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ABSTRACT

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Long-Memory Inflation Uncertainty: Evidence from the Term Structure of Interest Rates*

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Abstract

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1 Introduction

The term structure of interest rates links the academic fields of macroeconomics and finance. Depending on one's point of view, the level and slope of the yield curve are indicators of the current stance of monetary policy (Bernanke and Blinder 1992), predictors of future movements in real output (Estrella and Hardouvelis 1991), or reflections of the market's assessment of the risk and expected returns of bonds of different maturities (Brennan and Schwartz 1982, Cox, Ingersoll, and Ross 1985, and Vasicek 1977). We continue the tradition of linking finance and macroeconomics by connecting prices of bonds of different maturities to the stochastic process for the

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short-term rate of interest, as commonly done in finance, then going on to speculate about the macroeconomic origins of interest rate movements.

The primary focus of our analysis, though, is not the relation between interest rates and the macroeconomy, but the dynamics of interest rates. Roughly speaking, there are two dimensions to the dynamics of interest rates: the correlation between short rates at different points in time and the relation between yields on bonds of different maturities at the same date. These "time series" and "cross-section" features of interest rates are not the same, but we show that they are closely related in existing theory. For example, in many of the popular theories of bond pricing, the short rate process has the property that the correlation between short rates n periods apart goes to zero exponentially. We show, in this case, that the yield on an n-period bond converges to a constant: the variance of the yield on a long bond goes to zero exponentially, as well. In this way the time-series and cross-section properties of interest rates are closely linked.

The implication that long yields are constant seems to us to be at odds with the data. Although the yield curve generally flattens out as the maturity increases, there is considerable variation in long yields, even for yields on bonds with maturities up to ten years. We attempt to reconcile these two properties using the so-called fractional difference process introduced into economics by Granger and Joyeux (1980). With this process the variability of long yields approaches zero, but at a rate slower than exponential. With plausible parameter values, there is substantial variability in yields for maturities up to 20 years.

We develop these points in the remainder of the paper. In the next section we outline a theoretical framework that retains the simplicity of linearity but is general enough to include long-memory. We derive, for this framework, formulas for prices and yields of bonds of all maturities. In Section 3 we confront the central issue of the paper: the behavior of long yields. We argue that for many common short rate processes, the theoretical properties of long yields and forward rates differ significantly from what we see in US government bond data for the postwar period. This discrepancy between theory and data motivates the fractional difference model of Section 4.

Section 5 is concerned with the ability of the fractional difference model to mimic some of the features of short term interest rates, inflation, and money growth. With the possible exception of money growth, we find that the fractional difference model performs well relative to stationary ARMA or random walk models. Thus the model is able to reproduce important features of both the long end of the yield curve and the high-order autocorrelations of short rates and inflation. We speculate that the fractional short rate cum inflation process might be the result of heterogeneous agents responding to changes in monetary policy.

2 A Theoretical Framework

We begin by deriving prices of riskfree bonds in a log-linear theoretical framework. There are two common approaches to theoretical bond pricing. One approach, epitomized by Campbell (1986) and Hansen and Jagannathan (1991), is to start with an equilibrium price measure, or intertemporal marginal rate of substitution: given a stochastic process for one-period state-contingent claims prices, we construct prices of riskfree bonds of different maturities. A second approach, common in finance, is to start with the short rate: given a stochastic process for the short rate and an assumption about how risk is priced, we derive prices of bonds of longer maturities. Popular examples include the models of Cox, Ingersoll, and Ross (1985) and Vasicek (1977). The two approaches are closely related and, in some cases, equivalent. We follow the second approach in this paper.

To fix the notation, let b_t^n denote the dollar *price* at date t of an n-period discount bond: the claim to one dollar in all states at date $t+n$. By convention $b_t^0 = 1$. The *yield* on a bond of maturity n, for $n > 0$, is

$$y_t^n = -n^{-1} \log b_t^n. \tag{1}$$

The yield on a one-period bond is simply the short rate: $y_t^1 = r_t$. The n-period ahead *forward rate* is implicit in the prices of n- and $(n+1)$-period bonds:

$$f_t^n = \log(b_t^n / b_t^{n+1}). \tag{2}$$

From definitions (1) and (2) it is clear that yields are averages over forward rates:

$$y_t^n = n^{-1} \sum_{j=1}^{n} f_t^{j-1}. \tag{3}$$

Thus we can express the maturity structure of interest rates in two equivalent ways. The conventional yield curve is a plot of y_t^n versus the maturity n. The forward rate curve of f_t^n versus n contains the same information.

These definitions provide different ways of expressing bond prices at a point in time in terms of interest rates. Another way of doing this is to consider returns from holding a bond for one or more periods. We denote the one-period *return* from holding an $(n+1)$-period bond from date t to date $t+1$ by

$$R_{t+1}^{n+1} = \log(b_{t+1}^n / b_t^{n+1}). \tag{4}$$

This definition of the return, like that of the yield, retains the analytic convenience of log linearity.

With these definitions in hand, we can approach the theory of the term structure of interest rates. We follow Vasicek (1977), and others in the finance literature, in deriving prices of multi-period bonds from the stochastic process for the short rate and an assumption about how risk is priced. To make this as simple as possible, let us say that the short rate r is a stationary linear time series with iid normal increments. It can be expressed, then, as the moving average

$$r_t = \mu + \sum_{j=0}^{\infty} \alpha_j \epsilon_{t-j}, \tag{5}$$

for $\{\epsilon_t\}$ normally and independently distributed with mean 0 and variance σ^2. Stationarity requires that the coefficients be square summable: $\sum_{j=0}^{\infty} \alpha_j^2 < \infty$. We will see that this process guarantees positive bond prices but, since the increments are normal with constant variance, it permits interest rates to be negative with positive probability.

Our assumption about risk is that the expected excess return, or *term premium*, on a long bond is proportional to the conditional standard deviation of its return:

$$E_t R_{t+1}^{n+1} - r_t = \lambda (\, Var_t R_{t+1}^{n+1})^{1/2}, \tag{6}$$

where E_t denotes the expectation conditional on the history of bond prices and returns through date t and Var_t the analogous conditional variance. The parameter λ is referred to as the price of risk. Dependence of the term premium on the conditional variance, rather than a covariance, is justified by the one-factor structure of the model: since ϵ is the only shock each period, returns on bonds of all maturities are perfectly correlated. Since b_t^{n+1} is known at date t, we see from (4) that the variance of the return is

$$Var_t\ R_{t+1}^{n+1} = Var_t \log b_{t+1}^n .$$

Using the return relation, equation (6), we can express the price of an $(n+1)$-period bond in terms of moments of the n-period bond price:

$$-\log b_t^{n+1} = -E_t \log b_{t+1}^n + r_t + \lambda(Var_t \log b_{t+1}^n)^{1/2}. \qquad (7)$$

This equation is the basis of our derivation of equilibrium bond prices, and is analogous to the partial differential equation for the bond price in Vasicek's (1977) continuous time theory.

With our log-linear structure we can readily compute prices of bonds of all maturities. We show, by induction, that the log of the price of an n-period bond can be expressed as a moving average:

$$-\log b_t^n = \mu^n + \sum_{j=0}^{\infty} \beta_j^n\ \epsilon_{t-j}, \qquad (8)$$

for some choice of parameters $\{\mu^n, \beta_j^n\}$. For $n = 0$ we have, by convention, $\log b_t^0 = 0$, so

$$\mu^0 = \beta_j^0 = 0, \qquad (9)$$

for $j = 0, 1, \ldots$. Given the pricing function (8) for an n-period bond, for any $n \geq 0$, we use (7) to compute the parameters of the pricing function of an $(n+1)$-period bond. To do this we need the first two conditional moments of the one-period ahead price of an n-period bond:

$$-E_t \log b_{t+1}^n = \mu^n + \sum_{j=0}^{\infty} \beta_{j+1}^n\ \epsilon_{t-j},$$

and

$$Var_t \log b_{t+1}^n = (\beta_0^n \sigma)^2.$$

From these moments and equations (5,7,8), we find that the parameters of the $(n+1)$-period bond price satisfy

$$\mu^{n+1} = \mu^n + (\mu + \lambda\sigma\beta_0^n) \qquad (10)$$

and

$$\beta_j^{n+1} = \beta_{j+1}^n + \alpha_j, \tag{11}$$

for $n = 1, 2, \ldots$ and $j = 0, 1, \ldots$. In short, the log-linear structure makes it fairly easy to derive bond prices.

We summarize the behavior of bond prices in this economy in

Proposition 1 *The price of an n-period bond in this economy has the moving average representation,*

$$-\log b_t^n = \mu^n + \sum_{j=0}^{\infty} \beta_j^n \, \epsilon_{t-j},$$

with coefficients $\{\mu^n, \beta_j^n\}$ *given by the initial conditions (9) and the recursions (10) and (11), for all* $n, j \geq 0$.

We can get some intuition for these formulas by relating the parameters directly to economic fundamentals – the parameters of the short rate process and the price of risk. In doing this, it's useful to define partial sums of moving average coefficients,

$$A_{n+1} = \sum_{j=0}^{n} \alpha_j, \tag{12}$$

with $A_0 = 0$. Note from the recursion (11) and the initial value (9) that $\beta_0^n = A_n$ and, for $n, j \geq 1$,

$$\beta_j^n = \sum_{i=0}^{n-1} \alpha_{j+i} = A_{n+j} - A_j. \tag{13}$$

The intercept in the bond price function is then

$$\mu^n = n\mu + \lambda\sigma \sum_{j=1}^{n} \beta_0^j = n\mu + \lambda\sigma \sum_{j=0}^{n-1} A_j. \tag{14}$$

Thus the behavior of bond prices is governed by the partial sums of moving average coefficients, A_n.

Some of the mathematical structure is more obvious if we look at forward rates, rather bond prices. From Proposition 1 and equation (2), we see that forward rates can be expressed

$$f_t^n = (\mu^{n+1} - \mu^n) + \sum_{j=0}^{\infty}(\beta_j^{n+1} - \beta_j^n)\epsilon_{t-j}.$$

From (10) we find that the intercept in the forward rate function is

$$\mu^{n+1} - \mu^n = \mu + \lambda\sigma A_n$$

and the moving average coefficients are

$$\beta_j^{n+1} - \beta_j^n = A_{n+j+1} - A_{n+j} = \alpha_{n+j}.$$

Thus the forward rate is

$$f_t^n = \mu + \lambda\sigma A_n + \sum_{j=0}^{\infty} \alpha_{n+j}\epsilon_{t-j}. \tag{15}$$

The mean forward rate curve, which we get from setting the innovations equal to their unconditional means of zero, is simply the intercept term. Its behavior, for large n, mimics that of the partial sums A_n. If A_n converges, then so does the mean forward rate. Yields are similar: since they are averages over forward rates [see (3)], we can express them as

$$y_t^n = \mu + (\lambda\sigma/n)\sum_{j=0}^{n-1} A_j + (1/n)\sum_{j=0}^{\infty}(A_{n+j} - A_j)\epsilon_{t-j}. \tag{16}$$

The behavior of yields, like that of forward rates, is governed by the partial sums of moving average coefficients, A_n.

3 The Long End of the Yield Curve

Our interest in this model concerns the long end of the yield curve. Some of the salient properties of bond prices and yields in this economy are evident from two examples, which we use to motivate a more general result.

Example 1: MA(q) short rate. Suppose the short rate is a finite moving average:

$$r_t = \mu + \sum_{j=0}^{q} \alpha_j \epsilon_{t-j}, \tag{17}$$

for some finite integer q. Then the partial sums of moving average coefficients converge to

$$A_{q+1} = \sum_{j=0}^{q} \alpha_j. \tag{18}$$

This implies that the mean forward rate curve and mean yield curve are constant for large maturities n. The mean in the forward rate relation is $\mu + \lambda\sigma A_{q+1}$, for all $n \geq q$. The mean yield curve is

$$Ey_t^n = \mu + n^{-1}(\lambda\sigma) \sum_{j=0}^{n-1} A_j.$$

For large n this converges to $\mu + \lambda\sigma A_{q+1}$.

In this example and others, there is a stronger sense of convergence than that of mean forward rates and yields: the rates themselves converge. Consider forward rates. We have shown that the unconditional mean converges to a constant. We now show that the unconditional variance also goes to zero, thereby establishing that forwards converge to a constant with probability one. The variance of the forward rate is, from its moving average representation,

$$Var\ f_t^n = \sigma^2 \sum_{j=0}^{\infty} \alpha_{n+j}^2. \tag{19}$$

For $n > q$ the moving average coefficients are zero and the variance is therefore zero as well.

Example 2: AR(1) short rate. Suppose the short rate is a first-order autoregression with parameter ρ less than one in absolute value. Then its moving average representation is

$$r_t = \mu + \sum_{j=0}^{\infty} \rho^j \epsilon_{t-j}. \tag{20}$$

The partial sum of moving average coefficients is

$$A_{n+1} = (1 - \rho^{n+1})/(1 - \rho),$$

which approaches $1/(1-\rho)$ as n goes to infinity. Thus the mean forward rate converges to $\mu + \lambda\sigma/(1-\rho)$. The variance of the forward rate is

$$Var f_t^n = \sigma^2 \rho^{2n}/(1 - \rho^2),$$

which goes to zero with n. Thus the forward rate in this economy is effectively constant at long maturities. And since the yield is an average of forward rates, long yields are constant, too.

Both of these examples have the property that long forward rates, and hence yields, converge to a constant. In the MA case this convergence is sudden, in the AR case it is exponential. This convergence does not apply to all bond pricing theories, but it holds in most of the stationary models used in practice, including the popular examples from Cox, Ingersoll, and Ross (1985) and Vasicek (1977). We summarize this property in

Proposition 2 *Suppose the short rate process is stationary ARMA(p,q) for finite p and q. Then long forward rates and yields converge:*

$$f_t^n \rightarrow f^*$$

and

$$y_t^n \rightarrow y^*$$

where f^* and y^* are time-invariant constants.

We will not prove this proposition here, although its content should be clear from the examples. Proofs of related, and more general, propositions are reported in Backus, Gregory, and Zin (1989) and Dybvig, Ingersoll, and Ross (1987). Suffice it to say that the convergence property is not a consequence of the log linear structure we have imposed on the problem.

What we find interesting about this proposition is its apparent contrast with what we see in the data. In Figure 1 we report the mean yield curve for US treasury securities over the 1947-1986 period, based on monthly data from McCulloch (1990). The data suggest that the mean yield curve flattens out at the long end, as the theory suggests. But in Figure 2 we see that the variance of long yields is nowhere near zero: there is still a lot of variability of long yields, even out to ten years. Although the standard deviation of the yield declines somewhat for maturities over a year, the decline is modest: the standard deviation of one-year yields is 3.44 percent, that of ten-year yields 3.24 percent. The variability of long yields is inconsistent with Proposition 2 unless convergence occurs at maturities longer than ten years.

We get a clearer picture of the discrepancy between theory and data if we are willing to choose particular parameter values. In Figures 3 and 4 we plot the first two moments of the theoretical yield curve for an AR(1) short rate (Example 2). In Figure 3 we plot the theoretical mean yield curve for the AR(1) example with parameter values $\rho = 0.62$, $\sigma = 2.69$, and $\lambda = 0.117$. The computations use a period length of one year. The parameters were chosen from an informal moment matching exercise, which we call the Quattro Method of Moments: we varied the parameters until the graph of the theoretical mean yield curve was fairly close to the mean yields in the data. As you can see from the figure, the theory with these parameters matches quite well the observed mean yield curve, represented in the figure by black squares. The slope of the yield curve depends on the product $\lambda\sigma$ but its curvature depends only on ρ: for $\rho = 1$ the mean yield curve is linear, and for smaller values the yield curve gets progressively more concave. In this sense, the curvature of the mean yield curve suggests that ρ must be smaller than one. The data do not speak loudly on this point, but we feel the choice $\rho = 0.62$ is also consistent with estimates of short rate autocorrelations. This value corresponds to an autocorrelation larger than 0.9 at monthly time intervals.

What Figure 3 hides is the variability of yields, which we report in Figure 4. Although the standard deviation of the yield varies little with maturity in the data (again represented by black squares), in the theory it declines more than 70 percent between one and ten years. With these parameter values, at least, the effects of Proposition 2 produce predictions that are wildly at odds with what we see in the data. With larger values of ρ this decline is more gradual, but for any value less than one it occurs at an exponential rate. Thus values of short rate volatility σ chosen to match short rate behavior will generally understate volatility of long yields and prices. This feature of the theory is well understood by financial practitioners, who routinely choose volatility parameters for long bond prices that are larger than those implied by the volatility of short rates.

One way of resolving the discrepancy between theory and evidence is to consider nonstationary processes. We could, for example, consider a random walk short rate – that is, set $\rho = 1$. This is essentially what has been done in the theoretical models of Dothan (1978) and Ho and Lee (1986). We consider this suggestion later on. In the next section we consider an alternative in which the short rate is stationary, yet there is substantial variability of yields out to ten years and beyond.

4 A Fractional Short Rate Process

We have seen that with ARMA representations of the short rate, our theory produces long yields whose variance approaches zero at an exponential rate. In this section we consider a fractional difference process for the short rate. This process is stationary for appropriately chosen parameter values, but the autocorrelations damp out more slowly than with ARMA processes. The process is said to exhibit *long memory* for reasons that will be apparent shortly. More to the point, the theory based on this process generates yields whose variance approaches zero at a slower rate.

Let us say, then, that the short rate process is

$$(1 - L)^{\delta}(r_t - \mu) = \epsilon_t$$

or

$$r_t = \mu + (1 - L)^{-\delta}\epsilon_t, \tag{21}$$

where L is the lag operator, δ is the differencing parameter and, as before, the increments ϵ are normally and independently distributed with mean 0 and variance σ^2. The Taylor series expansion of eq (21) around $L = 0$ is the infinite moving average

$$r_t = \mu + \sum_{j=0}^{\infty} \alpha_j \epsilon_{t-j},$$

with

$$\alpha_j = \frac{\Gamma(\delta + j)}{\Gamma(\delta)\Gamma(j + 1)}, \tag{22}$$

where Γ is the gamma function. This process is stationary for $-1/2 < \delta < 1/2$. See Granger and Joyeux (1980) and Sowell (1990) for details.

The relevant features of this process stem, obviously, from those of the gamma function. Among these are

$$\Gamma(1) = 1$$

and, for all $x > 0$,

$$\Gamma(x+1) = x\Gamma(x). \tag{23}$$

These imply, for example, that for any positive integer n, $\Gamma(n+1) = n!$. With respect to the moving average coefficients we see that $\alpha_0 = 1$ and that successive coefficients are related by

$$\alpha_j = \left[1 - \left(\frac{1-\delta}{j}\right)\right] \alpha_{j-1}.$$

Since $\delta < 1/2$ we see that the moving average coefficients decline at rate $(1-\delta)/j < 1$. For large j this rate approaches zero, which is the source of the long memory. Finally, partial sums of moving average coefficients are

$$A_{n+1} = \sum_{j=0}^{n} \alpha_j = \sum_{j=0}^{n} \frac{\Gamma(\delta+j)}{\Gamma(\delta)\Gamma(j+1)} = \frac{\Gamma(\delta+n+1)}{\Gamma(\delta+1)\Gamma(n+1)}. \tag{24}$$

This property follows from successive application of (23) to (22).

Let us turn now to the behavior of bond prices, forward rates, and yields, which we compute using Proposition 1. Our interest is in the properties of long forward rates and yields which depend, as we have seen, on partial sums of moving average coefficients. The behavior of the mean forward rate is, from (15), governed by the partial sum A_n. In the fractional difference model this sum does not converge if δ is positive. The variances of long forward rates and yields approach zero at a hyperbolic rate, which is slower than the exponential rate of the ARMA models of Proposition 2.

We can see the impact of the fractional difference process on bond pricing by comparing yields in this model with those in the AR(1) example of Figures 3 and 4. In Figures 5 and 6 we report the mean and standard deviation of the yield curve with differencing parameter $\delta = 0.3$. With $\lambda = .17$ and $\sigma = 2.33$ this choice of δ matches the mean yields as well as the AR(1) model. The most interesting implication of the fractional model, however, is the relatively slow decay of the variability of yields across maturities. As with the AR(1) example, we choose σ to match the standard deviation of the one-year yield. We see in Figure 6 that the standard deviation of the yield also reproduces the variability of long yields. The AR(1) model, of course, is a miserable failure in this regard.

Another potential resolution of the variability anomaly is a random walk short rate, represented either by setting $\rho = 1$ in Example 2 or $\delta = 1$ in the fractional difference model. We know that estimates of short rate autocorrelations at short time intervals are close to one, and that these estimates are difficult to distinguish from one. With δ equal to one the model does not have unconditional moments, but we can nevertheless construct moments conditional on an initial value that are comparable to the sample estimates we compute from data. The conditional variance of bond yields, in this case, is flat: the conditional standard deviation is the same for all maturities. This is roughly what we see in the data (Figures 2 and 6). The theory implies, in this case, that shifts of the yield curve are parallel ones, so that yields of all maturities vary the same amount.

The random walk model does less well in replicating average yields. At least in our theoretical framework, a random walk short rate implies that the mean yield curve has no curvature: it has no tendency to flatten out at the long end as we see in the data. The fractional model performs much better in this regard. Both are pictured in Figure 7 where, again, the black squares are data.

In short, the fractional difference model, despite being relatively exotic, has implications for long yields that are closer to what we see in the data than low-order autoregressions, including the random walk.

5 Origins of a Fractional Short Rate

In this section we provide direct evidence on the fractional differencing paremeter for the short interest rate. We find modest evidence that short rate dynamics are closely approximated by a fractional process with δ of about 0.3. We also document somehwat stronger evidence of a fractional root in inflation and money growth. We conjecture that the long memory apparent in short rates is inherited from inflation, money growth, and monetary policy.

Estimates of fractional ARMA(p, δ, q) models for the short rate are presented in Table 1 and model selection criteria are reported in Table 2. The parameter p is the order of the autoregressive polynomial, q is the order of the moving average polynomial, and δ is the fractional difference parameter. We present estimates for all combinations of orders p and q up to three a total of 16 models per series. The estimates are maximum likelihood for normally distributed innovations, as described in Sowell (1992). The short rate series consists of monthly observations of the three-month treasury bill yield reported by McCulloch (1990).

The short rate data leave us in some doubt about the short rate process. With the exception of the models with $p = 1$, estimates of the fractional parameter, δ, are between zero and one-half, but the data are otherwise not very informative. The Akaike criterion, AIC, favors the model with $p = q = 3$, whereas the Schwartz criterion, SIC, favors the simple $p = 1$, $q = 0$ model. A likelihood ratio test across these models rejects at the 1 percent level but not at 5 percent. The direct evidence from the short rate, therefore, provides less than overwhelming support for a fractional model.

Curiously, the fractional root shows up more clearly in inflation and money growth (Tables 3 to 6). We measure inflation as the growth rate in the implicit price index for monthly consumption of nondurables and services. With this data we find that the fractional parameter is typically positive, significantly different from zero, and in the stationary region (less than one-half). The Akaike criterion favors a high-order model, $p = q = 3$, but the Schwartz criterion favors a parsimonious one, $p = 0$ and $q = 1$. A likelihood ratio test rejects the restrictions implicit in the parsimonious model. A closer look, however, suggests that the high-order model suffers from redundancy, given the apparent root cancellation in the lowest panel of the table. Our feeling is that a fractional model with no AR term and a single MA term is a fair characterization of inflation dynamics. This conclusion conforms with Granger and Joyeux's (1980) analysis of food prices over a shorter period.

We turn next to money growth, using the broad aggregate M2. Table 5 presents the clearest evidence of the presence of a fractional root in nominal variables. The estimate of the fractional parameter is remarkably stable across ARMA specifications and the model selection criteria agree on the simple fractional ARMA(0,1). The estimate of δ in this case is 0.31. This estimate provides some support for our choice of $\delta = 0.3$ in the previous section.

On the whole, we feel that this evidence is suggestive of a fractional unit root in the short rate arising from money growth and inflation. The fractional model, in this sense, fits both the time series behavior of the short rate and the cross sectional behavior of the yield curve. This emphasis on the inflation component of nominal yields is consistent with the weak evidence of fractional long memory in real output growth documented by Haubrich and Lo (1991) and Sowell (1992).

A deeper question is whether the fractional long memory apparent in inflation and money growth is the result of long memory in monetary policy or of the behavior of private agents. We outline a theory, adapted from Granger (1980), in which long memory in inflation is the result of aggregation across agents with heterogeneous beliefs. Haubrich and Lo (1991) tell an analogous story for a multisector real business cycle model.

Granger's (1980) aggregation result starts with N independent autoregressive time series,

$$x_{j,t+1} = \alpha_j x_{jt} + z_{j,t+1} \ ,$$

for $j = 1, 2, \ldots, N$, and z_{jt} dependent across j and t, the aggregate series, $\bar{x}_t = \sum_{j=1}^{N} x_{jt}$, has very different properties than any of the underlying series. In particular, if the individual parameters, α_j, are drawn from a Beta distribution with parameters γ_1 and γ_2, then the aggregate series approaches, for large N, a fractional process with differencing parameter $\delta = 1 - \gamma_2/2$. The smaller the value of γ_2, the larger the mean of the Beta distribution, and the more persistent is the aggregate series. Our choice of $\delta = 0.3$ in Section 4 corresponds to $\gamma_2 = 1.4$.

If the inflation expectations that affect nominal variables like interest rates are, in fact, an aggregation of individual beliefs, then in a situation in which monetary policy is subject to continual change, Granger's result may provide a useful description of the dynamics of aggregate beliefs. This line of reasoning is at this point strictly conjecture, but we feel that it is a direction worth further investigation. Perhaps it could be melded with a model of regime changes like that of Evans and Wachtel (1992).

6 Final Remarks

This paper has two purposes. The first is to show that one class of asset pricing theories, which we argue includes many of the popular bond pricing models, has the implication that long yields are constant. In fact the variability of long yields is not much different than that of short yields, a feature of the data emphasized by Shiller (1979) in the context of the expectations theory of long rates and by den Haan (1990) in a dynamic general equilibrium monetary economy. Our second purpose is to suggest a fractional differemce model as a resolution of the discrepancy between theory and evidence on long bond yields. This, too, is not completely new: Shea (1989) suggests a similar interpretation of Shiller's results. What our study adds is a theoretical structure that incorporates risk and, as a consequence, introduces the possibility of replicating the mean and variance of the yield curve and the time series properties of the short rate.

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Table 1
Parameter estimates for fractional ARMA models: Three-Month Short Rate
Monthly Short Rate: McCulloch 3-month zero-coupon yield; 1959:01-1986:12
t-statistics in parentheses

Model	δ	ϕ_1	ϕ_2	ϕ_3	θ_1	θ_2	θ_3
(0,δ,0)	.50						
	(3.12)						
(0,δ,1)	.48				.23		
	(23.85)				(4.85)		
(0,δ,2)	.44				.32	.16	
	(7.80)				(4.07)	(2.85)	
(0,δ,3)	.44				.32	.16	.00
	(7.39)				(4.87)	(2.77)	(.05)
(1,δ,0)	-.19	-.94					
	(-2.15)	(-23.79)					
(1,δ,1)	-.24	-.95			.03		
	(-1.16)	(-16.85)			(.23)		
(1,δ,2)	-.45	-.98			.22	.11	
	(-3.48)	(-52.56)			(1.70)	(1.44)	
(1,δ,3)	-.40	-.98			.18	.08	-.04
	(-2.09)	(-37.41)			(1.05)	(.77)	(-.51)
(2,δ,0)	.27	-.48	-.12				
	(1.76)	(-2.87)	(-2.10)				
(2,δ,1)	.32	.24	-.42		.69		
	(2.16)	(.82)	(-2.77)		(2.82)		
(2,δ,2)	.28	.22	-.56		.69	-.10	
	(1.82)	(1.08)	(-3.06)		(4.16)	(-.88)	
(2,δ,3)	.22	-.11	-.51		.43	-.17	-.10
	(1.11)	(-.23)	(-1.90)		(1.05)	(-1.27)	(-1.04)
(3,δ,0)	.27	-.48	-.12	.00			
	(2.03)	(-3.33)	(-1.92)	(.03)			
(3,δ,1)	.29	.33	-.50	-.05	.80		
	(1.84)	(1.31)	(-2.75)	(-.68)	(3.88)		
(3,δ,2)	.24	-.26	-.67	.23	.25	-.43	
	(1.19)	(-.29)	(-2.81)	(.51)	(.32)	(-.79)	
(3,δ,3)	.25	.27	.20	-.66	.79	.73	-.26
	(1.56)	(1.94)	(1.58)	(-5.57)	(6.68)	(5.48)	(-2.30)

Note: A fractional ARMA(p, δ, q) model for x_t has the form

$$(1 + \phi_1 L + \phi_2 L^2 + \cdots + \phi_p L^p)(1 - L)^\delta (x_t - \mu) = (1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)\varepsilon_t ,$$

where ε_t is white noise and L is the lag operator.

Table 2
Model Selection Criteria: $2\log(\mathcal{L})$, AIC, SIC
16 Fractional ARMA models from Table 1 (3-Month Short Rate)

Number of AR parameters (p)	Number of MA parameters (q)			
	0	1	2	3
0	-1258.921 -1260.921 -1264.739	-1113.901 -1117.901 -1125.535	-1105.099 -1111.099 -1122.551	-1105.097 -1113.097 -1128.366
1	-1103.573 -1107.573 -1115.207	-1103.533 -1109.533 -1120.984	-1101.885 -1109.885 -1125.154	-1101.615 -1111.615 -1130.700
2	-1102.065 -1108.065 -1119.516	-1100.762 -1108.762 -1124.030	-1100.101 -1110.101 -1129.187	-1099.121 -1111.121 -1134.023
3	-1102.064 -1110.064 -1125.333	-1100.310 -1110.310 -1129.395	-1099.685 -1111.685 -1134.588	-1091.393 -1105.393 -1132.113

Note: AIC is the Akaike Information Criterion, SIC is the Schwarz Information Criterion, and L is the normal likelihood function.

Table 3
Parameter estimates for fractional ARMA models: Inflation
Monthly Consumption Deflator: Citibase GMDC; 1959:01-1989:12
t-statistics in parentheses

Model	δ	ϕ_1	ϕ_2	ϕ_3	θ_1	θ_2	θ_3
(0,δ,0)	.31						
	(10.27)						
(0,δ,1)	.48				-.36		
	(15.80)				(-5.74)		
(0,δ,2)	.48				-.35	-.04	
	(19.57)				(-6.33)	(-.90)	
(0,δ,3)	.48				-.35	-.04	-.01
	(19.15)				(-6.04)	(-.77)	(-.13)
(1,δ,0)	.41	.24					
	(11.20)	(4.37)					
(1,δ,1)	.48	-.11			-.46		
	(19.10)	(-.92)			(-4.19)		
(1,δ,2)	.48	-.04			-.39	-.03	
	(18.83)	(-.07)			(-.65)	(-.11)	
(1,δ,3)	.48	.67			.32	-.27	-.05
	(19.40)	(1.39)			(.66)	(-1.50)	(-.95)
(2,δ,0)	.45	.31	.11				
	(11.55)	(4.91)	(2.03)				
(2,δ,1)	.48	-.09	.01		-.44		
	(19.01)	(-.51)	(.17)		(-2.56)		
(2,δ,2)	.06	-1.67	.68		-1.63	.69	
	(.44)	(-29.06)	(12.51)		(-15.83)	(7.23)	
(2,δ,3)	-.12	-1.75	.76		-1.53	.54	.06
	(-.53)	(-16.65)	(7.37)		(-7.18)	(2.19)	(.75)
(3,δ,0)	.48	.34	.16	.09			
	(16.85)	(5.93)	(2.81)	(1.67)			
(3,δ,1)	.48	.06	.07	.05	-.29		
	(19.85)	(.20)	(.54)	(.75)	(-.88)		
(3,δ,2)	.46	-1.29	.46	.16	-1.64	.88	
	(11.86)	(-12.82)	(4.28)	(2.02)	(-25.70)	(9.68)	
(3,δ,3)	-.15	-2.30	1.90	-.59	-2.84	2.90	-.94
	(-1.28)	(-17.77)	(5.98)	(-3.11)	(-30.11)	(11.70)	(-3.27)

Table 4
Model Selection Criteria: $2\log(\mathcal{L})$, AIC, SIC
16 Fractional ARMA models from Table 3 (Inflation)

Number of AR parameters (p)	Number of MA parameters (q): 0	1	2	3
	3712.112	3732.059	3732.853	3732.868
0	3710.112	3728.059	3726.853	3724.868
	3706.134	3720.101	3714.917	3708.953
	3726.636	3732.845	3732.858	3733.121
1	3722.636	3726.845	3724.858	3723.121
	3714.678	3714.909	3708.943	3703.227
	3730.447	3732.873	3744.431	3744.743
2	3724.447	3724.873	3734.431	3732.743
	3712.510	3708.957	3714.536	3708.870
	3732.972	3733.413	3741.075	3746.166
3	3724.972	3723.413	3729.075	3732.166
	3709.056	3703.519	3732.166	3704.313

Table 5
Parameter estimates for fractional ARMA models: Money Growth Rates
Monthly M2: Citibase FM2; 1959:01-1989:12
t-statistics in parentheses

Model	δ	ϕ_1	ϕ_2	ϕ_3	θ_1	θ_2	θ_3
(0,δ,0)	.50						
	(3.17)						
(0,δ,1)	.31				.38		
	(6.46)				(5.97)		
(0,δ,2)	.37				.32	-.08	
	(5.12)				(3.59)	(-1.14)	
(0,δ,3)	.34				.35	-.05	.04
	(3.89)				(3.37)	(-.50)	(.67)
(1,δ,0)	.29	-.34					
	(8.14)	(-7.96)					
(1,δ,1)	.37	.27			.58		
	(6.52)	(1.51)			(4.50)		
(1,δ,2)	.34	.78			1.13	.23	
	(6.03)	(3.48)			(4.34)	(1.62)	
(1,δ,3)	.35	.81			1.15	.22	-.02
	(4.64)	(3.83)			(5.27)	(1.47)	(-.23)
(2,δ,0)	.40	-.27	.14				
	(5.42)	(-3.15)	(2.60)				
(2,δ,1)	.33	.38	-.10		.73		
	(4.22)	(1.73)	(-.70)		(3.40)		
(2,δ,2)	.35	.90	.07		1.24	.32	
	(5.00)	(1.76)	(.24)		(2.63)	(.89)	
(2,δ,3)	.35	1.24	.34		1.58	.71	.08
	(5.40)	(1.21)	(.41)		(1.52)	(.58)	(.29)
(3,δ,0)	.32	-.36	.15	-.08			
	(2.74)	(-2.74)	(2.60)	(-1.20)			
(3,δ,1)	.36	.52	-.10	.05	.84		
	(4.18)	(2.55)	(-.91)	(.67)	(5.27)		
(3,δ,2)	.35	.99	.11	-.02	1.33	.39	
	(4.32)	(1.43)	(.31)	(-.12)	(1.90)	(.67)	
(3,δ,3)	.35	1.24	.34	.00	1.58	.71	.08
	(3.06)	(.25)	(.07)	(.00)	(.31)	(.11)	(.05)

Table 6
Model Selection Criteria: $2\log(\mathcal{L})$, AIC, SIC
16 Fractional ARMA models from Table 3 (Money Growth)

Number of AR parameters (p)	Number of MA parameters (q) 0	1	2	3
0	3576.667 3574.667 3570.688	3630.659 3626.659 3618.702	3631.850 3625.850 3613.914	3632.302 3624.302 3608.386
1	3622.668 3618.668 3610.710	3632.338 3626.338 3614.401	3633.201 3625.201 3609.285	3633.254 3623.254 3603.359
2	3629.052 3623.052 3611.115	3632.863 3624.863 3608.948	3633.256 3623.256 3603.362	3633.275 3621.275 3597.401
3	3630.584 3622.584 3606.669	3633.202 3623.202 3603.307	3633.264 3621.264 3597.391	3633.275 3619.275 3591.423

Figure 1
Mean Yield Curve, 1947-86

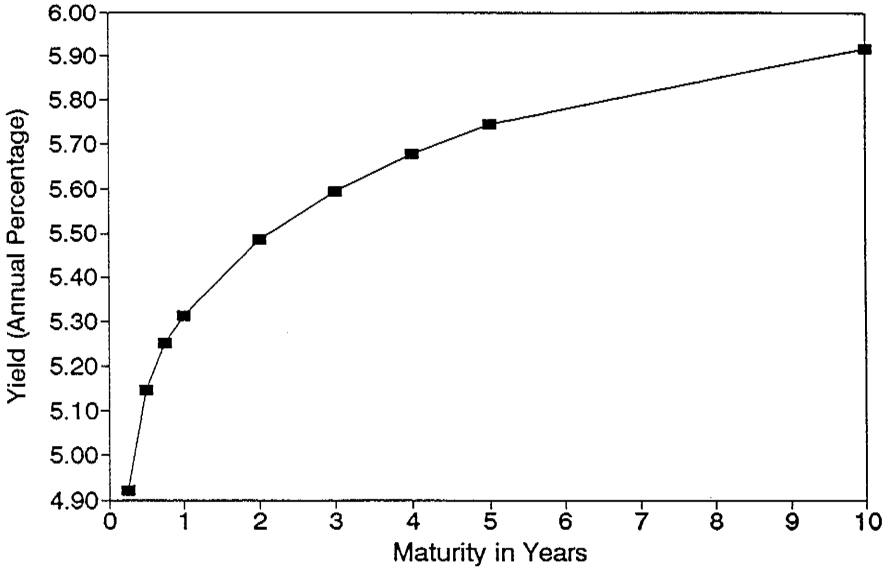

Figure 2
Monthly Yield Variability, 1947-86

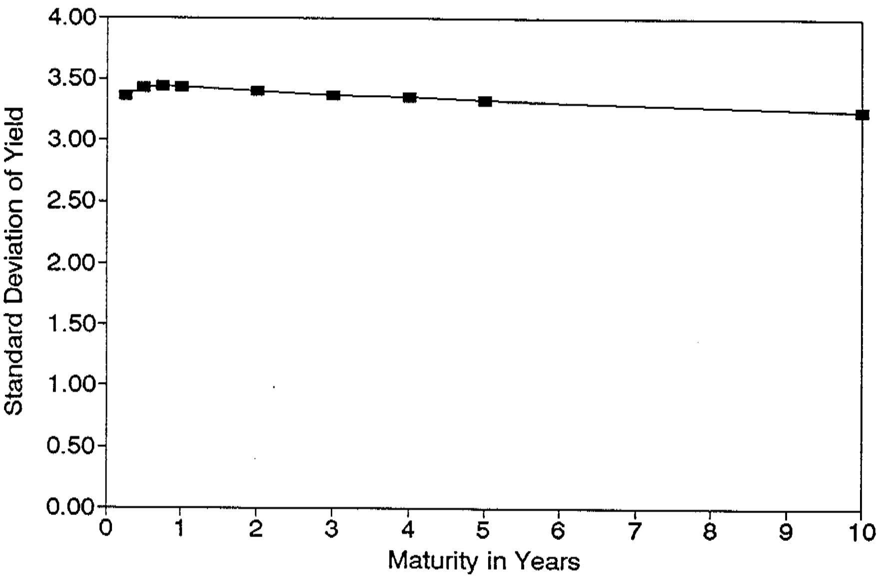

Figure 3
Mean Yield Curve with AR(1) Short Rate

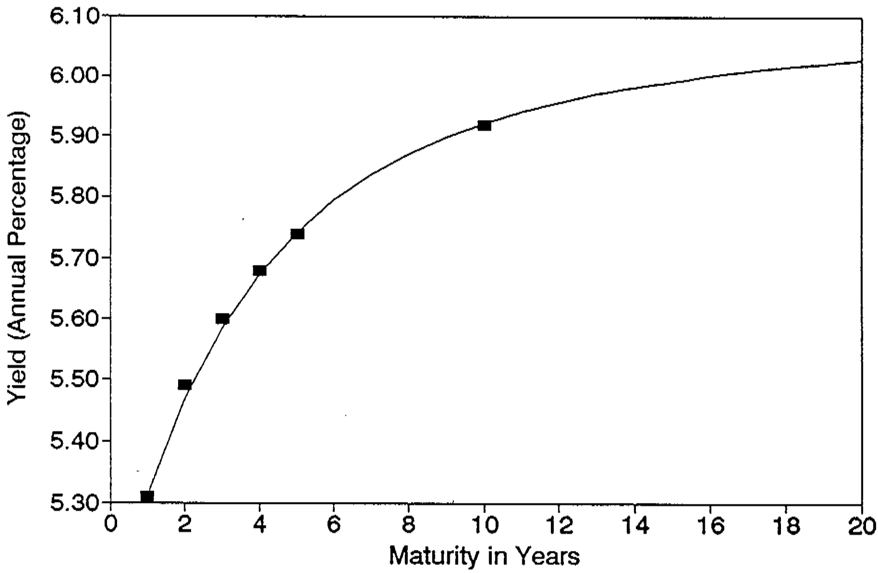

Figure 4
Yield Variability with AR(1) Short Rate

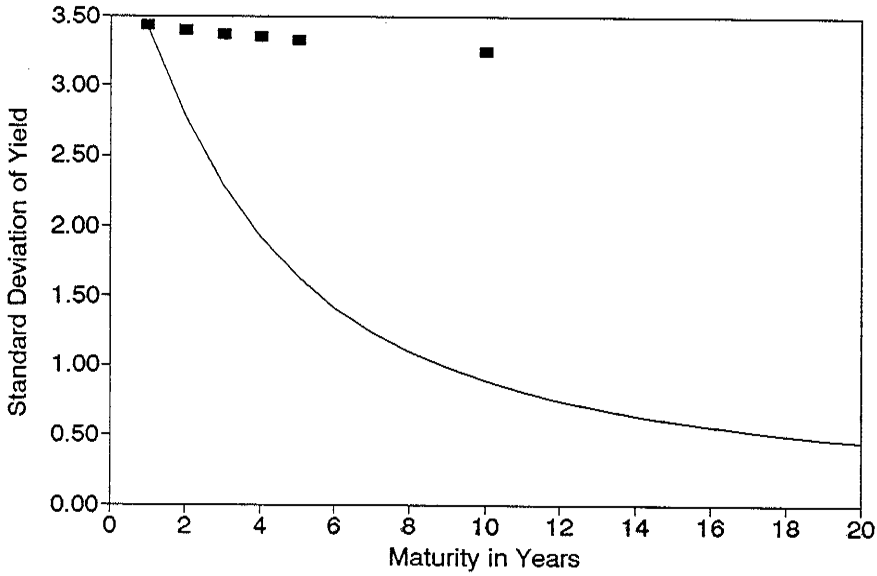

Figure 5
Mean Yield Curve: Fractional Short Rate

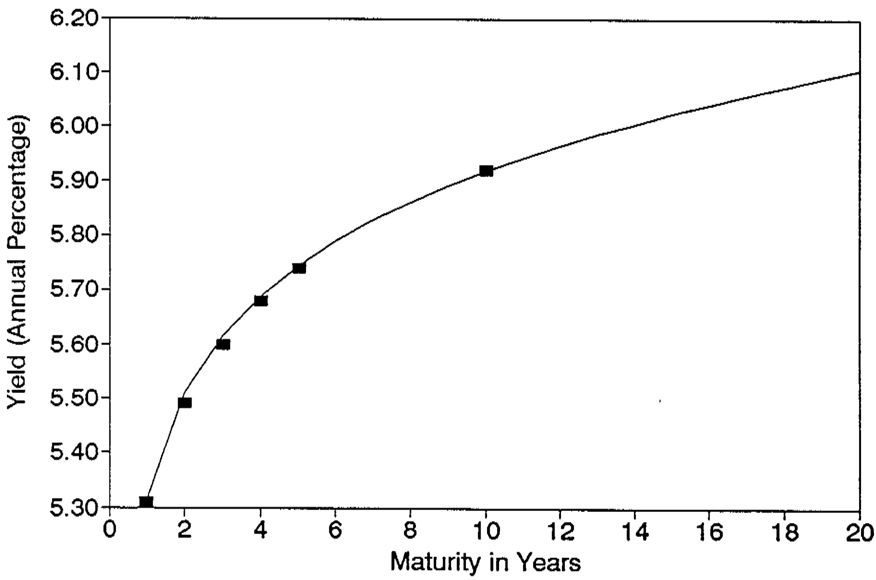

Figure 6
Yield Variability: Fract'l Short Rate

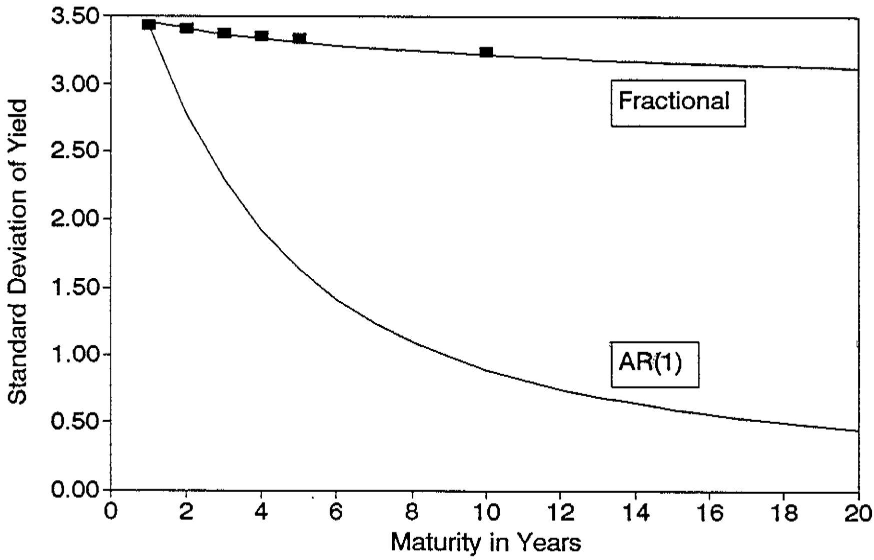

Figure 7
Mean Yields: Fractional vs Random Walk