**Theorem 2. Uniqueness versus multiplicity of long run equilibrium as a function of strength of complementarities**

For each index set  $\Delta_{k,l}$  with at least one of  $k$  or  $l$  nonzero, there exist numbers  $\bar{\Theta}_{k,l}$  and  $\underline{\Theta}_{k,l}$ ,  $0 < \underline{\Theta}_{k,l} < \bar{\Theta}_{k,l} < 1$  such that

A. If  $\Theta_{k,l}^{\min} \geq \bar{\Theta}_{k,l}$ , then  $Prob(\omega_{i,\infty} = 1 | \omega_{-1} = -\underline{1}) = 1$ .

B. If  $\Theta_{k,l}^{\max} \leq \underline{\Theta}_{k,l}$ , then

i.  $Prob(\omega_{i,\infty} = 1 | \omega_{-1} = -\underline{1}) < 1$ .

ii.  $Prob(\omega_\infty = \underline{1} \mid \omega_{-1} = -\underline{1}) = 0$ .

The main result of the model is that when production decisions are interdependent, then a low level production trap is produced. In terms of the underlying probability structure, the model (and its associated properties) is a generalization of the Stavskaya-Shapiro model, described in Shnirman (1968) and Stavskaya and Pyatetskii-Shapiro (1968).

This model can be rewritten in the canonical discrete choice form as follows. For each industry  $i$  at  $t$ , reinterpret  $u(\omega_{i,t}, \cdot, \cdot)$  as the expected discounted profits, which depends on current technique choice.<sup>11</sup> Following eqs. (5) and (6),  $h(\underline{X}_{i,t})$  measures the relevant individual- and time-specific deterministic private payoff parameter associated with a technique choice. Further, assume that the only relevant characteristics is determining the relative profitability of the two techniques for a given industry are the past technique choices of technologically similar industries,

$$h(\underline{X}_{i,t}) = h(\omega_{i-k,t-1}, \dots, \omega_{i+l,t-1}). \quad (37)$$

Restrict this function so that it is increasing in all arguments and positive when all previous technique choices were equal to 1,

$$h(1, \dots, 1) > 0. \quad (38)$$

Finally, assume that  $\beta(\underline{X}_{i,t})$  has the property that

$$\beta(\underline{X}_{i,t}) = \infty \text{ if } \omega_{i-k,t-1} = \dots = \omega_{i+l,t-1} = 1 \quad (39)$$

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<sup>11</sup>Recall that the structure of industries is such that technique choices do not reflect any consequences for future spillover effects, so that an industry always chooses a technique to maximize one-period ahead profits.

Subject to these restrictions, there will exist a discrete choice specification which replicates the probabilities in the interacting industries model. This specification reveals an important feature of the particular specification studied in Durlauf (1993): a strong nonlinearity in the case where all influences are positive at  $t - 1$  versus all other cases. This is not surprising given the unbounded support of the logistic density for finite  $\beta$  combined with the probability 1 assumption on technique choice under the conditioning in eq. (35). This suggests that the particular case studied in Durlauf (1993) is knife-edge in terms of parameter values under the general discrete choice framework. While this does not affect the theoretical interest of the model, it does suggest that it should be reparameterized if one is to use it in empirical work.

Finally, it is worth noting that when interactions between decisions are all intertemporal, then the assumption of extreme-valued random utility increments can be dropped. The equilibrium properties of the dynamic models in this section can be recomputed under alternative probability densities such as probit which are popular in the discrete choice work. In fact, under the mean field analysis of global interactions, alternative specifications incorporate probit or other densities as well. In both cases, the large scale properties of models under alternative error distributions are largely unknown.

## 5. Stochastic interaction structures

A third area of work has focused on cases where the interaction environments are themselves stochastic. Three approaches to this have been taken. The first is to allow for heterogeneity within a group. The second allows for random group formation. The third treats group membership a

choice variable.

**i. multiple agent types**

Glaeser, Sacerdote, and Scheinkman (1996) provide a model of local interactions and crime which fits cleanly into the general discrete choice framework. They consider a model in which individuals are arrayed on a one-dimensional lattice. Each individual has one of three kinds of preferences with reference to the decision to commit a crime. The decision to commit is coded as  $\omega_i = 1$ . Preference type 1 is such that the agent always decides to commit a crime. Preference type 2 is such that an agent will choose to commit a crime only if the agent to his left does as well. Preference type 3 is such that the agent never commits a crime. Representing the preferences of agents as  $U_j(\omega_i, \omega_{i-1})$  where  $j$  denotes the preference type,

$$U_1(1,1) > U_1(-1,1); U_1(1,-1) > U_1(-1,-1) \quad (40)$$

$$U_2(1,1) > U_2(-1,1); U_2(-1,-1) > U_2(1,-1) \quad (41)$$

$$U_3(-1,1) > U_3(1,1); U_3(-1,-1) > U_3(1,-1) \quad (42)$$

The distribution of preference types is *i.i.d.* across agents. The interaction structure described here is a variant of the so-called voting model in statistical mechanics; see Liggett (1985) for a detailed description.

From the discrete choice perspective, these preference assumptions can be thought of as doing the following. Each agent possesses a neighborhood consisting of the agent to his left, so that

$$J_j(\underline{X}_i) = J(\underline{X}_i) \text{ if } j = i - 1, 0 \text{ otherwise} \quad (43)$$

Further, each agent is associated with a latent variable  $\phi_i$ , with support  $\{\phi^l, \phi^m, \phi^h\}$ . The latent variable is the only personal characteristic which influences individual utility, so that the different preference types can be thought of as induced by  $\phi_i$ . These influences work through the deterministic private and social utility functions according to

$$h(\phi^l) < 0, J(\phi^l) = 0 \quad (44)$$

$$h(\phi^m) = 0, J(\phi^m) > 0 \quad (45)$$

$$h(\phi^h) > 0, J(\phi^h) = 0. \quad (46)$$

The joint probability measure for choices in this model is

$$Prob(\underline{\omega} | \phi_1, \dots, \phi_I) \sim \prod_i \exp(\beta h(\phi_i) \omega_i + \beta J(\phi_i) \cdot \omega_i \omega_{i-1}) \quad (47)$$

The unconditional probability measure can be computed once an assumption is made on the form of  $dF_{\phi}$ . When  $\beta = \infty$ , the model reduces to the deterministic choice structure of Glaeser, Sacerdote, and Scheinkman.

A nice feature of this model is that by preserving nonoverlapping neighborhoods, the problems created by the contemporaneous determination of choices are avoided in the perfect foresight case. Further, notice that if the interactions are intertemporal and past behavior influences current behavior (as clearly seems natural in this context), then joint behavior will follow

$$Prob(\underline{\omega}_t) \sim \exp\left(\sum_i h_{i,t} \omega_{i,t}\right) \quad (48)$$

where

$$h_{i,t} = h(\omega_{i-1,t-1}, \omega_{i,t-1}, \phi_{i,t}) \quad (49)$$

In this case, the interaction structure is special case of that studied in Durlauf (1993).

## ii. Random communication structures

Kirman (1983), Ioannides, (1990) and Durlauf (1996b), have studied economic environments in which the structure of bilateral interactions has important aggregate consequences. In their models, random communication links exist between any pair of agents  $i$  and  $j$ . Coalitions emerge across any grouping of agents such that a path of direct bilateral communication links can be formed between any pair of members in a coalition. As all agents within a coalition communicate whereas members of different coalitions do not, this structure can illustrate the role of group membership in phenomena ranging from fluctuations in the prices of a particular good across trading regions to the role of the degree of specialization of labor in explaining business cycles. A rich set of results from random graph theory illustrate how the distribution of coalition sizes will depend sensitively on the probability of the bilateral links. In particular, when the bilateral links are conditionally *i.i.d.* (in the sense that the probability that any pair of agents is directly linked is independent of whether any other pair is linked), then as  $I \Rightarrow \infty$  (1) if the probability of any link is less than  $1/I$ , then the largest coalition will be of order  $\log I$ , (2) if the probability equals  $c/I$  for  $c > 1$ , then the largest coalition will be of order  $I$ , (3) if the probability is greater than  $c \log I / I$  for  $c > 1$ , then all agents will be members of a common coalition.

Previous papers employing random graph formulations have been

interested in the size distribution of coalitions. However, for socioeconomic environments involving networks of contacts or friends, the approach can enrich the exogenous interaction structures which are typically assumed. The discrete choice structure can accommodate random interactions by assigning a probability measure to  $J_{i,j}$ 's such that

$$J_{i,j} \in \{0,1\} \quad (50)$$

$$J_{i,j} = J_{j,i} \quad (51)$$

$$\text{If } J_{i,l_1} J_{l_1 l_2} \cdots J_{l_m, j} = 1, \text{ then } J_{i,j} = 1 \quad (52)$$

Any assumptions about the distribution of the bilateral interactions can be mapped in a straightforward fashion to the  $J_{i,j}$  weights subject to these restrictions.

This particular structure is related to the Mattis model in statistical mechanics, which is described in Fischer and Hertz (1991). In the Mattis model, the individual weights are defined by

$$J_{i,j} = J \xi_i \xi_j \quad (53)$$

where  $\xi_i$  has support  $\{1, -1\}$  and is distributed *i.i.d.* across  $i$ . Using the transformation

$$\zeta_{i,j} = \frac{\xi_i \xi_j + 1}{2} \quad (54)$$

then the Mattis model will produce a random graph structure and associated interactions weights with the formula

$$J_{i,j} = J \zeta_{i,j} \quad (55)$$

Notice that in this formulation, the bilateral links between any two agents will no longer be conditionally independent, as they are for the random graph structures which have been studied in the economics literature. Dependence is a natural assumption if agents with common attributes communicate with each other.

### iii. Self-organizing neighborhood composition

Bénabou (1993,1996a) and Durlauf (1995,1996a) have emphasized the importance of endogeneity in neighborhood structure. This class of models embodies interactions of the type which have already been surveyed, with an essential difference. While agents interact noncooperatively within a neighborhood, they choose neighborhoods in recognition of these interactions.

Evolving neighborhood composition can be accommodated in the statistical mechanics approach through the  $J_j(\mathcal{X}_{i,t})$ . This can be seen in two steps. First, suppose that at time  $t$ , each agent is assigned to a neighborhood  $n$ ,  $n = 1 \dots N$ . By allowing the interaction weights to depend on whether one is in a common or different neighborhood as other agents, one can replicate the interactive structure of the endogenous neighborhoods models. The endogenous neighborhood model will be complete once a neighborhood assignment rule is specified. Since such a rule will presumably depend upon the attributes of all agents in the economy, this implies that

$$n_{i,t} = \Lambda(\mathcal{X}_{1,t}, \dots, \mathcal{X}_{I,t}) \tag{56}$$

Now, the specification of this function is far from trivial, as it must embody factors such as house price or rental differences which endogenously determine the assignment of individuals. What is relevant to the current

discussion is that there is no reason in principle that the statistical mechanics approach cannot accommodate a rich neighborhood structure.

Different specifications of the neighborhood determination will correspond to different models in the literature. For example, if

$$J_j(\underline{X}_{i,t}) = 1 \text{ if } n_{i,t} = n_{j,t}, 0 \text{ otherwise} \quad (57)$$

then general form (10) will produce a Brock-Durlauf (1995) model specification for each of the neighborhoods.

While Bénabou and Durlauf have emphasized the role of economic segregation in neighborhood formation, observe that neighborhoods can be given alternative definitions. For example, by treating neighborhood membership as determined by ethnicity, one can use the model to study differences in ethnic group behavior; more generally, individual interaction weights can depend on a multiplicity of factors which reflect the different communities or reference groups which characterize individuals. Alternatively, one can allow for random interactions in the way discussed above, where the individual probabilities are determined by individual attributes. This can capture some of the ideas on social connections analyzed in Montgomery (1991) and Chwe (1996).

The introduction of heterogeneous and time dependent social utility weights  $J_j(\underline{X}_{i,t})$  in the random interaction and endogenous neighborhood models links the analysis of socioeconomic interactions with the frontier of research in statistical mechanics. Specifically, statistical mechanics has in the last two decades focused on the behavior of systems in which the configuration of elements obeys the canonical probability structure

$$Prob(\underline{\omega}) \sim \exp\left(\sum_i \sum_j J_{i,j} \omega_i \omega_j\right) \quad (58)$$

when the  $J_{i,j}$  terms are a function of something other than  $|i-j|$ , this

structure is known as an anisotropic ferromagnet. When  $J_{i,j} < 0$  for some  $i$  and  $j$ , the system is known as a spin glass. See Fischer and Hertz (1991) and Mézard, Parisi, and Virasoro (1987) for introductions to these models.

Anisotropic ferromagnets and spin glasses can exhibit phase transitions and complex pattern formation. Unlike the models which have been used in economics thus far, these phase transitions can apply to moments of the equilibrium probability measure other than the population mean. Spin glass formulations can (in particular formulations) exhibit phase transitions with respect to the variance of individual choices as well as spatial and intertemporal correlations. These models thus hold the promise of allowing the study of multiple equilibria and multiple steady states in the distribution of social and economic activity. In addition, the spin glass formulation will permit the modelling of interactions within and across neighborhoods. Suppose that each member of a neighborhood assigns a weight to conforming with others in the economy which depends upon the neighborhood in which the individual lives. In this case, there will exist interdependence across the neighborhood mean choices so that the equilibrium mean choice level of neighborhood  $n$  is

$$m_n^* = \int \tanh(\beta h(\underline{X}) + \beta \sum_r J_{n,r} m_r^*) dF_{n,\underline{X}} \quad n = 1 \dots N \quad (59)$$

where the weights  $J_{n,r}$  have been reindexed to reflect interaction weights between and within neighborhoods and  $dF_{n,\underline{X}}$  refers to the distribution of individual characteristics within neighborhood  $n$ . These equations may be used to analyze the consequences of different neighborhood formation rules on the population-wide mean choice level, thus providing a way to study the aggregate effects of integration and segregation, whose importance in different context has been studied by Bénabou (1996b).

Finally, it is important to recognize that endogenous neighborhood formation represents a class of statistical mechanics models whose properties

have yet to be directly explored in the physics or complex systems literatures. Economic models usually will impose  $J_{i,j}$  weights which are at least partially determined by individual choices. In fact, it is clear that new forms of phase transition are likely to occur in these systems. The reason is the following. Endogenous neighborhood models typically result in the stratification of neighborhoods by some observable attribute such as income; recent examples include Benabou (1993,1996), Durlauf (1996a,1996b) and Fernandez and Rogerson (1996). These attributes will correlate with the  $h_i$  and  $J_{i,j}$  terms which distinguish individuals. What this means is that when one considers cross-group inequality, differences can be explained by the presence of multiple equilibrium invariant measures as well as by different characteristics of the populations. One example where this endogeneity matters is the Brock-Durlauf model, where an integrated community mixing agents with high and low  $|h_i|$  values can exhibit a unique equilibrium whereas segregated communities can exhibit multiple equilibria for those communities in which members are characterized by low  $|h_i|$  values.

## 6. Limitations

While the statistical mechanics approach have yielded numerous insights into phenomena ranging from out of wedlock births to crime, the literature suffers from a number of limitations at this time.

First, the use of statistical mechanics methods in social science has exclusively focused on binary choices, which omits an important range of interaction environments. Bénabou (1993,1996b) and Durlauf (1996a,1996b) for example, explicitly focus on the level of education investment, which is naturally thought of as continuous. While there is a small literature on  $n$ -state spin systems for  $n > 2$  (see Yeomans (1992) for examples), it not clear that the systems which have been studied are rich enough for social science

applications.

Second, there has been relatively little success at this stage in developing rational expectations variants of these models. Dynamic approaches such as Durlauf (1991,1993,1994) have imposed linear technologies precisely in order to avoid the need for firms to forecast future prices. The difficulty with developing rational expectations versions of these models is that the interaction structures embody sufficiently complicated nonconvexities to render the standard fixed point arguments invalid. Important recent work by Blume (1995) makes progress on this issue.

Third, there has been little formal econometric work on statistical mechanics models.<sup>12</sup> Manski (1993a,b) provides a general framework which suggests that even the little empirical work which attempted to uncover interaction effects is flawed by identification problems. Hence, the interaction effects which underlie the theoretical literature, while plausible, are unproven. Brock and Durlauf (1996) attempt to address this limitation by providing a general estimation theory for their global interactions framework.

## 7. Conclusion

The goal of this paper has been to provide a unified perspective on the ways in which economists have employed statistical mechanics to model socioeconomic interactions. My main claim is that the statistical mechanics approach is compatible with good microeconomic reasoning, and thus represents a valuable additional tool for a range of research questions. This

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<sup>12</sup>A recent exception is Topa (1996). This is not to say that statistical mechanics models have not been used to interpret empirical findings. For example, Glaeser, Sacerdote, and Scheinkman (1996) show how their local interactions model can explain wide discrepancies between crime rates in cities.

claim has been made on the basis of unifying a number of applications in a discrete choice framework. This framework encompasses a range of approaches in the literature, and shares a common form with the logistic likelihood function. This common form holds the promise that the theoretical insights which have been generated by the statistical mechanics approach can be matched by empirical evidence.

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