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DYNAMIC EQUILIBRIUM ECONOMIES: A FRAMEWORK FOR COMPARING MODELS AND DATA

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ABSTRACT

We propose a constructive, multivariate framework for assessing agreement between (generally misspecified) dynamic equilibrium models and data. The framework enables a complete second-order comparison of the dynamic properties of models and data in both graphical and numerical ("goodness-of-fit") form. We propose bootstrap algorithms to evaluate the significance of deviations between models and data. We use the goodness-of-fit criteria to produce estimators that optimize economically-relevant loss functions and whose finite-sample properties are again approximated using bootstrap procedures. We provide a detailed illustrative application to modeling the U.S. cattle cycle.

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1. Introduction

Dynamic equilibrium models are now used routinely in many fields. Such models, for example, have been used to address a variety of macroeconomic issues, including business-cycle fluctuations, economic growth, and the effects of government policies.[1] Additional prominent fields of application include, among others, agricultural economics, international macroeconomics, and public economics.[2]

But in economics, as in all the sciences, one counts on the interplay between theory and data to fuel progress. At present, many important questions regarding the empirical implementation of dynamic equilibrium models remain unasked, unanswered, or incompletely answered. The questions fall roughly into two methodological groups. The first group involves issues related to assessing model adequacy, and the second involves issues related to model estimation. We propose and illustrate a framework that deals squarely with these issues. Some parts of the framework are new, while others exploit established theory. Our approach has a number of key distinguishing features. It is:

A. Based upon the realistic assumption that all models are misspecified. We regard all of the models we entertain as false, in which case traditional statistical methods lose some of their appeal.

B. Graphical and constructive. Our procedures permit one to assess visually and quickly the dimensions along which a model performs well, and the dimensions along which it performs poorly.

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[1] Among many others, see Kydland and Prescott (1982), Hansen (1985), and Christiano and Eichenbaum (1992) (business cycles), Lucas (1988), Jones and Manuelli (1990), Rebelo (1991), and Krussel (1992) (growth), and Lucas (1990), Cooley and Hansen (1992), and Ohanian (1993) (policy effects).

[2] Among many others, see Rosen, Murphy and Scheinkman (1994) (agricultural economics), Backus, Kehoe and Kydland (1992) (international macroeconomics), and Auerbach and Kotlikoff (1987) (public economics).

C. Capable of providing a common set of tools that can be used by researchers with potentially very different objectives and research strategies. The framework can be used to evaluate strictly calibrated models, and it can also be used formally to estimate and test models.

D. Based on a full second-order comparison of model and data dynamics. This is in contrast to the common approach of comparing only a few variances and covariances of detrended variables from the model economy and the actual economy.

E. Frequency-domain and multivariate. Working in the frequency domain enables us to decompose variation across frequencies, which is often useful. The multivariate nature of our framework facilitates simple examination of cross-variable correlations and lead-lag relationships, at the frequencies of interest.

F. Focused on finite-sample distributions of objects estimated from the data, including spectra, goodness-of-fit measures, model parameters, and test statistics. Whenever possible, we eschew reliance on asymptotics, which may be unreliable in samples of the size arising in many applications. Instead, we propose and use new bootstrap procedures.

G. Capable of delivering goodness-of-fit measures and estimators that are invariant to the method of trend removal adopted.

Related literature includes Christiano and Eichenbaum (1992), Gregory and Smith (1990, 1991), Canova (1991), Hansen and Sargent (1993), Hansen, Sargent and Tallarini (1994), Hansen, McGrattan, and Sargent (1994), Kim and Pagan (1994), Leeper and Sims (1994), Cogley and Nason (1994), and especially Watson (1993) and King and Watson (1992, 1994).

In many respects we pick up where Watson (1993) leaves off, and we make progress along a number of dimensions. Our methods can be used both to *assess* the performance of a model (for a given set of parameters), to *estimate* the model's parameters, and to *test hypotheses* about parameters or models. All of this is done using a loss function selected by the user, which might involve only certain frequencies of interest and might be very different from the loss function implicit in Gaussian maximum-likelihood (ML) estimation.

We focus for the most part neither on calibration nor on ML, however, because in many applications neither may be satisfactory. Our approach is considerably broader, involving the comparison of models and data from a complete second-order vantage point, with the loss function used to gauge "closeness" specified by the user. That is, we proceed by comparing model and data spectral density functions.[3] The frequency-domain approach is attractive for a number of reasons. First, the spectral density function is continuous, 2π-periodic and symmetric around the origin, all of which makes for convenient manipulation. Second, and more importantly, the spectral density function represents a convenient orthogonal decomposition of variance across frequencies, which facilitates analysis of dynamics. For example, all of the classical ideas of business-cycle analysis discussed by Lucas (1977) have spectral analogs, ranging from univariate persistence ("typical spectral shape") to multivariate issues of comovement ("coherence") and lead-lag relationships ("phase shifts") at business-cycle frequencies. In fact, in order to economize on discussion, we draw mostly upon the business-cycle literature for motivation in the methodological sections 2 and 3. In section 4, however, we present a complementary microeconometric application to modeling the well-known cycle in U.S. cattle consumption and stock. We conclude in section 5.

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[3] The spectral density function provides a complete summary of Gaussian time series dynamics and an approximate summary of non-Gaussian time series dynamics.

2. Assessing Agreement Between Model and Data

Our basic strategy is to assess models by comparing model spectra to data spectra. Our goal is provision of a graphical framework that facilitates visual comparisons of model spectra to interval estimates of data spectra. We compute *model* spectra exactly (either analytically or numerically); thus, they have no sampling uncertainty. Sampling error does, however, affect the sample *data* spectra, which are of course just estimates of true but unknown (population) data spectra. We exploit well-established procedures for estimating spectra, and we propose new bootstrap techniques for assessing the sampling uncertainty of estimated spectra.[4]

2a. Univariate

We focus first on the univariate case, in order to fix ideas and establish notation. We also assume covariance stationarity. In subsequent sections of the paper, we analyze the objects of ultimate interest: trending, multivariate processes. Consider the linearly regular covariance-stationary stochastic process

$$y_t = \mu + B(L)\varepsilon_t = \mu + \sum_{i=-\infty}^{\infty} b_i \varepsilon_{t-i}, \tag{1}$$

where $b_0 = 1$, $\sum_{i=-\infty}^{\infty} b_i^2 < \infty$, and $\varepsilon_t \sim WN(0, \sigma^2)$. The autocovariance function is

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[4] Alternatively, one could fix the data spectrum, and assess sampling error in the model spectrum by simulating repeated realizations from the model. The two approaches are essentially complementary, motivated by the usual "Wald" and "Lagrange multiplier" testing perspectives. See, for example, Gregory and Smith (1991). For our purposes, however, it proves fruitful to view all uncertainty as associated with the data spectrum, because we don't want to condition on any particular model when performing our bootstrap procedures.

$\gamma(\tau) = \sigma^2 \sum_{i=-\infty}^{\infty} b_i b_{i+\tau}$, and the spectral density function is $f(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \gamma(\tau) e^{-i\omega\tau}$, $-\pi < \omega < \pi$.

Given a sample path $\{y_t\}_{t=1}^{T}$, we estimate μ consistently by the sample mean, $\bar{y} = \frac{1}{T} \sum_{t=1}^{T} y_t$. From this point onward, we assume that all sample paths have been centered around this sample mean. We estimate the autocovariance function with $\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=1}^{T-|\tau|} y_t \, y_{t+|\tau|}$, $\tau = 0, \pm 1, \ldots, \pm(T-1)$. This construction ensures that $\hat{\gamma}(\tau)$ is a positive definite function of τ, which we know holds true in population. We estimate the spectral density using the Blackman-Tukey lag-window approach in which the sample spectral density function, $\hat{f}(\omega_j) = \frac{1}{2\pi} \sum_{\tau=-(T-1)}^{T-1} \hat{\gamma}(\tau) e^{-i\omega_j\tau}$ $(\omega_j = \frac{2\pi j}{T}, \; j = 1, \ldots, \frac{T}{2}-1)$ is replaced with one involving the "windowed" sample autocovariance sequence, $f^*(\omega_j) = \frac{1}{2\pi} \sum_{\tau=-(T-1)}^{T-1} \lambda(\tau) \, \hat{\gamma}(\tau) e^{-i\omega_j\tau}$ $(\omega_j = \frac{2\pi j}{T}, \; j = 1, \ldots, \frac{T}{2}-1)$. This results in a consistent estimator if the lag window $\lambda(\tau)$ is adjusted with sample size in such a way that variance and bias are reduced simultaneously.[5]

A key issue for our purposes is how to ascertain the sampling variability of the estimated spectral density function. Asymptotic results have been available for some time, but they may be unreliable guides in samples of the size that concern us. Therefore, we shall consider various bootstrapping algorithms. Our first bootstrap approach is based directly on frequency-domain considerations. It is well-known that in large samples $\frac{2\hat{f}(\omega_j)}{f(\omega_j)} \equiv \varepsilon_j \overset{iid}{\sim} \chi^2_2$, or equivalently, $\hat{f}(\omega_j) = \frac{1}{2} f(\omega_j) \chi^2_2$, $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, \frac{T}{2}-1$. Franke and Härdle (1991) suggest replacing the population spectral density on the right side with a consistent estimator, drawing from the distribution of the ε_j's to form a drawing of

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[5] Alternatively, of course, one may smooth the sample spectral density function directly. The duality between the two approaches, for appropriate window choices, is well known. See Priestley (1981).

$\hat{f}(\omega_j)'s$ which are then smoothed, and repeating many times to build up an approximation to the distribution of the $f^*(\omega_j)'s$.

As always, one may perform either a parametric or a nonparametric bootstrap. The parametric bootstrap proceeds as follows. At bootstrap replication (i), we draw $\{\varepsilon_j^{(i)}\}_{j=1}^{\frac{T}{2}-1}$ from a χ_2^2 distribution and convert them into $\{\hat{f}^{(i)}(\omega_j)\}_{j=1}^{\frac{T}{2}-1}$ via

$$\hat{f}^{(i)}(\omega_j) = \frac{1}{2} f^*(\omega_j)\, \varepsilon_j^{(i)}, \tag{2}$$

which we smooth to obtain $f^{*(i)}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$. At the end we have $f^{*(i)}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, $i = 1, \ldots, R$, where R denotes the number of bootstrap replications performed. Then we form confidence intervals for $f(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, using the percentile method.[6]

The nonparametric bootstrap, on the other hand, is motivated by the fact that the χ_2^2 distribution for ε_j obtains only asymptotically, so that it may be preferable to sample with replacement directly from the empirical distribution of the "observed" $\varepsilon_j^* = \frac{2\hat{f}(\omega_j)}{f^*(\omega_j)}$, $j = 1, \ldots, \frac{T}{2}-1$.[7] We proceed as follows. At bootstrap replication (i) we draw $\{\varepsilon_j^{*(i)}\}_{j=1}^{\frac{T}{2}-1}$ by sampling with replacement from $\{\varepsilon_j^*\}_{j=1}^{\frac{T}{2}-1}$ and convert them into $\{\hat{f}^{(i)}(\omega_j)\}_{j=1}^{\frac{T}{2}-1}$ via $\hat{f}^{(i)}(\omega_j) = \frac{1}{2} f^*(\omega_j)\, \varepsilon_j^{*(i)}$, $j = 1, 2, \ldots, \frac{T}{2}-1$. We then smooth to obtain $f^{*(i)}(\omega_j)$, $j = 1, 2, \ldots, \frac{T}{2}-1$. At the end we have $f^{*(i)}(\omega_j)$, $i = 1, \ldots, R$, from which confidence

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[6] The percentile method proceeds by simply using the empirical percentiles (across bootstrap replications) of the $f^*(\omega_j)$'s. For details and refinements, see Efron and Tibshirani (1993). The method as described here produces intervals with asymptotically correct coverage for any fixed frequency, ω_j. Construction of confidence intervals for the entire spectral density function requires additional refinements, which we discuss later.

[7] Following standard practice, the $\varepsilon_j^{*'}s$ are first rescaled to have mean equal to their known asymptotic population mean, 2, by the transformation $\varepsilon_j^* = 2\varepsilon_j^*/\bar{\varepsilon}^*$.

intervals are formed for $f(\omega_j)$ using the percentile method, $j = 1, 2, \ldots, \frac{T}{2}-1$.

Our second bootstrap approach is more general, and of independent interest, in that it can be used to bootstrap a variety of objects. This contrasts with the Franke-Härdle bootstrap, which works only for the spectral density function. As before, let $y = \{y_t\}_{t=1}^T$ denote a T-period sample path of a covariance-stationary time series with population mean μ and sample mean $\bar{y}$. The population covariance matrix associated with the sample path is Toeplitz; call it Σ, where $\Sigma_{ij} = \gamma(|i-j|)$. By symmetry and positive definiteness of the covariance matrix, we can write $\Sigma = PP'$, where the unique Cholesky factor P is lower triangular. Similarly, the sample covariance matrix associated with the sample path is Toeplitz; call it $\hat{\Sigma}$, where $\hat{\Sigma}_{ij} = \hat{\gamma}(|i-j|)$. By symmetry and positive definiteness of the sample covariance matrix, we can write $\hat{\Sigma} = \hat{P}\hat{P}'$, where the unique Cholesky factor $\hat{P}$ is lower triangular. Now let $\{\lambda_{|i-j|}\}_{|i-j|=0}^{T-1}$ be a set of decreasing weights applied to the successive diagonal elements of $\hat{\Sigma}$, and call the resulting matrix Σ^*. Finally, let P^* be the Cholesky factor of Σ^*.

First consider a parametric bootstrap. We draw $e^{(i)} = \{e_t\}_{t=1}^T \sim N(0, I_T)$, and form[8]

$$y^{(i)} = \bar{y} + P^* e^{(i)} \sim N(\bar{y}, \Sigma^*). \tag{3}$$

Then we compute the estimate $f^{*(i)}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, $i = 1, \ldots, R$, after which we construct confidence intervals.

Alternatively, we may use a nonparametric bootstrap, which proceeds as follows. In

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[8] Following Efron (1982), the e vectors should be centered around their mean to reflect the known population mean of 0. The same argument suggests standardizing them, to ensure that the variance matches the known population variance of 1.

population we have that $y = \mu + P\varepsilon$, where $\varepsilon \sim (0, I)$, so that $\varepsilon = P^{-1}(y - \mu)$. The sample analog is that $y = \bar{y} + P^{*}e$, so that $e = P^{*-1}(y - \bar{y})$. These are the "residuals" from which we bootstrap. We proceed by drawing $e^{\circledast}$ with replacement from e. Then we construct

$$y^{\circledast} = \bar{y} + P^{*}e^{\circledast} \sim (\bar{y}, \Sigma^{*}), \tag{4}$$

from which we compute $f^{*\circledast}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, $i = 1, \ldots, R$, and then we construct confidence intervals.

2b. Multivariate

In the multivariate case, the analysis becomes somewhat more complicated, but our general approach remains intact. Consider the N-variate linearly regular covariance stationary stochastic process,

$$y_t = \mu + B(L)\,\varepsilon_t = \mu + \sum_{i=-\infty}^{\infty} B_i\,\varepsilon_{t-i} \tag{5}$$

$$E(\varepsilon_t\varepsilon_s') = \begin{cases} \Sigma & \text{if } t = s \\ 0 & \text{otherwise,} \end{cases}$$

where $E(\varepsilon_t) = 0$, $B_0 = I$, and the coefficients are square summable (in the matrix sense). The autocovariance function is $\Gamma(\tau) = \sum_{i=-\infty}^{\infty} B_i\,\Sigma\,B_{i-\tau}'$ and the spectral density function is $F(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\Gamma(\tau)\,e^{-i\omega\tau}$, $-\pi < \omega < \pi$.

Consider now a generic off-diagonal element of $F(\omega)$, $f_{kl}(\omega)$. In polar form, the cross-spectral density is $f_{kl}(\omega) = ga_{kl}(\omega)\exp[i\,ph_{kl}(\omega)]$, where $ga_{kl}(\omega) = [re^2(f_{kl}(\omega)) + im^2(f_{kl}(\omega))]^{1/2}$ is the gain or amplitude, and where $ph_{kl}(\omega) = \arctan\{im^2(f_{kl}(\omega)) / re^2(f_{kl}(\omega))\}$ is the phase. As is well known, the gain tells how the amplitude of y_t is multiplied in

contributing to the amplitude of y_k at frequency ω, and phase measures the lead of y_k over y_l at frequency ω. (The phase shift in time units is $ph(\omega)/\omega$.) We shall often find it convenient to examine coherence rather than gain, where the coherence is defined as $coh_{kl}(\omega) = \frac{ga_{kl}^2(\omega)}{f_{kk}(\omega)\, f_{ll}(\omega)}$, which measures the squared correlation between y_k and y_l at frequency ω.

We estimate the Nx1 mean vector with $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)'$. We estimate the autocovariance function with $\hat{\Gamma}(\tau) = [\hat{\gamma}_{kl}(\tau)]$, where $\hat{\gamma}_{kl}(\tau) = \frac{1}{T} \sum_{t=1}^{T-|\tau|} y_{kt} y_{l,t+\tau}$. We estimate the spectral density matrix with $F^*(\omega_j) = \frac{1}{2\pi} \sum_{\tau=-(T-1)}^{(T-1)} \Lambda(\tau)\, \hat{\Gamma}(\tau)\, e^{-i\omega\tau}$, $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, \frac{T}{2}-1$, where $\Lambda(\tau)$ is a matrix of lag windows. This amounts to smoothing the sample spectral density function, $\hat{F}(\omega_j) = \frac{1}{2\pi} \sum_{\tau=-(T-1)}^{(T-1)} \hat{\Gamma}(\tau)\, e^{-i\omega\tau}$, $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, \frac{T}{2}-1$. The sample gain and phase at any frequency ω_j are then obtained by transforming the appropriate elements of $F^*(\omega_j)$.

To approximate the finite-sample distribution of the smoothed spectral density estimator with the bootstrap, we first generalize the Franke-Härdle procedure to the multivariate case. The parametric bootstrap proceeds by making use of the result that $\hat{F}(\omega_j) \xrightarrow{d} W_N^c(1, F(\omega_j))$, an N-dimensional complex Wishart, $j = 1, \ldots, \frac{T}{2}-1$.[9] Thus, in parallel with our univariate discussion, we have that in large samples $F(\omega_j)^{-1/2}\, \hat{F}(\omega_j)\, F(\omega_j)^{-1/2} = \varepsilon_j \overset{iid}{\sim} W_N^c(1, I)$, or equivalently, $\hat{F}(\omega_j) = F(\omega_j)^{1/2}\, W_N^c(1, I)\, F(\omega_j)^{1/2}$, $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, \frac{T}{2}-1$. This suggests drawing a bootstrapped sample spectral density as

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[9] See Brillinger (1981).

$$\hat{F}^{(i)}(\omega_j) = F^*(\omega_j)^{1/2}\, W_N^{c(i)}(1, I)\, F^*(\omega_j)^{1/2}, \quad j = 1, \ldots, \frac{T}{2}-1, \; i = 1, \ldots, R, \tag{6}$$

where $W_N^{c(i)}(1, I)$ is a draw from an N-dimensional complex Wishart distribution, and $F^*(\omega_j)$ is the smoothed estimate of the spectral density function. We then smooth to obtain $F^{*(i)}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, $i = 1, \ldots, R$, from which we compute confidence intervals.

To make the bootstrap nonparametric, we obtain $\{\varepsilon_j^{*(i)}\}_{j=1}^{\frac{T}{2}-1}$ by sampling with replacement from $\{\varepsilon_j^*\}_{j=1}^{\frac{T}{2}-1}$, where $\varepsilon_j^* = F^*(\omega_j)^{-1/2}\, \hat{F}(\omega_j)\, F^*(\omega_j)^{-1/2}$, $j = 1, \ldots, \frac{T}{2}-1$. We then form $\hat{F}^{(i)}(\omega_j) = F^*(\omega_j)^{1/2}\, \varepsilon_j^{*(i)}\, F^*(\omega_j)^{1/2}$, and we smooth to obtain $F^{*(i)}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, $i = 1, \ldots, R$, from which we compute confidence intervals.

The Cholesky factor procedure is readily generalized to the multivariate case. Consider the sample path $\{y_{1t}, \ldots, y_{Nt}\}_{t=1}^{T}$. Let $z_t = (y_{1t}, \ldots, y_{Nt})'$. Then $z = (z_1', z_2', \ldots, z_T')' \sim (1 \otimes \mu, \Sigma)$, where 1 is an N-dimensional column vector of ones, and $\Sigma = \text{Band}(\Gamma(0), \Gamma(1), \ldots, \Gamma(T-1))$. In parallel to the univariate case, we estimate μ by $\bar{y}$ and Σ by $\hat{\Sigma} = \text{Band}(\hat{\Gamma}(0), \hat{\Gamma}(1), \ldots, \hat{\Gamma}(T-1))$, where $\hat{\Gamma}(\tau) = \frac{1}{T} \sum_{t=1}^{T-|\tau|} z_t\, z_{t+|\tau|}'$, $\tau = 0, \pm 1, \ldots, \pm(T-1)$. This construction ensures that $\hat{\Sigma}$ is symmetric and positive semidefinite, enabling the Cholesky factorization $\hat{\Sigma} = PP'$. As before, let $\{\lambda_{|i-j|}\}_{|i-j|=0}^{T-1}$ be a set of decreasing weights applied to the successive (block) diagonal elements of $\hat{\Sigma}$; call the resulting matrix Σ^*, with Cholesky factor P^*.

$$z^{(i)} = \bar{z} + P^* e^{(i)} \sim N(\bar{z}, \Sigma^*) \tag{7}$$

To perform a parametric bootstrap, we draw $e^{(i)} = \{e_t\}_{t=1}^{T} \sim N(0, I_{NT})$, and form where $\bar{z} = 1 \otimes \bar{y}$. Finally, we compute the estimate $F^{*(i)}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, $i = 1, \ldots, R$, after which confidence intervals are formed.

The multivariate nonparametric bootstrap also parallels the univariate case. In population we have that $z = 1 \otimes \mu + P\varepsilon$, $\varepsilon \sim (0, I_{NT})$, so that $\varepsilon = P^{-1}(z - 1 \otimes \mu)$. The sample analog is $z = \bar{z} + P^{*}e$, so that $e = P^{*-1}(z - \bar{z})$. These are the "residuals" from which we bootstrap. We proceed by drawing $e^{(i)}$ with replacement from e. Then we constructct

$$z^{(i)} = \bar{z} + P^{*}e^{(i)} \sim (\bar{z}, \Sigma^{*}), \tag{8}$$

from which we compute $F^{*(i)}(\omega_j)$, $j = 1, \ldots, \frac{T}{2}-1$, $i = 1, \ldots, R$, and then we construct confidence intervals.

2c. Constructing Confidence "Tunnels"

Consider first a univariate spectral density function. If interest centers on only one frequency, we simply use the bootstrap distribution at that frequency to construct the usual bootstrap confidence interval. That is, we find q_T^L, q_T^U such that $P(f^{*(\cdot)}(\omega) \leq q_T^U) = 1-\frac{\alpha}{2}$ and $P(f^{*(\cdot)}(\omega) \geq q_T^L) = 1-\frac{\alpha}{2}$, where $(1-\alpha)$ is the desired confidence level, "L" stands for lower, "U" stands for upper, the "T" subscript indicates that the band is tailored to the finite-sample size, T, and the (.) superscript indicates that the probability is taken under the bootstrap distribution. The $(1-\alpha)$% two-sided confidence interval is $[q_T^L, q_T^U]$.[10]

However, one often wants to assess the sampling variability of the entire spectral density *function* over many frequencies (e.g., business-cycle frequencies, or perhaps all frequencies) to learn about the broad agreement between data and model. One approach is to form the pointwise bootstrap confidence intervals described above, and then to "connect the

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[10] Note that, in general, the confidence interval is not symmetric around the point estimate.

dots." But obviously, a set of $(1-\alpha)\%$ confidence intervals constructed for each of n ordinates will not achieve $(1-\alpha)\%$ joint coverage probability. Rather, the actual confidence level will be closer to $(1-\alpha)^n\%$, which holds exactly if the pointwise intervals are independent. A better approach is to use the Bonferroni method to approximate the desired coverage level, by assigning $(1 - \alpha/n)\%$ coverage to each ordinate.[11] The resulting tunnel has coverage of at least $(1 - \alpha)\%$.[12]

A third approach to confidence tunnel construction is the supremum method of Woodroofe and van Ness (1967) and Swanepoel and van Wyk (1986), which uses an estimate of the (standardized) distribution of

$$\sup_{0<\omega_j<\pi} |f^*(\omega_j) - f(\omega_j)|, \ \omega_j = \frac{2\pi j}{T}, \ j = 1, \ldots, \frac{T}{2}-1.$$

to construct a confidence tunnel for the curve. Specifically,[13]

(1) Calculate $f^{*(\omega_j)}(\omega_j)$, $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, \frac{T}{2}-1$..

(2) Find c such that:

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[11] In the univariate case, typically $n = T/2 - 1$. In the multivariate case, the question arises as to "how wide to cast the net" in forming confidence tunnels. One might view each element of the spectral density matrix in isolation, for example, in which case each of the respective confidence tunnels would use $n = T/2 - 1$. At the other extreme, one could use $n = N^2(T/2-1)$, effectively forming a tunnel for the entire matrix.

[12] Bonferroni tunnels achieve the desired coverage only for (1) independent values of the estimated function across ordinates, which is clearly violated in spectral density estimation as the smoothing required for consistency results in averaging across frequencies, and (2) large n, (because $(1 - \alpha/n)^n \geq (1 - \alpha)$, for any finite n.

[13] This procedure is similar to the one advocated in Gallant, Rossi and Tauchen (1993).

$$P\left[\sup_{0<\omega_j<\pi}\left(\frac{|f^{*(\cdot)}(\omega_j) - f^{*}(\omega_j)|}{\sqrt{2/T}\, f^{*}(\omega_j)}\right) \leq c\right] = 1-\alpha,$$

where the probability is evaluated with respect to the bootstrap distribution.

(3) Construct the confidence band, $f^{*}(\omega_j) \pm c\sqrt{2/T}\, f^{*}(\omega_j)$,

$$\omega_j = \frac{2\pi j}{T}, j = 1, \ldots, \frac{T}{2}-1.$$

In the multivariate case, one can bootstrap coherence and phase in identical fashion.

Unlike the Bonferroni tunnels, the supremum tunnels attain asymptotically correct coverage rates even with statistical dependence among ordinates. Little is known, however, about the comparative finite-sample performance of the Bonferroni and supremum tunnels, and the supremum tunnels may require very large samples for accurate coverage.[14]

3. Estimation: Maximizing Agreement Between Model and Data

Now we consider estimation, together with the related issues of goodness-of-fit and hypothesis testing.

3a. Univariate

Estimation requires a loss function, or goodness-of-fit measure, for assessing closeness between model and data. Many loss functions may be entertained; the particular loss function adopted reflects the user's preferences. In most cases it would seem that a function of the form

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[14] See Hannan (1970), p. 294.

$$C_{gw}(\theta) = \int_0^{\pi} g(f_m(\omega;\ \theta),\ f^*(\omega))\ w(\omega)\ d\omega \tag{9}$$

will be adequate. The function g measures the divergence between $f_m(\omega;\ \theta)$ (model spectrum) and $f^*(\omega)$ (estimate of data spectrum).[15] This divergence is weighted across frequencies by the function $w(\omega)$. In practice, the integral is replaced by a sum over frequencies $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, \frac{T}{2}-1$. Quadratic loss with uniform weighting over all frequencies, for example, corresponds to $g(a, b) = (a - b)^2$ and $w(\omega) = 1$, yielding $C_{gw}(\theta) = \sum_j (f_m(\omega_j;\ \theta) - f^*(\omega_j))^2$.

The goodness-of-fit measure may readily be transformed into an estimation criterion by taking

$$\hat{\theta}_{gw} = \underset{\theta}{\text{argmin}}\ C_{gw}(\theta). \tag{10}$$

ML, for example, is (asymptotically) of this form, for a particular and potentially restrictive choice of g, f^*, and w; it is

$$\underset{\theta}{\text{argmax}} \left[-\frac{1}{2} \sum_j \ln f_m(\omega_j;\ \theta) - \frac{1}{2} \sum_j \frac{\hat{f}(\omega_j)}{f_m(\omega_j;\ \theta)} \right]. \tag{11}$$

We also use bootstrap procedures to compute standard errors and interval estimates for parameters of interest, and to test hypotheses about the elements of $\hat{\theta}_{gw}$. We proceed as follows:

(1) At bootstrap replication (i), draw a bootstrap sample of size T using the Cholesky

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[15] Note that the model spectrum is either computable analytically or numerically to any desired degree of accuracy. The data spectrum, on the other hand, is consistently estimable.

factor algorithm.

(2) Numerically minimize $C_{gw}^{@}(\theta)$ to get $\hat{\theta}_{gw}^{@}$.

(3) Repeat R times.

(4) Compute standard errors, form interval estimates, implement bias corrections, or test hypotheses using the distribution of $\hat{\theta}_{gw}^{@}$, i = 1, ..., R.

Note that, unlike typical implementations of the bootstrap, ours *does not* involve conditioning on the model; instead, the bootstrap samples are generated directly from the autocovariance matrix of the data. This is important in our environment, in which *all* models are best regarded as false.

3b. Multivariate

The multivariate analog of our earlier loss function is

$$C_{ow}(\theta) = \int_0^{\pi} G\left(F_m(\omega;\, \theta),\, F^*(\omega)\right) \odot W(\omega)\, d\omega, \tag{12}$$

where $\odot$ denotes component-by-component multiplication. The multivariate analog of our earlier univariate quadratic loss function, for example, is

$C_{ow}(\theta) = \sum_j tr\left(D'(\omega_j;\, \theta)\, D(\omega_j;\, \theta)\right)$, where $D(\omega_j;\, \theta) = F_m(\omega_j;\, \theta) - F^*(\omega_j)$, $\omega_j = \frac{2\pi j}{T}$, $j = 1, \ldots, \frac{T}{2}-1$.[16]

The estimation criterion function has the same form as in the univariate case,

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[16] Other matrix norms are of course possible, such as $\sup_l \left(\sum_k |D_{kl}(\omega_j;\, \theta)|\right)$, which has a minimax flavor.

$$\hat{\theta}_{GW} = \underset{\theta}{\operatorname{argmin}}\ C_{GW}(\theta). \tag{13}$$

It is worth emphasizing how all parts of the spectrum contribute to loss in the multivariate case. Consider, for example, a bivariate model (variables x and y) under quadratic loss. Then

$$D(\omega_j;\ \theta) = \begin{bmatrix} d_{xx}(\omega_j;\ \theta) & d_{xy}(\omega_j;\ \theta) \\ d_{yx}(\omega_j;\ \theta) & d_{yy}(\omega_j;\ \theta) \end{bmatrix},$$

where

$$d_{xx}(\omega_j;\ \theta) = f_{xx,m}(\omega_j;\ \theta) - f_{xx}^{*}(\omega_j)$$

$$d_{yy}(\omega_j;\ \theta) = f_{yy,m}(\omega_j;\ \theta) - f_{yy}^{*}(\omega_j)$$

$$d_{xy}(\omega_j;\ \theta) = f_{xy,m}(\omega_j;\ \theta) - f_{xy}^{*}(\omega_j)$$

$$d_{yx}(\omega_j;\ \theta) = f_{yx,m}(\omega_j;\ \theta) - f_{yx}^{*}(\omega_j) = \overline{f_{xy,m}(\omega_j;\ \theta)} - \overline{f_{xy}^{*}(\omega_j)} = \overline{d_{xy}}(\omega_j;\ \theta).$$

Thus,

$$\operatorname{tr}(D'(\omega_j;\ \theta)D(\omega_j;\ \theta)) = [d_{xx}^2(\omega_j;\ \theta) + d_{xy}(\omega_j;\ \theta)d_{yx}(\omega_j;\ \theta)] + [d_{yy}^2(\omega_j;\ \theta) + d_{xy}(\omega_j;\ \theta)d_{yx}(\omega_j;\ \theta)]$$

$$= d_{xx}^2(\omega_j;\ \theta) + 2|d_{xy}(\omega_j;\ \theta)|^2 + d_{yy}^2(\omega_j;\ \theta)$$

$$= [f_{xx,m}(\omega_j;\ \theta) - f_{xx}^{*}(\omega_j)]^2 + 2[\operatorname{re}(f_{xy,m}(\omega_j;\ \theta)) - \operatorname{re}(f_{xy}^{*}(\omega_j))]^2$$

$$+ 2[\operatorname{im}(f_{xy,m}(\omega_j;\ \theta)) - \operatorname{im}(f_{xy}^{*}(\omega_j))]^2 + [f_{yy,m}(\omega_j;\ \theta) - f_{yy}^{*}(\omega_j)]^2.$$

This expression shows clearly how the goodness of fit of both univariate spectra, as well as

both the real and imaginary parts of the cross spectrum, contribute to loss.

The multivariate ML estimator is

$$\underset{\theta}{\text{argmax}} \left[-\frac{1}{2}\sum_j \ln|F_m(\omega_j;\ \theta)| \ - \ \frac{1}{2}\text{tr}\sum_j F_m^{-1}(\omega_j;\ \theta)\ \hat{F}(\omega_j) \right]. \tag{14}$$

Some economic models, such as stylized real business cycle models in which one shock drives the evolution of a higher-dimensional system, have a singular spectral density F_m. This presents a problem for implementation of the ML estimator, which involves F_m^{-1}. We have found that this problem can be satisfactorily skirted in practice, because the model spectrum is typically obtained by simulating a long realization from the model, whose spectrum is then consistently estimated. This has the effect of introducing just enough "measurement error" to avoid a singular spectrum.[17]

The bootstrap approaches to computing standard errors, confidence intervals, and hypothesis testing parallel the univariate case precisely.

3c. On Detrending[18]

If loss functions involving *ratios* of data spectra to model spectra are used, and if the data and the model are first passed through identical linear filters, then any results obtained by comparing filtered data spectra to filtered model spectra will match those obtained from *unfiltered* data spectra and model spectra. In light of the common practice in

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[17] Note that our "measurement error" is just a numerical device to facilitate matrix inversion, in contrast to the approaches of Altug (1989) and Hansen and Sargent (1990), in which measurement error plays a key role.

[18] A 1993 conversation with Tim Cogley influenced the development of this section, which was written contemporaneously and independently of Cogley and Nason (1995), but which was nonetheless influenced by their ideas.

macroeconomics of subjecting both the data and the model to identical detrending filters, this result may be of interest, because it indicates that the contentious issue of detrending may be less important than commonly believed. Instead, it may be unnecessary to detrend at all.

To see this, consider first the univariate case. Consider filtered "model data," and filtered "real data," $y_{tf}^{m} = H(L)A^{m}(L)\varepsilon_t$ and $y_{tf}^{d} = H(L)A^{d}(L)\nu_t$. Immediately, $f_{y_t}^{m}(\omega_j) = \frac{\sigma_\varepsilon^2}{2\pi}|H(e^{-i\omega_j})|^2|A^{m}(e^{-i\omega_j})|^2$ and $f_{y_t}^{d}(\omega_j) = \frac{\sigma_\nu^2}{2\pi}|H(e^{-i\omega_j})|^2|A^{d}(e^{-i\omega_j})|^2$. Thus, the ratio of filtered model spectrum to filtered data spectrum is the same as the ratio of the (pseudo) spectra of the original series. This suggests the attractiveness of loss functions involving only the ratio of model spectrum to data spectrum,

$$C_{gw}(\theta) = \sum_j g\left[\frac{f_m(\omega_j;\ \theta)}{f_d^{*}(\omega_j)}\right] w(\omega_j), \tag{15}$$

so as to achieve invariance to H(L). Our earlier quadratic loss function, for example, achieves invariance if recast in terms of *log* differences,

$$C_{gw}(\theta) = \sum_j \left(\ln f_m(\omega_j;\ \theta) - \ln f_d^{*}(\omega_j)\right)^2.$$

Similar results hold for the multivariate case. In fact, they are even stronger, in the sense that coherences and phases are always invariant to linear filtering, quite apart from the particular loss function adopted. First consider an arbitrary pairwise coherence of filtered series, $coh_{x_t y_t}(\omega_j) = \frac{|f_{x_t y_t}(\omega_j)|^2}{f_{x_t}(\omega_j)\ f_{y_t}(\omega_j)}$. Making use of the fact that $f_{x_t y_t}(\omega_j) = |H(e^{-i\omega_j})|^2 f_{xy}(\omega_j)$, rewrite the coherence of the filtered series as

$$coh_{x_t y_t}(\omega_j) = \frac{|\ |H(e^{-i\omega_j})|^2\ f_{xy}(\omega_j)\ |^2}{|H(e^{-i\omega_j})|^2\ f_x(\omega_j)\quad |H(e^{-i\omega_j})|^2\ f_y(\omega_j)} = \frac{|f_{xy}(\omega_j)|^2}{f_x(\omega_j)\ f_y(\omega_j)},$$

which of course is just the coherence of the unfiltered series.

Next, consider an arbitrary pairwise phase of filtered series, $ph_{x,y_f}(\omega_j) = \tan^{-1}\left[\frac{im(f_{x,y_f}(\omega_j))}{re(f_{x,y_f}(\omega_j))}\right]$. The cross spectrum is $re(f_{x,y_f}(\omega_j)) + i\ im(f_{x,y_f}(\omega_j))$, or $re(f_{x,y_f}(\omega_j)) + i\ im(f_{x,y_f}(\omega_j)) = |H(e^{-i\omega_j})|^2\ re(f_{xy}(\omega_j)) + i|H(e^{-i\omega_j})|^2\ im(f_{xy}(\omega_j))$. Thus, $ph_{x,y_f}(\omega_j) = \tan^{-1}\left[\frac{|H(e^{-i\omega_j})|^2\ im(f_{xy}(\omega_j))}{|H(e^{-i\omega_j})|^2\ re(f_{xy}(\omega_j))}\right] = \tan^{-1}\left[\frac{im(f_{xy}(\omega_j))}{re(f_{xy}(\omega_j))}\right]$, which is the phase of the unfiltered series.

In closing this subsection, we note that although we think the results reported here are of interest, we don't regard them as resolving the many difficult issues associated with trends and trend removal in economic time series. First, as we have emphasized repeatedly, a loss function should be chosen that reflects the investigator's goals, regardless of whether it is "convenient." Second, since *model data* are usually stationary time series, a user may reasonably decide not to filter them. In contrast, the *real data* clearly display trend, so that detrending the real data may be desirable. Typically, there is no one "best" way to detrend, and one may want to check robustness of results to alternative detrending procedures.

4. Application: The U.S. Cattle Cycle

A cycle of roughly ten years in U.S. cattle consumption and stock ("the cattle cycle") is well-known among agricultural economists. In this section, we provide a detailed illustration of the use of our assessment and estimation techniques by applying them to an important model of the cattle cycle developed by Rosen, Murphy, and Scheinkman (RMS, 1994). The cattle cycle is of intrinsic interest, and it also provides a convenient vehicle for illustrating our framework.

4a. The Data

The data are annual U.S. cattle consumption and stock, 1900-1990.[19] The series are plotted in Figures 1 and 2, in which the cycle is visually apparent. Moreover, the series are clearly trending. The trends may be removed in a variety of ways, and it is not clear how "best" to remove the trends, or, in light of our earlier discussion, whether the trends need be removed at all. Also in Figures 1 and 2, we superimpose three estimated trends: linear, kinked-linear, and Hodrick-Prescott. Linear trend is widely-used, kinked-linear trend is used by RMS and allows for some nonlinearity in the trend shape, and Hodrick-Prescott trend is popular and allows a great deal of nonlinearity in the trend shape.[20] Clearly, the three procedures produce very different trend estimates.

We make use--here and throughout--of a matrix graphic with univariate spectra plotted on the main diagonal, coherence in the upper-right corner, and phase in the lower-left corner. This estimated data spectrum is presented in Figure 3. Not all frequencies are of equal interest, however. The frequencies most relevant to an investigation of the cattle cycle, typically thought to have a period of roughly ten years, are not those in the entire $[0, \pi]$ range, but rather those in a *subset* that excludes very low and very high frequencies. This presents no problem for our procedures and in fact provides a good opportunity to illustrate the ease with which they can be tailored to specific applications. Thus, for much of our

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[19] The data were kindly supplied by Sherwin Rosen and were originally obtained from *Historical Statistics: Colonial Times to 1970* and *Agricultural Statistics*, published by the U.S.D.A.

[20] The kink in the kinked-linear trend is in 1930, following RMS. The smoothing parameter for the Hodrick-Prescott trend is 400, following Englund, Persson, and Swensson (1992).

analysis, we concentrate on the frequency band corresponding to periods of 34 years to 4 years, indicated by the vertical dashed lines in Figure 3. Interestingly, it turns out that the various detrending procedures have very little effect on the estimated spectra within that band.[21] Thus, although we have performed the analysis for all three detrending methods (as well as on non-detrended data), we report results almost exclusively for kinked-linear detrending.

Two features of the point estimates of the data spectra stand out. First, consumption (and to a lesser extent, stock) clearly displays a power concentration at roughly a ten-year cycle. Second, each spectrum otherwise has Granger's (1966) "typical spectral shape," with high power at low frequencies, and declining power throughout the frequency range.

As for the point estimates of the cross spectrum, the coherence between consumption and stock is generally high and varies across frequencies, ranging from a maximum of about .95 (at roughly a ten-year cycle), to a minimum of about .60 (at roughly a three-year cycle). The phase indicates that, for periods in the band of interest, consumption consistently leads stock.[22] At the ten-year cycle, the phase lead is roughly one year.

Figure 4 presents the data spectra, coherence and phase shift, along with 90% confidence tunnels computed using the Bonferroni technique in conjunction with the Cholesky-factor bootstrap. To facilitate evaluation, the spectra are plotted on a logarithmic scale. All of the point estimates are subject to substantial uncertainty, as manifest in the

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[21] The different detrending procedures do, of course, have different effects on behavior at the excluded very low frequencies.

[22] Phase shift is measured in years by which consumption leads stock.

90% confidence tunnels. This is typical of economic time series, although it often goes unacknowledged.

4b. The Model

We begin with some accounting identities. The head count of all animals (y_t) is the sum of the adult breeding stock (x_t), the stock of calves (assumed equal to gx_{t-1}), and the stock of yearlings (assumed equal to gx_{t-2}), where g is a fertility parameter. That is,

$$y_t = x_t + gx_{t-1} + gx_{t-2}.$$

The adult breeding stock consists of surviving stock from the previous period (assumed equal to $(1-\delta)x_{t-1}$) and the yearlings from t-1 entering the adult herd (gx_{t-3}) less the number that are marketed (c_t),

$$x_t = (1-\delta)x_{t-1} + gx_{t-3} - c_t.$$

We are concerned with the equilibrium determination of c_t and y_t. The risk-neutral rancher maximizes the present discounted value of expected profits, which involves equating the expected marginal benefit of marketing an animal for consumption to the expected marginal benefit of holding the animal for breeding. First, suppose that the rancher markets the animal for consumption. He receives net revenue $q_t = p_t - m_t$, where p_t is price and m_t is finishing cost. Alternatively, suppose the rancher holds an animal for breeding. Expected discounted net revenue is the sum of expected discounted revenue from selling tomorrow plus expected discounted revenue from marketing its offspring, less expected total holding costs (z_t), $E_t[\beta(1-\delta)q_{t+1} + \beta^3 g q_{t+3} - z_t]$. Total holding costs are equal to the sum of time t holding costs (h_t), discounted holding costs of the resultant time t+1 calves, and discounted holding

costs of the resultant time t+2 yearlings. That is, $z_t = h_t + \beta g \gamma_0 h_{t+1} + \beta^2 g \gamma_1 h_{t+2}$ (assuming proportional costs for calves and yearlings, γ_0 and γ_1).

In equilibrium, the expected marginal net revenue from marketing for consumption equals the expected marginal discounted net revenue from holding for breeding; that is,

$$E_t[q_t] = E_t[\beta(1-\delta)q_{t+1} + \beta^3 g q_{t+3} - z_t].$$

The model is closed by specifying the exogenous processes $\{m_t, h_t, d_t\}$ as first-order autoregressions.[23] Following RMS, we assume that each of the three shocks has common serial-correlation parameter ρ.

The model structure implies that the reduced-form equations for c_t and y_t can be expressed in terms of a single disturbance, ω_t, which is a linear combination of the independent innovations from the three AR(1) driving processes. In particular, $c_t \sim$ ARMA(2,1) and $y_t \sim$ ARMA(4,2):

$$(1-\lambda_1 L)(1-\rho L)\, c_t = -(1 - \phi_1 L)\, \omega_t$$

$$(1-\lambda_1 L)(1-\phi_2 L)(1-\phi_3 L)(1-\rho L)\, y_t = (1 + gL + gL^2)\, \omega_t,$$

where ϕ_1 is the one unstable root and ϕ_2 and ϕ_3 are the two stable roots of

$$\phi^3 - (1-\delta)\phi^2 - g = 0,$$

and λ_1 is the one stable root of

The associated univariate spectra are

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[23] d_t is a preference shock. We have not discussed the demand side of the model, because it is not used in estimation.

$$g\beta^3\lambda^3 + (1-\delta)\beta\lambda - 1 = 0.$$

$$f_\varepsilon(\omega) = \sigma_u^2 \frac{|(1 - \phi_1 e^{i\omega})|^2}{|(1 - \lambda_1 e^{i\omega})(1 - \rho e^{i\omega})|^2}$$

$$f_y(\omega) = \sigma_u^2 \frac{|(1 + ge^{i\omega} + ge^{2i\omega})|^2}{|(1 - \lambda_1 e^{i\omega})(1 - \phi_2 e^{i\omega})(1 - \phi_3 e^{i\omega})(1 - \rho e^{i\omega})|^2},$$

and the cross spectrum is

$$f_{cy}(\omega) = \frac{-(1 - \phi_1 e^{i\omega})(1 - \phi_2 e^{i\omega})(1 - \phi_3 e^{i\omega})}{(1 + ge^{i\omega} + ge^{2i\omega})} f_y(\omega).$$

These equations provide a full description of the model in the frequency domain. σ_u^2 is a complicated function of the structural parameters, including some from the demand side of the model. All of the parameters of present interest, however, may be identified from the other reduced-form parameters, with the exception of γ_0 and γ_1. We therefore treat σ_u^2 as a free parameter and estimate it subject to no restrictions.

4c. Assessing, Estimating, and Testing the Model

We first compare the data spectrum to the model spectrum evaluated at the band-restricted maximum likelihood parameter estimates (Band-ML). Band-ML estimates are obtained by using a loss function that coincides with ML, but with the important difference that periods of more than 34 years or less than 4 years are excluded. Figure 5 displays the model spectra, coherence, and phase for the Band-ML parameter configurations. Interestingly, neither the consumption nor the stock model spectrum has a peak corresponding to a ten-year cycle. Similarly, model phase shift fails to peak at a ten-year cycle. The model coherence reminds us of an additional limitation of the model. As

presently specified with a single shock, the model is singular, resulting in unit coherence at all frequencies, regardless of the parameter configuration. Although the data coherence is estimated very imprecisely, it seems unnecessarily restrictive to force the model coherence to be unity at all frequencies.

Figure 6 displays the same information in log form and adds the earlier-discussed 90% confidence tunnels. The diagonal elements provide comparative assessments of model and data univariate dynamics, and the off-diagonal elements provide comparative assessments of cross-variable dynamics. The model spectrum lies in the 90% confidence tunnels for the data spectrum at almost all frequencies.

The Band-ML estimation results appear in Table 1; several features are noteworthy. First, the estimated parameters are remarkably similar across detrending methods. The only exception is ρ, the serial correlation parameter. The variation in the estimates of ρ occurs because the three detrending procedures differ in terms of how much low-frequency variation is removed from the data. For example, removal of a linear trend leaves considerable power at low frequencies, which results in a relatively high value of ρ (.72). The HP procedure, on the other hand, removes more power at lower frequencies and accordingly yields a lower estimate of ρ (.20). The kinked-linear detrending procedure is intermediate and results in an intermediate value of ρ (.52).

Second, the estimates of the fertility rate g and the death rate δ accord with the biological considerations discussed by RMS, but the estimate of the discount factor β

appears low.[24] This conveys information as to the workings and possible limitations of the model.[25] The low estimated discount factor may perhaps be interpreted as follows. The model requires that $\beta = \frac{1}{\lambda_1 \phi_1}$, but capturing the persistence in the dynamics, which is a key feature of the data, requires that λ_1 and ϕ_1 be large. (Recall that λ_1 and ϕ_1 are the coefficients in the ARMA(1,1) representation for consumption.) This works to produce a small estimated discount factor.

Third, the RMS values for g, δ and ρ are close to our point estimates.[26] Our methods let us go farther, however, and assess the uncertainty associated with the estimates. Standard errors are of some use in that regard; we compute them using 200 replications of the Cholesky factor bootstrap procedure, and we report them in parentheses below the estimated parameters. Moreover, our bootstrap procedures allow us to examine the entire sampling distribution of the estimated parameters; they are shown in Figure 7. It is evident that all of the sampling distributions are highly non-Gaussian. Those of the discount and death rates are asymmetric, while that of the fertility parameter appears roughly uniform. Interestingly, the sampling distribution of the persistence parameter appears bimodal, with one mode at the point estimate and one much closer to unity. This may be indicative of multiple local

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[24] Others have also had trouble estimating β in the RMS model. Hansen, McGrattan and Sargent (1994), for example, fail to obtain convergence unless β is held fixed at a prespecified value.

[25] To gain more insight into this finding, we imposed a higher value for the discount factor (.91); and left the other parameters free. This specification led to an optimized loss function that was considerably larger than the original optimized loss function, and it resulted in implausible estimates for fertility and depreciation rates.

[26] The variance of the exogenous shock σ_ω^2 is not reported in the table, because it is not reported by RMS.

minima of the loss function in that dimension, and if so, the bootstrap would be performing a useful service by highlighting it. This is clearly an interesting direction for future research.

Finally, we can examine the joint distribution of the estimated parameters. Table 2 presents the correlations between the estimated parameters. Perhaps the most interesting relationship is the strong negative correlation between the discount factor and the fertility rate. This is because the discount factor and the fertility rate enter multiplicatively in one of the cubic equations that define the ARMA polynomials, so that the loss function trades high fertility rates for low discount factors.

5. Concluding Remarks

We have tried to develop a flexible framework that will be useful to a variety of applied economists--a constructive framework that provides a comprehensive comparison of model and data, while nevertheless taking seriously the user's preferences, model misspecification, and small sample sizes. Such a framework can facilitate communication between researchers with potentially very different research objectives and strategies. As economists use richer and more complicated models to understand a wider variety of data, the procedures discussed in this paper can help significantly in understanding the dimensions along which models are consistent (and inconsistent) with data. The information provided can in turn be used to construct new and improved models. We hope the framework will help bring modern dynamic economic theory into closer and more frequent contact with data.

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Table 1
Parameter Estimates
Band-Restricted Maximum Likelihood

	β	g	δ	ρ	σ_u^2
Linear Detrending	.72	.80	.07	.72	1.14
	(.06)	(.24)	(.05)	(.14)	(1.31)
Kinked-Linear Detrending	.72	.79	.07	.52	1.15
	(.06)	(.23)	(.05)	(.27)	(1.29)
Hodrick-Prescott Detrending	.73	.77	.09	.20	1.37
	(.06)	(.23)	(.08)	(.23)	(1.26)
Rosen-Murphy-Scheinkman	.909	.85	.10	.60	
	(NA)	(NA)	(NA)	(NA)	

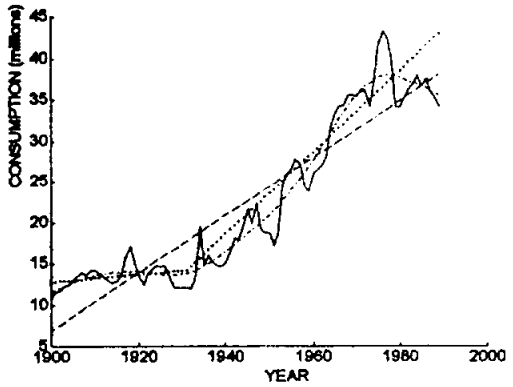
Notes to Table: β = discount factor, g = fertility rate, δ = death rate, ρ = serial correlation coefficient. The frequency band used for estimation corresponds to periods from 34 to 4 years. Standard errors, based on 200 bootstrap replications, appear in parentheses. The Rosen-Murphy-Scheinkman parameters are included for comparison. (They have no standard errors, because they were not estimated.)

Table 2
Estimated Parameter Correlations
Band-Restricted Maximum Likelihood

	β	g	δ	ρ	σ
β	1.00	-.92	.07	.02	.25
g	-.92	1.00	.25	.11	-.17
δ	.07	.25	1.00	.25	.11
ρ	.02	.11	.25	1.00	.13
σ	.25	-.17	.11	.13	1.00

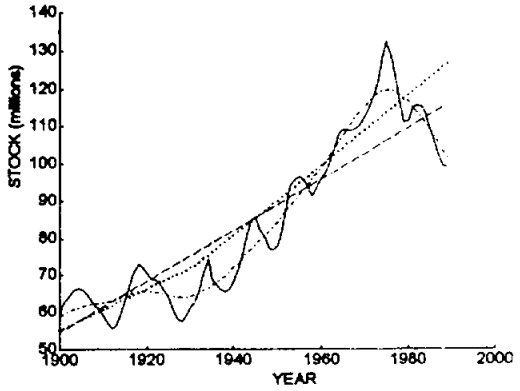
Notes to Table: β = discount factor, g = fertility rate, δ = death rate, ρ = serial correlation coefficient. Estimated parameter correlations are based on 200 bootstrap replications. The frequency band used for estimation corresponds to periods from 34 to 4 years.

Figure 1
U.S. Cattle Consumption, 1900-1990
Actual and Three Estimated Trends



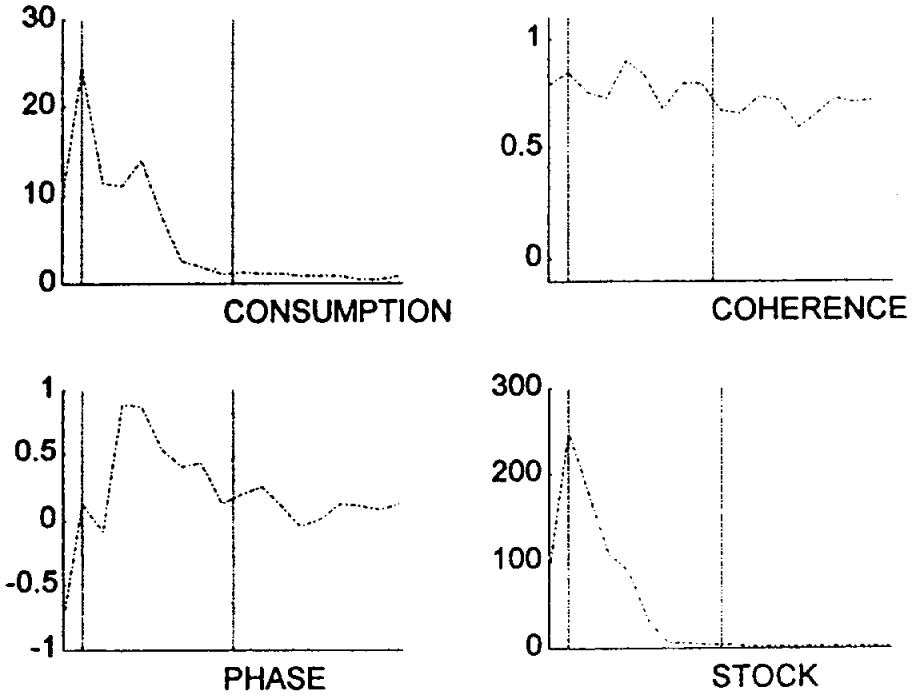
Notes to Figure: We show cattle consumption with a solid line. We superimpose three estimated trends: linear (dash), kinked-linear (dot), and Hodrick-Prescott (dash-dot).

Figure 2
U.S Total Stock of Cattle, 1900-1990
Actual and Three Estimated Trends



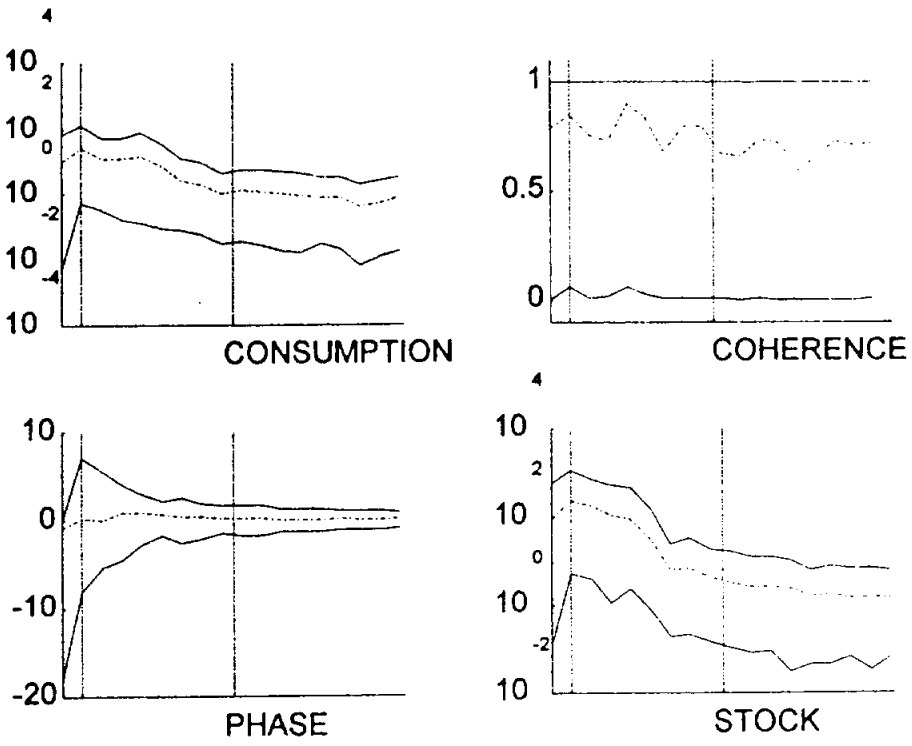
Notes to Figure: We show cattle stock with a solid line. We superimpose three estimated trends: linear (dash), kinked-linear (dot), and Hodrick-Prescott (dash-dot).

Figure 3
Estimated Spectral Density Matrix
U.S. Cattle Consumption and Stock



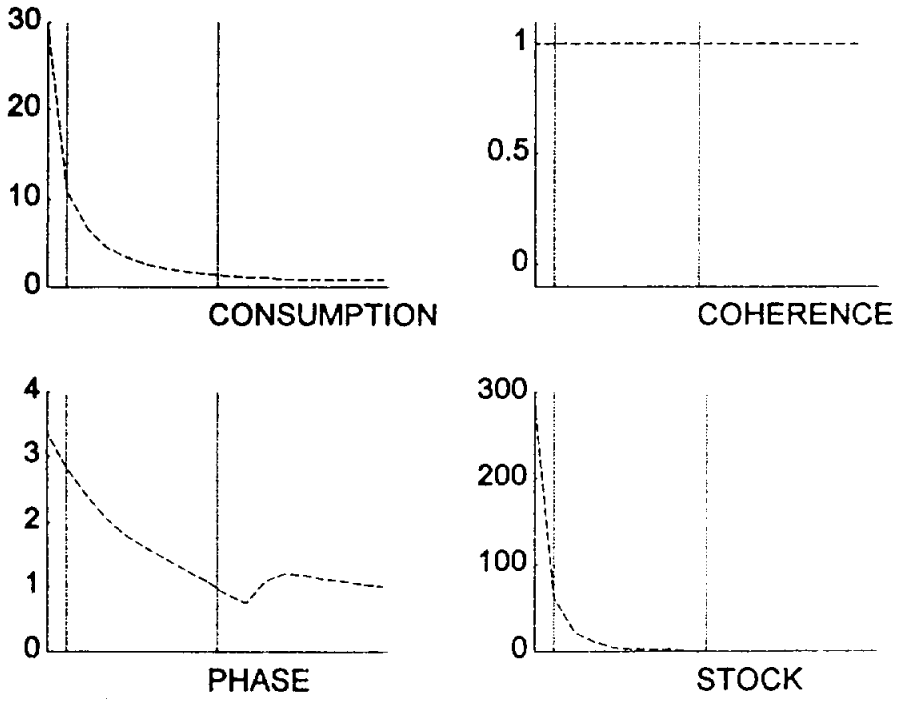
Notes to Figure: We detrend using the kinked-linear method. We show the point estimate of each element of the spectral density matrix, obtained by smoothing the periodogram. The frequency band indicated by vertical dashed lines corresponds to cycles with periods of 34-4 years and is the band of primary relevance for studying cattle cycles. See text for details.

Figure 4
Estimated (Log) Spectral Density Matrix and Confidence Tunnels
U.S. Cattle Consumption and Stock



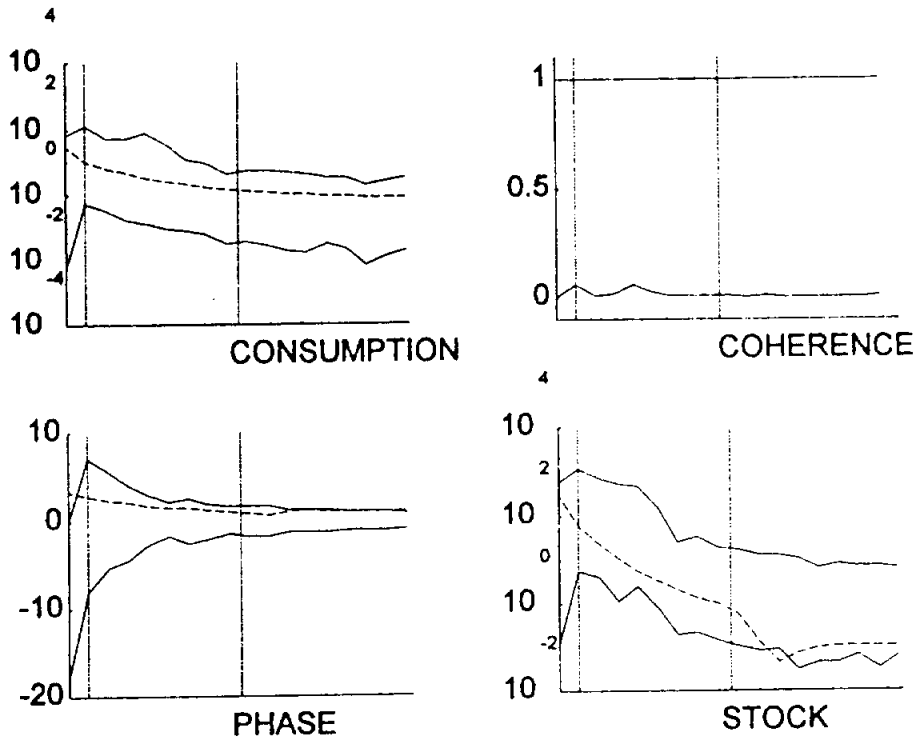
Notes to Figure: We detrend using the kinked-linear method. We show the point estimate together with a 90% confidence tunnel for each element of the spectral density matrix. We obtain the point estimates by smoothing the periodogram. We construct confidence tunnels by the Bonferroni technique in conjunction with the bootstrap. The frequency band indicated by vertical dashed lines corresponds to cycles with periods of 34-4 years and is the band of primary relevance for studying cattle cycles. See text for details.

Figure 5
Model Spectrum Evaluated at Band-ML Estimates
U.S. Cattle Consumption and Stock



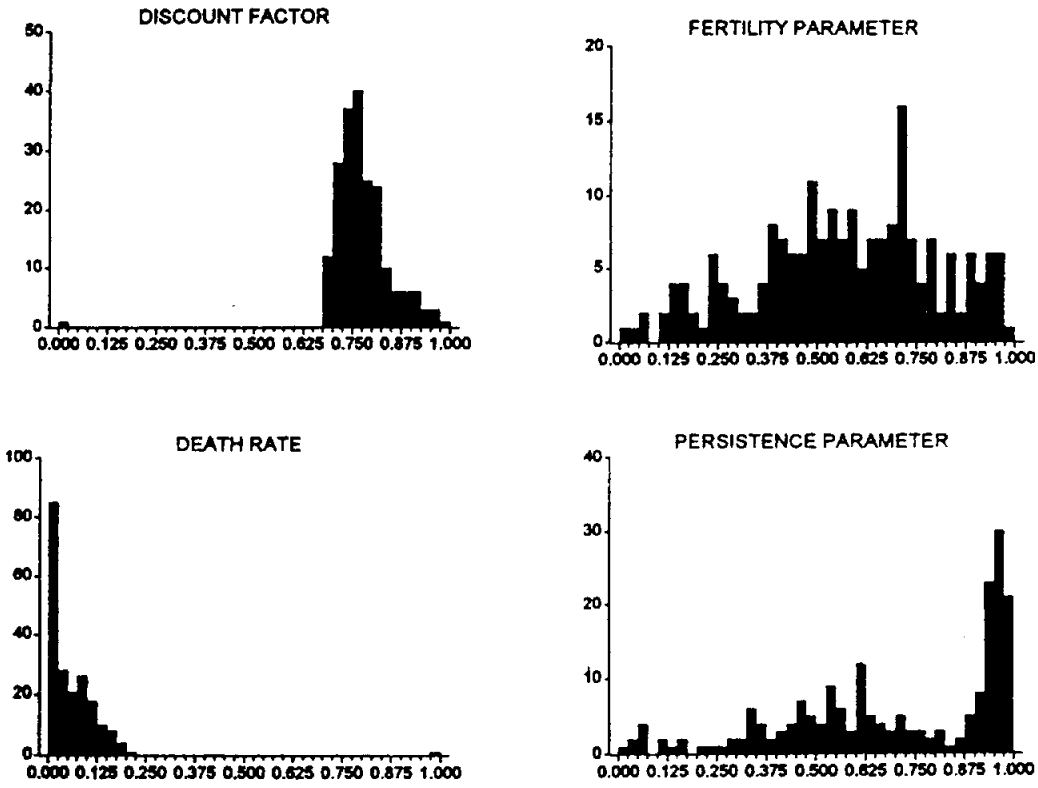
Notes to Figure: We show the model spectrum evaluated at the band-restricted maximum likelihood parameter values, for each element of the spectral density matrix.

Figure 6
(Log) Model Spectra, and Data Spectra Confidence Tunnels
U.S. Cattle Consumption and Stock



Notes to Figure: We show the 90% confidence tunnel for the data spectrum, together with the model spectrum evaluated at the band-restricted maximum likelihood parameter values, for each element of the spectral density matrix.

Figure 7
Sampling Distributions of Parameter Estimates



Notes to Figure: Sampling distributions are based on 100 bootstrap replications. We detrend using the kinked-linear method.