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STANDARD RISK AVERSION

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ABSTRACT

This paper introduces the concept of standard risk aversion. A von Neumann-Morgenstern utility function has standard risk aversion if any risk makes a small reduction in wealth more painful (in the sense of an increased reduction in expected utility) also makes any undesirable, independent risk more painful. It is shown that, given monotonicity and concavity, the combination of decreasing absolute risk aversion and decreasing absolute prudence is necessary and sufficient for standard risk aversion. Standard risk aversion is shown to imply not only Pratt and Zeckhauser's "proper risk aversion" (individually undesirable, independent risks always being jointly undesirable), but also that being forced to face an undesirable risk reduces the optimal investment in a risky security with and independent return. Similar results are established for the effect of broad class of increases in one risk on the desirability of (or optimal investment in) a second, independent risk.

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I. Introduction

The idea of John Pratt (1964) and Kenneth Arrow (1965) that "absolute risk aversion" should decrease with wealth has turned out to be surprisingly powerful. It unifies such apparently diverse magnitudes as the wealth elasticity of risky investment,[1] the effect of wealth on risk premia,[2] the effect of compensated increases in risk on saving[3] and the strength of the precautionary saving motive relative to the strength of risk aversion.[4] The additional structure provided by the assumption of decreasing absolute risk aversion often allows models that otherwise yield ambiguous results to give clear predictions consistent with economic intuition.[5]

In view of the success of decreasing absolute risk aversion in imposing sensible structure on the von Neumann-Morgenstern theory of expected utility maximization,[6] it is natural to ask if any further structure can sensibly be imposed. Addressing this question—and cognizant of the potential importance of background risk in influencing economic decisions—Pratt and Richard Zeckhauser (1987) suggest in a pathbreaking paper that a utility function should exhibit "proper risk aversion." Simply stated, proper risk aversion means that an agent forced to accept an undesirable risk will be less willing to accept other, statistically independent risks. Proper risk aversion is one way of formalizing the notion that, aside from the effects of any statistical relationship, risks should tend to crowd each other out. In their pathbreaking paper, Pratt and Zeckhauser prove that a sufficient condition for proper risk aversion is for all odd derivatives of a utility function to be positive and all even derivatives negative,[7] and that the much weaker condition $a''(w) \geq a'(w)a(w)$, where $a(w)$ is absolute risk aversion, is necessary for proper risk aversion.

As I will show, the simpler and less restrictive assumption of *decreasing absolute prudence*—

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[1] See Arrow (1965) and Pratt (1964).

[2] See Pratt (1964).

[3] See Jacques Drèze and Franco Modigliani (1972).

[4] See Kimball (1990a).

[5] For example, the assumption of decreasing absolute risk aversion allows one to show that third-order stochastic dominance guarantees that a distribution will be preferred (Jonathan E. Ingersoll, 1987, pp. 138–139) and that any synchronization of stock market declines with large movements in consumption should raise the equity premium (N. Gregory Mankiw, 1986).

[6] Though many recent articles (*e.g.*, Machina (1982)) question expected utility maximization, expected utility maximization is likely to remain important in applications and as a standard of comparison for non-expected-utility theories, much as the often-questioned notion of perfect competition remains important in applications and as a standard of comparison for theories of imperfect competition. Therefore, even if one discounts defenses of expected utility maximization such as Samuelson (1988), it seems worthwhile to study the implications of adding additional structure to the von Neumann-Morgenstern theory of expected utility maximization. Pratt and Zeckhauser (1987) ably defend the approach we share of analyzing normatively attractive properties of utility functions as an important complement to the approach of analyzing the "chronic violations of the traditional axioms" that often show up in experiments.

[7] As they point out, this is equivalent to being able to construct the utility function as a positive linear combination of exponential utility functions.

$\frac{-u'''(w)}{u''(w)}$ being decreasing in w or $\ln(-u''(w))$ being convex in w—added to the usual assumptions of *monotonicity, concavity,* and *decreasing absolute risk aversion,* is a sufficient condition for proper risk aversion.

Moreover, given monotonicity and concavity of the utility function, the combination of decreasing absolute risk aversion and decreasing absolute prudence is necessary and sufficient for *standard risk aversion.*[8] By my definition, a utility function exhibits "standard risk aversion" or "standardness" if any risk that makes a small reduction in wealth more painful also makes any undesirable, statistically independent risk more painful—where I measure "pain" by the reduction in expected utility associated with a change. This paper makes the case for standard risk aversion as the most logical amplification of decreasing absolute risk aversion.

In the context of a two-period model with additive time-separability, Kimball (1990a) shows that absolute prudence $\frac{-u'''(w)}{u''(w)}$ measures the strength of the precautionary saving motive, so that decreasing absolute prudence can be interpreted as a precautionary saving motive that decreases in intensity with wealth.[9] This paper shows that decreasing absolute prudence has important implications for multiple risk bearing even in a timeless model—implications that carry over with little alteration when that timeless model of risk bearing is embedded in the two-period model of the joint saving-risky investment decision in the presence of background risk (Elmendorf and Kimball (1991)).

The arrangement of the paper is determined by the thread of the mathematical argument. Section II continues the introduction by contrasting standardness and properness. Section III presents formal definitions of properness and standardness and formal statements of three central propositions, leading into the argument of Sections IV–VI about necessity and the effect of background risks. Section VII establishes key lemmas about the comparative statics of risk, leading into the sufficiency argument of Section VIII and its extension to increases in risk in Section IX. Sections X and XI indicate in complementary ways that decreasing absolute prudence, rather than decreasing absolute risk aversion, is the key property behind standard risk aversion. Section X shows that—given monotonicity and concavity—decreasing absolute prudence over a semi-infinite interval implies decreasing absolute risk aversion over that interval. Section XI shows that even on a finite interval, decreasing absolute prudence alone goes most of the way toward guaranteeing standard risk aversion. Section XII is a brief conclusion.

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[8] Throughout the paper, "decreasing" means "weakly decreasing," except when otherwise noted.

[9] Kimball (1990b) presents a number of arguments for decreasing absolute prudence beyond those given here. Most commonly used utility functions have decreasing absolute prudence.

II. Standard Risk Aversion versus Proper Risk Aversion

Though standard risk aversion and proper risk aversion are both about negative interactions between statistically independent risks, the two concepts differ both mathematically and in the economic behavior they imply.

Mathematically, one might hope that an amplification of decreasing absolute risk aversion would share some of the virtues of decreasing absolute risk aversion as a theoretical construct. First, there is a simple analytical condition that is a necessary and sufficient for decreasing absolute risk aversion. The analytical condition makes it possible to establish many consequences of decreasing absolute risk aversion that are not obvious from its definition. Second, decreasing absolute risk aversion applies equally to risk premia and the optimal level of risky investment.[10] Third, decreasing absolute risk aversion is preserved under expectations, making propositions involving decreasing absolute risk aversion valid even in the presence of independent background risk. (Pratt (1964) establishes the first and second virtues of decreasing absolute risk aversion, while Kihlstrom, Romer and Williams (1981) and Nachman (1982) establish the third.) Standard risk aversion shares these virtues; Proper risk aversion does not.

As shown below, standard risk aversion has an analytical necessary and sufficient condition that, beyond incorporating the condition for decreasing absolute risk aversion, is just as simple as the condition for decreasing absolute risk aversion. The analytical condition for standardness allows one to establish many consequences of standardness that are not obvious from its definition. For example, standard risk aversion applies equally to the effect of one risk on the risk premium of another independent risk and to its effect on the optimal level of risky investment in an independent risk. Standard risk aversion also has natural implications for behavior in reaction to a broad class of increases in risk as well as to initiations of risk. Standardness is preserved under expectations, and standardness is the same property whether defined in a way that assumes nonstochastic initial wealth in a way that allows for independent background risk.

By contrast, for proper risk aversion, the analytical necessary and sufficient condition is quite complex.[11] It is unlikely to be useful in establishing many consequences of properness that are not obvious from its definition. Proper risk aversion has no discernible implications about the effect of an undesirable risk on the optimal level of investment in a statistically independent risk. Finally,

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[10] Decreasing absolute risk aversion guarantees both that risk premia for discrete chunks of risk are decreasing in wealth and that the optimal level of risky investment is increasing in wealth.

[11] For fixed-wealth properness, the analytical necessary and sufficient condition involves three universal quantifiers. For (random-wealth) properness, the analytical necessary and sufficient condition would involve five universal quantifiers.

two variant notions of properness—fixed-wealth properness, which depends on nonstochastic initial wealth, and unqualified properness, which allows for independent background risk—do not seem to be equivalent (though it is difficult to establish whether properness and fixed wealth properness are equivalent or not).

Economically, the difference between proper risk aversion and standard risk aversion has to do with the set of risks guaranteed to have a negative interaction with other statistically independent, undesirable risks. Standard risk aversion guarantees that any risk that increases the pain of a small reduction in wealth—or equivalently, *any risk that raises expected marginal utility*—will make an agent less willing to accept another independent risk. Proper risk aversion guarantees that *any undesirable risk* will make an agent less willing to accept another independent risk.

Standard risk aversion differs from proper risk aversion because the set of risks that raise expected marginal utility is not the same as the set of undesirable risks. Drèze and Modigliani (1972) and Kimball (1990a) demonstrate that the decreasing absolute risk aversion entailed by both standard risk aversion and proper risk aversion implies that *any undesirable risk raises expected marginal utility.* However, the reverse is not true. When absolute risk aversion is decreasing, there are *many desirable risks* that *raise expected marginal utility* and thereby *increase the pain of a small reduction in wealth.* The main issue between properness and standardness is whether the *desirability of a risk* $\tilde{x}$, or *the effect of* $\tilde{x}$ *on the pain of a small reduction in wealth* is a better guide to the effect of $\tilde{x}$ on the pain of another independent risk $\tilde{y}$.

It seems sensible to argue that the effect of a risk, $\tilde{x}$, on the pain of a small reduction in wealth is a better guide than the desirability of $\tilde{x}$ to its effect on the pain of a second risk, $\tilde{y}$. The effect of $\tilde{x}$ on the pain of facing $\tilde{y}$ is an interaction between two changes. The effect of $\tilde{x}$ on the pain of a small reduction in wealth is also an interaction between two changes, while the desirability of $\tilde{x}$ is a property of one change in isolation.[12]

[12] When the timeless model of this paper is embedded in a two-period model (with additively time-separable utility) as in Elmendorf and Kimball (1991), an increase in expected marginal utility in the second period corresponds to a reduction in first-period consumption. Therefore, one can speak heuristically of risks that raise expected marginal utility as risks that cause an agent to cut back on consumption. (Similarly, one can speak heuristically of risks that increase the pain of other independent risks as risks that make an agent reduce exposure to other independent risks.) Decreasing absolute risk aversion allows one to rank three types of risks from most negative to most positive: risks that reduce both total expected utility and consumption, risks that raise total expected utility but reduce consumption, and risks that raise both total expected utility and consumption. To be more pointed, on the bottom there are risks that make an agent feel poorer and act poorer; on the top there are risks that make an agent feel richer and act richer; in the middle, there are risks that make an agent *feel richer, but act poorer.* The main issue between properness and standardness is whether the effect of a risk on total expected utility or its effect on consumption is a better guide to its effect on exposure to other independent risks. Standard risk aversion guarantees that an agent will *act* poorer by reducing exposure to independent risks whenever the agent *acts* poorer by reducing consumption. Proper risk aversion requires that an agent *feel* poorer as well as *act* poorer by reducing consumption before guaranteeing that she will reduce her exposure to independent risks. It seems sensible to argue that the effect of a risk on consumption should

III. Formalities

Standard risk aversion can be defined formally as follows:

Definition of Standardness: *The utility function $u(\cdot)$ is standard iff for any triple of mutually independent random variables $\tilde{w}$, $\tilde{x}$ and $\tilde{y}$,*

$$\mathrm{E}\, u'(\tilde{w} + \tilde{x}) \geq \mathrm{E}\, u'(\tilde{w}) \tag{1}$$

and

$$\mathrm{E}\, u(\tilde{w} + \tilde{y}) \leq \mathrm{E}\, u(\tilde{w}) \tag{2}$$

imply

$$\mathrm{E}\, u(\tilde{w} + \tilde{x}) - \mathrm{E}\, u(\tilde{w} + \tilde{x} + \tilde{y}) \geq \mathrm{E}\, u(\tilde{w}) - \mathrm{E}\, u(\tilde{w} + \tilde{y}). \tag{3}$$

Replacing (1) with the inequality $\mathrm{E}\, u(\tilde{w} + \tilde{x}) \leq \mathrm{E}\, u(\tilde{w})$ yields the corresponding definition of proper risk aversion.

Given monotonicity and concavity, there are two ways of characterizing *decreasing absolute risk aversion:*

$\mathcal{A}$. *Any undesirable change in wealth makes any undesirable risk more painful.*

$\mathcal{B}$. *Any change in wealth that makes a small reduction in wealth more painful also makes any undesirable risk more painful.*[13]

Substituting "statistically independent risk" for "change in wealth" in $\mathcal{A}$ yields a[14] verbal definition of *proper risk aversion*:

$\mathcal{A}'$. *Any undesirable statistically independent risk makes any undesirable risk more painful.*

The same substitution in $\mathcal{B}$ yields a verbal definition of *standard risk aversion*:

$\mathcal{B}'$. *Any statistically independent risk that makes a small reduction in wealth more painful also makes any undesirable risk more painful.*

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be a better guide than its effect on total expected utility to its effect on exposure to other risks. Standard risk aversion predicts the cross-effect of a risk on one decision variable (the level of exposure to an independent risk) from the cross-effect of a risk on another decision variable (consumption). Proper risk aversion predicts the cross-effect of a risk on one decision variable (exposure to independent risk) from the direct effect of the risk on expected utility.

[13] Appendices A and D show that $\mathcal{A}$ and $\mathcal{B}$ are equivalent to other, more familiar definitions of decreasing absolute risk aversion.

[14] Appendix G shows that $\mathcal{A}'$ is equivalent to Pratt and Zeckhauser's definitions of proper risk aversion.

Since a certain change in wealth is a risk with a degenerate probability distribution which is (trivially) independent of all other risks, $\mathcal{A}'$ implies $\mathcal{A}$ and $\mathcal{B}'$ implies $\mathcal{B}$; both proper risk aversion and standard risk aversion entail decreasing absolute risk aversion. Given monotonicity and concavity, the two characterizations of decreasing absolute risk aversion, $\mathcal{A}$ and $\mathcal{B}$, are equivalent, since the set of undesirable changes in wealth is the same as the set of changes in wealth that make a small reduction in wealth more painful. It is shown in Appendix G that each of the four statements above is equivalent to the corresponding statement in which "makes any undesirable risk more painful" is replaced by "makes any undesirable risk remain undesirable."

As discussed above, the definition of proper risk aversion, $\mathcal{A}'$, is not equivalent to the definition of standard risk aversion, $\mathcal{B}'$, because the set of undesirable risks is not the same as the set of risks that make a small reduction in wealth more painful. The decreasing absolute risk aversion entailed by both properness and standardness implies that the set of undesirable risks is a subset of the set of risks that increase the pain of a small reduction in wealth, so that $\mathcal{B}'$ implies $\mathcal{A}'$. Thus, standardness is stronger than properness.

It may not be obvious that (1) can be interpreted as saying that $\tilde{x}$ makes a small reduction in wealth more painful. Appendix F shows that the definition of standardness above is equivalent to a definition in which (1) is replaced by the statement that for the particular risk $\tilde{x}$, there is some h for which

$$\mathrm{E}\,u(\tilde{w}+\tilde{x})-\mathrm{E}\,u(\tilde{w}+\tilde{x}-\epsilon)\geq \mathrm{E}\,u(\tilde{w})-\mathrm{E}\,u(\tilde{w}-\epsilon) \tag{4}$$

for all $\epsilon \in [0,h]$.

One way to read (4) and (1) is "$\tilde{x}$ increases the pain of a small (or infinitesimal) reduction in wealth." Another way to read (4) and (1) is "a small (or infinitesimal) reduction in wealth increases the pain (or reduces the pleasantness) of the risk $\tilde{x}$. To see this, rewrite (4) as

$$\mathrm{E}\,u(\tilde{w}-\epsilon)-\mathrm{E}\,u(\tilde{w}+\tilde{x}-\epsilon)\geq \mathrm{E}\,u(\tilde{w})-\mathrm{E}\,u(\tilde{w}+\tilde{x}) \tag{5}$$

and (1) as

$$\frac{\partial}{\partial\epsilon}\big(u(\tilde{w}-\epsilon)-u(\tilde{w}+\tilde{x}-\epsilon)\big)\Big|_{\epsilon=0}\geq 0. \tag{6}$$

Similarly, (3) can be read to say that "$\tilde{x}$ makes $\tilde{y}$ more painful," *or* to say that $\tilde{y}$ makes $\tilde{x}$ more painful, as can be seen by rearranging the terms in (3) to get

$$\mathrm{E}\,u(\tilde{w}+\tilde{y})-\mathrm{E}\,u(\tilde{w}+\tilde{y}+\tilde{x})\geq \mathrm{E}\,u(\tilde{w})-\mathrm{E}\,u(\tilde{w}+\tilde{y}). \tag{7}$$

Thus, the following is an alternative statement characterizing standardness:

C'. Any risk that is made more painful (or less pleasant) by a small reduction in wealth is also made more painful (or less pleasant) by any statistically independent, undesirable risk.[15]

Rephrasing the argument for standardness as opposed to mere properness one more time in terms of the C' may make the case clearer. Once one accepts the fact that (assuming decreasing absolute risk aversion) there are many desirable risks $\tilde{x}$ that are made less pleasant by a small reduction in wealth, it seems sensible to argue that the effect of a small reduction in wealth on the pleasantness of a risk $\tilde{x}$ is a better guide than the desirability of $\tilde{x}$ to the effect of a second risk $\tilde{y}$ on the pleasantness of $\tilde{x}$.

The main analytic proposition of the paper is Proposition 1:

Proposition 1: *If $u'(w) \geq 0$ and $u''(w) \leq 0$ over the entire domain of u, then u is standard if and only if both absolute risk aversion $\frac{-u''(w)}{u'(w)}$ and absolute prudence $\frac{-u'''(w)}{u''(w)}$ are monotonically decreasing over the entire domain of u.*

Proposition 2 is a corollary of Proposition 1:

Proposition 2: *If $u'(\cdot) \geq 0$ and $u''(\cdot) \leq 0$ and both absolute risk aversion $\frac{-u''(\cdot)}{u'(\cdot)}$ and absolute prudence $\frac{-u'''(\cdot)}{u''(\cdot)}$ are monotonically decreasing over the entire domain of u, then u is proper.*

Proposition 3 broadens the message of Proposition 1:

Proposition 3: *If $u'(w) \geq 0$ and $u''(w) \leq 0$ over the entire domain of u, then any expected-marginal-utility-increasing risk (that is, a risk satisfying (11)) always lowers the absolute value of the optimal level of investment in any other independent risk if and only if u is standard—that is, if and only if it has decreasing absolute risk aversion and decreasing absolute prudence.*

As shown in Appendix D, decreasing absolute risk aversion, monotonicity and concavity guarantee that u is once-differentiable. These three properties plus decreasing absolute prudence guarantee that u is twice-differentiable.[16] When u is not thrice-differentiable, re-express *monotonically decreasing* $\frac{-u'''(w)}{u''(w)}$ in the above propositions as *convex* $\ln(-u''(w))$.

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[15] Statement C' parallels the following verbal characterization of decreasing absolute risk aversion:

C *Any change in wealth that is made more painful by a small reduction in wealth is also made more painful by any undesirable risk.* (Once stated mathematically, it is clear that C is equivalent to B, just as C' is equivalent to B'.)

[16] Both of these guarantees of differentiability have an exception at the lower limit of the domain of u.

IV. Standardness and Fixed-Wealth Standardness

Replacing the arbitrary stochastic initial wealth $\tilde{w}$ in the definitions of properness and standardness by certain, but otherwise arbitrary initial wealth w yields the definitions of fixed-wealth properness and fixed-wealth standardness.

Clearly, properness implies fixed-wealth properness and standardness implies fixed-wealth standardness.

It is not known whether properness and fixed-wealth properness are equivalent or whether fixed-wealth properness is strictly weaker than properness. Pratt and Zeckhauser (1987) were unable either to prove that fixed-wealth properness implies properness or to find an example of a utility function that was fixed-wealth proper without being fully proper.

On the other hand, it will be shown that (given monotonicity and concavity) the combination of decreasing absolute risk aversion and decreasing absolute prudence is necessary for fixed-wealth standardness and sufficient for unqualified standardness. Therefore, standardness and fixed-wealth standardness are equivalent.

V. Necessity of Decreasing Absolute Risk Aversion and Decreasing Absolute Prudence

The proof that, given $u'(\cdot) > 0$ and $u''(\cdot) < 0$, decreasing absolute risk aversion and decreasing absolute prudence are necessary for fixed-wealth standardness is relatively straightforward.

To prove that decreasing absolute risk aversion is necessary for u to be fixed-wealth standard, specialize the first risk $\tilde{x}$ in the definition of fixed-wealth standardness to a certain negative quantity, $-\epsilon$, where $\epsilon \geq 0$. Concavity of u ensures that $\tilde{x} = -\epsilon$ will satisfy (1) when $\tilde{w}$ is replaced by w. It is obvious that w, $-\epsilon$ and $\tilde{y}$ are mutually independent. Then

$$\mathrm{E}\, u(w + \tilde{y}) \leq u(w) \tag{8}$$

implies

$$\mathrm{E}\, u(w - \epsilon + \tilde{y}) \leq u(w - \epsilon) \tag{9}$$

for any $\epsilon \geq 0$. This statement is itself one definition of decreasing absolute risk aversion. Appendix D shows that it implies the familiar analytical definition of decreasing absolute risk aversion.[17] Appendix G shows that it is equivalent to the characterizations $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of decreasing absolute risk aversion.

[17] If u is already known to be twice-differentiable, one can argue following Pratt (1964) that if $\tilde{y}$ is small, (8) implies

$$\mu = \frac{-u''(w)}{u'(w)}\frac{\sigma^2}{2} + o(\sigma^2), \tag{a}$$

To prove that decreasing absolute prudence is necessary, specialize the second risk $\tilde{y}$ to a certain negative quantity, $-\epsilon$, where $\epsilon \geq 0$. Monotonicity ensures that $\tilde{y} = -\epsilon$ will satisfy (2) with $\tilde{w}$ replaced by w. It is obvious that w, $\tilde{x}$ and $-\epsilon$ are mutually independent. Therefore, if u is fixed-wealth standard,

$$\mathrm{E}\, u(w + \tilde{x}) - \mathrm{E}\, u(w + \tilde{x} - \epsilon) \geq u(w) - u(w - \epsilon) \tag{10}$$

for all w, $\epsilon \geq 0$ and $\tilde{x}$ satisfying

$$\mathrm{E}\, u'(w + \tilde{x}) \geq u'(w). \tag{11}$$

Equivalently, for all w, $\epsilon \geq 0$ and $\tilde{x}$ satisfying (11),

$$\int_{w-\epsilon}^{w} [\mathrm{E}\, u'(\omega + \tilde{x}) - u'(\omega)]\, d\omega. \geq 0 \tag{12}$$

This means among other things that there cannot be an interval $[w-\epsilon, w]$ on which $\mathrm{E}\, u'(\omega+\tilde{x}) - u'(\omega)$ is monotonically increasing from a negative value to zero.

The decreasing absolute risk aversion proved first implies that $u'(w)$ is continuous, except perhaps at some lower boundary below which utility is $-\infty$.[18] As long as the support of $w + \tilde{x} - \epsilon$ is bounded strictly away from such a lower boundary, $\mathrm{E}\, u'(\omega + \tilde{x}) - u'(\omega)$ is continuous as well. Therefore, since there is no interval on which $\mathrm{E}\, u'(\omega + \tilde{x}) - u'(\omega)$ is monotonically increasing from a negative value to zero, $\mathrm{E}\, u'(\omega + \tilde{x}) - u'(\omega)$ cannot increase from a negative value to zero or a positive value at all. In other words,

$$\mathrm{E}\, u'(w + \tilde{x}) \geq u'(w) \tag{13}$$

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where μ is the mean of $\tilde{y}$ and σ^2 is the variance of $\tilde{y}$; while (5) implies

$$\mu \leq \frac{-u''(w-\epsilon)}{u'(w-\epsilon)} \frac{\sigma^2}{2} + o(\sigma^2). \tag{b}$$

Combining (a) and (b) and dividing by $\sigma^2/2$, one finds that

$$\frac{-u''(w-\epsilon)}{u'(w-\epsilon)} \geq \frac{-u''(w)}{u'(w)} + \frac{o(\sigma^2)}{\sigma^2}. \tag{c}$$

If one chooses smaller and smaller risks $\tilde{y}$ so that $\sigma^2 \to 0$, then $\frac{o(\sigma^2)}{\sigma^2} \to 0$ as well. Therefore

$$\frac{-u''(w-\epsilon)}{u'(w-\epsilon)} \geq \frac{-u''(w)}{u'(w)}. \tag{d}$$

[18] A proof can be found toward the end of Appendix D.

implies

$$\mathrm{E}\, u'(w + \tilde{x} - \epsilon) \geq u'(w - \epsilon) \tag{14}$$

for all $\epsilon \geq 0$. The statement that (13) implies (14) attests to decreasing absolute prudence just as the analogous statement that (8) implies (9) attests to decreasing absolute risk aversion.[19]

VI. Handling Random Initial Wealth

The proof that decreasing absolute risk aversion and decreasing absolute prudence (along with monotonicity and concavity) are sufficient to ensure standardness is more involved. To show that these properties are sufficient to ensure standardness instead of only fixed-wealth standardness, I will borrow a technique from Kihlstrom, Romer and Williams (1981) and consider the utility function v derived by taking an expectation over $\tilde{w}$:

$$v(x) = \mathrm{E}\, u(\tilde{w} + x). \tag{15}$$

As long as $\tilde{x}$ is independent of $\tilde{w}$,

$$\mathrm{E}\, u(\tilde{w} + \tilde{x}) = \mathrm{E}_x\, (\mathrm{E}_w\, u(\tilde{w} + \tilde{x})) = \mathrm{E}\, v(\tilde{x}). \tag{16}$$

A statement corresponding to (16) can be made about any combination of $\tilde{x}$, $\tilde{y}$ and ϵ that is independent of $\tilde{w}$. Therefore, the definition of standardness can be restated as follows in terms of the derived utility function v: the original utility function u is standard if for any two mutually independent random variables $\tilde{x}$ and $\tilde{y}$,

$$\mathrm{E}\, v'(\tilde{x}) \geq v'(0) \tag{17}$$

and

$$\mathrm{E}\, v(\tilde{y}) \leq v(0) \tag{18}$$

imply

$$\mathrm{E}\, v(\tilde{x}) - \mathrm{E}\, v(\tilde{x} + \tilde{y}) \geq \mathrm{E}\, v(0) - v(\tilde{y}). \tag{19}$$

Clearly, the original utility function u is standard if the derived utility function v is fixed-wealth standard.

Given the objective of showing that v is fixed-wealth standard, it is helpful that v inherits every important property assumed for u: monotonicity, concavity, decreasing absolute risk aversion

[19] The details of the proof are in Appendix D. If u is already known to be thrice-differentiable, one can use the argument of footnote 17 with u replaced by $-u'$.

and decreasing absolute prudence. That v inherits monotonicity and concavity from u is easy to verify. Kihlstrom, Romer and Williams (1981) and Nachman (1980) prove that v inherits decreasing absolute risk aversion from u as well.[20] Since decreasing absolute prudence of a utility function is decreasing absolute risk aversion of the negative of marginal utility,[21] the same proof shows that $-v'$—which satisfies the equation $-v'(x) = \mathrm{E}\left[-u'(\tilde{w} + x)\right]$—inherits decreasing absolute prudence from $-u'$ and therefore that v inherits decreasing absolute prudence from u. The inheritance of monotonicity, concavity, decreasing absolute risk aversion and decreasing absolute prudence by v means that proving these four properties are enough to ensure fixed-wealth standardness proves they are enough to ensure standardness.

VII. Diffidence and Central Risk Aversion

Properness and standardness can be seen as setting forth conditions on $\tilde{x}$ under which the doubly-derived utility function $\hat{v}$ defined by

$$\hat{v}(y) = \mathrm{E}\, v(\tilde{x} + y) = \mathrm{E}\, u(\tilde{w} + \tilde{x} + y). \tag{20}$$

is—in some sense—more risk averse than v *for risks that are undesirable under* v. Of course, distinguishing between desirable and undesirable risks gives a special status to the level of utility at the initial wealth, which the definition of the derived utility function v allows one to assume is zero. Therefore, I prefer the statement that the definitions of properness and standardness set forth conditions under which $\hat{v}$ is—in some sense—more risk averse than v *around zero*.

I will define two notions of being more risk averse around a specific initial wealth: being *more diffident* and being *centrally more risk averse*.[22]

Definition of Greater Diffidence: *v_2 is more diffident around κ than v_1 iff* $\mathrm{E}\, v_2(\tilde{y}) \leq \mathrm{E}\, v_2(\kappa)$ *whenever* $\mathrm{E}\, v_1(\tilde{y}) \leq \mathrm{E}\, v_1(\kappa)$.

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[20] The preservation of decreasing absolute risk aversion under expectations can be seen as a consequence of Pratt's (1964) theorem that a positive linear combination of functions with decreasing absolute risk aversion also has decreasing absolute risk aversion. Appendix E gives a proof of the preservation of decreasing absolute risk aversion under expectations that assumes for differentiability only the once-differentiability of u that Appendix D shows is implied by decreasing absolute risk aversion.

[21] This is apparent from straightforward calculation.

[22] These two notions of being more risk averse around a specific point are of some independent interest, beyond their application in this paper. For example, in a two-good model with ordinary consumption and leisure, being forced to precommit to a certain labor supply before the resolution of uncertainty about wealth makes the indirect utility of wealth more diffident—but not always centrally more risk averse—around the level of wealth that would lead an agent to choose that labor supply *ex post* if he or she were not precommitted (Kimball (1990c)).

Definition of Greater Central Risk Aversion: *v_2 is centrally more risk averse around κ than v_1 iff for any pair of nonnegative real numbers α and ϵ, $\mathrm{E}\, v_2(\kappa+(\alpha+\epsilon)\tilde{z}) \leq \mathrm{E}\, v_2(\kappa+\alpha\tilde{z})$ whenever $\mathrm{E}\, v_1(\kappa+(\alpha+\epsilon)\tilde{z}) \leq \mathrm{E}\, v_1(\kappa+\alpha\tilde{z})$.*

(In these definitions, $\tilde{y}$ and $\tilde{z}$ represent any random variable.)

Since one can choose $\alpha = 0$ and $\epsilon = 1$, *greater central risk aversion implies greater diffidence.* Greater diffidence means that *starting from the initial wealth κ,* any risk that is undesirable under v_1 is also undesirable under v_2. Greater central risk aversion means that *starting from the initial wealth κ,* any *increase in the scale of a risk* (including an increase in scale from zero to one) that is undesirable under v_1 is also undesirable under v_2. Lemma 1 states that, given concavity, greater central risk aversion is necessary and sufficient for optimal level of investment in any risky asset to be lower (in absolute value) when starting from the initial wealth κ.

Lemma 1: *If both v_1 and v_2 are concave, then $\arg\max_\theta \mathrm{E}\, v_2(\kappa+\theta\tilde{z})$ is closer to zero than $\arg\max_\theta \mathrm{E}\, v_1(\kappa+\theta\tilde{z})$ for any random variable $\tilde{z}$ if and only if v_2 is centrally more risk averse around κ than v_1.*

Lemma 1 is proved in Appendix C.

Lemma 2 offers a simple criterion for judging when one utility function is more diffident around a given point than another.

Lemma 2: *v_2 is more diffident around κ than v_1 on the domain $\mathcal{D}$ if and only if either* (i) *κ is a misery point of v_1 (i.e., $v_1(\xi) \geq v_1(\kappa)$ for all $\xi \in \mathcal{D}$) and $v_2(\xi) \leq v_2(\kappa)$ whenever $v_1(\xi) = v_1(\kappa)$ or* (ii) *there is an $m \geq 0$ such that*

$$v_2(\xi) - v_2(\kappa) \leq m\,(v_1(\xi) - v_1(\kappa)) \qquad \textit{for all } \xi \in \mathcal{D}. \tag{21}$$

Condition (i) can be seen as a variant of condition (ii) with $m = +\infty$.

Lemma 2 has the following corollary:

Lemma 3: *When both $v_1'(\kappa)$ and $v_2'(\kappa)$ exist and are strictly greater than zero, v_2 is more diffident around κ than v_1 on the domain $\mathcal{D}$ if and only if*

$$\frac{v_2(\xi) - v_2(\kappa)}{v_2'(\kappa)} \leq \frac{v_1(\xi) - v_1(\kappa)}{v_1'(\kappa)} \qquad \textit{for all } \xi \in \mathcal{D}. \tag{22}$$

Lemma 3 says that v_2 is more diffident than v_1 around κ if and only if v_2 is always below v_1 once v_1 and v_2 have been normalized to be tangent to each other at κ.

Proof of Lemma 2. The sufficiency of condition (i) in Lemma 2 for v_2 to be more diffident around κ than $v_1(\kappa)$ is obvious. Condition (ii) is sufficient for v_2 to be more diffident around κ since if $\mathrm{E}\, v_1(\tilde{y}) \leq \mathrm{E}\, v_1(\kappa)$ then by condition (ii),

$$\mathrm{E}\left[v_2(\tilde{y}) - v_2(\kappa)\right] \leq m\mathrm{E}\left[v_1(\tilde{y}) - v_1(\kappa)\right] \leq 0. \tag{23}$$

Necessity of one or the other of the conditions in Lemma 2 for v_2 to be more diffident around κ than $v_1(\kappa)$ is established in Appendix A.∎

Proof of Lemma 3. The condition in Lemma 3 is an instance of condition (ii) and so by Lemma 2 is sufficient for v_2 to be more diffident around κ. To see the necessity of the condition in Lemma 3, note that when both $v_1'(\kappa)$ and $v_2'(\kappa)$ exist and are strictly greater than zero, condition (i) is impossible, making condition (ii) necessary. Inequality (21) of condition (ii) indicates that the function

$$(v_2(\xi) - v_2(\kappa)) - m\,(v_1(\xi) - v_1(\kappa))$$

reaches its maximum when $\xi = \kappa$, implying that its derivative with respect to ξ is zero at κ:

$$v_2'(\kappa) = mv_1'(\kappa). \tag{24}$$

Substituting $\frac{v_2'(\kappa)}{v_1'(\kappa)}$ for m in (21) leads to (22).∎

Lemma 4 offers a simple criterion for judging when one utility function is centrally more risk averse around a given point than another.

Lemma 4: *If v_1 and v_2 are piecewise differentiable on the domain $\mathcal{D}$, and $\mathcal{D}^*$ is the set of points at which both v_1 and v_2 are differentiable, then v_2 is centrally more risk averse around κ than v_1 if and only if either* (i) *for all $\xi \in \mathcal{D}^*$, $(\xi - \kappa)v_1'(\xi) \geq 0$ (making κ a misery point of v_1), and $(\xi - \kappa)v_2'(\xi) \leq 0$ wherever $v_1'(\xi) = 0$ for an interval of positive length or* (ii) *there is an $m \geq 0$ such that for $\xi \in \mathcal{D}^*$,*

$$v_2'(\xi) \geq mv_1'(\xi) \quad when\ \xi < \kappa \tag{25}$$

and

$$v_2'(\xi) \leq mv_1'(\xi) \quad when\ \xi > \kappa. \tag{26}$$

As was the case for Lemma 2, condition (i) can be seen as a variant of condition (ii) with $m = +\infty$.

Lemma 4 has the following corollary:

Lemma 5: *If v_1 and v_2 are continuous and piecewise differentiable on the domain $\mathcal{D}$, and the set of points $\mathcal{D}^*$ at which both v_1 and v_2 are differentiable includes κ, with $v_1'(\kappa) > 0$ and $v_2'(\kappa) > 0$, then v_2 is centrally more risk averse around κ than v_1 if and only if for $\xi \in \mathcal{D}^*$,*

$$\frac{v_2'(\xi)}{v_2'(\kappa)} \geq \frac{v_1'(\xi)}{v_1'(\kappa)} \quad \text{when } \xi < \kappa \tag{27}$$

and

$$\frac{v_2'(\xi)}{v_2'(\kappa)} \leq \frac{v_1'(\xi)}{v_1'(\kappa)} \quad \text{when } \xi > \kappa. \tag{28}$$

Lemma 5 says that v_2 is centrally more risk averse than v_1 around κ if and only if v_2' is always above v_1' to the left of κ and always below v_1' to the right of κ once v_1' and v_2' have been normalized so that they cross at κ.

Proof of Lemma 4. The sufficiency of condition (ii) in Lemma 4 for v_2 to be centrally more risk averse around κ than v_1 can be established as follows. First, (25) and (26) imply

$$zv_2'(\kappa + \theta z) \leq mzv_1'(\kappa + \theta z) \tag{29}$$

for any z and any $\theta \geq 0$ except at isolated points where the derivatives fail to exist. Since

$$\frac{\partial}{\partial \alpha} \mathbf{E}\, v(\kappa + \alpha\tilde{z}) = \mathbf{E}\, \tilde{z} v'(\kappa + \alpha \tilde{z}), \tag{30}$$

(25) and (26) imply that

$$\begin{aligned} \mathbf{E}\, v_2(\kappa + (\alpha + \epsilon)\tilde{z}) - \mathbf{E}\, v_2(\kappa + \alpha\tilde{z}) &= \mathbf{E} \int_\alpha^{\alpha+\epsilon} [\tilde{z} v_2'(\kappa + \theta\tilde{z})]\, d\theta \\ &\leq m\mathbf{E} \int_\alpha^{\alpha+\epsilon} [\tilde{z} v_1'(\kappa + \theta\tilde{z})]\, d\theta \\ &= m\left(\mathbf{E}\, v_1(\kappa + (\alpha+\epsilon)\tilde{z}) - \mathbf{E}\, v_1(\kappa + \alpha\tilde{z})\right). \end{aligned} \tag{31}$$

The sufficiency of condition (i) in Lemma 4 for v_2 to be centrally more risk averse around κ than v_1 follows because $\mathbf{E}\, v_1(\kappa + (\alpha + \epsilon)\tilde{z}) \leq \mathbf{E}\, v_1(\kappa + \alpha\tilde{z})$ and condition (i) together imply that $\tilde{z}v_1'(\kappa + \theta\tilde{z}) = 0$ on the interval $\theta \in (\alpha, \alpha + \epsilon)$, except on a set of measure zero—and therefore that $\tilde{z}v_2'(\kappa + \theta\tilde{z}) \leq 0$ except on a set of measure zero, ensuring that $\mathbf{E}\, v_1(\kappa + (\alpha + \epsilon)\tilde{z}) \leq \mathbf{E}\, v_1(\kappa + \alpha\tilde{z})$.

The necessity of the conditions in Lemma 4 for v_2 to be centrally more risk averse around κ than v_1 is established in Appendix B.∎

Proof of Lemma 5. The condition in Lemma 5 is a particular instance of condition (ii); therefore sufficiency of the condition in Lemma 5 is guaranteed by Lemma 4. As for the necessity of the condition in Lemma 5, continuity of v_1' and v_2' at κ together with (25) and (26) implies that

$$v_2'(\kappa) = mv_1'(\kappa). \tag{32}$$

Substituting $\frac{v_2'(\kappa)}{v_1'(\kappa)}$ for m in (25) and (26) leads to (27) and (28).■

Although it is defined in terms of increases in the scale of risks around κ, greater central risk aversion implies that any increase in risk that only moves probability mass away from κ will be undesirable under the centrally more risk averse utility function v_2 if it is undesirable under the centrally less risk averse utility function v_1. Formally, define a *central stretch* around κ as follows:

Definition of a Central Stretch: *$\tilde{Y}$ is a central stretch of $\tilde{y}$ around κ iff $\tilde{Y} =^d \tilde{y} + \tilde{\nu}$ where the distribution of $\tilde{\nu}$ conditional on the realization of $\tilde{y}$, $\tilde{\nu}(y_1)$, satisfies $\tilde{\nu}(y_1) \geq 0$ with probability one whenever $y_1 \geq \kappa$ and $\tilde{\nu}(y_1) \leq 0$ with probability one whenever $y_1 \leq \kappa$.*

Then one can state the following proposition:

Lemma 6: *Given two piecewise differentiable utility functions v_1 and v_2, if v_2 is centrally more risk averse than v_1 around κ and $\tilde{Y}$ is a central stretch of $\tilde{y}$ around κ, then $\mathrm{E}\, v_2(\tilde{Y}) \leq \mathrm{E}\, v_2(\tilde{y})$ whenever $\mathrm{E}\, v_1(\tilde{Y}) \leq \mathrm{E}\, v_1(\tilde{y})$.*

The proof relies on the necessary conditions for greater central risk aversion. Suppose first that condition (ii) of Lemma 4 is satisfied. Then

$$\begin{aligned} v_2(\tilde{Y}) - v_2(\tilde{y}) - m[v_1(\tilde{Y}) - v_1(\tilde{y})] &= \int_0^1 \mathrm{E}\,\tilde{\nu}[v_2'(\tilde{y}+\theta\tilde{\nu}) - m v_1'(\tilde{y}+\theta\tilde{\nu})]\,d\theta \\ &= \int_0^1 \mathrm{E}_y\,\mathrm{E}_\nu\,[\tilde{\nu}(\tilde{y})[v_2'(\tilde{y}+\theta\tilde{\nu}(\tilde{y})) - m v_1'(\tilde{y}+\theta\tilde{\nu}(\tilde{y}))]\,d\theta \\ &\leq 0. \end{aligned} \tag{33}$$

If instead, condition (i) of Lemma 4 is satisfied, a similar proof using these integral expressions for $v_2(\tilde{Y}) - v_2(\tilde{y})$ and $v_1(\tilde{Y}) - v_1(\tilde{y})$ establishes the desired result.■

VIII. Sufficiency of Decreasing Absolute Risk Aversion and Decreasing Absolute Prudence

The machinery is now in place to prove that—given monotonicity and concavity—the combination of decreasing absolute risk aversion and decreasing absolute prudence is sufficient to ensure standardness. The proof hinges on showing that if $\mathrm{E}\, v'(\tilde{x}) \geq v'(0)$, then $\hat{v}(\xi) = \mathrm{E}\, v(\tilde{x}+\xi)$ is centrally more risk averse around zero than $v(\xi)$.

First, since decreasing absolute risk aversion guarantees that v' is continuous and concavity guarantees that v' is monotonically decreasing, one can define the precautionary premium $\psi(\tilde{x},\xi)$ (where $\tilde{x}$ represents the entire distribution of $\tilde{x}$ rather than a specific realization) implicitly by

$$v'(\xi - \psi(\tilde{x},\xi)) = \mathrm{E}\, v'(\xi + \tilde{x}). \tag{34}$$

Then

$$\hat{v}'(\xi) = v'(\xi - \psi(\tilde{x}, \xi)). \tag{35}$$

For the same reason that a risk premium is decreasing in initial wealth if absolute risk aversion is decreasing, the *precautionary* premium $\psi(\tilde{x}, \xi)$ is decreasing in ξ if absolute prudence is decreasing.[23] The hypothesis $\mathrm{E}\, v'(\tilde{x}) \geq 0$ in the definition of standardness, together with the fact that v' is monotonically decreasing, ensures that $\psi(\tilde{x}, 0) \geq 0$. Therefore, when $\xi < 0$,

$$\frac{\hat{v}'(\xi)}{\hat{v}'(0)} = \frac{v'(\xi - \psi(\tilde{x}, \xi))}{v'(-\psi(\tilde{x}, 0))} \geq \frac{v'(\xi - \psi(\tilde{x}, 0))}{v'(-\psi(\tilde{x}, 0))} \geq \frac{v'(\xi)}{v'(0)}, \tag{36}$$

where the first inequality is a consequence of decreasing absolute prudence and the fact that v' is monotonically decreasing, and the second inequality is a consequence of decreasing absolute risk aversion. [24] Similarly, when $\xi > 0$,

$$\frac{\hat{v}'(\xi)}{\hat{v}'(0)} = \frac{v'(\xi - \psi(\tilde{x}, \xi))}{v'(-\psi(\tilde{x}, 0))} \leq \frac{v'(\xi - \psi(\tilde{x}, 0))}{v'(-\psi(\tilde{x}, 0))} \leq \frac{v'(\xi)}{v'(0)}. \tag{37}$$

Inequalities (36) and (37) together imply, by Lemma 4, that $\hat{v}$ is centrally more risk averse around zero than v.

Being centrally more risk averse, $\hat{v}$ is also more diffident than v around zero. By Lemma 5[25] greater central risk aversion means that

$$\mathrm{E}\, \hat{v}(\tilde{y}) - \hat{v}(0) \leq \left(\frac{\hat{v}'(0)}{v'(0)}\right) (\mathrm{E}\, v(\tilde{y}) - \mathrm{E}\, v(0)). \tag{38}$$

Since $\hat{v}'(0) = \mathrm{E}\, v'(\tilde{x}) \geq v'(0)$ by the hypothesis in the definition of standardness, (38) implies that

$$\mathrm{E}\, \hat{v}(\tilde{y}) - \hat{v}(0) \leq \mathrm{E}\, v(\tilde{y}) - \mathrm{E}\, v(0) \tag{39}$$

for any risk $\tilde{y}$. Inequality (39) directly establishes that the derived utility function v is fixed-wealth standard, and establishes indirectly that the original utility function u is standard without qualification.∎

[23] See Kimball (1990a).

[24] Using $a(\omega)$ for absolute risk aversion as before,

$$\frac{v'(\xi - \psi(\tilde{x}, 0))}{v'(-\psi(\tilde{x}, 0))} = \exp\left(-\int_0^{\xi} a(\omega - \psi(\tilde{x}, 0))\, d\omega\right) \leq \exp\left(-\int_0^{\xi} a(\omega)\, d\omega\right) = \frac{v'(\xi)}{v'(0)}.$$

Note that monotonicity and decreasing absolute risk aversion rule out $v'(0) = 0$ or $\hat{v}'(0) = 0$.

[25] Lemma 5 is applicable since decreasing absolute risk aversion ensures that v and $\hat{v}$ are both differentiable.

Proof of Proposition 3. Moreover, the fact that $\hat{v}$ is centrally more risk averse around zero than v (and the fact that both are concave) implies that the absolute value of the optimal level of risky investment under $\hat{v}$ is less than the optimal level of risky investment under v, when starting from an initial position of zero. Thus, given monotonicity and concavity, the combination of decreasing absolute risk aversion and decreasing absolute prudence which is equivalent to standardness is sufficient to ensure that any expected-marginal-utility-increasing risk lowers the absolute value of the optimal level of risky investment in any other independent risk, establishing sufficiency in Proposition 3. Conversely, if any expected-marginal-utility-increasing risk $\tilde{x}$ lowers the absolute value of the optimal level of risky investment in any other independent risk, then by Lemma 1, $\hat{v}$ must be centrally more risk averse than v around zero, which in turn[26] implies (39) and the standardness of the original utility function u. This establishes necessity in Proposition 3, completing the proof of Proposition 3. Proposition 3 confirms that though standardness is defined in terms of the interaction of discrete risks, its consequences extend to the continuous choice of the optimal level of risky investment.∎

IX. The Effect of Increases in One Risk on the Pain of Another

In many of the applications in which one is concerned about the effect of one risk on another one is not concerned as much with the effect of the presence or absence of one risk on the desirability of another as with a greater or lesser amount of one risk on the desirability of another. For example, Elmendorf and Kimball (1991) are concerned with the effect of labor income taxes—which alter the amount of unmarketable human capital risk agents face—on investment in freely traded risky assets. I will show that standardness implies that a wide range of increases in one risk will make a second, statistically independent risk more painful.

Ross (1981) shows that it is difficult to establish seemingly reasonable comparative statics results involving globally greater risk aversion that will hold for all mean-preserving spreads. Ross sees this as a problem with the Pratt-Arrow idea of "globally more risk averse," but Pratt (1990) argues convincingly that this is a problem with mean-preserving spreads as a way of modeling increases in risk. Ross defines "strongly more risk averse" in such a way that appealing comparative statics properties will hold. Unfortunately this notion of "strongly more risk averse" is so strong that it is seldom applicable. The alternative is to define a stronger notion of riskier. I will define

[26] Decreasing absolute risk aversion again guarantees the differentiability required to apply Lemma 5: the proof in Section V of the necessity of decreasing absolute risk aversion for standardness can easily be adapted to show that u—and therefore v and $\hat{v}$—must have decreasing absolute risk aversion if any expected-marginal-utility-increasing risk lowers the absolute value of the optimal level of risky investment in any other independent risk.

"patently more risky" as follows:

Definition of Patently Greater Risk: *$\tilde{X}$ is patently more risky than $\tilde{x}$ iff for any two monotonic, concave utility functions of which at least one has decreasing absolute risk aversion, v_2 being globally more risk averse than v_1 implies that for any initial wealth w,*

$$\Pi_2(\tilde{X}, \tilde{x}, w) > \Pi_1(\tilde{X}, \tilde{x}, w), \tag{40}$$

where $\Pi_j(\tilde{X}, \tilde{x}, w)$ is the solution to

$$\mathbf{E}\, v_j(w + \tilde{X}) = \mathbf{E}\, v_j(w + \tilde{x} - \Pi_j(\tilde{X}, \tilde{x}, w)) \tag{41}$$

for $j = 1, 2$.

Note that saying $\tilde{X}$ is patently more risky than $\tilde{x}$ says nothing about the relative desirability of $\tilde{X}$ and $\tilde{x}$; in fact, being patently more risky is a relationship that is invariant to any alteration of the difference in means between $\tilde{x}$ and $\tilde{X}$ by the addition of different constants to the two distributions.

A full characterization of patently greater risk must be left to the future, but the results of Pratt (1988) make it clear that $\tilde{X}$ is patently more risky than $\tilde{x}$ if $\tilde{X}$ can be obtained from $\tilde{x}$ by adding to $\tilde{x}$ a random variable ν that is positively related to $\tilde{x}$ in the sense of having a distribution conditional on $\tilde{x}$ which improves according to third-order stochastic dominance for higher realizations of $\tilde{x}$. This sufficient condition for patently greater risk includes as polar special cases ν perfectly correlated with $\tilde{x}$—which makes the movement from $\tilde{x}$ to $\tilde{X}$ a simple change of location and increase in scale—and ν statistically independent of $\tilde{x}$.

One obvious consequence of the definition of patently greater risk is that the risk premium $\Pi(\tilde{X}, \tilde{x}, w)$ will be decreasing in w if the utility function exhibits monotonicity, concavity and decreasing absolute risk aversion and $\tilde{X}$ is patently more risky than $\tilde{x}$. Less obvious is the following proposition.

Proposition 4: *If the monotonically increasing, concave utility function v is standard and $\tilde{X}$ is patently more risky than $\tilde{x}$, then $\hat{v}_2(w) = \mathbf{E}\, v(w + \tilde{X})$ is centrally more risk averse than $\hat{v}_1(w) = \mathbf{E}\, v(w + \tilde{x})$ around any point w at which $\mathbf{E}\, v_2'(w + \tilde{X}) \geq \mathbf{E}\, v_1'(w + \tilde{x})$.*

Proof of Proposition 4. The result of Proposition 4 follows from the decreasing absolute risk aversion and decreasing absolute prudence that are jointly equivalent to standardness in much the same way that standardness itself follows from these properties. Defining the three-argument precautionary premium $\Psi(\tilde{X}, \tilde{x}, \xi)$ implicitly by

$$\mathbf{E}\, v'(\xi + \tilde{X}) = \mathbf{E}\, v'(\xi + \tilde{x} - \Psi(\tilde{X}, \tilde{x}, \xi)), \tag{42}$$

the fact that decreasing absolute prudence of v is decreasing absolute risk aversion of $-v'$, that concavity of v is monotonicity of $-v'$, and that decreasing absolute risk aversion guarantees concavity of $-v'$, means that the precautionary premium $\Psi(\tilde{X}, \tilde{x}, \xi)$ will be decreasing in ξ when v has monotonicity, concavity, decreasing absolute risk aversion and decreasing absolute prudence and $\tilde{X}$ is patently more risky than $\tilde{x}$. By the definitions of $\hat{v}_1$, $\hat{v}_2$ and $\Psi(\tilde{X}, \tilde{x}, \xi)$,

$$\hat{v}_2'(\xi) = \mathrm{E}\, v'(\xi + \tilde{X}) = \mathrm{E}\, v'(\xi + \tilde{x} - \Psi(\tilde{X}, \tilde{x}, \xi)) = \hat{v}_1'(\xi - \Psi(\tilde{X}, \tilde{x}, \xi)). \tag{43}$$

Therefore, when $\xi < w$,

$$\frac{\hat{v}_2'(\xi)}{\hat{v}_2'(w)} = \frac{\hat{v}_1'(\xi - \Psi(\tilde{X}, \tilde{x}, \xi))}{\hat{v}_1'(w - \Psi(\tilde{X}, \tilde{x}, w))} \geq \frac{\hat{v}_1'(\xi - \Psi(\tilde{X}, \tilde{x}, w))}{\hat{v}_1'(w - \Psi(\tilde{X}, \tilde{x}, w))} \geq \frac{\hat{v}_1'(\xi)}{\hat{v}_1'(w)}. \tag{44}$$

When $\xi > w$, the inequalities in (44) are reversed. Thus, by Lemma 4, $\hat{v}_2$ is centrally more risk averse then $\hat{v}_1$.∎

Proposition 4 has several obvious corollaries. First, since decreasing absolute risk aversion guarantees that $-v'$ is globally more risk averse than v, one can see that

$$\Psi(\tilde{X}, \tilde{x}, w) \geq \Pi(\tilde{X}, \tilde{x}, w), \tag{45}$$

which means that $\mathrm{E}\, v'(w + \tilde{X}) \geq \mathrm{E}\, v'(w + \tilde{x})$ whenever $\mathrm{E}\, v(w + \tilde{X}) \leq \mathrm{E}\, v(w + \tilde{x})$. Therefore $\hat{v}_2$ will be centrally more risk averse than $\hat{v}_1$ around any point w at which $\tilde{X}$ is less desirable than $\tilde{x}$. Second, $\hat{v}_2$ being centrally more risk averse than $\hat{v}_1$ plus the fact that

$$\hat{v}_2'(w) = \mathrm{E}\, v'(w + \tilde{X}) \geq \mathrm{E}\, v'(w + \tilde{x}) = \hat{v}_1'(w) \tag{46}$$

has all the previously established consequences of greater risk premia, lower risky investment and central spreads being less desirable under $\hat{v}_2$ than under $\hat{v}_1$. Third, since the inequality

$$\mathrm{E}\left[v(w + \tilde{X} + \tilde{Y}) - v(w + \tilde{x} + \tilde{Y}) - v(w + \tilde{X} + \tilde{y}) + v(w + \tilde{x} + \tilde{y})\right] \leq 0 \tag{47}$$

is symmetric for an interchange of $\tilde{x}$ with $\tilde{y}$ and $\tilde{X}$ and $\tilde{Y}$, the fact that an expected-marginal-utility-increasing patent increase in risk makes any undesirable, independent central stretch more painful coincides with the fact that an undesirable central stretch will make any expected-marginal-utility-increasing, independent patent increase in risk more painful.

X. Proof that Decreasing Absolute Prudence Begets Decreasing Absolute Risk Aversion

It is shown above that—given monotonicity and concavity—decreasing absolute prudence and decreasing absolute prudence are together sufficient to guarantee standard risk aversion. In this section I will show that—given monotonicity and concavity—decreasing absolute prudence on a semi-infinite interval $(w_0, +\infty)$ implies decreasing absolute risk aversion on that same interval, and therefore that *monotonicity, concavity and decreasing absolute prudence on a semi-infinite interval* $(w_0, +\infty)$ *is enough to guarantee standard risk aversion on that interval.*

Proposition 5: *If the von Neumann-Morgenstern utility function u has decreasing absolute prudence over the interval* $(w_0, +\infty)$*—that is, if* $\ln(-u'')$ *is convex over that interval—and* $u'(\cdot) > 0$ *and* $u''(\cdot) < 0$ *over the interval* $(w_0, +\infty)$*, then u has decreasing absolute risk aversion (convex* $\ln u'$*) over the interval* $(w_0, +\infty)$.

Proof of Proposition 5. The proof that decreasing absolute prudence begets decreasing absolute risk aversion is an adaptation of a proof by Mark Bagnoli and Ted Bergstrom (1991) about log-concave probability. Mathematically, I must show that if $\ln(-u'')$ is convex (decreasing absolute prudence), then $\ln(u')$ is convex (decreasing absolute risk aversion), which is similar to showing that the probability distribution function is log-concave if the density function is log-concave.

To show that u has decreasing absolute risk aversion, I must show that $-\frac{u''(w+\epsilon)}{u'(w+\epsilon)} \leq \frac{-u''(w)}{u'(w)}$ for any $\epsilon > 0$, or equivalently, that

$$u''(w+\epsilon)u'(w) - u''(w)u'(w+\epsilon) \geq 0 \tag{48}$$

for any $\epsilon > 0$. But

$$\begin{aligned} u''(w+\epsilon)u'(w) - u''(w)u'(w+\epsilon) &= u''(w+\epsilon)\left\{u'(w+N) - \int_0^N u''(w+\xi)d\xi\right\} \\ &\quad - u''(w)\left\{u'(w+\epsilon+N) - \int_0^N u''(w+\epsilon+\xi)d\xi\right\} \\ &= \int_0^N \left[u''(w)u''(w+\epsilon+\xi) - u''(w+\epsilon)u''(w+\xi)\right] d\xi \\ &\quad + u'(w+N)[u''(w+\epsilon) - u''(w)] \\ &\quad - u''(w)[u'(w+\epsilon+N) - u'(w+N)]. \end{aligned} \tag{49}$$

In the end, I will let N go to infinity, but allowing N to be any positive number for now will help make it clear how the semi-infinite interval figures into the proof.

For any $\xi > 0$ and $\epsilon > 0$, decreasing absolute prudence guarantees that $u''(w)u''(w+\epsilon+\xi) - u''(w+\epsilon)u''(w+\xi)$ is positive, since convexity of $\ln(-u''(w))$ implies that

$$\ln(-u''(w)) + \ln(-u''(w+\epsilon+\xi)) \geq \ln(-u''(w+\epsilon)) + \ln(-u''(w+\xi)). \qquad (50)$$

Therefore, decreasing absolute prudence over the interval $(w, w+N+\epsilon)$ guarantees that the integral on the first line of the lower right-hand side of (49) is positive.

The second term on the lower right-hand side of (49) is positive if $u''(w+\epsilon) - u''(w) > 0$. Decreasing absolute prudence on the interval $(w, +\infty)$ guarantees that $u''(w+\epsilon) - u''(w) > 0$, since if $u''(w+\epsilon) \leq u''(w)$, then $\ln(-u''(w+\epsilon)) \geq \ln(-u''(w))$, and concavity of $\ln(-u'')$, would guarantee that $\ln(-u''(\zeta)) \geq \ln(-u''(w))$ for all $\zeta \geq w+\epsilon$. But then $u''(\zeta) \leq u''(w)$ for all $\xi \geq w+\epsilon$ and u itself would be asymptotically bounded above by a quadratic utility function, and u' could not be positive on the entire semi-infinite interval.

The third term on the lower right-hand side of (49) is negative for finite N, since $-u''(w) > 0$ and $u'(w+\epsilon+N) - u'(w+N) < 0$. But $u'(w+\epsilon+N) - u'(w+N) \rightarrow 0$ as $N \rightarrow +\infty$ since a decreasing function bounded below by zero like u' must have a limit as its argument goes to $+\infty$.

Combining the foregoing arguments yields the result that if $u'(w) > 0$, $u''(w) \leq 0$ and absolute prudence is decreasing on the interval $(w_0, +\infty)$, then (48) is satisfied for any $w > w_0$, so that absolute risk aversion is decreasing on the interval $(w_0, +\infty)$.■

XI. The Consequences of Decreasing Absolute Prudence Alone

In order to more clearly distinguish the contribution of decreasing absolute prudence to standardness from the contribution of decreasing absolute risk aversion *on a finite interval over which decreasing absolute prudence does not guarantee decreasing absolute risk aversion*, one can ask to what extent decreasing absolute prudence alone implies a negative interaction between different risks. The answer is given by Proposition 6.

Proposition 6: *If the utility function u is twice-differentiable and $u''(w) < 0$ for any w, then*

$$\mathrm{E}\ [u'(\tilde{w}+\tilde{x}) - u'(\tilde{w})][u'(\tilde{w}+\tilde{y}) - u'(\tilde{w})] \geq 0 \qquad (51)$$

implies

$$\mathrm{E}\ [u(\tilde{w}+\tilde{x}+\tilde{y}) - u(\tilde{w}+\tilde{y}) - u(\tilde{w}+\tilde{x}) + u(\tilde{w})] \leq 0 \qquad (52)$$

for any $\tilde{w}$ independent of the joint distribution of $\tilde{x}$ and $\tilde{y}$ if and only if u has decreasing absolute prudence.

Treating the two risks symmetrically, Proposition 6 indicates that (given the stipulation that u'' exists and is strictly negative), decreasing absolute prudence alone guarantees a negative interaction between any two independent risks which individually raise expected marginal utility. Thus (other than helping to prove differentiability) the only contribution of decreasing absolute risk aversion to standardness is in ensuring that any undesirable risk raises expected marginal utility.

When absolute risk aversion *is* decreasing, Proposition 6 extends the main result of section VIII by allowing an independent second risk $\tilde{y}$ to satisfy the weaker condition of raising expected marginal utility rather than the condition of being undesirable, and by *allowing $\tilde{y}$ to be statistically related to $\tilde{x}$ as long as inequality (51) is satisfied.*

Proof of Proposition 6. The necessity of decreasing absolute prudence for (51) to imply (52) can be shown in essentially the same way as the necessity of decreasing absolute prudence for standardness, after specializing $\tilde{y}$ to the negative constant $-\epsilon$ and specializing to nonstochastic initial wealth.

The sufficiency of decreasing absolute prudence for (51) to imply (52) can be shown as follows. First, to handle the stochastic, but independent initial wealth, define $v(x) = \mathrm{E}\, u(\tilde{w} + x)$ as above. The derived utility function v inherits decreasing absolute prudence from u. The convexity of $\ln(-v'')$ which is one way of expressing decreasing absolute prudence guarantees that

$$\ln(-v''(x+y)) - \ln(-v''(y)) \geq \ln(-v''(x)) - \ln(-v''(0)) \tag{53}$$

if $xy \geq 0$, with the inequality reversed if $xy \leq 0$. Equivalently,

$$\frac{v''(x+y)}{v''(0)} \geq \left(\frac{v''(x)}{v''(0)}\right)\left(\frac{v''(y)}{v''(0)}\right) \tag{54}$$

if $xy \geq 0$, with the inequality reversed if $xy \leq 0$. Since the direction of integration cancels out the direction of the inequality, (54) can be integrated to yield

$$\int_0^x \int_0^y \frac{v''(\chi+\xi)}{v''(0)}\, d\xi d\chi \geq \left(\int_0^x \frac{v''(\chi)}{v''(0)}\, d\chi\right)\left(\int_0^y \frac{v''(\xi)}{v''(0)}\, d\xi\right) \tag{55}$$

for any pair of x and y. Performing the integration in (55) yields

$$\frac{v(x+y) - v(x) - v(y) + v(0)}{v''(0)} \geq \left(\frac{v'(x) - v'(0)}{v''(0)}\right)\left(\frac{v'(y) - v'(0)}{v''(0)}\right) \tag{56}$$

for any pair of x and y. Taking expectations of both sides of (56) and multiplying both sides by $v''(0)$ (which changes the direction of the inequality),

$$\mathrm{E}\left[v(\tilde{x}+\tilde{y}) - v(\tilde{x}) - v(\tilde{x}) + v(0)\right] \leq \frac{\mathrm{E}\left[\left(v'(\tilde{x}) - v'(0)\right)\left(v'(\tilde{y}) - v'(0)\right)\right]}{v''(0)} \tag{57}$$

for any pair of random variables $\tilde{x}$ and $\tilde{y}$. This means that

$$\mathrm{E}\left[v(\tilde{x}+\tilde{y})-v(\tilde{x})-v(\tilde{x})+v(0)\right] \leq 0 \tag{58}$$

whenever

$$\mathrm{E}\left[\left(v'(\tilde{x})-v'(0)\right)\left(v'(\tilde{y})-v'(0)\right)\right] \geq 0. \tag{59}$$

Given the definition of v and the independence of $\tilde{w}$ from the joint distribution of $\tilde{x}$ and $\tilde{y}$, (58) is equivalent to (52) and (59) is equivalent to (51).■

XI. Conclusion

The empirical content of an economic theory is in its ability to connect two or more different observable phenomena. In this paper, expected utility theory—with the notion of additive time-separability in the background to help lend meaning to changes in marginal utility—is used to establish an unsuspected connection between a negative interaction of independent risks and a precautionary saving motive decreasing in intensity with wealth. I have also established a connection between a negative interaction of risks and a negative interaction of increases in risk.

One important task necessary to round out the account of standard risk aversion is to place the interaction between various risks in an explicitly intertemporal context. This is a task taken up in Kimball (1990b) and Elmendorf and Kimball (1991). Other important tasks are to relate the results here to non-expected-utility theories of choice under uncertainty and to study the interaction of risks in two or more qualitatively different variables.

The primary mode of proof used above to show that decreasing absolute prudence, in conjunction with decreasing absolute risk aversion, implies various types of negative interaction between risks is a mode of proof that promises to be easily adaptable to other conditions that might arise in applications.[27] The notion of the precautionary premium is the key to this mode of proof. Intuitively, a decreasing precautionary premium indicates that a risk is causing expected marginal utility *conditional* on the outcome of another risk to decline more rapidly, thereby making an agent act more risk averse toward that other risk.[28] The mode of proof used in Section XI, making direct use of the concavity of $\ln(-v'')$ that is equivalent to decreasing absolute prudence, is also easily adaptable to other circumstances.

[27] See for example the proof in Elmendorf and Kimball (1991) that a risk which increases expected marginal utility *in the presence of an optimal amount of a risky asset* will lower the optimal level of investment of that risky asset.

[28] This intuition is also discussed in Eeckhoudt and Kimball (forthcoming) and Kimball (forthcoming).

What may seem a weakness of the concept of standard risk aversion—that it deals with the properties of higher derivatives of the utility function—may be one of its great strengths. The concept of standard risk aversion clarifies the economic meaning of the curvature of the second derivative of the utility function allowing one to interpret the fourth derivative of the utility function, when it exists. On finding that a key comparative statics result depends on the fourth derivative of the utility function one need no longer give up the problem as insoluble. This territory has been claimed.

Appendix A

Conditions necessary for v_2 to be more diffident around κ than v_1 on $\mathcal{D}$

I will prove that one or the other of conditions (i) or condition (ii) of Lemma 2 is necessary for v_2 to be more diffident around κ than v_1 by assuming that $\mathrm{E}\, v_2(\tilde{y}) \leq \mathrm{E}\, v_2(\kappa)$ whenever $\mathrm{E}\, v_1(\tilde{y}) \leq \mathrm{E}\, v_1(\kappa)$ and then showing that if condition (i) is not satisfied, then there is an $m \geq 0$ such that

$$v_2(\xi) - v_2(\kappa) \leq m\,(v_1(\xi) - v_1(\kappa)). \qquad (A.1)$$

First, define the following four sets:

$$\Xi^- = \{\xi \in \mathcal{D} | v_1(\xi) \leq v_1(\kappa)\}, \qquad (A.2)$$

$$\Xi^+ = \{\xi \in \mathcal{D} | v_1(\xi) > v_1(\kappa)\}, \qquad (A.3)$$

$$M^- = \left\{m \in \mathcal{R}^+ |\; v_2(\xi) - v_2(\kappa) \leq m\,(v_1(\xi) - v_1(\kappa))\;\; \forall \xi \in \Xi^- \right\}, \qquad (A.4)$$

and

$$M^+ = \left\{m \in \mathcal{R}^+ |\; v_2(\xi) - v_2(\kappa) \leq m\,(v_1(\xi) - v_1(\kappa))\;\; \forall \xi \in \Xi^+ \right\}, \qquad (A.5)$$

where $\mathcal{R}^+$ is the set of nonnegative reals. The set Ξ^- is the set of points indifferent to or worse than κ under v_1 while Ξ^+ is the set of points strictly better than κ under v_1. The set M^- is the set of values for m that satisfy (A.1) when $\xi \in \Xi^-$ while M^+ is the set of values for m that satisfy (A.2) when $\xi \in \Xi^+$. I need to prove that given the hypothesis that undesirable risks under v_1 are undesirable risks under v_2, then if (i) is not satisfied, M^- and M^+ will have a nonempty intersection.

If $m^* \in M^-$ and $m^* \geq m^{**} \geq 0$ then $m^{**} \in M^-$ since if $v_1(\xi) - v_1(\kappa) \leq 0$ and $v_2(\xi) - v_2(\kappa) \leq m^*(v_1(\xi) - v_1(\kappa))$ then

$$m^{**}(v_1(\xi) - v_1(\kappa)) \geq m^*(v_1(\xi) - v_1(\kappa)) \geq v_2(\xi) - v_2(\kappa). \qquad (A.6)$$

Therefore, if nonempty, M^- must be an interval with zero included as its lower limit. By similar reasoning, if nonempty, M^+ must be a semi-infinite interval with ∞ as its upper limit.

By specializing the hypothesis that $\mathrm{E}\, v_2(\tilde{y}) \leq \mathrm{E}\, v_2(\kappa)$ whenever $\mathrm{E}\, v_1(\tilde{y}) \leq \mathrm{E}\, v_1(\kappa)$ to degenerate random variables $\tilde{y}$ equal to ξ almost surely, one finds that $v_2(\xi) \leq v_2(\kappa)$ whenever $v_1(\xi) \leq v_1(\kappa)$. This has two consequences. First, it means that $0 \in M^-$, so that M^- is nonempty. Second, it means that if condition (i) is not satisfied, it cannot be because there is a point that is indifferent to κ under v_1 but strictly preferred to κ under v_2; instead there must be a point τ at which $v_1(\tau) < v_1(\kappa)$. This implies in turn that $m \notin M^-$ if $m > \frac{v_2(\tau)-v_2(\kappa)}{v_1(\tau)-v_1(\kappa)}$, so that M^- is an interval with a finite upper limit.

Writing out the definitions of the complements of M^- and M^+ in the nonnegative reals $\mathcal{R}^+$,

$$\bar{M}^- = \mathcal{R}^+ \backslash M^- = \left\{ m \in \mathcal{R}^+ | \exists \xi \in \Xi^- \ s.t. \ v_2(\xi) - v_2(\kappa) > m\left(v_1(\xi) - v_1(\kappa)\right)\right\} \qquad (A.7)$$

and

$$\bar{M}^+ = \mathcal{R}^+ \backslash M^+ = \left\{ m \in \mathcal{R}^+ | \exists \xi \in \Xi^+ \ s.t. \ v_2(\xi) - v_2(\kappa) > m\left(v_1(\xi) - v_1(\kappa)\right)\right\}, \qquad (A.8)$$

reveals that both of the complementary sets $\bar{M}^-$ and $\bar{M}^+$ are open, since if $v_2(\xi) - v_2(\kappa) > m^*(v_1(\xi) - v_1(\kappa))$ then for small enough $\delta > 0$, $v_2(\xi) - v_2(\kappa) > m^{**}(v_1(\xi) - v_1(\kappa))$ for any $m^{**} \in (m^* - \delta, m^* + \delta)$. If the complementary sets $\bar{M}^-$ and $\bar{M}^+$ are open, then M^- and M^+ must be closed. Therefore, since it is nonempty and has a finite upper limit, $M^- = [0, p]$ for some $q \geq 0$, and either M^+ is empty or $M^+ = [q, \infty)$.

To see that M^- and M^+ must have a nonempty intersection, think of what would happen if they were disjoint. For M^- and M^+ to be disjoint would require $p < q$, which in turn would mean that any m strictly between p and q would be outside both M^- and M^+ and therefore in both $\bar{M}^-$ and $\bar{M}^+$. From the definitions of $\bar{M}^-$ and $\bar{M}^+$ ((A.7) and (A.8)) there would therefore be an m, a $\xi^- \in \Xi^-$ and a $\xi^+ \in \Xi^+$ for which

$$v_2(\xi^-) - v_2(\kappa) > m\left(v_1(\xi^-) - v_1(\kappa)\right) \qquad (A.9)$$

and

$$v_2(\xi^+) - v_2(\kappa) > m\left(v_1(\xi^+) - v_1(\kappa)\right). \qquad (A.10)$$

But then considering the two point distribution $\tilde{y}$ equal to ξ^- with probability $\frac{v_1(\xi^+)-v_1(\kappa)}{(v_1(\xi^+)-v_1(\kappa))+(v_1(\kappa)-v_1(\xi^-))}$ and equal to ξ^+ with probability $\frac{v_1(\kappa)-v_1(\xi^-)}{(v_1(\xi^+)-v_1(\kappa))+(v_1(\kappa)-v_1(\xi^-))}$ reveals the contradiction, since then $\mathrm{E}\, v_1(\tilde{y}) \leq \mathrm{E}\, v_1(\kappa)$ by construction but (A.9) and (A.10) would imply that $\mathrm{E}\, v_1(\tilde{y}) > \mathrm{E}\, v_1(\kappa)$.

Appendix B

Conditions necessary for v_2 to be centrally more risk averse around κ than v_1 on $\mathcal{D}$

Define the sets

$$\Xi^- = \{\xi \in \mathcal{D}^* | \ (\xi - \kappa)v_1'(\xi) \leq 0\}, \tag{B.1}$$

$$\Xi^+ = \{\xi \in \mathcal{D}^* | \ (\xi - \kappa)v_2'(\xi) > 0\}, \tag{B.2}$$

$$M^- = \left\{m \in \mathcal{R}^+ | \ (\xi - \kappa)v_2'(\xi) \leq m(\xi - \kappa)v_1'(\xi) \ \forall \xi \in \Xi^-\right\}, \tag{B.3}$$

and

$$M^+ = \left\{m \in \mathcal{R}^+ | \ v_2'(\xi) \leq m(\xi - \kappa)v_1'(\xi) \ \forall \xi \in \Xi^+\right\}. \tag{B.4}$$

With these definitions, the proof that either (i) or (ii) in Lemma 4 is necessary for v_2 to be centrally more risk averse around κ than v_1 is essentially identical to the proof of Appendix A that either (i) or (ii) in Lemma 2 is necessary for v_2 to be more diffident around κ than v_1. By virtually identical arguments, if they are nonempty, M^- must be an interval with zero as its lower limit and M^+ must be an interval with ∞ as its upper limit. By specializing the hypothesis that $\mathrm{E}\, v_2(\kappa + (\alpha + \epsilon)\tilde{z}) \leq \mathrm{E}\, v_2(\kappa + \alpha\tilde{z})$ whenever $\mathrm{E}\, v_1(\kappa + (\alpha + \epsilon)\tilde{z}) \leq \mathrm{E}\, v_1(\kappa + \alpha\tilde{z})$ by choosing $\alpha = 1$ and $\tilde{z}$ equal to $\xi - \kappa$ with probability one, it can be shown that $0 \in M^-$ *and* that if condition (i) fails it cannot be because there is a point $\xi \in \mathcal{D}^*$ around which $v_1'(\xi) = 0$ on an interval of positive length but $(\xi - \kappa)v_2'(\xi) > 0$, but must be because there is a point $\tau \in \mathcal{D}^*$ at which $(\tau - \kappa)v_1'(\tau) < 0$. This in turn means that $m \notin M^-$ if $m > \frac{v_2'(\tau)}{v_1'(\tau)}$, so that M^- has a finite upper limit. The same reasoning as in Appendix A shows that M^- and M^+ must be closed so that if they were disjoint there would be a point outside both sets. But this is impossible, since if there were an $m \in \mathcal{R}^+$ a $\xi^- \in \Xi^-$ and a $\xi^+ \in \Xi^+$ for which

$$(\xi^- - \kappa)v_2'(\xi^-) > m(\xi^- - \kappa)v_1'(\xi^-) \tag{B.5}$$

and

$$(\xi^+ - \kappa)v_2'(\xi^+) > m(\xi^+ - \kappa)v_1'(\xi^+) \tag{B.6}$$

then if $\tilde{z}$ were equal to $\xi^- - \kappa$ with probability $\frac{(\xi^+ - \kappa)v_1'(\xi^+) + \delta}{(\xi^+ - \kappa)v_1'(\xi^+) + \delta - (\xi^- - \kappa)v_1'(\xi^-)}$ and $\xi^+ - \kappa$ with probability $\frac{-(\xi^- - \kappa)v_1'(\xi^+)}{(\xi^+ - \kappa)v_1'(\xi^+) + \delta - (\xi^- - \kappa)v_1'(\xi^-)}$, then for small enough δ and ϵ and $\alpha = 1$, one would have $\mathrm{E}\, v_1(\kappa + (\alpha + \epsilon)\tilde{z}) \leq \mathrm{E}\, v_1(\kappa + \alpha\tilde{z})$ but $\mathrm{E}\, v_2(\kappa + (\alpha + \epsilon)\tilde{z}) > \mathrm{E}\, v_2(\kappa + \alpha\tilde{z})$.

Appendix C

Proof of Lemma 1

Sufficiency of greater central risk aversion for the optimal level of risky investment to be lower in absolute value for any risk. Concavity of v_1 and v_2 guarantees concavity of

$$\phi_1(\theta) = \mathrm{E}\, v_1(\kappa + \theta\tilde{z}) \tag{C.1}$$

and

$$\phi_2(\theta) = \mathrm{E}\, v_2(\kappa + \theta\tilde{z}) \tag{C.2}$$

as functions of θ. Consider the graph of $\phi_1(\theta)$ and $\phi_2(\theta)$. The definition of v_2 being centrally more risk averse around κ than v_1 guarantees that, to the right of zero, ϕ_2 is nonincreasing wherever ϕ_1 is nonincreasing. The same definition with $\tilde{z}$ replaced by $-\tilde{z}$ guarantees that, to the left of zero, ϕ_2 is nondecreasing wherever ϕ_1 is nondecreasing.

If ϕ_1 is always strictly increasing, $\arg\max_\theta \phi_1(\theta)$ is equal to $-\infty$ (or some other boundary value) and $\arg\max_\theta \phi_2(\theta)$ is obviously closer to zero. If ϕ_1 is always strictly decreasing, $\arg\max_\theta \phi_1(\theta)$ is equal to $+\infty$ (or some other boundary value) and $\arg\max_\theta \phi_2(\theta)$ is obviously closer to zero.

If ϕ_1 is not always strictly increasing or strictly decreasing, its graph consists, from left to right, of a (possibly empty) increasing section, a (possibly single point) constant section, then a (possibly empty) decreasing section. The constant section of ϕ_1 is $\arg\max_\theta \phi_1(\theta)$. If the constant section of ϕ_1 includes zero, the constant section of ϕ_2 (which is $\arg\max_\theta \phi_2(\theta)$) must also include zero, so that the distance of both $\arg\max_\theta \phi_1(\theta)$ and $\arg\max_\theta \phi_2(\theta)$ to zero is zero. If the constant section of ϕ_1, is on an interval $[a_1, b_1]$ entirely to the right of zero, ϕ_2 must be nonincreasing on $[a_1, b_1]$ and nondecreasing to the left of zero. Therefore $\arg\max_\theta \phi_2(\theta)$ must include a point somewhere in the interval $[0, a_1]$ as close or closer to zero as any point in $\arg\max_\theta \phi_1(\theta)$. Similarly, if the constant section of ϕ_1, is on an interval $[a_1, b_1]$ entirely to the left of zero, ϕ_2 must be nondecreasing on $[a_1, b_1]$ and nonincreasing to the right of zero. Therefore $\arg\max_\theta \phi_2(\theta)$ must include a point somewhere in the interval $[b_1, 0]$, as close or closer to zero as any point in $\arg\max_\theta \phi_1(\theta)$.

Necessity of greater central risk aversion for the optimal level of risky investment to be lower in absolute value for any risk. The optimal level of risky investment being lower in absolute value for any risk can be shown to imply the analytical necessary and sufficient condition for greater central risk aversion given in Lemma 4 using almost exactly the same proof as the proof in Appendix B that greater central risk aversion implies that analytical condition. The only change is the way in which one proves that $0 \in M^-$ *and* that if condition (i) fails it cannot be because there is a point $\xi \in \mathcal{D}^*$ around which $v_1'(\xi) = 0$ on an interval of positive length but $(\xi - \kappa)v_2'(\xi) > 0$, but must be because there is a point $\tau \in \mathcal{D}^*$ at which $(\tau - \kappa)v_1'(\tau) < 0$. Namely, the function v_2 cannot go strictly up when moving away from κ while v_1 stays the same or goes down since that would unavoidably put the point of $\arg\max_\theta \phi_2(\theta)$ nearest to zero further away from zero than the point of $\arg\max_\theta \phi_2(\theta)$ nearest to zero.

Appendix D

Proof that if $\mathrm{E}\, u(w+\tilde{x}) \leq u(w)$ implies $\mathrm{E}\, u(w-\epsilon+\tilde{x}) \leq u(w-\epsilon)$ for any $\epsilon \geq 0$ then $\ln(u'(w))$ is convex.

i. **Proof that if $\mathrm{E}\, u(w+\tilde{x}) \leq u(w)$ implies $\mathrm{E}\, u(w-\epsilon+\tilde{x}) \leq u(w-\epsilon)$ for any $\epsilon \geq 0$ then $u(w-\epsilon)$ is a concave function of $u(w)$ for any $\epsilon \geq 0$.**

As long as the agent is not totally indifferent to everything, one can choose a w that is not a misery point of u. Then by the results of Appendix A, with $v_1(w) = u(w)$ and $v_2(w) = u(w-\epsilon)$, there is an m such that for every w and ξ and every $\epsilon \geq 0$,

$$u(w-\epsilon+\xi) - u(w-\epsilon) \leq m\,(u(w+\xi) - u(w)) \tag{D.1}$$

whenever all quantities are well defined.

Continuing to use the notation $v_1(w) = u(w)$ and $v_2(w) = u(w-\epsilon)$, what I need to show is that if $v_2(w)$ is more diffident than $v_1(w)$ around any point, then the function $v_2(v_1^{-1}(\zeta)$ is concave, implying that v_2 is globally more concave than v_1. Formally, I must show[1] that for any ζ^* and any ζ^{**} in the range of v_1, and any $\lambda \in [0,1]$,

$$\lambda v_2(v_1^{-1}(\zeta^*)) + (1-\lambda)v_2(v_1^{-1}(\zeta^{**})) \leq v_2(v_1^{-1}(\lambda\zeta^* + (1-\lambda)\zeta^{**})). \tag{D.2}$$

To that purpose, let

$$\xi^* = v_1^{-1}(\zeta^*), \tag{D.3}$$

$$\xi^{**} = v_1^{-1}(\zeta^{**}) \tag{D.4}$$

and

$$\kappa = v_1^{-1}(\lambda\zeta^* + (1-\lambda)\zeta^{**}). \tag{D.5}$$

[1] Note that continuity of v_1 (which is implied by concavity) and restriction to a common domain of v_1 and v_2 on which both are defined ensures that $v_1^{-1}(\lambda\zeta^* + (1-\lambda)\zeta^{**})$ and $v_2(v_1^{-1}(\lambda\zeta^* + (1-\lambda)\zeta^{**}))$ will exist if ζ^* and ζ^{**} are in the range of v_1.

Then by construction

$$\lambda v_1(\xi^*) + (1-\lambda)v_1(\xi^{**}) = v_1(\kappa), \tag{D.6}$$

or

$$\lambda(v_1(\xi^*) - v_1(\kappa)) + (1-\lambda)(v_1(\xi^{**}) - v_1(\kappa)) \le 0. \tag{D.7}$$

Inequality (D.7), together with (D.1), implies

$$\lambda(v_2(\xi^*) - v_2(\kappa)) + (1-\lambda)(v_2(\xi^{**}) - v_2(\kappa)) \le 0, \tag{D.8}$$

which is equivalent to the desired (D.2).

ii. **Proof that monotonicity, concavity and decreasing absolute prudence of u together imply differentiability of u, except perhaps at the lower limit of the domain of u**

Let $S(w)$ be the set of slopes of tangents to u at w:

$$S(w) = \{s | u(\xi) - u(w) \le s(\xi - w)\ \forall \xi\}. \tag{D.9}$$

Concavity guarantees that $S(w)$ is a nonempty interval. Since globally greater risk aversion implies greater diffidence around every point, decreasing absolute risk aversion and the monotonicity of u which prevents the existence of a misery point, guarantee that for every w there is an $m \ge 0$ for which

$$u(\xi - \epsilon) - u(w - \epsilon) \le m(u(\xi) - u(w)) \le ms(\xi - w) \tag{D.10}$$

for any ξ, any $s \in S(w)$ and any $\epsilon > 0$. Therefore,

$$mS(w) \subset S(w - \epsilon). \tag{D.11}$$

Also, min $S(w-\epsilon) \ge$ max $S(w)$ if $\epsilon > 0$ since $s^* \in S(w-\epsilon)$, $s^{**} \in S(w)$ and $s^{**} > s^*$ implies the contradiction $u(w) - u(w-\epsilon) \le s^*\epsilon$ and $u(w) - u(w-\epsilon) \ge s^{**}\epsilon$.

Now suppose that $S(w)$ is an interval with positive length. Monotonicity then guarantees that max $S(w) > 0$ and therefore that $0 \notin S(w-\epsilon)$ for any $\epsilon > 0$. As a consequence, the value of m in (D.10) and (D.11) cannot be zero. Thus, for any $\epsilon > 0$, $S(w-\epsilon)$ must have positive length whenever $S(w)$ does. Combined with the fact that for any $\epsilon > 0$, $S(w)$ and $S(w-\epsilon)$ cannot overlap at more than a single point, this implies that if $S(w)$ has positive length, $S(w-\epsilon) = -\infty$ for any $\epsilon > 0$.

iii. **Proof that $\ln u'(w)$ is convex.**

Since $u(w-\epsilon)$ is globally more risk averse as a function of w than $u(w)$, $u(w-\epsilon)$ is centrally more risk averse around every point than $u(w)$, and if $\xi^{**} \geq \xi^*$ then

$$\frac{v'(\xi^{**})}{v'\left(\frac{\xi^*+\xi^{**}}{2}\right)} \geq \frac{v'\left(\frac{\xi^*+\xi^{**}}{2}\right)}{v'(\xi^*)} \tag{D.12}$$

whenever $v'(\xi^*) > 0$ and $v'\left(\frac{\xi^*+\xi^{**}}{2}\right) > 0$. Inequality (D.12) means first, that either $u'(w)$ is zero everywhere above the lower limit of the domain of $u(w)$ or that $u'(w)$ is never zero. If $u'(w)$ is zero everywhere above the lower limit of the domain of $u(w)$, it is obvious that $\ln u'(w)$ is convex, even if u' is considered to have a value of $+\infty$ below the lower limit of the domain of $u(w)$. If $u'(w)$ is never zero, then one can take logarithms of both sides of (D.12) and rearrange to find that

$$\ln v'\left(\frac{\xi^*+\xi^{**}}{2}\right) \leq \frac{\ln v'(\xi^*) + \ln v'(\xi^{**})}{2} \tag{D.13}$$

for any ξ^* and ξ^{**}. By iterative application of (D.13), it is clear

$$\ln v'\left(\lambda\xi^* + (1-\lambda)\xi^{**}\right) \leq \lambda \ln v'(\xi^*) + (1-\lambda)\ln v'(\xi^{**}) \tag{D.14}$$

for any $\lambda = n2^{-j} \in [0,1]$ where n and j are nonnegative integers. By the continuity of $\ln u'(w)$ induced by the continuity of $u'(w)$ when $u'(w)$ is always strictly positive, if (D.14) is true for any $\lambda = n2^{-j} \in [0,1]$ where n and j are nonnegative integers, (D.14) must be true for any $\lambda \in [0,1]$. Therefore, $u'(w)$ is convex.

iv. Proof that if $\mathrm{E}\,u'(w+\tilde{x}) \geq u'(w)$ implies $\mathrm{E}\,u'(w-\epsilon+\tilde{x}) \geq u'(w-\epsilon)$ for any $\epsilon \geq 0$ then $\ln(-u''(w))$ is convex.

The proof is the same as the proof above that $\ln(u'(w))$ is convex under analogous conditions, but with u replaced by $-u'$ and u' replaced by $-u''$ throughout. Concavity of $u(w)$ implies that $u'(w)$ is decreasing, while the convexity of $\ln u'(w)$ implied by decreasing absolute risk aversion implies that $-u'(w)$ is concave.

Appendix E

Proof that Convexity of $\ln u'(w)$ is Preserved Under Expectations.

If one defines

$$u_i(\xi) = u(w_i + \xi) \tag{E.1}$$

for various initial wealth levels indexed by a (possibly continuous) index i, then the objective is to show that if $\ln u_i'(\xi)$ is convex in ξ for all i, then $\ln(\mathbf{E}_i\, u_i'(\xi))$ is convex in ξ, where $\mathbf{E}_i$ is an expectation over the index i; or formally that

$$\ln(\mathbf{E}_i\, u_i'(\lambda\xi^* + (1-\lambda)\xi^{**})) \leq \lambda \ln(\mathbf{E}_i\, u_i'(\xi^*)) + (1-\lambda)\ln(\mathbf{E}_i\, u_i'(\xi^{**})) \tag{E.2}$$

for any ξ^* and ξ^{**} and any $\lambda \in [0,1]$.

The concavity of u_i for all i implies that

$$\begin{aligned} \ln(\mathbf{E}_i\, u_i'(\lambda\xi^* + (1-\lambda)\xi^{**})) &= \ln(\mathbf{E}_i\, \exp(\ln u_i'(\lambda\xi^* + (1-\lambda)\xi^{**}))) \\ &\leq \ln(\mathbf{E}_i\, \exp(\lambda u_i'(\xi^*) + (1-\lambda)\ln u_i'(\xi^{**}))). \end{aligned} \tag{E.3}$$

To complete the proof of (E.2), note that

$$\ln(\mathbf{E}_i\, \exp(\lambda u_i'(\xi^*) + (1-\lambda)\ln u_i'(\xi^{**}))) \leq \lambda \ln(\mathbf{E}_i\, u_i'(\xi^*) + (1-\lambda)\ln(\mathbf{E}_i\, u_i'(\xi^{**})) \tag{E.4}$$

because of the concavity of

$$f(\varphi) = \ln(\mathbf{E}_i\, \exp(\varphi u_i'(\xi^*) + (1-\varphi)\ln u_i'(\xi^{**})))(E.5)$$

in φ: straightforward differentiation shows that

$$\begin{aligned} f''(\varphi) = &\{\mathbf{E}_i\, \exp(\varphi v_i'(\xi^*) + (1-\varphi)v_i'(\xi^{**}))\}^{-2} \\ &\cdot \{\mathbf{E}_i\, [(v_i'(\xi^*) - v_i'(\xi^{**}))^2 \exp(\varphi v_i'(\xi^*) + (1-\varphi)v_i'(\xi^{**})]\mathbf{E}_i\, [\exp(\varphi v_i'(\xi^*) + (1-\varphi)v_i'(\xi^{**})] \\ &- (\mathbf{E}_i\, [(v_i'(\xi^*) - v_i'(\xi^{**})] \exp(\varphi v_i'(\xi^*) + (1-\varphi)v_i'(\xi^{**})])^2\Big\} \\ \geq 0 \end{aligned} \tag{E.6}$$

by the Cauchy-Schwartz inequality and (E.4) can be written in terms of the function $f(\cdot)$ as

$$f(\lambda) \leq \lambda f(0) + (1-\lambda)f(1). \tag{E.7}$$

Appendix F

Proof that two different definitions of standardness are equivalent

Let the statement that (4) and (5) imply (6) (the main definition of standardness) be S_1. Let the statement that (7) and (5) imply (6) (the alternative definition of standardness be S_2. It is easy to show that S_2 as well as S_1 implies decreasing absolute risk aversion, which in turn ensures that the utility function u will be differentiable.

Since (7) implies (4) (by continuity as $\epsilon \to 0$), S_1 implies S_2.

Conversely, assume S_2 and suppose that $\tilde{y}$ satisfies (6). The set of $\tilde{x}$ that satisfy (4) is the closure (by convergence in distribution) of the set of $\tilde{x}$ that satisfy (7). Given any $\tilde{x}_0$ that satisfies (4) one can find a sequence of random variables $\{\tilde{x}_n\}_{n=1}^{\infty}$, all satisfying (7), which converge to $\tilde{x}_0$. By construction, $\tilde{y}$ and $\tilde{x}_n$ satisfy (7) for every n. By continuity, this means that $\tilde{y}$ and $\tilde{x}_0$ must satisfy (7). Therefore, S_2 implies S_1.

Appendix G

Equivalence of statements of decreasing absolute risk aversion, proper risk aversion and standard risk aversion with either the clause "makes any undesirable risk more painful" or the clause "makes any undesirable risk remain undesirable."

Let an asterisk (*) denote statements in which the clause "makes any undesirable risk more painful" has been replaced by "makes any undesirable risk remain undesirable":

$\mathcal{A}^*$. Any undesirable change in wealth makes any undesirable risk more painful.

$\mathcal{B}^*$. Any change in wealth that makes a small reduction in wealth more painful also makes any undesirable risk remain undesirable.

$\mathcal{A}'^*$. Any undesirable statistically independent risk makes any undesirable risk remain undesirable.

$\mathcal{B}'^*$. Any statistically independent risk that makes a small reduction in wealth more painful also makes any undesirable risk remain undesirable.

Since "undesirable" means "positively painful," making an undesirable risk *more* painful always makes it remain undesirable. Therefore the clause "makes any undesirable risk more painful" always implies "makes any undesirable risk remain undesirable." But the clause "makes any undesirable risk remain undesirable" does not in general imply "makes any undesirable risk more painful," since an initially undesirable risk could be made *less* painful but still remain undesirable. Individualized proofs are needed to show that $\mathcal{A}^*$ and $\mathcal{B}^*$ imply $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}'^*$ implies $\mathcal{A}'$, and $\mathcal{B}'^*$ implies $\mathcal{B}'$.

The proofs below are written with nonstochastic initial wealth, but each is valid in the presence of independent background risk since they can be applied to the Kihlstrom-Romer-Williams derived utility function defined by $v(x) = \mathrm{E}\, u(\tilde{w} + x)$, which inherits any of the four properties of monotonicity, concavity, decreasing absolute risk aversion or decreasing absolute prudence exhibited by $u(\cdot)$.

Proof that $\mathcal{A}^*$ and $\mathcal{B}^*$ imply $\mathcal{A}$ and $\mathcal{B}$. Both $\mathcal{A}$ and $\mathcal{B}$ reduce to the statement that

$$u(w-\epsilon) - \mathrm{E}\,u(w-\epsilon+\tilde{y}) \geq u(w) - \mathrm{E}\,u(w+\tilde{y}) \tag{G.1}$$

whenever $\epsilon \geq 0$ and $u(w+\tilde{y}) \leq u(w)$. Both $\mathcal{A}^*$ and $\mathcal{B}^*$ reduce to the statement that

$$u(w-\epsilon+\tilde{y}) \leq u(w-\epsilon) \tag{G.2}$$

in the same circumstances. Inequality (G.1) is equivalent to

$$\begin{aligned} &\mathrm{E}\,[u(w-\epsilon+\tilde{y}) - u(w+\tilde{y}) - u(w-\epsilon) + u(w)] \qquad\qquad (G.3)\\ &\qquad = \mathrm{E}\,[u(w-\epsilon+\tilde{y}) - u(w+\tilde{y}) - u(w-\epsilon+\tilde{y}+\pi^*) + u(w+\tilde{y}+\pi^*)]\\ &\qquad\quad + \mathrm{E}\,[u(w-\epsilon+\tilde{y}+\pi^*) - u(w+\tilde{y}+\pi^*) - u(w-\epsilon) + u(w)]\\ &\qquad \geq 0, \end{aligned}$$

where $\pi^* \geq 0$ is the compensating risk premium satisfying $\mathrm{E}\,u(w+\tilde{y}+\pi^*) = u(w)$. (Concavity guarantees that $u(\cdot)$ is continuous, except possibly at the lower limit of its domain, so π^* will exist for all risks $\tilde{y}$ with finite support as long as the domain of $u(\cdot)$ not bounded above and $\mathrm{E}\,u(w+\tilde{y})$ is defined.) The first line on the right-hand side of (G.3) is negative as a consequence of concavity since, given concavity, it is an expectation of a random variable that is always negative. Either $\mathcal{A}^*$ or $\mathcal{B}^*$ implies that the second line is negative since $\mathrm{E}\,u(w+\tilde{y}+\pi^*)] = u(w)$ by construction and therefore $\mathrm{E}\,u(w-\epsilon+\tilde{y}+\pi^*) \leq u(w-\epsilon)$ by $\mathcal{A}^*$ or $\mathcal{B}^*$.

Proof that $\mathcal{A}'^*$ implies $\mathcal{A}'$. An undesirable risk $\tilde{x}$ increases the pain of another statistically independent, undesirable risk $\tilde{y}$ in accordance with $\mathcal{A}'$ if and only if

$$\begin{aligned} &\mathrm{E}\,[u(w+\tilde{x}+\tilde{y}) - u(w+\tilde{y}) - u(w+\tilde{x}) + u(w)] \qquad\qquad (G.4)\\ &\qquad = \mathrm{E}\,[u(w+\tilde{x}+\tilde{y}) - u(w+\tilde{y}) - u(w+\tilde{x}+\tilde{y}+\pi^*) + u(w+\tilde{y}+\pi^*)]\\ &\qquad\quad + \mathrm{E}\,[u(w+\tilde{x}+\tilde{y}+\pi^*) - u(w+\tilde{y}+\pi^*) - u(w+\tilde{x}) + u(w)]\\ &\qquad \geq 0, \end{aligned}$$

where $\pi^* \geq$ is the compensating risk premium satisfying $\mathrm{E}\,u(w+\tilde{y}+\pi^*) = u(w)$, as above.

To see that $\mathcal{A}'^*$ guarantees that the first line on the right-hand side of (G.4) is negative, note first that $\mathcal{A}'^*$ implies decreasing absolute risk aversion for $u(\cdot)$ (because it implies $\mathcal{A}^*$) and therefore decreasing absolute risk aversion for the function $\hat{u}(\cdot)$ defined by

$$\hat{u}(w) = \mathrm{E}\,u(w+\tilde{y}+\pi^*) \tag{G.5}$$

(Kihlstrom, Romer and Williams (1981) and Nachman (1982)). Statement $\mathcal{A}'^*$ and the hypotheses $\mathrm{E}\, u(w+\tilde{x}) \le u(w)$ and $\mathrm{E}\, u(w+\tilde{y}+\pi^*) = u(w)$ imply that

$$\mathrm{E}\, u(w+\tilde{x}+\tilde{y}+\pi^*) \le \mathrm{E}\, u(w+\tilde{y}+\pi^*), \tag{G.7}$$

or that

$$\mathrm{E}\, \hat{\hat{u}}(w+\tilde{x}) \le \hat{\hat{u}}(w). \tag{G.7}$$

By (G.3), the decreasing absolute risk aversion of $\hat{u}(\cdot)$ implies that

$$\begin{aligned} &\mathrm{E}\left[\hat{u}(w+\tilde{x}-\pi^*) - \hat{\hat{u}}(w-\pi^*) - \hat{\hat{u}}(w+\tilde{x}) + u(w)\right] \\ &\qquad = \mathrm{E}\left[u(w+\tilde{x}+\tilde{y}) - u(w+\tilde{y}) - u(w+\tilde{x}+\tilde{y}+\pi^*) + u(w+\tilde{y}+\pi^*)\right] \\ &\qquad \le 0. \end{aligned} \tag{G.8}$$

Statement $\mathcal{A}'^*$ implies that the second line on the right-hand side of (G.4) is negative since $\mathrm{E}\, u(w+\tilde{y}+\pi^*) = u(w)$ by construction and

$$\mathrm{E}\, u(w+\tilde{x}+\tilde{y}+\pi^*) \le \mathrm{E}\, u(w+\tilde{x}) \tag{G.9}$$

as a direct consequence of $\mathcal{A}'^*$, together with the hypotheses $\mathrm{E}\, u(w+\tilde{x}) \le u(w)$ and $\mathrm{E}\, u(w+\tilde{y}+\pi^*) = u(w)$.

Note that $\mathcal{A}'^*$ corresponds closely to Pratt and Zeckhauser's (1987) primary definition of properness. Statement A' represents an addition to Pratt and Zeckhauser's list of equivalent definitions of properness.

Proof that $\mathcal{B}'^*$ implies $\mathcal{B}'$.

Statement $\mathcal{B}'^*$ clearly implies decreasing absolute risk aversion (which in turn implies that the utility function is continuously differentiable). I will show here that $\mathcal{B}'^*$ also implies decreasing absolute prudence. The combination of decreasing absolute risk aversion and decreasing absolute prudence implies $\mathcal{B}'$ (as shown in Section VII).

Define

$$\hat{u}(w) = \mathrm{E}\, u(w+\tilde{x}). \tag{G.10}$$

Let $\tilde{x}$ be any risk satisfying

$$\mathrm{E}\, u'(w+\tilde{x}) = \hat{u}'(w) \ge u'(w). \tag{G.11}$$

Then B'^* and (G.11) imply that $\hat{u}(\cdot)$ is more diffident than $u(\cdot)$ around w. By the results of Appendix A, this means that

$$\frac{\hat{u}(w+y)-\hat{u}(w)}{\hat{u}'(w)} \le \frac{u(w+y)-u(w)}{u'(w)} \tag{G.12}$$

for any y. Subtracting y from both sides of (G.12) yields

$$\frac{\hat{u}(w+y)-\hat{u}(w)-\hat{u}'(w)y}{\hat{u}'(w)} \le \frac{u(w+y)-u(w)-u'(w)y}{u'(w)}. \tag{G.13}$$

Concavity of $u(\cdot)$ guarantees that $u(w+y)-u(w)-u'(w)y \le 0$. Multiplying both sides by $\hat{u}'(w)$ reveals therefore that

$$\begin{aligned}\hat{u}(w+y)-\hat{u}(w)-\hat{u}'(w)y &\le \left(\frac{\hat{u}'(w)}{u'(w)}\right)[u(w+y)-u(w)-u'(w)y] \\ &\le u(w+y)-u(w)-u'(w)y.\end{aligned} \tag{G.14}$$

Specializing y to $-\epsilon$, where $\epsilon \ge 0$, (G.14) implies that

$$\hat{u}(w-\epsilon)-\hat{u}(w) \le u(w-\epsilon)-u(w), \tag{G.15}$$

since $\hat{u}'(w) \ge u(w)$. But (G.14) is equivalent to (10), and it is shown in section V that (10) being true whenever $\mathbf{E}\, u'(w+\tilde{x}) \ge u'(w)$ implies decreasing absolute prudence.

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