NBER TECHNICAL WORKING PAPERS SERIES

MEASURES OF FIT FOR CALIBRATED MODELS

Mark W. Watson

Technical Working Paper No. 102

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
May 1991

This research has benefitted from constructive comments by many seminar participants; in particular I thank John Cochrane, Marty Eichenbaum, Jon Faust, Lars Hansen, Robert Hodrick, Robert King and Robert Lucas. The first draft of this paper was written while I was visiting the University of Chicago, whose hospitality is gratefully acknowledged. This research was supported by the National Science Foundation through grant SES-89-10601. This paper is part of NBER's research program in Economic Fluctuations. Any opinions expressed are those of the author and not those of the National Bureau of Economic Research.

NBER Technical Working Paper #102
May 1991

MEASURES OF FIT FOR CALIBRATED MODELS

ABSTRACT

This paper develops a new procedure for assessing how well a given dynamic economic model describes a set of economic time series. To answer the question, the variables in the model are augmented with just enough error so that the model can exactly mimic the second moment properties of the actual data. The properties of this error provide a useful diagnostic for the economic model, since they show the dimensions in which model fits the data relatively well and the dimensions in which it fits the data relatively poorly.

Mark W. Watson
Department of Economics
Northwestern University
Evanston, IL 60208
and
Federal Reserve Bank of Chicago
Chicago, IL 60604
and
NBER

1. Introduction

The appropriate method for assessing the empirical relevance of economic models has been debated by economists for many years. The standard econometric approach can be traced back to Haavelmo (1944), who argued that an economic model should be embedded within a complete probability model and analyzed using statistical methods designed for conducting inference about unknown probability distributions. The appeal of this approach follows from interpreting the probability distribution as a likelihood function, which in turn provides the basis for a unified theory of estimation and inference. In the modern literature, this approach is clearly exemplified in work like that of Hansen and Sargent (1980) or McFadden (1981). However, many economic models do not provide a realistic and complete probability structure for the variables under consideration. Using the standard econometric approach these models must be discarded as empirically irrelevant, or augmented in some way with additional random components. Inferences drawn from these augmented models are meaningful only to the extent that the additional random components do not mask or change the salient features of the original economic models.

Another econometric approach, markedly different from the one advocated by Haavelmo, is becoming increasingly popular in empirical macroeconomics. This approach, which I'll call calibration/simulation, is most clearly articulated in the work of Kydland and Prescott (1982) and Prescott (1986). In a general sense, calibration/simulation asks whether data from a real economy share certain characteristics with data generated by the artificial economy described by an economic model. There is no claim that the model explains all

of the characteristics of the actual data, nor is there any attempt to augment the model with additional random components to more accurately describe the data. Because of this, calibration/simulation results are often easier to interpret than results form traditional econometric analysis, since the economic model is not complicated by additional random elements added solely for statistical convenience. Yet, inference procedures for calibration/simulation lack statistical foundations and are necessarily *ad hoc*, since the economic model does not provide a complete probability structure. For example, a researcher may determine that a model fits the data well because it implies moments for the variables under study that are "close" to the moments of the actual data, even though the metric used to determine the distance between the moments is left unspecified.

This paper is an attempt to put the latter approach on a less *ad hoc* foundation by developing goodness of fit measures for the class of dynamic econometric models whose endogenous variables follow covariance stationary processes. It is not assumed that the model accurately describes data from an actual economy; the economic model is not a null hypothesis in the statistical sense. Rather, the economic model is viewed as an approximation to the stochastic processes generating the actual data, and goodness of fit measures are proposed to measure the quality of this approximation. A standard device -- stochastic error -- is used to motivate the goodness of fit measures. These measures answer the question: "How much random error would have to be added to data generated by the model so that the autocovariances implied by the model+error match the autocovariances of the observed data?"

The error represents the degree of abstraction of the model from the data. Since the error can't be attributed to a data collection procedure or to a

forecasting procedure, etc., it is difficult *a priori* to say much about its properties; in particular its covariance with the observed data cannot be restricted by *a priori* reasoning. Rather than making a specific assumption about the error's covariance properties, a representation is constructed which minimizes the contribution of the error in the complete model. Thus, in this sense, the error process is chosen to make the model as close to the data as possible.

Many of the ideas in this paper are close to, and were motivated by, ideas in Altug (1989) and Sargent (1989). Altug (1989) showed how a one-shock real business cycle model, similar to the model developed in Kydland and Prescott (1982), could be analyzed using standard dynamic econometric methods, by augmenting each variable in the model with an idiosynchratic error. This produced a restricted version of the dynamic factor analysis or unobserved index models developed by Sargent and Sims (1977) and Geweke (1977). Sargent (1989) discusses two models of measurement error; in the first the measurement error is uncorrelated with the data generated by the model, and in the second it is uncorrelated with the sample data.[1] While similar in spirit, the approach taken in this paper differs from that of Altug and Sargent in two respects. First, in this paper, the error process is not assumed to be uncorrelated with the model's artificial data or with the actual data. Rather, the correlation properties of the error process are determined by the requirement that the variance of the error is as small as possible. Second, the joint data-error process is introduced to motivate goodness of fit measures; it is not introduced to describe a statistical model that can be used to estimate unknown parameters or to test statistical hypotheses.[2]

The minimum approximation error representation motivates two sets of statistics that can be used to evaluate the goodness of fit of the economic model. First, the variance of the approximation error can be used, like the variance of the error in a regression model, to form an "R^2" measure for each variable in the model. This provides an overall measure of fit. Moreover, since the model is dynamic, spectral methods can be used to calculate the R^2 measure for each frequency. These can be used, for example, to measure the fit over the "business cycle" or "growth" frequencies. Second, the minimum measurement error representation can be used to form fitted values of the variables in the economic model using actual data. These fitted values show how well the model explains specific historical episodes; for example, can a real business cycle model simultaneously explain the growth in the 1960's and the 1981-1982 recession?

The plan of the paper is as follows. The next section develops the minimum approximation error representation and goodness of fit measures. The third section calculates these goodness of fit statistics for a standard real business cycle model using post-war U.S. macroeconomic data on output, consumption, investment and employment. The fourth section discusses a variety of statistical issues, and the fifth section concludes.

2. Measures of Fit

Consider an economic model that describes the evolution of an $n \times 1$ vector of variables x_t. Assume that the variables in the model have been transformed, say by first differencing or forming ratios, so that x_t is covariance

stationary. As a notational device, it is useful to introduce the autocovariance generating function (ACGF) of x_t, $A_x(z)$, which completely summarizes the unconditional second moment properties of the process. In what follows "economic model" and "$A_x(z)$" will be used interchangeably: the analysis considers only the unconditional second moment implications of the model. Nonlinearities and variation in conditional second and higher moments are ignored to keep the problem tractible. The analysis will also ignore the unconditional first moments of x_t; modifying the measures of fit for differences in the means of the variables is straightforward.

The empirical counterparts of x_t are denoted y_t. These variables differ from x_t in an important way. The variables making up x_t correspond to the variables appearing in the theorist's simplification of reality; in a macroeconomic model they are variables like "output," "money" and the "interest rate." The variables making up y_t are functions of raw data collected in a real economy; they are variables like "Per capita Gross National Product in the United States in 1982 dollars," and "U.S. M2" and "the yield on 3 Month U.S. Treasury Bills."

The question of interest is whether the model generates data with characteristics similar to those of the data from the real economy. Below, goodness of fit measures are proposed to help answer this question. Before introducing these new measures, it is useful to review standard statistical goodness of fit measures to highlight their deficiencies for answering the question at hand.

Standard measures of fit use the size of sampling error to judge the coherence of the model with the data. They are based on the following:

First, $A_y(z)$, the population ACGF of the data, is unknown but can be estimated from sample data. Discrepancies between the estimator $\hat{A}_y(z)$ and $A_y(z)$ arise solely from sampling error in $\hat{A}_y(z)$, and the likely size of this error can be deduced from the stochastic process that generated the sample. Now, *if* $A_y(z)=A_x(z)$, sampling error also accounts for the differences between $\hat{A}_y(z)$ and $A_x(z)$. Standard goodness of fit measures show how likely it is that $A_y(z)=A_x(z)$, based on the probability that differences between $\hat{A}_y(z)$ and $A_x(z)$ arise solely from sampling error. If the differences between $\hat{A}_y(z)$ and $A_x(z)$ are so large as to be unlikely, standard measures of fit suggest that the model fits the data poorly, and vice versa if the differences between $\hat{A}_y(z)$ and $A_x(z)$ are not so large as to be unlikely. The key point is that the differences between $\hat{A}_y(z)$ and $A_x(z)$ are judged by how informative the sample is about the population moments of y_t. This is a sensible procedure for judging the coherence of a null hypothesis, $A_y(z)=A_x(z)$, with the data, but is arguably less sensible when this null hypothesis is known to be false.

Rather than rely on sampling error, the meaures of fit proposed here are based on the size of the stochastic error required to reconcile the autocovariances of x_t with those of y_t. In particular, letting u_t denote an $n\times 1$ error vector, the importance of the difference between $A_x(z)$ and $A_y(z)$ will be determined by asking: "How much error would have to be added to x_t so that the autocovariances of x_t+u_t are equal to the autocovariances of y_t?" If the variance of the required error is large then the discrepancy between $A_x(z)$ and $A_y(z)$ is large, and conversely if the variance of u_t is small. The vector u_t is the approximation error in the economic model interpreted as a stochastic process. It captures the (second moment) characteristics of the

observed data that are not captured by the model. Loosely speaking, it is analogous to the error term in a regression where the set of regressors is interpreted as the economic model. The economic model might be deemed a good approximation to the data if the variance of the error term is small (i.e. the R^2 of the regression is large) and might be deemed a poor approximation if the variance of the error term is large (i.e. the R^2 of the regression is small.)

To be more precise, assume that x_t and y_t are jointly covariance stationary and define the error u_t by the equation

$$(2.1) \quad u_t = y_t - x_t,$$

so that

$$(2.2) \quad A_u(z) = A_y(z) + A_x(z) - A_{xy}(z) - A_{yx}(z)$$

where $A_u(z)$ is the autocovariance generating function of u_t, $A_{xy}(z)$ is the cross autocovariance generating function between x_t and y_t, etc. From the right hand side of (2.2), three terms are needed to calculate $A_u(z)$. The first, $A_y(z)$, can be consistently estimated from sample data, the second, $A_x(z)$, is completely determined by the model, but the third, $A_{xy}(z)$, is not determined by the model nor can it be estimated from the data, since this would require a sample drawn from the *joint* (x_t,y_t) process. To proceed, an assumption is necessary.

A common assumption used in econometric analysis is that $A_{xy}(z)=A_x(z)$ so that x_t and u_t are uncorrelated at all leads and lags. Equation (2.1) can then

be interpreted as the dynamic analogue of the classical errors-in-variables model. Sargent (1989) discusses this assumption and an alternative assumption, $A_{xy}(z)=A_y(z)$. He points out that under this latter assumption, u_t can be interpreted as signal extraction error, with y_t an optimal estimate of the unobserved "signal" x_t.[3] In many applications, these covariance restrictions follow from the way that the data were collected or the way expectations are formed. For example, if x_t represented the true value of the U.S. unemployment rate and y_t the value published by the U.S. Department of Labor, then y_t would differ from x_t because of the sampling error inherent in the monthly Current Population Survey (CPS) from which y_t is derived. The sample design underlying the CPS implies that the error, u_t, is statistically independent of x_t. Similarly, if y_t denoted a rational expectation of x_t, then the error would be uncorrelated with y_t. Neither of these assumptions seems appropriate in the present context. The error isn't the result of imprecise measurement; it isn't a forecast or signal extraction error. Rather, it represents approximation or abstraction error in the economic model. Any restriction used to identify $A_{xy}(z)$, and hence $A_u(z)$, is arbitrary.[4]

Is it possible, however, to calculate a lower bound for the variance of u_t without imposing any restrictions on $A_{xy}(z)$. When this lower bound on the variance of u_t is large, then under any assumption about $A_{xy}(z)$, the model fits the data poorly. If the lower bound on the variance of u_t is small, then there are possible assumptions about $A_{xy}(z)$ that imply that the model fits the data well. Thus, this bound is potentially useful for rejecting models based on their empirical fit. Needless to say, models that appear to fit the data well using this bound require further scrutiny.

The bound is calculated by choosing $A_{xy}(z)$ to minimize the variance of u_t subject to the constraint that the implied joint autocovariance generating function of x_t and y_t is positive semi-definite. Equivalently, since the spectrum is proportional to the autocovariance generating function evaluated at $z=e^{-i\omega}$, the cross spectrum between x_t and y_t, $A_{xy}(e^{-i\omega})$, must be chosen so that the spectral density matrix of $(x_t' \; y_t')'$ is positive semi-definite at all frequencies.

Since the measures of fit proposed in this paper are based on the solution to this minimization problem and the implied minimum approximation error representation of the (x_t,y_t) process, it is useful to discuss the problem and its solution in detail. This is done by considering a few simple models before proceding to the general case. Four models are considered. The first model is very simple, and the solution follows by inspection. The second model is more complicated than the first, the third more complicated than the second, etc. In the first model, x_t and y_t are scalar serially uncorrelated random variables. In the second model, x_t and y_t are serially uncorrelated random vectors with non-singular covariance matrices. Since many economic models contain fewer sources of noise than variables, x_t is allowed to have a singular covariance matrix in the third model. Finally in the last model, x_t and y_t are allowed to be serially correlated. After discussing these four models in general terms, an example is presented.

Model 1:

Suppose that x_t, y_t and u_t are scalar serially uncorrelated random variables. The problem is to choose σ_{xy} to minimize the variance of $\sigma_u^2=\sigma_x^2+\sigma_y^2-2\sigma_{xy}$, subject to the constraint that the covariance matrix of

x_t and y_t remains positive semidefinite, i.e., $|\sigma_{xy}| \leq \sigma_x \sigma_y$ The solution sets $\sigma_{xy} = \sigma_x \sigma_y$ and yields $\sigma_u^2 = (\sigma_x - \sigma_y)^2$ as the minimum. Since $\sigma_{xy} = \sigma_x \sigma_y$, x_t and y_t are perfectly correlated with

$$(2.3) \quad x_t = \gamma y_t,$$

where $\gamma = \sigma_x / \sigma_y$. Equation (2.3) is important because it shows how to calculate fitted values of x_t, given data on y_t. Variants of equation (2.3) will hold for all of the models considered. In each model, the minimum approximation error representation makes (x_t) perfectly correlated with (y_t). In each model, the analogue of (2.3) provides a formula for calculating the fitted values of the variables in the model given data from the actual economy.

Model 2:

Now suppose that x_t and y_t are serially uncorrelated random vectors with nonsingular covariance matrices Σ_x and Σ_y respectively. Let $\Sigma_u = \Sigma_x + \Sigma_y - \Sigma_{xy} - \Sigma_{yx}$ denote the covariance matrix of u_t. Since Σ_u is a matrix, there is no unique definition of a "small" variance for u_t. Any metric comparing Σ_u with 0 will do. A convenient measure of the size of the variance of u_t is the trace of Σ_u, $\mathrm{tr}(\Sigma_u) = \sum_{i=1}^{n} \Sigma_{u,ii}$, where $\Sigma_{u,ij}$ denotes the ij'th element of Σ_u. While convenient, this measure is not always ideal, since it weights all variables equally. When the units of the variables are different, or when the researcher cares about certain variables more than others, unequal weighting might be preferred, say:

$$(2.4) \quad \sum_{i=1}^{n} \Sigma_{u,ii} w_i,$$

where w_i, i=1,...n, are a set of nonzero constants or weights.

The appendix shows how Σ_{xy} can be chosen to minimize (2.4) subject to the constraint that the covariance matrix for $(x_t' \; y_t')'$ is positive semi-definite. There it is shown that the solution sets $\Sigma_{xy}=C_x'R'C_y$, where C_x and C_y are arbitrary "square roots" of Σ_x and Σ_y (i.e., $\Sigma_x=C_x'C_x$ and $\Sigma_y=C_y'C_y$; so for example, C_x and C_y can be the Cholesky factors of Σ_x and Σ_y). The orthonormal matrix R is a function of $C=C_xWC_y'$, where W is a diagonal matrix with w_i as the i'th diagonal element. In particular, writing $C'C=D\Delta D'$, where the columns of D contain the orthonormal eigenvectors of C'C and Δ is a diagonal matrix with the corresponding eigenvalues on the diagonal, the matrix R can be written as $R=D\Delta^{-\frac{1}{2}}D'C'$.

One important implication of this solution is that, like the scalar example, the joint covariance matrix $(x_t' \; y_t')'$ is singular and x_t can be represented as

$$(2.5) \quad x_t = \Gamma y_t,$$

where $\Gamma=C_x'R'C_y^{-1}$. (Since R is orthonormal, this simplifies to the scalar result when x_t and y_t are scalars.)

Model 3:

In many economic models, the number of variables exceeds the number of shocks. In this case Σ_x is singular, and the solution derived in the appendix

for non-singular Σ_x is not immediately applicable. The solution can be applied to a slightly modified problem however. Suppose that Σ_x has rank $k \leq n$. Then the analysis for Model 2 can be applied to a kxl subset of the elements of x_t and y_t. In particular, let S be a k×n matrix, such that $S\Sigma_x S'$ has full rank. Let $\tilde{x}_t = Sx_t$, $\tilde{y}_t = Sy_t$, $\Sigma_{\tilde{x}} = S\Sigma_x S'$ and $\Sigma_{\tilde{y}} = S\Sigma_y S'$. The results for Model 2 can then be used to find the value of $\Sigma_{\tilde{x}\tilde{y}} = \text{cov}(\tilde{x}_t, \tilde{y}_t)$ that minimizes the (weighted trace of the) variance of $\tilde{u}_t = \tilde{x}_t - \tilde{y}_t$. Moreover, from (2.5), the solution of minimum variance problem implies that

$$(2.6) \quad \tilde{x}_t = \tilde{\Gamma}\tilde{y}_t = \tilde{\Gamma}Sy_t,$$

where $\tilde{\Gamma}$ is the analogue of Γ in (2.4) constructed using $\Sigma_{\tilde{x}}$ and $\Sigma_{\tilde{y}}$ in place of Σ_x and Σ_y.

Now, since Σ_x and $S\Sigma_x S'$ both have rank k, it is possible to express x_t as a linear combination of the elements of $\tilde{x}_t$. In particular $x_t = B\tilde{x}_t$, where the n×k matrix B is easy to compute from Σ_x and the matrix S.[5] Thus,

$$(2.7) \quad x_t = B\tilde{x}_t = B\tilde{\Gamma}\tilde{y}_t = B\tilde{\Gamma}Sy_t,$$

so that $\Sigma_{xy} = B\tilde{\Gamma}S\Sigma_{yy}$.

Model 4:

This same approach can be used in a dynamic multivariate model with slight modifications; when u_t is serially correlated, the weighted trace of the spectral density matrix, rather that the covariance matrix can be minimized.

To motivate the approach, it is useful to use the Cramer representations for x_t, y_t and u_t (see e.g. Brillinger [1981], section 4.6). Assume that x_t, y_t and u_t are jointly covariance stationary with mean zero; the Cramer representation can be written as:

$$
\begin{aligned}
x_t &= \int_0^{2\pi} e^{i\omega t} dz_x(\omega) \\
y_t &= \int_0^{2\pi} e^{i\omega t} dz_y(\omega) \\
u_t &= \int_0^{2\pi} e^{i\omega t} dz_u(\omega),
\end{aligned}
\tag{2.8}
$$

where $dz(\omega)=(dz_x(\omega)'\ dz_y(\omega)'\ dz_u(\omega)')'$ is a complex valued vector of orthogonal increments, with $E(dz(\omega)\overline{dz(\lambda)}')= \delta(\omega-\lambda)S(\omega)d\omega d\lambda$, where $\delta(\omega-\lambda)$ is the dirac delta and $S(\omega)$ is the spectral density matrix of $(x_t'\ y_t'\ u_t')'$ at frequency ω. Equation (2.8) represents x_t, y_t, and u_t as the integral (sum) of increments $dz_x(\omega)$, $dz_y(\omega)$ and $dz_u(\omega)$ which are uncorrelated across frequencies and have variances and covariances given by the spectra and cross spectra of x_t, y_t, and u_t. Since the spectra are proportional to the autocovariance generating functions evaluated at $z=e^{-i\omega}$, $E(dz_x(\omega)\overline{dz_x(\omega)}')$ is proportional to $A_x(e^{-i\omega})$, $E(dz_x(\omega)\overline{dz_y(\omega)}')$ is proportional to $A_{xy}(e^{-i\omega})$, etc.

Now consider the problem of choosing $A_{xy}(z)$ to minimize the variance of u_t. Since u_t can be written as the integral of the uncorrelated increments $dz_u(\omega)$, the variance of u_t can be minimized by minimizing the variance of $dz_u(\omega)$ for each ω. Since the increments are uncorrelated across frequency, the minimization problems can be solved independently for each frequency. Thus, the analysis carried out for Models 1-3 carries over directly, with spectral

density matrices replacing covariance matrices. The minimum trace problems for Models 2 and 3 are now solved frequency by frequency using the spectral density matrix. In principle this introduces additional flexibility into the representation since the weights, w_i, in the objective function (2.4) can depend on frequency as can the matrix S used for Model 3 to select the variables of interest.

Like Models 1-3, the solution yields:

$$(2.9) \quad dz_x(\omega) = \Gamma(\omega)dz_y(\omega)$$

where $\Gamma(\omega)$ is the complex analogue of Γ from (2.5) when the spectral density matrix of x_t is non-singular, and the analogue of $B\Gamma S$ from (2.7) when the spectral density matrix of x_t is singular. Equation (2.9) implies

$$(2.10) \quad A_{xy}(e^{-i\omega}) = \Gamma(\omega)A_y(e^{-i\omega}), \text{ and}$$

$$(2.11) \quad A_u(e^{-i\omega}) = A_x(e^{-i\omega}) + A_y(e^{-i\omega}) - A_{xy}(e^{-i\omega}) - A_{xy}(e^{i\omega})'.$$

The variance and covariances of u_t and all autocovariance follow directly from (2.11). Moreover, since $dz_x(\omega)$ and $dz_y(\omega)$ are perfectly correlated from (2.9), x_t can be expressed as a function of leads and lags of y_t:

$$(2.12) \quad x_t = \beta(L)y_t,$$

where $\beta(L)=\sum_{-\infty}^{\infty}\beta_j L^j$, with $\beta_j=\int_{-\pi}^{\pi}\Gamma(\omega)e^{i\omega j}d\omega$. Thus, fitted values of x_t can be calculated from leads and lags of y_t.

An Example:

The model considered in the next section describes the dynamic properties of output, consumption, investment and labor supply as functions of a single productivity shock. The mechanics of the minimum approximation error representation for that model can be demonstrated in a model in which x_t and y_t are bivariate, and the elements of x_t are driven by a single iid(0,1) shock ϵ_t. Letting x_t^1, x_t^2, y_t^1, and y_t^2, denote the elements of x_t and y_t, suppose

$$(2.13) \qquad \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} = \begin{bmatrix} \alpha_1(L) \\ \alpha_2(L) \end{bmatrix} \epsilon_t,$$

where $\alpha_1(L)$ and $\alpha_2(L)$ are scalar polynomials in the lag operator. Thus,

$$(2.14) \qquad A_x(z) = \begin{bmatrix} A_{x,11}(z) & A_{x,12}(z) \\ A_{x,21}(z) & A_{x,22}(z) \end{bmatrix} = \begin{bmatrix} \alpha_1(z)\alpha_1(z^{-1}) & \alpha_1(z)\alpha_2(z^{-1}) \\ \alpha_2(z)\alpha_1(z^{-1}) & \alpha_2(z)\alpha_2(z^{-1}) \end{bmatrix}.$$

Assume that the data y_t have a full rank ACGF, given by

$$(2.15) \qquad A_y(z) = \begin{bmatrix} A_{y,11}(z) & A_{y,12}(z) \\ A_{y,21}(z) & A_{y,22}(z) \end{bmatrix}.$$

Since the spectrum of x_t has rank 1, the procedure outlined for Model 3 (modified for serially correlated data) is appropriate. Let S=[1 0], so that $\tilde{x}_t = x_t^1$. This choice of S means that $A_{xy}(z)$ will be chosen to minimize the variance of $u_t^1 = x_t^1 - y_t^1$. Let $dz_{x1}(\omega)$, $dz_{y1}(\omega)$ and $dz_{u1}(\omega)$ denote the first

elements of $dz_x(\omega)$, $dz_y(\omega)$ and $dz_u(\omega)$. Since $dz_{u1}(\omega)$ is a scalar, the solution to the minimum variance problem is the complex analogue of the solution described for Model 1. In particular, the solution sets:

(2.16) $\quad dz_{x1}(\omega) = \delta(\omega) dz_{y1}(\omega)$

where $\delta(\omega) = [A_{x,11}(e^{-i\omega})/A_{y,11}(e^{-i\omega})]^{\frac{1}{2}}$. Since the x_t process is singular, $dz_{x2}(\omega)$ is perfectly correlated with $dz_{x1}(\omega)$; in particular, from (2.13):

(2.17) $\quad dz_x(\omega) = B(\omega)\, dz_{x1}(\omega)$

where $B(\omega) = [\, 1 \quad \alpha_2(e^{-i\omega})/\alpha_1(e^{-i\omega})]'$. Thus:

(2.18) $\quad dz_x(\omega) = B(\omega)\delta(\omega) S dz_y(\omega)$,

so that

(2.19) $\quad A_{xy}(e^{-i\omega}) = B(\omega)\delta(\omega) S A_y(e^{-i\omega})$, and

$A_u(e^{-i\omega})$ follows from (2.11).

Relative Mean Square Approximation Error:

A bound on the relative mean square approximation error for the economic model can be calculated directly from (2.11). This bound -- analogous to a lower bound on $1-R^2$ from a regression -- is:

$$(2.20) \quad r_j(\omega)=[A_u(z)]_{jj}/[A_y(z)]_{jj}, \; z=e^{-i\omega},$$

where $[A_u(z)]_{jj}$ and $[A_y(z)]_{jj}$ are the j'th diagonal elements of $A_u(z)$ and $A_y(z)$ respectively. Thus, $r_j(\omega)$ is the variance of the j'th component of $dz_u(\omega)$ relative to the j'th component of $dz_y(\omega)$, i.e. the variance of the error relative to the variance of the data for each frequency. A plot of $r_j(\omega)$ against frequency shows how well the economic model fits the data over different frequencies. Integrating the numerator and denominator of $r_j(\omega)$ provides an overall measure of fit. Note that since u_t and x_t are correlated, $r_j(\omega)$ can be larger than 1, i.e. the R^2 of the model can be negative.

One advantage of $r_j(\omega)$ is that it is unaffected by time invariant linear filters applied to the variables. Filtering merely multiplies both the numerator and denominator of $r_j(\omega)$ by the same constant, the squared gain of the filter. So for example, $r_j(\omega)$ is invariant to Hodrick-Prescott filtering (see Hodrick and Prescott [1980] and King and Rebelo [1989]) or standard seasonal adjustment filters.[6] The integrated version of the relative mean square approximation error is not invariant to filtering, since it is a ratio of averages of both the numerator and denominator across frequencies. When the data are filtered, the integrated verson of $r_j(\omega)$ changes because the weights implicit in the averaging change. Frequencies for which the filter has a large gain are weighted more heavily than frequencies with a small gain.

3. Measures of Fit for a RBC Model

In this section we investigate the coherence of a standard real business cycle model with post-war U.S. data using the measures of fit developed in the last section. The model, which derives from Kydland and Prescott (1982) is the "baseline" model detailed in King, Plosser, and Rebelo (1988b). It is a one sector neoclassical growth model driven by an exogenous stochastic trend in technology.[7]

This baseline model is analyzed, rather than a more complicated variant, for several reasons. First, the calibration/simulation exercise reported in King, Plosser and Rebelo suggest that the model explains the relative variability of aggregate output, consumption and investment, and produces series with serial correlation properties broadly similar to the serial correlation properties of post-war U.S. data. Second, King, Plosser, Stock, and Watson (1991) show that the low-frequency/cointegration implications of the model are broadly consistent with similar post-war U.S. data. Finally, an understanding of the where this baseline model fits the data and where it doesn't fit, may suggest how the model should be modified.

Only a brief sketch of the model is presented; a thorough discussion is contained in King, Plosser, and Rebelo (1989a,1989b). The details of the model are as follows:

Preferences:

$$E_0\sum_{t=0}^{\infty}\beta^t u(C_t,L_t), \text{ with}$$
$$u(C_t,L_t) = \log(C_t) + \theta\log(L_t)$$

Technology:

$$Q_t = K_t^{1-\alpha}(A_t N_t)^{\alpha}, \text{ with}$$
$$\log(A_t) = a_t = \gamma_a + a_{t-1} + \epsilon_t, \quad \epsilon_t \text{ iid}(0,\sigma_\epsilon^2)$$
$$K_{t+1} = (1-\delta)K_t + I_t$$

Constraints:

$$Q_t = C_t + I_t$$
$$1 = N_t + L_t$$

where C_t denotes consumption, L_t is leisure, Q_t is output, K_t is capital, N_t is labor input, I_t is investment, and A_t is the stock of technology, which is assumed to follow a random walk with drift γ_a and iid innovation ϵ_t.

To analyze the model's empirical predictions, the equilibrium of the model is calculated as a function of the parameters β, θ, α, γ_a, σ_ϵ^2 and δ. This equilibrium implies a stochastic process for the variables C_t, L_t, N_t, K_t, I_t and Q_t, and these stochastic processes can then be compared to the stochastic processes characterizing U.S. post-war data. As is well known, the equilibrium can be calculated by maximizing the representative agent's utility function subject to the technology and the resource constraints. In general, a closed form expression for the equilibrium does not exist and numerical methods must be used to calculate the stochastic process for the variables corresponding to the equilibrium. A variety of numerical approximations have been proposed (see Taylor and Uhlig (1989) for a survey); here I use the log-linearization of the Euler equations proposed by King, Plosser, and Rebelo (1987).[8] A formal justification for approximating the equilibrium of this stochastic nonlinear model near its deterministic steady state using linear methods is provided in Woodford (1986, Theorem 2).

The approximate solution yields a VAR for the logarithms of Q_t, C_t, K_t, I_t and N_t. (Following the standard convention, these logarithms will be denoted by lower case letters.) Each of the variables except n_t is nonstationary, but can be represented as stationary deviations about a_t, the logarithm of the stock of technology, which by assumption follows an integrated process. Thus, q_t, c_t, i_t, and k_t are cointegrated with a single common trend, a_t. Indeed, the variables in the VAR are not only cointegrated, they are singular; the singularity follows since ϵ_t is the only shock to the system. The coefficients in the VAR are complicated functions of the structural parameters β, θ, α, γ_a, σ_ϵ^2 and δ. Values for these parameters are the same as those used by King, Plosser, and Rebelo (1989b) and the reader is referred to their work for a detailed discussion of the values chosen for these parameters. Assuming that the variables are measured quarterly, the parameter values are: α=.58, δ=.025, γ_a=.004, σ_ϵ=.010, β=.988, and θ is chosen so that the steady state value of N is 0.20. Using these values for the parameters, the VAR describing the equilibrium can be calculated and the autocovariance generating function of x_t=(Δq_t Δc_t Δi_t n_t)' follows directly.[9]

These autocovariances will be compared to the autocovariances of post-war data for the United States. The data used here are the same data used by King, Plosser, Stock, and Watson (1989). The output measure is total real private GNP, defined as total real GNP less government purchases of goods and services. The measure of consumption is total real consumption expenditures and the measure of investment is total real fixed investment. The measure of labor input is total labor hours in private nonagricultural establishments. All variables are in per capita terms using the total civilian

noninstitutional population over the age of 16.[10] Letting $\tilde{q}_t$ denote the log of per capita private output, $\tilde{c}_t$ the log of per capita consumption expenditures etc., the data used in the analysis will be written as $y_t=(\Delta\tilde{q}_t \quad \Delta\tilde{c}_t \quad \Delta\tilde{i}_t \quad \tilde{n}_t)'$.

The analysis presented in the last section assumed that the autocovariance generating function/spectrum of y_t was known. In practice of course this is not the case, and the spectrum must be estimated. In this work, the spectrum of y_t was estimated in two different ways. First, an autoregressive spectral estimator was used, calculated by first estimating a VAR for the variables and then forming the implied spectral density matrix. Following King, Plosser, Stock and Watson (1989) the VAR was estimated imposing the constraint that output, consumption and investment were cointegrated. Thus, the VAR was specified as the regression of y_t onto a constant, six lags of y_t, and the error-correction terms $\tilde{q}_{t-1}-\tilde{c}_{t-1}$ and $\tilde{q}_{t-1}-\tilde{I}_{t-1}$. The parameters of the VAR were estimated using data from 1950 through 1988. (Values before 1950 were used as lags in the regression for the initial observations.) Second, a standard nonparametric spectral estimator was also calculated. The spectrum was estimated by a simple average of 10 periodgram ordinates after pre-whitening employment with the filter (1-.95L). These two estimators yielded similar values for the measures of fit, and to conserve space only the results for the autoregressive spectral estimator are reported.

For each variable, Figure 1 presents the spectrum implied by the model, the spectrum of the data, and the spectrum of the error required to reconcile the model with the data. Since the spectral density matrix of variables in the model has rank one, the joint error process is determined by minimizing the

variance of only one of the errors. The error spectra shown in Figure 1 were calculated by minimizing the error associated with output growth, $\Delta q_t - \Delta \tilde{q}_t$. For output, consumption and investment, the model sectra and the data spectra are similar for very low frequencies (periods greater than 50 quarters) and, for output and investment, at high frequencies (periods less than 5 quarters). There are significant differences between the spectra for periods typically associated with the business cycle; the largest differences occur at a frequency corresponding to approximately 10 quarters. The spectra of n_t and $\tilde{n}_t$ are quite different. The employment data have much more low frequency movement than is predicted by the model.[11]

The figure implies that relatively little measurement error is needed to reconcile the model and the data for output, consumption and investment over the very low frequencies. On the other hand, measurement error with a variance on the order of 30% to 55% of the magnitude of the variance or the series is needed for the output, consumption and investment components with periods in the 6-32 quarter range. At higher frequencies, this representation is able to match movements in output, but not in the other variables.

Table 1 provides a summary of the relative mean square approximation error for a variety of minimum error representations and filters. Each panel shows the relative mean square error (mse) for each variable constructed from four different minimum error representations. The first column of each panel provides a summary of the minimum output error representation, the second column presents results from the representation that minimizes the consumption error, the third column shows the results from the minimum investment error representation, and the final column shows the results from the minimum

employment error representation. The top panel presents the results for the first differences of the data integrated across all frequencies; the middle panel shows the results for the levels of the series detrended by the Hodrick-Prescott filter integrated across all frequencies, and the bottom panel presents the results for the levels of the series integrated over business cycle frequencies (6-32 quarters). The tradeoff inherent in the different representations is evident in all panels. For example in the top panel, using the minimum output error representation, the relative mse for output growth is 26%, while the relative mse for consumption growth is 78%; when the minimum consumption error representation is chosen, the relative mse of consumption growth can be reduced to 30%, but the relative mse for output growth increases to 76%. The bottom two panels show that, at least for output, consumption and investment, most of this tradeoff occurs at the high frequencies: for the business cycle frequencies the relative mse's are generally in the 40%-60% range.[12]

Given the minimum measurement error representation developed in section 2, it is possible to calculate x_t from the realization $(\ldots, y_{-1}, y_0, y_1, \ldots)$. Since the measurement error model represents y_t as x_t plus error, standard signal extraction formula can be used to extract $\{x_t\}$ from $\{y_t\}$. In general, of course, signal extraction methods will yield an estimate of x_t, say $\dot{x}_t$, that is not exact in the sense that $E[(x_t-\dot{x}_t)^2]\neq 0$. In the present context, the estimate will be exact since the measurement error process is chosen so that $dz_x(\omega)$ and $dz_y(\omega)$ are perfectly correlated for all ω.[13] Figure 2 shows the realizations of the data, and the realizations of the variables in the model calculated from the data using the minimum output error representation.[14]

Looking first at Figure 2a which shows the results for output, the model seems capable of capturing the long swings in the post-war U.S. data, but not capable of capturing all of the cyclical variability in the data. Using the standard NBER peak and trough dates, U.S. private per capita GNP fell by 8.4% from the peak in 1973 to the trough in 1975 and by 7.9% from the peak in 1979 to the trough in 1982. In contrast, the corresponding drops in Q_t -- output in the model -- were 2.7% and 3.3% respectively. The dampened cyclical swings in consumption and fixed investment, shown in Figures 2b and 2c are even more dramatic. Finally, figure 2d shows that model predicts changes in labor input that have little to do with the changes observed in the U.S. during the post-war period.

Before leaving this section six additional points deserve mention. First, the fitted values in figure 2 are quantitatively and conceptually similar to figures presented in Christiano (1988) and Plosser (1989). They calculated the Solow residual from actual data and then simulated the economic model using this residual as the forcing process. Implicitly, they assumed that the model and data were the same in terms of their Solow residual, and then asked whether the model and data were similar in other dimensions. Figure 2 is constructed by making the model and data as close as possible in one dimension (in this case the variance of output growth) and then asking whether the model and data are similar in other dimensions. The difference between the two approaches can be highlighted by considering the circumstance in which they would produce exactly the same figure. If the Solow residual computed from the actual data followed exactly the same stochastic process as the change in productivity in the model, and if the approximation error representation was

constructed by minimizing the variance of the difference between the Solow residual in the data and productivity growth in the model, then the two figures would be identical. Thus, the figures will differ if the stochastic process for the empirical Solow residual is not the same as assumed in the model, or the approximation error representation is chosen to make the model and data close in some dimension other than productivity growth.

Second, the inability of the model to capture the business cycle properties of the data is not an artifact of the minimum measurement error representation used to form the projection of x_t onto y_τ, $\tau=1,\ldots,n$. Rather, it follows directly from a comparison of the spectra of x_t and y_t. The fitted values are constrained to have an ACGF/spectra given by the economic model. Figure 1 shows that, for all the variables, the spectral power over the business cycle frequencies is significantly less for the model than for the data. Therefore, fitted values from the model are constrained to have less cyclical variability than the data.

Third, the ability of the model to mimic the behavior of the data depends critically on the size of the variance of the technology shock. The value of σ_ϵ used in the analysis above is two and half times larger than the drift in the series. Thus, if ϵ_t were approximately normally distributed, the stock of technology A_t would, on average, fall in 1 out of 3 quarters. Reducing the standard deviation of the technology shock so that it equals the average growth in a_t drastically increases the size of the measurement error necessary to reconcile the model with the data. For example, integrated across all frequencies, the size of the measurement error variance relative to the variance of observed data increases to 63% for output.

Fourth, there is nothing inherent in the structure of the model that precludes the use of classical statistical procedures. Altug (1990) used maximum likelihood methods to study a version of the model which is augmented with serially correlated classical measurement errors. Singleton (1988) and Christiano and Eichenbaum (1990) pointed out that generalized method of moments procedures can be used to analyze moment implications of models like the one presented above. In the empirical work of Christiano and Eichenbaum the singularity in the probability density function of the data that is implied by the model was finessed in two ways. First, limited information estimation and testing methods were used, and second, the authors assumed that their data on labor input was measured with error.

Fifth, many if not all of the empirical shortcomings of this model have been noted by other researchers. King, Plosser, and Rebelo clearly show that the model is not capable of explaining the variation in labor input that is observed in the actual data. The implausibility of the large technology shocks is discussed in detail in Mankiw (1989), McCallum (1989) and Summers (1986).

Finally, the analysis above has concentrated on the ability of the model to explain the variability in output, consumption, investment and employment across different frequencies. While it is possible to analyze the covariation of these series using the cross spectrum of the measurement error, such an analysis has not been carried out here. This is a particularly important omission, since this is the dimension in which the baseline real business cycle model is typically thought to fail. For example, Christiano and Eichenbaum (1990) and Rotemberg and Woodford (1989) use the model's

counterfactual implication of a high correlation between average productivity and output growth as starting points for their analysis, and the empirical literature on the ICAPM beginning with Hansen and Singleton (1982) suggests that the asset pricing implications of the model are inconsistent with the data. It would be useful to derive simple summary statistics based on the cross spectra of the measurement error and the data to highlight the ability of the model to explain covariation among the series.

4. Statistical Issues

The empirical analysis in the last section highlights two related statistical issues. First, how can uncertainty about the parameters of the economic model and uncertainty about the ACGF of the data be incorporated in the analysis, and second, when the parameters of the economic model are unknown, does it make sense to estimate these parameters by minimizing the relative mean square approximation error?

It is conceptually straightforward to incorporate uncertainty about $A_x(z)$ and $A_y(z)$. Let $\hat{r}_j(\omega)$ be an estimator of $r_j(\omega)$ constructed from $\hat{A}_x(z)$ and $\hat{A}_y(z)$, estimators of $A_x(z)$ and $A_y(z)$ respectively. Given the joint distribution of $\hat{A}_x(z)$ and $\hat{A}_y(z)$ the distribution of $\hat{r}_j(\omega)$ can be readily deduced. This distribution can be used to construct confidence intervals for $r_j(\omega)$ or to carry out other standard inference procedures. This exercise would be like constructing the confidence interval for a regression R^2, which is possible (see Anderson [1984]), but almost never done.

The second issue, using the relative mean square approximation error as a criteria for choosing parameters is more subtle. Dropping the standard

statistical assumption that the economic model is correctly specified raises a number of important issues. Foremost among these is the meaning of the parameters. If the model doesn't necessarily describe the data, then what do the parameters measure? Presumably, the model is meant to describe certain characteristics of the data's stochastic process (the business cycle or the growth properties, for example), while ignoring other characteristics. It is then sensible to define the model's parameters as those that minimize the differences between the model and the data's stochastic process in the dimensions that the model is attempting to explain. So, for example, it seems sensible to define the parameters of a growth model as those that minimize $r_j(\omega)$ over very low frequencies or parameters of a business cycle model as those that minimize $r_j(\omega)$ over business cycle frequencies. Given this definition of the parameters, constructing an analog estimator (see Manski [1987]), by minimizing $\hat{r}_j(\omega)$ corresponds to a standard statistical practice.

The parameters may also be defined using other characteristics of the model and the stochastic process describing the data. If the model is meant to describe certain moments of the data, then the parameters are implicitly defined in terms of these moments and can be efficiently estimated using GMM techniques (see Hansen [1982]).[15] In any event, the important point is that the parameters must be defined in terms of the stochastic process for y_t before properties of estimators of the parameters can be discussed.

Discussion

Two final points deserve mention. First, while this paper has concentrated on measures of fit motivated by a model of measurement error, other measures are certainly possible. For instance, one measure, which like the measures in this paper uses only the autocovariances implied by the model and the data, is the expected log likelihood ratio using the gaussian probability density function of the data and the model. More precisely, if $g(x)$ denotes the gaussian pdf constructed from the autocovariances of the data, $f(x)$ denotes the gaussian pdf constructed for the autocovariances implied by the model, and E_g is the expectation operator taken with respect to $g(x)$, the expected log likelihood ratio $I(g,f)=E_g(\log[g(x)/f(x)])$ can be used to measure the distance between the densities $f(.)$ and $g(.)$. $I(g,f)$ is the Kullback-Leibler Information Criterion (KLIC) which plays an important role in the statistical literature on model selection (e.g. Akaike [1973]) and quasi-maximum likelihood estimation (White [1982]). Unfortunately, the KLIC will not be defined when $f(x)$ is singular and $g(x)$ is not; the KLIC distance between the two densities is infinite. Thus for example, it would add no additional information on the fit of the real business cycle model analyzed in Section 3 beyond pointing out the singularity.

Finally, since the measures of fit developed in this paper are based on a representation that minimizes the discrepancy between the model and the data, they only serve as a bound on the fit of the model. Models with large relative mean square approximation errors don't fit the data well. Models with small relative mean square approximation errors fit the data well given

certain assumptions about the correlation properties of the noise that comes between the model and the data, but may fit the data poorly given other assumptions about this noise.

Footnotes

1. Also see Hansen and Sargent (1988).

2. The spirit of the analysis in this paper is similar to the analysis in Campbell and Shiller (1989), Cochrane (1989), Durlauf and Hall (1989), and Hansen and Jaganathan (1991). Each of these papers uses a different approach to judge the goodness of fit of an economic model by calculating a value or an upper bound on the variance of an unobserved "noise" or a "marginal rate of substitution" or a "discount factor" in observed data.

3. The reader familiar with work on data revisions will recognize these two sets of assumptions as the ones underlying the "news" and "noise" models of Mankiw, Runkle, and Shapiro (1984) and Mankiw and Shapiro (1986).

4. Interestingly, it is possible to determine whether the dynamic errors-in-variable model or the signal extraction error model is consistent with the model and the data. The dynamic errors-in-variables model implies that $A_y(z)-A_x(z)\geq 0$ for $|z|=1$, so that the spectrum of y_t lies everywhere above the spectrum of x_t; the signal extraction error model implies the converse. If the spectrum of x_t lies anywhere above the spectrum of y_t, the errors-in-variables model is inappropriate; if the spectrum of y_t lies anywhere above the spectrum of x_t, the signal extraction model is inappropriate. If the spectra of x_t and y_t cross, neither model is appropriate.

5. Since Σ_x has rank k, there exists an (n-k)×n matrix $\bar{S}$, with full row rank, such that $\bar{S}x_t=0$. (The rows of $\bar{S}$ can be computed as the eigenvectors of Σ_x corresponding to zero eigenvalues.) Thus,

$$\begin{bmatrix} \tilde{x}_t \\ 0 \end{bmatrix} = \begin{bmatrix} S \\ \bar{S} \end{bmatrix} x_t.$$

Since $S\Sigma_x S'$ has rank k, $(S'\ \bar{S}')'$ is non-singular, which implies that $x_t=B\tilde{x}_t$, where the n×k matrix B contains the first k columns of $[(S'\ \bar{S}')']^{-1}$.

6. Standard seasonal adjustment filters, such as the linear approximations to Census X-11 have zeros at the seasonal frequencies, so that $r_j(\omega)$ is undefined at these frequencies for filtered data.

7. This model is broadly similar to the model analyzed in Kydland and Prescott (1982). While the baseline model does not include the complications of time to build, inventories, time non-separable utility, and a transitory component to technology contained in the original Kydland and Prescott model, these have been shown to be reasonably unimportant for the empirical predictions of the model (see Hansen [1985]). Moreover, the King, Plosser and Rebelo baseline model appears to fit the data better at the very low frequencies than the original Kydland and Prescott model since it incorporates a stochastic trend rather than the deterministic trend present in the Kydland and Prescott formulation.

8. Sergio Rebelo kindly provided the computer software to calculate the approximate solution.

9. Of course, this is not the only possible definition of x_t. The only restriction on x_t is covariance stationarity, so for example the log ratios $c_t - q_t$ and $i_t - q_t$ could be included as elements.

10. All data are taken from Citibase. Using the Citibase labels, the precise variables used were gnp82-gge82 for output, gc82 for consumption, and gif82 for investment. The measure of total labor hours was constructed as total employment in nonagricultural establishments (lhem) less total government employment (lpgov) multiplied by average weekly hours (lhch). The population series was P16.

11. Figure 1 is reminiscent of figures in Howrey (1971) (1972) who calculated the spectra implied by the Klein-Goldberger and Wharton Models. A similar exercise is carried out in Soderlind (1991), who compares the spectra of variables in the Kydland-Prescott model to the spectra of post-war U.S. data.

12. Using the notation introduced in Section 2 (see equation 2.6), Table 1 shows the relative mean squared approximation errors for four different

choices of S. Lars Hansen has suggested that it would be useful to graphically present the results for *all* values of S, which would trace out the complete set of possible rmse combinations and more effectively show the tradeoff.

13. More precisely, the estimate is exact in the sense that $P(x_t|y_{t-j},\ldots,y_{-1}, y_0, y_1,\ldots,y_{t+j})$ converges in mean square to x_t as $j \to \infty$.

14. The estimates of x_t were calculated as the inverse fourier transform of the fourier transform of y_t multiplied by the estimated gain from equation (2.10), i.e., x_t is calculated as the inverse fourier transform of $\Gamma(\omega)d\hat{z}_y(\omega)$, where $\Gamma(\omega)$ is given in equation (2.9) and $d\hat{z}_y(\omega)$ is the finite fourier transform of y_t, $t=1,\ldots,n$. This procedure induces slight errors near the beginning and end of the sample. However, because the lead/lag coefficients in the projection of x_t onto y_τ, $\tau=1,\ldots n$, are small for this model, this error is not expected to be large.

15. A careful analysis of a more complicated version of the model discussed in the last section is carried out by Christiano and Eichenbaum (1990) using GMM methods.

Appendix

Derivation of (2.5):

The function to be minimized is:

$$(A.1) \quad \sum_{i=1}^{n} \Sigma_{u,ii} w_i,$$

where $\Sigma_{u,ii}$ is the ii'th element of $\Sigma_u = \Sigma_x + \Sigma_y - \Sigma_{xy} - \Sigma_{yx}$, and w_i are a set of non-zero constants. Since Σ_x and Σ_y are given, (A.1) can be minimized by maximizing the function:

$$(A.2) \quad \sum_{i=1}^{n} \Sigma_{xy,ii} w_i,$$

where $\Sigma_{xy,ii}$ is the ii'th element of Σ_{xy}.

It is convenient to parameterize the covariance matrices as $\Sigma_x = F'F$, $\Sigma_y = G'G + H'H$, and $\Sigma_{xy} = F'G$, where the matrices F, G and H will be chosen to maximize (A.2). This parameterization imposes the constraint that the resulting covariance matrix for $(x_t' \; y_t')'$ is positive semi-definite. The minimum approximation error representation can be found by choosing F, G, and H to maximize (A.2) subject to the constraints: $\Sigma_x = F'F$ and $\Sigma_y = G'G + H'H$.

Letting F_i, G_i and H_i denote the i'th column of F, G and H respectively, the Lagrangian is:

$$(A.3) \quad L = \sum_{i=1}^{n} F_i' G_i w_i - \sum_{i=1}^{n} [\sum_{j=1}^{i-1} \lambda_{ij} (F_i' F_j - \Sigma_{x,ij}) + \tfrac{1}{2} \lambda_{ii} (F_i' F_i - \Sigma_{x,ii})]$$
$$- \sum_{i=1}^{n} [\sum_{j=1}^{i-1} \theta_{ij} (G_i' G_j + H_i' H_j - \Sigma_{y,ij}) + \tfrac{1}{2} \theta_{ii} (G_i' G_i + H_i' H_i - \Sigma_{y,ii})]$$

where λ_{ij} and θ_{ij} are the Lagrange multipliers for the constraints. The first order conditions are:

$$\text{(A.4.i)} \quad \partial L/\partial F_i = G_i w_i - \sum_{j=1}^{n} \lambda_{ij} F_j = 0, \qquad i=1,\ldots,n,$$

$$\text{(A.5.i)} \quad \partial L/\partial G_i = F_i w_i - \sum_{j=1}^{n} \theta_{ij} G_j = 0, \qquad i=1,\ldots,n,$$

$$\text{(A.6.i)} \quad \partial L/\partial H_i = \sum_{j=1}^{n} \theta_{ij} H_j = 0, \qquad i=1,\ldots,n,$$

$$\text{(A.7)} \quad \Sigma_x = F'F, \text{ and}$$

$$\text{(A.8)} \quad \Sigma_y = G'G + H'H.$$

Horizontally concantenating (A.4.i), (A.5.i) and (A.6.i) for i=1,...,n, yields:

$$\text{(A.9)} \quad GW = F\Lambda$$

$$\text{(A.10)} \quad FW = G\Theta$$

$$\text{(A.11)} \quad 0 = H\Theta,$$

where W is a diagonal matrix with w_i on the diagonal, and Λ and Θ are symmetric matrices with typical elements λ_{ij} and θ_{ij}, respectively.

Since F and W are non-singular, (A.7)-(A.11) imply that H=0. The first order conditions can then be solved by finding factors of Σ_x and Σ_y, F and G, such that $F^{-1}GW$ and $G^{-1}FW$ are symmetric. Equivalently, F and G must chosen so that FWG' is symmetric.

Let C_x and C_y denote (arbitrary) matrix square roots of Σ_x and Σ_y, i.e., $\Sigma_x = C_x'C_x$ and $\Sigma_y = C_y'C_y$, and let $C = C_x W C_y'$. Notice that C'C can be decomposed

as $C'C=D\Delta D'$, where the columns of D are the orthonormal eigenvectors of $C'C$ and Δ is a diagonal matrix with the eigenvalues of $C'C$ on the diagonal.

The solution to the problem sets $G=C_y$ and $F=RC_x$, where $R=D\Delta^{-\frac{1}{2}}D'C'$. This solution can be verified by noting that $FWG'=RC=D\Delta^{\frac{1}{2}}D'$ is symmetric and that $RR'=R'R=I$, so that $F'F=C_x'C_x=\Sigma_x$. Note that both $F=RC_x$ and $F=-RC_x$ satisfy the first order conditions. The first, $F=RC_x$, corresponds to the value of F that maximizes the weighted covariance between the elements of x_t and y_t (and minimizes the weighted sum of the approximation error variance). The second, $F=-RC_x$, corresponds to the value of F that minimizes the weighted covariance between the elements of x_t and y_t (and maximizes the weighted sum of the approximation error variance).

References

Altug, S. (1989), "Time-to-Build and Aggregate Fluctuations: Some New Evidence," International Economic Review, 30, 889-920.

Anderson, T.W. (1984), An Introduction to Multivariate Analysis, Second Edition, John Wiley and Sons, New York.

Brillinger, D. (1981), Time Series , Data Analysis and Theory, Holden Day, San Francisco.

Campbell, J.Y. and R.J. Shiller (1989), "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors," The Review of Financial Studies, 1, 195-228.

Christiano, L.J. (1988), "Why Does Inventory Fluctuate So Much," Journal of Monetary Economics, 21, 247-80.

Christiano, L.J. and M. Eichenbaum (1990), "Current Real Business Cycle Theories and Aggregate Labor Market Fluctuations," Discussion Paper 24, Institute for Empirical Macroeconomics.

Cochrane, J. H. (1989), "Explaining the Variance of Price Dividend Ratios," manuscript, University of Chicago.

Durlauf, S.N. and R.E. Hall (1989), "Measuring Noise is Stock Prices," manuscript, Stanford University.

Geweke, J. (1977) "The Dynamic Factor Analysis of Economic Time Series," in D.J. Aigner and A.S. Goldberger eds., Latent Variables in Socio-Economic Models, North-Holland, Amsterdam, Ch. 19.

Haavelmo, Trygve (1944), "The Probability Approach in Econometrics," Econometrica, Vol. 12, Supplement.

Hansen, G. D. (1985), "Indivisible Labor and the Business Cycle," Journal of Monetary Economics, 16, 309-327.

Hansen, G.D. and T. Sargent (1988), "Straight Time and Overtime in Equilibrium" Journal of Monetary Economics, 21.

Hansen, L.P. and R. Jaganathan (1991), "Implications of Security Market Data for Models of Dynamic Economies," Journal of Political Economy, 99, 225-262.

Hansen, L.P and T. Sargent (1980), "Formulating and Estimating Dynamic Linear Rational Expectations Models," Journal of Economic Dynamics and Control, 2, 7-46.

Hansen, L.P. and K.J. Singleton (1982), "Generalized Instrumental Variable Estimation of Nonlinear Rational Expectations Models," Econometrica, 50, 1269-1286.

Hodrick, R. and E.C. Prescott (1980), "Post-War U.S. Business Cycles: An Empirical Investigation," manuscript, Carnegie-Mellon University.

Howrey, E.P. (1971), "Stochastic Properties of the Klein-Goldberger Model," Econometrica, 39, 73-87.

Howrey, E.P. (1972), "Dynamic Properties of the Wharton Model," in Econometric Models of Cyclical Behavior, volume 2, edited by B. Hickman, Columbia University Press, New York.

King, R. G., C.I. Plosser and S.T. Rebelo (1987), "Production, Growth, and Business Cycles: Technical Appendix," Manuscript, University of Rochester.

King, R. G., C.I. Plosser and S.T. Rebelo (1988a), "Production, Growth, and Business Cycles: I. The Basic Neoclassical Model," Journal of Monetary Economics, 21, 195-232.

King, R. G., C.I. Plosser and S.T. Rebelo (1988b), "Production, Growth, and Business Cycles: II. New Directions," Journal of Monetary Economics, 21, 309-342.

King, R. G., C.I. Plosser, J.H. Stock and M.W. Watson (1991), "Stochastic Trends and Economic Fluctuations," forthcoming American Economic Review.

King, R. G. and S.T. Rebelo (1989), "Low Frequency Filtering and Real Business Cycle Models," Working Paper No. 205, Rochester Center for Economic Research.

Kydland, F.E. and E.C. Prescott (1982), "Time To Build and Aggregate Fluctuations," Econometrica, 50, 1345-1370.

Mankiw, N.G. (1989), "Real Business Cycles: A New Keynesian Perspective," Journal of Economic Perspectives, 3, 79-90.

Mankiw, N.G., D.E. Runkle and M.D. Shapiro (1984), "Are Preliminary Announcements of the Money Supply Rational Forecasts," Journal of Monetary Economics, 13, 15-27.

Mankiw, N.G. and M.D. Shapiro (1986), "News or Noise: An Analysis of GNP Revisions," Survey of Current Business, 66, 20-25.

Manski, C. F. (1987), Analogue Estimation Methods in Econometrics, Chapman and Hall, New York.

McCallum, Bennett T. (1989), "Real Business Cycle Models," in Modern Business Cycle Theory, edited by Robert J. Barro, Cambridge: Harvard University Press.

McFadden, D. (1981), "Econometric Models of Probabilistic Choice," in Structural Analysis of Discrete Data, edited by C. Manski and D. McFadden, 198-272, Cambridge: MIT Press.

Plosser, C. I. (1989), "Understanding Real Business Cycles," Journal of Economic Perspectives, 3, 51-77.

Prescott, E. C. (1986), "Theory Ahead of Business Cycle Measurement," Carnegie-Rochester Conference Series on Public Policy, 25 (Autumn), 11-44.

Rotemberg, J. and M. Woodford (1989), "Oligopolistic Pricing and the Effects of Aggregate Demand on Economic Activity," NBER Working Paper 3206.

Sargent, T. J. (1989), "Two Models of Measurements and the Investment Accelerator," Journal of Political Economy, 97, 251-287.

Sargent, T.J, and C.A. Sims (1977), "Business Cycle Modeling without Pretending to have Too Much a-priori Economic Theory," in C. Sims et al., New Methods in Business Cycle Research, Minneapolis: Federal Reserve Bank of Minneapolis.

Singleton, K.J. (1988), "Econometric Issues in the Analysis of Equilibrium Business Cycle Models," Journal of Monetary Economics, 21, 361-386.

Soderlind, Paul (1991), "Is There a Cycle in Real Business Cycle Models?" manuscript, Stockholm University.

Summers, L.H. (1986), "Some Skeptical Observations of Real Business Cycle Theory," Federal Reserve Bank of Minneapolis Quarterly Review, 10, 23-27.

Taylor, J.B. and H. Uhlig (1990), "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods," Journal of Business and Economic Statistics, 8, 1-18.

White, H. (1982), "Maximum Likelihood Estimation of Misspecified Models," Econometrica, 50, 1-26.

Woodford, Michael (1986), "Stationary Sunspot Equilibria, The Case of Small Fluctuations Around a Deterministic Steady State," mimeo, The University of Chicago.

Table 1

Relative Mean Squared Approximation Error
Baseline Real Business Cycle Model
Minimum Variance Representations

A. First Differences -- all frequencies

	Error Minimized with respect to			
Variable	Output	Consumption	Investment	Employment
Output	.26	.76	.64	.79
Consumption	.78	.30	.75	.98
Investment	.63	.76	.28	.78
Employment	.71	.79	.71	.56

B. HP Levels -- all frequencies

	Error Minimized with respect to			
Variable	Output	Consumption	Investment	Employment
Output	.38	.61	.51	.66
Consumption	.62	.36	.66	.89
Investment	.50	.66	.38	.65
Employment	.74	.86	.73	.61

C. Levels -- 6 - 32 quarters

	Error Minimized with respect to			
Variable	Output	Consumption	Investment	Employment
Output	.40	.57	.44	.60
Consumption	.58	.40	.60	.81
Investment	.48	.61	.43	.61
Employment	.73	.85	.72	.61

Notes: Output, Consumption, and Investment are log first differences of quarterly values. Employment is the log of quarterly labor input. See the text for precise definitions. Each column presents the relative mean square of the row variable constructed from the representation that minimizes the measurement error variance for the column variable. Relative mean square approximation error is the lower bound on the variance of the approximation error divided by the variance of the data.

Figure 1
Decomposition of Spectra
Data, Model and Approximation Error
(Variance of Output Error Minimized)

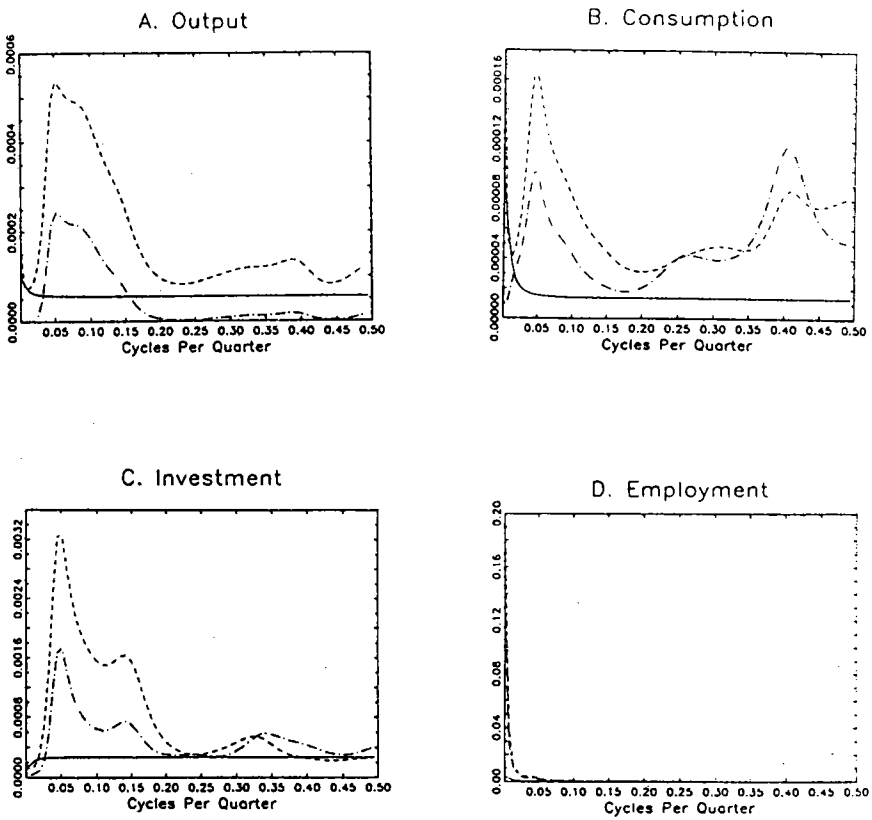
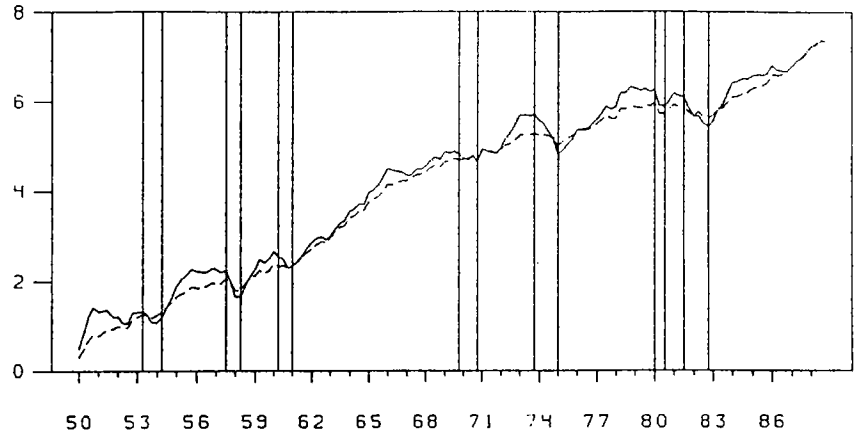
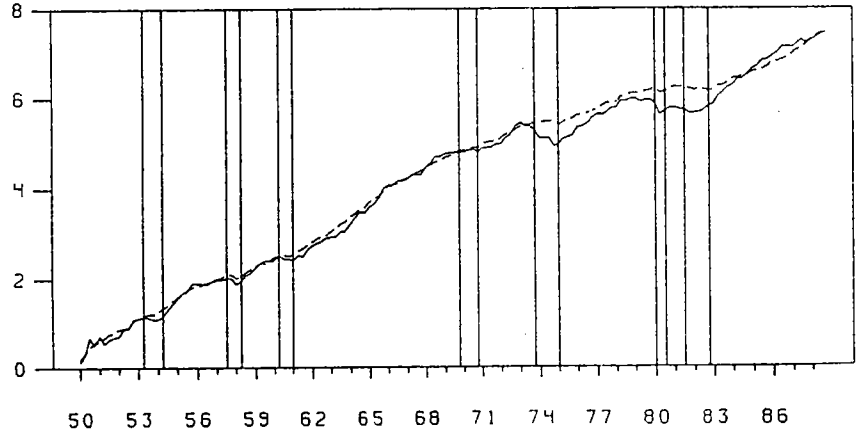


Figure 2
Historical Series
Actual Data and Realization from Model
(Variance of Output Error Minimized)
(Log Scale)

A. Output



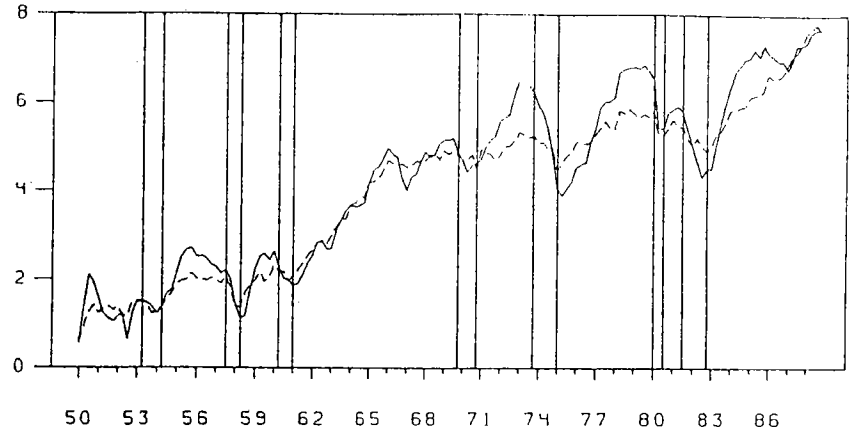
B. Consumption



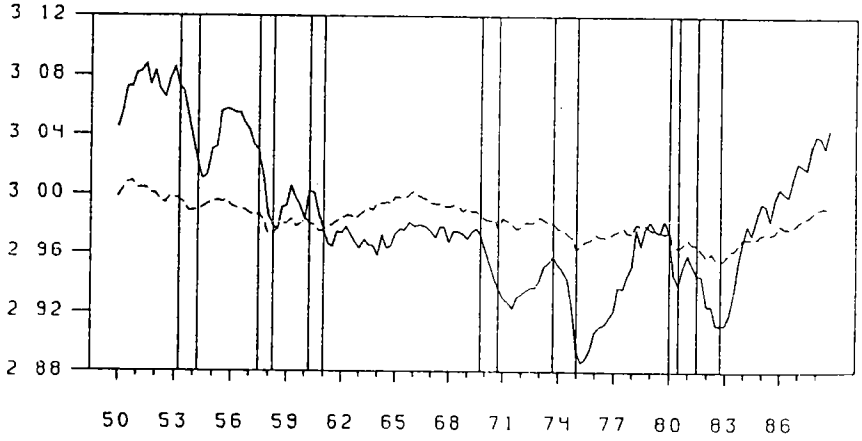
Realization from U.S. Economy ————————
Realization from Model — — — — — —

Figure 2
(Continued)

C. Investment



D. Employment



Realization from U.S. Economy ————————
Realization from Model — — — — — —