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EXCHANGE-RATE DYNAMICS AND OPTIMAL ASSET ACCUMULATION REVISITED

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ABSTRACT

It has recently been observed that when equations of motion for state variables are nonautonomous, optimal control problems involving Uzawa's endogenous rate of time preference cannot be solved using the change-of-variables method common in the literature. Instead, the problem must be solved by explicitly adding an additional state variable that measures the motion of time preference over time. This note reassesses earlier work of my own on exchange rate dynamics, which was based on a change-of-variables solution procedure. When the correct two-state-variable solution procedure is used, the model's qualitative predictions are unchanged. In addition, the analysis yields an intuitive interpretation of the extra costate variable that arises in solving the individual's maximization problem.

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In a recent paper, Kompas and Abdel-Razeq (1987) analyze Uzawa's (1968) intertemporal consumption model, in which the individual's subjective time-preference rate is endogenous. The Uzawa specification introduces a second state variable (in addition to individual wealth) into the standard model with a constant time-preference rate; Uzawa (1968) suggested that the extra state variable could be eliminated by a change of variables that shifts the control problem from calendar time to "psychological" time. Kompas and Abdel-Razeq show that this transformation is valid only when the equation of motion for the first state variable (wealth) is time-independent (as Uzawa indeed assumed). The purpose of this note is to reassess some earlier work of my own, which used Uzawa's transformation, in the light of their results.

I focus below on the model of exchange-rate dynamics with optimal asset accumulation explored in Obstfeld (1981b).[1] Solving the model as a two-state-variable problem reveals that the steady-state properties of the model, and the qualitative nature of the model's dynamics near the steady state, are the same as those claimed in my paper. Accordingly, none of the paper's conclusions about the effects of macroeconomic policies needs to be altered.

An additional result of the analysis is an intuitive economic interpretation of the extra costate variable that arises in the two-state-variable solution procedure.

The Model

Details of the model can be found in Obstfeld (1981b). Briefly, agents maximize

[1]Similar results can be derived for the simpler crawling-peg model in Obstfeld (1981a). Notice that by defining individual wealth as is done below, that model can be written so that the equation of motion for wealth

$$V = \int_0^{\infty}[u(c_t) + v(m_t)]e^{-\Delta_t}dt \tag{1}$$

subject to

$$\dot{a}_t = ra_t - c_t - (\pi_t + r)m_t,\ a_0 \text{ given}, \tag{2}$$

$$\dot{\Delta}_t = \delta[u(c_t) + v(m_t)],\ \Delta_0 = 0, \tag{3}$$

where, for any time t,

c_t = consumption,

m_t = real money balances,

a_t = real individual wealth,

r = constant world real interest rate,

π_t = inflation rate (expected and actual under perfect foresight).

Real wealth is defined as

$$a_t \equiv \frac{y}{r} + \int_t^{\infty} \tau_s e^{-r(s-t)}ds + m_t + F_t$$

where, for any time t,

y = fixed flow of output,

τ_t = real transfers from the government,

F_t = foreign assets (measured in output).

Above, $u(\cdot)$ and $v(\cdot)$ are positive and satisfy standard assumptions. Also, $\delta(\cdot)$ satisfies the Uzawa postulates: $\delta > 0$, $\delta' > 0$, $\delta'' > 0$, and, for positive z, $\delta(z) - z\delta'(z) > 0$.

The presence of π_t in (2) makes the equation for the state variable a_t nonautonomous, necessitating a two-state-variable solution procedure. To solve the individual's problem, write the current-value Hamiltonian as

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is autonomous. Application of Uzawa's transformation still requires the eventual introduction of an additional dynamic variable, equal to the present discounted value of government transfers.

$$H_t = u(c_t, m_t) + \sigma_t[ra_t - c_t - (\pi_t + r)m_t] + \phi_t\delta[u(c_t) + v(m_t)].$$

As observed by Kompas and Abdel-Razeq, necessary conditions for an optimum are (for all t):

$$u'(c_t) = \frac{\sigma_t}{1 + \phi_t\delta'[u(c_t) + v(m_t)]} , \tag{4}$$

$$v'(m_t) = \frac{\sigma_t(\pi_t + r)}{1 + \phi_t\delta'[u(c_t) + v(m_t)]} , \tag{5}$$

$$\dot{\sigma}_t = \sigma_t(\delta[u(c_t) + v(m_t)] - r) , \tag{6}$$

$$\dot{\phi}_t = \phi_t\delta[u(c_t) + v(m_t)] + u(c_t) + v(m_t). \tag{7}$$

The additional costate variable ϕ_t has an informative interpretation. The saddle-path solution for equation (7) is

$$\phi_t = - \int_t^{\infty}[u(c_s) + v(m_s)]e^{-\int_t^s \delta[u(c_\tau) + v(m_\tau)]d\tau} ds. \tag{8}$$

Thus, $-\phi_0$ equals the maximized value of V [given by (1)] along the optimal path.

Equation (4) can be rewritten (at least locally) as

$$c_t = c(\sigma_t, m_t, \phi_t)$$

where

$$c_\sigma = 1/(u''[1 + \phi\delta'] + \phi\delta''(u')^2),$$

$$c_m = -u'v'\phi\delta''c_\sigma,$$

$$c_\phi = -u'\delta'c_\sigma .$$

Let overbars denote stationary-state values. Then (6) and (7) imply

$$\bar{\delta} = r,$$

$$\bar{\phi} = -(\bar{u} + \bar{v})/\bar{\delta}.$$

Uzawa's assumptions on $\delta(\cdot)$ thus imply that at the stationary state,

$$\bar{c}_\sigma = \bar{\delta}/(u''[\bar{\delta} - (\bar{u} + \bar{v})\bar{\delta}'] - (\bar{u} + \bar{v})\bar{\delta}''(\bar{u}')^2) < 0, \tag{9}$$

$$\bar{c}_m = \bar{u}'\bar{v}'\bar{\delta}''(\bar{u} + \bar{v})\bar{c}_\sigma/\bar{\delta} < 0, \tag{10}$$

$$\bar{c}_\phi = -\bar{u}'\bar{\delta}'\bar{c}_\sigma > 0. \tag{11}$$

Equilibrium

Now let the government adjust its transfer payments so that $\tau_t = \mu m_t + rR - g$, where

μ = monetary growth rate (assumed $> -r$),

R = central-bank reserves,

g = government consumption.

Define $x(c,m) \equiv v'(m)/u'(c)$. Then in perfect foresight equilibrium, the economy's evolution is described by four equations of motion:

$$\dot{c} = c_\sigma\dot{\sigma} + c_m\dot{m} + c_\phi\dot{\phi}, \tag{12}$$

$$\dot{m} = [\mu + r - x(c, m)]m, \tag{13}$$

$$\dot{F} = y + r(F + R) - c - g, \tag{14}$$

$$\dot{\phi} = \phi\delta[u(c) + v(m)] + u(c) + v(m). \tag{15}$$

In my earlier paper (1981b), I claimed that the model's equilibrium is characterized by two key features that determine the adjustment to macroeconomic shocks:

(A) Near the stationary state, the linear Taylor approximation to the model has a positive characteristic root corresponding to each of the jump variables c and m, and a negative characteristic root corresponding to the predetermined variable F.

(B) The eigenvector belonging to the system's negative root is such that c, m, and F rise or fall together along the system's saddle-path.

These features are also true of the system (12) - (15). (In addition, the system has an additional positive characteristic root associated with the jumping costate variable ϕ.) Since the system's stationary state is as described in my earlier paper, the qualitative effects of macroeconomic disturbances are the same.

Local Saddle-path Dynamics

Near the stationary state $(\bar{c}, \bar{m}, \bar{F}, \bar{\phi})$, the dynamics of (12) - (15) are approximated by the matrix equation

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{F} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -\bar{c}_m \bar{x}_c \bar{m} & -\bar{c}_m \bar{x}_m \bar{m} & 0 & \bar{c}_\phi r \\ -\bar{x}_c \bar{m} & -\bar{x}_m \bar{m} & 0 & 0 \\ -1 & 0 & r & 0 \\ (1+\bar{\phi}\bar{\delta}')\bar{u}' & (1+\bar{\phi}\bar{\delta}')\bar{v}' & 0 & r \end{bmatrix} \begin{bmatrix} c - \bar{c} \\ m - \bar{m} \\ F - \bar{F} \\ \phi - \bar{\phi} \end{bmatrix} . \quad (16)$$

The determinant of system (16) -- the product of its characteristic roots -- is

$$\det = \bar{m}r^2\bar{c}_\phi[1 - (\bar{u}+\bar{v})(\bar{\delta}'/\bar{\delta})][\bar{v}'' + (\mu+r)^2\bar{u}''] < 0.$$

Since det < 0, the linearized system must have three roots with negative real part or one (real) negative root. To rule out the first possibility, observe that one of the system's positive roots is r; however the system's trace is

$$\text{trace} = -(\bar{c}_m\bar{x}_c + \bar{x}_m)\bar{m} + 2r > r$$

(because $\bar{x}_c > 0$, $\bar{x}_m < 0$). Since the trace is the sum of the characteristic roots, the system must have exactly one real negative root, denoted θ. This establishes property (A) above.

To establish property (B), let $[\omega_1, \omega_2, \omega_3, -1]'$ be an eigenvector belonging to θ. Property (B) is true if ω_1, ω_2, and ω_3 all have the same sign.

Direct calculation shows that this is so, since

$$\omega_1 = \frac{-\bar{c}_\phi r(\theta + \bar{x}_m\bar{m})}{\theta^2 + \theta(\bar{x}_m\bar{m} + \bar{c}_m\bar{x}_c\bar{m})} > 0,$$

$$\omega_2 = \frac{-\bar{x}_c\bar{m}}{\theta + \bar{x}_m\bar{m}}\,\omega_1 > 0,$$

$$\omega_3 = \frac{\omega_1}{r - \theta} > 0.$$

The saddle-path dynamics are given by $\dot{c}_t = \theta(\omega_1/\omega_3)(F_t-\bar{F})$, $\dot{m}_t = \theta(\omega_2/\omega_3)(F_t-\bar{F})$, $\dot{F}_t = \theta(F_t-\bar{F})$; so c, m, and F rise or fall together, as stated in (B).

Notice, finally, that along the saddle-path

$$\dot{\phi}_t = (-1/\omega_3)(F_t-\bar{F}),$$

so that ϕ is falling when c, m, and F are rising and is rising in the opposite case. This makes intuitive sense given the interpretation of the costate ϕ in (8). The higher the economy's foreign assets F, other things equal, the higher the equilibrium lifetime welfare V of a representative agent. Thus, as F rises, ϕ, which is minus the equilibrium value of V, must fall.

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