NBER TECHNICAL PAPER SERIES

TEMPORAL AGGREGATION AND STRUCTURAL INFERENCE IN MACROECONOMICS

Lawrence J. Christiano

Martin Eichenbaum

Technical Working Paper No. 60

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 1986

We have benefited from useful discussions with Bennett McCallum, Allan Meltzer, Jim Stock, and especially Lars Hansen. We acknowledge the research assistance of Tony Braun and David Marshall. The research reported here is part of the NBER's research program in Economic Fluctuations. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Technical Paper #60
September 1986

Temporal Aggregation and Structural Inference in Macroeconomics

ABSTRACT

This paper examines the quantitative importance of temporal aggregation bias in distorting parameter estimates and hypothesis tests. Our strategy is to consider two empirical examples in which temporal aggregation bias has the potential to account for results which are widely viewed as being anomalous from the perspective of particular economic models. Our first example investigates the possibility that temporal aggregation bias can lead to spurious Granger causality relationships. The quantitative importance of this possibility is examined in the context of Granger causal relations between the growth rates of money and various measures of aggregate output. Our second example investigates the possibility that temporal aggregation bias can account for the slow speeds of adjustment typically obtained with stock adjustment models. The quantitative importance of this possibility is examined in the context of a particular class of continuous and discrete time equilibrium models of inventories and sales. The different models are compared on the basis of the behavioral implications of the estimated values of the structural parameters which we obtain and their overall statistical performance. The empirical results from both examples provide support for the view that temporal aggregation bias can be quantitatively important in the sense of significantly distorting inference.

Lawrence J. Christiano
Research Department
Federal Reserve Bank
250 Marquette Avenue
Minneapolis, Mn.
(612) 340-2368

Martin Eichenbaum
Graduate School of
Industrial Administration
Carnegie-Mellon University
Pittsburgh, PA.
(412) 268-3683

1. Introduction

In order to analyze the effects of changes in the economic environment it is necessary to identify and estimate the parameters of structural relationships. One approach to this problem is to interpret economic time series as the outcome of a well specified dynamic equilibrium in which rational economic agents solve stochastic optimization problems.

Despite their emphasis on the need to uncover structure, proponents of this approach to empirical research typically model economic agents as making decisions at fixed, exogenously specified intervals of time. Presumably, this modeling strategy does not reflect a belief that the timing of economic decisions is invariant to macroeconomic policy interventions. Instead it reflects the difficulty of endogenizing timing decisions in dynamic equilibrium models.[1.1] In general we would expect decision intervals to be time varying and different across heterogeneous agents. Suppose however that, for technical reasons, we accept the need to proceed under the assumption that agents make decisions at common, fixed, prespecified intervals of time. Does it necessarily follow that this interval of time should be thought of as coinciding with the data sampling interval? Unfortunately, the answer to this question is no. There is simply no reason to believe that the frequency at which economic time series are collected coincides with the frequency at which economic agents make decisions.

In this paper we proceed under the assumption, which corresponds to standard practice in applied econometric research, that agents make decisions at fixed intervals of time. However we abandon the assumption that this interval of time coincides with the data sampling interval. The purpose of our paper is to examine the consequences of the specification error that results when agents' true decision interval is finer than the data sampling interval. We call the resulting distortion to parameter estimates and hypothesis tests temporal aggregation bias.

It is not surprising that temporal aggregation bias could lead the analyst astray. In principle any specification error or measurement error could distort inference. The question addressed in this paper is whether temporal aggregation bias is important in practice. Clearly we cannot hope to provide a definitive answer to this question which will be applicable under all circumstances. Accordingly, our strategy is to consider two empirical examples in which temporal aggregation bias has the potential to account for results which are widely viewed as being anomalous from the perspective of particular economic models. In both cases, we find evidence of substantial temporal aggregation bias.

Our first example illustrates Sims' (1971) observation that temporal aggregation bias can generate spurious Granger causality relationships. In particular, the use of temporally aggregated data can make a bivariate system in which there is one way Granger causality appear to display bidirectional Granger causality. Our example focuses upon the Granger causality relationships between growth rates of money and output. Using post - war U.S. monthly data Eichenbaum and Singleton (1986) provide evidence that growth rates of nominal aggregates do not Granger cause output growth. These results are used in conjunction with a monetary model of the business cycle to argue that exogenuous shocks to the monetary growth rate were not an important source of variation in output growth in the U.S. postwar period. More generally their results imply that <u>any</u> monetary model in which monetary growth rates Granger cause output growth is inconsistent with post - war U.S. data. Since GNP figures are not available on a monthly basis, Eichenbaum and Singleton use an index of industrial production as their measure of real economic activity. A natural question that emerges is whether their Granger causality findings are sensitive to the use of quarterly real GNP data. We show that, at least for some time periods, there is somewhat more evidence that monetary growth rates Granger cause quarterly growth rates in real GNP. One interpretation of this result is that quarterly real GNP figures represent more temporally aggregated measures of real activity than monthly industrial output. Consequently, Granger causality orderings between real GNP growth and monetary growth could be spurious in the sense that they reflect the effects of temporal aggregation. In order to explore this possibility we constructed quarterly industrial output figures by taking the appropriate averages of the monthly data. Using this quarterly data, we find that monetary growth appears to Granger cause industrial production. We conclude that Granger causality tests can in practice, as well as in principle, be strongly affected by temporal aggregation bias.

Our second example investigates conjectures of Mundlak (1961) and Zellner (1968) that temporal aggregation bias can account for the slow speeds of adjustment reported in the empirical literature on the stock adjustment model. Our strategy for investigating this conjecture is as follows. First, we construct a continuous time equilibrium rational expectations model of inventories and sales. The model rationalizes a continuous time inventory stock adjustment equation. Using techniques developed by Hansen and Sargent (1980a, 1981) we estimate the model using monthly data on inventories and sales in the nondurable manufacturing sector. The parameter estimates from the continuous time model imply that firms close 95 percent of the gap between actual and "desired" inventories in seventeen days. We then estimate an analogous discrete time model using monthly, quarterly and annual data. The parameter estimates obtained using monthly data imply that it takes firms forty six days to close 95 percent of the gap between actual and "desired" inventories. The analogous figure obtained using quarterly data is two hundred and eleven days. The point estimates obtained with annual data imply that it takes firms one thousand nine hundred and eighty days to close ninety five percent of the gap between actual and "desired" inventories. In our view these results provide support for Mundlak and Zellner's conjectures. More generally they indicate just how sensitive structural inference can be to temporal aggregation bias. Unfortunately, we cannot claim that temporal aggregation effects account for the statistical shortcomings of existing stock adjustment models. Both the discrete and continuous time versions of our equilibrium stock adjustment model impose strong over identifying restrictions on the data. Using a variety of tests and diagnostic devices, we find substantial evidence against these restrictions. In addition, we find no evidence that the overall fit for the continuous time better is superior to that of the discrete time model.

Our empirical examples illustrate two distinct approaches taken in the literature to the study of temporal aggregation bias: the "reduced form" and "structural" approaches, respectively.[1,2] The reduced form approach is concerned with properties of the mapping from the continuous time statistical representation of a stochastic process to the representation of the sampled and possibly averaged data. For example, Hansen and Sargent (1984) and Marcet (1985) focus on the relationship between continuous and discrete time moving average representations of covariance stationary stochastic processes. Sims (1971b) studies the mapping from the continuous time regression of one variable onto another and its sampled counterpart. The results in this literature have an important role to play in the model selection and evaluation stages of empirical research. An illustration of this is provided by our first empirical example, where it is argued that the observed bidirectional Granger causality pattern between money growth and GNP growth may reflect spurious temporal aggregation effects rather than supporting evidence for monetary models of the business cycle.

The structural approach to the study of temporal aggregation bias focuses on distortions to parameter estimates and hypothesis tests. This approach to the temporal aggregation problem is typified by the work of Hansen and Sargent (1983) and Christiano (1984,1985). Our second example is very much in the spirit of this approach. In particular we use the apparatus developed by Hansen and Sargent (1980a,1981) to illustrate empirically the ways in which temporal aggregation bias can lead the analyst astray in making structural inferences based on temporally aggregated data.

For the most part, this paper proceeds under the assumption that the economic system evolves in continuous time. This does not necessarily reflect a belief on our part that economic agents are best modeled as making decisions continuously. Instead we adopt that framework because it is an interesting limiting case which provides us with a useful benchmark. In addition it is the standard framework in the temporal aggregation literature.

Some of the material discussed in this paper is unavoidably technical. In order to alleviate this problem we make extensive use of footnotes and references. In addition we refer the reader to Christiano and Eichenbaum (1985) which is essentially a technical appendix to this paper.

Unfortunately, this strategy does not allow us to completely circumvent the inevitable tradeoff between theoretical rigor and ease of exposition. When faced with this tradeoff, we chose to sacrifice rigor so as to provide the reader with intuitive interpretations of the main results.

The remainder of this of the paper is organized as follows. Section 2 discusses some reduced form effects of temporal aggregation, and reports our money and output growth example. In addition, some basic characteristics of the class of continuous time statistical models we use are described there. Section 3 describes a continuous time rational expectations model of inventories and sales. In addition we report the empirical results obtained using that model. Readers anxious for the empirical results can proceed directly to subsections 2.D and 3.C. In section 4 we provide some concluding remarks.

2. The Effects of Temporal Aggregation on a Reduced Form Time Series Representation.

In this section we discuss the temporal aggregation problem from the "reduced form" point of view. In doing so, we accomplish three tasks. First, we briefly review certain theoretical results on the impact of time aggregation bias on reduced form representations of time series data. Second, we present two empirical examples which are designed to shed light on the practical importance of these theoretical results. Third, we set up the necessary background for our analysis of the structural model of section three.

In our opinion, the "reduced form" approach to the study of temporal aggregation bias has important contributions to make at both the model selection and model evaluation stages of structural empirical work. At the model selection stage, the analyst chooses from the class of models under consideration a variant which maps into a set of reduced form characteristics qualitatively similar to those found in the data being studied. In the context of business cycle models, the analyst might be occupied at this stage in choosing among different propagation mechanisms, such as costs of adjusting output, serial correlation in the exogenous shocks, or sticky prices and wages. A standard unexamined assumption made at this stage is that the model timing interval and the data sampling interval coincide. If the analyst is not committed to this assumption, then understanding the reduced form effects of time aggregation is important. This follows from the fact that temporal aggregation affects the qualitative properties of the mapping from a particular structural model to implications for the dynamic properties of the data at hand.

After the model selection stage the analyst uses some procedure, perhaps the method of maximum likelihood, to assign values to the parameters of the model selected. Once this is accomplished, the model evaluation stage begins, during which the analyst considers the time series implications of his model and verifies whether these are consistent with those of the data. When they are inconsistent, the structural model is rejected, at which point the analyst considers different classes of structural models. Viewed in this way the model selection and evaluation stages are really part of one ongoing process.

In this section we emphasize two kinds of temporal aggregation effects. The first was pointed out by Working (1960) and Telser (1967), who showed that time averaging and sampling can increase the MA order of a time series representation. A consequence of this is that the temporal aggregation effects induced by shrinking the model timing interval can play a qualitatively similar role, in improving model fit, as increasing the serial correlation in shock terms. A different reduced form effect of temporal aggregation was emphasized by Sims (1971b) who noted that time aggregation can convert a one way causal system into bidirectional causality. One example of the potential practical importance of this observation is reported in Christiano (forthcoming). That paper studies the model in Taylor (1980), which implies that output fails to Granger cause prices, an implication which is not consistent with the data. One reponse to this inconsistency, pursued by Taylor (1980), is to introduce serial correlation into the exogenous shocks, while preserving the assumption that the model timing interval and data sampling interval coincide. Christiano (forthcoming) shows that another way to accomodate the bidirectional causality between prices and output in Taylor's model is to preserve the serial independence of the exogenous shocks, but shrink the model timing interval. This change induces the temporal aggregation effects described by Sims(1971b). A second example, which is examined in detail below, concerns the empirical relation between post war U.S. output and money growth.

The remainder of this section is organized as follows. In subsection 2.A we discuss some basic ideas about continuous time models which are used in the rest of the paper. In subsections 2.B and 2.C we discuss the impact of time sampling and averaging on MA orders of time series models. This discussion is illustrated with the use of data on the Japanese-U.S. exchange rate. Section 2.D examines the impact of temporal aggregation on Granger causality patterns.

2.A Some Notation and Concepts.

In this subsection we describe some basic features of the class of continuous time statistical models that we work with in this paper. A more careful (though still very informal) version of what follows appears in Appendix A.[2.1]

Let z(t) denote an n dimensional, linearly indeterministic, continuous time, covariance stationary, stochastic process.[2.2] According to the continuous time version of Wold's decomposition theorem, z(t) can be represented as,

$$(2.1)\qquad z(t) = \int_0^{\infty} f(\tau)\epsilon(t-\tau)d\tau,$$

where $\epsilon(t)$ is a continuous time n dimensional vector white noise process with $E\epsilon(t)\epsilon(t-k)' = \delta(k)V$, and δ is the Dirac delta function which can be thought of as satisfying $\delta(k) = 0$ for all k not equal to zero. The vector $\epsilon(t)$ is the innovation in z(t) and satisfies,

$$(2.2)\qquad z(t+k) - E[z(t+k))|z(t-s),s\geq 0] = \int_0^{k} k(\tau)\epsilon(t+k-\tau)d\tau,$$

for any $k > 0$. Here, E is the linear least squares projection operator.

For many purposes, it is convenient to write (2.1) in operator notation as follows:

$$(2.3)\qquad z(t) = F(D)\epsilon(t),$$

where,

$$(2.4)\qquad F(D) = \int_0^{\infty} e^{\tau D} f(\tau)d\tau.$$

Here, D denotes the time derivative operator, i e., $Dx(t) \equiv dx(t)/dt$, and $e^{\tau D}$ is the continuous time lag operator, ie., $e^{\tau D}x(t) \equiv x(t+\tau)$. It can be shown that there is a one-to-one relation between f and F. Consequently, there is no substantive difference between parameterizing the Wold representation at the level of f or F. We find it convenient to parameterize F.

While Wold's theorem does not require P to be a rational function of D, we impose this assumption for computational reasons. Accordingly we assume that F(D) is of the form,

(2.5) $F(D) = C(D)/\theta(D)$.

Here, C is an n x n matrix valued, q-th ordered polynomial in D, and θ is a scalar, p-th ordered polynomial in D, with $p,q < \infty$. Using this notation, we can write (2.1) in operator notation as,

(2.6) $\theta(D)z(t) = C(D)\epsilon(t)$

Exploiting the obvious analogy with discrete time models of time series, we say that (2.6) is a continuous time ARMA(p,q) model for z(t).[2.3]

Wold's theorem restricts the polynomials C(D) and θ(D) in several respects. First, the assumption that $\{z(t), t \in (-\infty,\infty)\}$ is a covariance stationary stochastic process requires the zeroes of θ to be negative in real part. This is reminiscent of the analogous condition for discrete time models, where covariance stationarity requires the zeroes of the AR component to be greater than one in modulus. In addition, the condition that $\epsilon(t)$ is the innovation in z(t) restricts the zeroes of detC(D) to be nonpositive in real part. Again, there is an analogy with the discrete time case, where Wold's theorem requires the zeroes of the determinant of the MA component to be equal to or greater than one in modulus. A restriction which we impose on θ and C which has no counterpart in the discrete time case is $q \leq p-1$. If this condition is violated, then θ and C do not correspond to an "ordinary" $p(\tau)$ function via (2.3) - (2.5). This condition is discussed further in Appendix A. Finally, as in the discrete time case, econometric identification requires some normalization of the coefficients on θ and C. We adopt the normalization that the coefficient on D^p in θ be unity and that the coefficient matrix on D^0 in C be the identity matrix. Accordingly we write θ(D) and C(D) as, (2.7a) $\theta(D) = \theta_0 + \theta_1 D + \theta_2 D^2 + \ldots + \theta_{p-1} D^{p-1} + D^p$

(2.7b) $C(D) = I + C_1 D + C_2 D^2 + \ldots + C_{q-1} D^{q-1} + C_q D^q$.

With the exception of example 2 below, all of the models which we consider in this paper have continuous time reduced form time series representations of the form given by (2.6) and (2.7).

2.B Sampling Point-in-Time From a Continuous Time Process.

Given a continuous time process $\{z(t), t \in (-\infty,\infty)\}$, we can define the discrete time process $\{z_t, t \in (0,\pm 1,\pm 2,\ldots)\}$ by setting $z_t = z(t)$ for integer values of t. In this case, z_t is said to be z(t) sampled point-in-time. Since z(t) is covariance stationary and linearly indeterministic, so is z_t. Therefore, by the discrete time version of Wold's theorem, it is possible, without loss of generality, to represent z_t as follows:

(2.8) $z_t = \sum_{i=0}^{\infty} f_i \epsilon_{t-i}$,

were

$f_0 = I$, ϵ_t is white noise with $E\epsilon_t\epsilon_t'$ positive semidefinite

$z_{t+k} - E_t z_{t+k} = \sum_{i=0}^{k-1} f_i \epsilon_{t+k-i}$,

for $k \in (1,2,3,\ldots)$. Here, E_t denotes the linear least squares projection operator on the space formed by z_{t-s}, $s \in (0,1,2,\ldots)$.[2.4] Using the operator notation, $Lx_t \equiv x_{t-1}$, (2.7) can be written as follows:

$$z_t = [\sum_{I=0}^{\infty} f_i L^i]\epsilon_t.$$

When the parent process is given by the continuous time model (2.6), then the polynomial in L above is rational and can be written as,

$$(2.9a)\quad \sum_{i=0}^{\infty} f_i L^i = C^c(L)/\theta^c(L),$$

where

$$(2.9b)\quad \theta^c(L) = 1 + \theta_1{}^c L + \theta_2{}^c L^2 + \ldots + \theta_p{}^c L^p$$

$$(2.9c)\quad C^c(L) = I + C_1{}^c L + C_2{}^c L^2 + \ldots + C_{p-1}^c L^{p-1}.$$

Hansen and Sargent (1984) and Marcet (1985) analyze in great detail the relationship between the innovations to the $z(t)$ and z_t processes as well as the moving average representations of these two processes. For our purposes it is more convenient to focus upon the relationships summarized by the following theorem.

Theorem 1

If

(i) $\{z(t), t \in (-\infty,\infty)\}$ is generated by (2.6) and (2.7),

(ii) the roots of θ are distinct and negative in real part, and $p < q$,

(iii) $z_t = z(t)$ for $t \in (0,\pm1,\pm2,\ldots)$

Then,

(iv) z_t has the representation given by (2.8)-(2.9),

(v) $\theta(\lambda) = 0$ if and only if $\theta^c(e^{-\lambda}) = 0$.

Proof: see Appendix A.

The result in Theorem 1 which we wish to focus upon is (iv), according to which a point-in-time sampled representation of a continuous time ARMA(p,q) model is ARMA(p,p-1) with C_{p-1}^c in general not equal to zero. This result does not depend on the assumption that sampling is being done from a continuous time "parent" model. The result holds whenever a fine interval model with $q < p$ is sampled. To motivate this assertion, consider the following example.

Example 1: Point-In-Time Sampling From a Discrete Time ARMA(2,0) Model.

Suppose the data generating mechanism is given by

$$(2.10)\qquad (1 - \lambda_1 L^{\frac{1}{2}})(1 - \lambda_2 L^{\frac{1}{2}})z_t = \epsilon_t,$$

where $L^{\tau}x_t \equiv x_{t-\tau}$ and $\{\epsilon_t : t=0, \pm 1/2, \pm 2/2, \pm 3/2, \ldots\}$ is the white noise forecast error in linearly predicting z_t using z_{t-s}, $s=0, 1/2, 2/2, 3/2, \ldots$. Also $|\lambda_i| < 1$ for $i = 1,2$. Evidently, (2.10) defines an ARMA(2,0) representation for $\{z_t : t=0, \pm 1/2, \ldots\}$. Now, multiply both sides of (2.10) by the operator $(1 + \lambda_1 L^{\frac{1}{2}})(1 + \lambda_2 L^{\frac{1}{2}})$, and exploit the fact $(1 - \lambda_i^2 L) = (1 - \lambda_i L^{\frac{1}{2}})(1 + \lambda_i L^{\frac{1}{2}})$ for $i=1,2$, to obtain the representation,

$$(2.11)\qquad (1 - \lambda_1^2 L)(1 - \lambda_2^2 L)z_t = \epsilon_t + (\lambda_1 + \lambda_2)\epsilon_{t-\frac{1}{2}} + \lambda_1\lambda_2\epsilon_{t-1}.$$

Since the expression on the right hand side of (2.11) is autocorrelated at lag one it is not surprising that the unit sampled representation of z_t can be shown to be ARMA(2,1)[2.5] .

Theorem 1 implies that the order of the MA component of the ARMA representation of z_t is independent of the order of the MA component of the ARMA representation of z(t). Even if q is equal to zero, temporal aggregation induces a non-trivial MA component to z_t provided that $p \geq 2$. Consequently, temporal aggregation can be an important source of serial persistance in discrete time series data. At the model building stage, this implies the existence of an interesting tradeoff between the temporal aggregation effects induced by shrinking the model timing interval and adding factors such as costs of adjustment and serially correlated shocks to the model. Each of these has a qualitatively similar effect on the reduced form dynamics of the model for the sampled data. For example, in a model such as the one in section 3 of the paper, the reduced form for inventories and sales is vector AR(2). If the econometrician implements empirically the discrete time version of the model, he may find evidence of first order autocorrelation in the fitted residuals. One way to respond to this situation would be to preserve the discrete time specification and introduce an extra MA term in the exogenous taste and/or technology shock processes. Introducing costs of adjusting output, by adding higher order AR lags, may also accommodate serial correlation "missed" by the model. Theorem 1 suggests that a possible alternative strategy is to preserve the basic structure of the model, but formulate it at a finer timing interval.

2.C Sampling Averages From a Continuous Time Process.

We now consider the impact of the use of time averaged data on the ARMA representation of a time series. Define the average of z(t) over the unit interval as follows:

$$\bar{z}(t) = \int_0^1 z(t-\tau)d\tau. \tag{2.12}$$

Again, it is possible to define the sampled process, $\bar{z}_t \equiv \bar{z}(t)$ for $t = 0, \pm 1, \pm 2 \ldots$. The following theorem shows that when $\bar{z}(t)$ is generated by (2.6), then the discrete time representation of z_t is ARMA(p,p). Thus, the effect of averaging is to increase the order of the moving average of the sampled representation by one.

Theorem 2

If conditions (i) through (iii) of Theorem 1 are satisfied, then the Wold representation of $\bar{z}_t$ has the following ARMA(p,p) form:

$$(1 + \theta_1^c L + \theta_2^c L^2 + \ldots + \theta_p^c L^p) z_t = (I + \bar{C}_1^c L + \bar{C}_2^c L^2 + \ldots + \bar{C}_p^c L^p)\epsilon_t$$

where the θ^c's match those referred to in Theorem 1.

Proof see Appendix A).

In order to provide the reader some intuition for this result we now present an example, taken from Working (1960), of the way in which averaging induces an extra moving average term in a time series representation.

Example 2:

Suppose z(t) has the representation:

$$(2.13) \qquad z(t) = \int_0^1 \epsilon(t-\tau)d\tau = F(D)\epsilon(t),$$

so that z(t) is the integral of white noise disturbances over the unit interval. It is easy to verify that $F(D) = (1 - e^{-D})/D$. Since this F(D) function is not rational, it does not satisfy the condition of Theorem 2. Nevertheless, the example neatly illustrates the fact that averaging introduces an extra moving average term in the sampled representation.

Note first that z_t is a white noise process and therefore has a discrete time ARMA(0,0) representation. Now consider the stochastic process $\bar{z}(t)$ defined by,

$$(2.14) \qquad \bar{z}(t) \equiv \int_0^1 \left[\int_0^1 \epsilon(t-\nu-\tau)d\tau \right] d\nu = \int_0^1 \tau\epsilon(t-\tau)d\tau + \int_1^2 (2 - \tau)\epsilon(t-\tau)d\tau.$$

It is easy to verify that,

$$(2.15) \qquad r_k \equiv \mathrm{cov}(\bar{z}(t),\bar{z}(t-k))/\mathrm{var}(\bar{z}(t)) = \begin{cases} 0 & k > 1 \\ \frac{1}{4} & |k| = 1 \end{cases}$$

(The result in (2.15) can be found in Working (1960).) Thus, the effect of averaging is to convert the white noise, z_t, into the first order serially correlated process, $\bar{z}_t$.

To illustrate the potential practical importance of this observation, we analyzed the monthly log difference of the Japanese - U.S. exchange rate for the period February 1974 to February 1986. That is, we set z(t) = log[s(t)]-log[s(t-1)], where s(t) is the exchange rate at date t. As the first row of

Table 2.1 indicates, when the observations are point-in-time, the sample correlogram of the z_t's conforms to that of a white noise. This result is consistent with an important class of economic models which predicts that real asset returns ought to be serially uncorrelated. We also computed the following two measures of average z(t) which correspond to measures of exchange rates and asset returns that might be used in empirical work when only time averaged data are available,[2.6]

$$\bar{z}(t) = \int_0^1 \log[s(t-\tau)]d\tau - \int_0^1 \log[s(t-1-\tau)]d\tau, \qquad \text{and,}$$

$$\tilde{z}(t) = \log[\int_0^1 s(t-\tau)d\tau] - \log[\int_0^1 s(t-1-\tau)d\tau]$$

The first of these measures is the one for which the analytic result, (2.15), was derived. The second is a measure that is commonly employed in actual empirical work. Let $\tilde{z}_t$ and $\bar{z}_t$ denote the monthly sampled $\tilde{z}(t)$'s and $\bar{z}(t)$'s, respectively. The second row of Table 2.1 reports the first 11 sample correlations of $\bar{z}_t$, while the third reports results for $\tilde{z}_t$. The results are virtually indistinguishable. Note that the null hypothesis that the averaged data are a white noise can be rejected. Moreover, they are consistent with the implications of (2.15), since the implied 90% confidence interval for the lag one autocorrelation is (.21,.47), which includes $\frac{1}{4}$ in its interior.

An analyst who was not aware of the effects of time averaging on the reduced form time series representation of z(t) would be led to incorrectly reject the class of economic models which predict that exchange rates and asset returns ought to be serially uncorrelated if he used $\bar{z}_t$ or $\tilde{z}_t$ rather than z_t. In this empirical example, the fact that the exchange rate movements are serially correlated is purely an artifact of time averaging[2.7].

Table 2.1

Autocorrelations, Log Difference of U.S. - Japanese Exchange Rate
February 1974 - February 1986

Lag	1	2	3	4	5	6	7	8	9	10	11
End-of-Month*	.10	.03	.14	.07	.07	-.05	-.01	.06	-.10	-.06	.04
Average of Logs**	.34	.02	.09	.19	.07	-.06	-.03	.02	-.06	-.06	.01
Log of Average	.34	.02	.09	.19	.07	-.06	-.03	.03	-.06	-.06	.00

* Standard error (s.e.) under null hypothesis of white noise: .08.

** Under (2.15), s.e. of $\hat{r}_k$, for $k>1$ is $\{[1+2(\frac{1}{4})^2]/145\}^{\frac{1}{2}} = .09$ and s.e. of $\hat{r}_1 = \{[1 - 3(\frac{1}{4})^2 + 4(\frac{1}{4})^4]/145\}^{\frac{1}{2}} = .08$ (see Box and Jenkins[1976,pp.34-35].) Here, $\hat{r}_k$ denotes the sample estimate of r_k.

2.D The Impact of Temporal Aggregation on Tests of Granger Causality.

Since the work of Sims (1971b) and Geweke (1978), it has been well known that temporal aggregation can convert a one way Granger-causal relation into bidirectional Granger causality. The intuition underlying this result is a simple omitted variables argument. Suppose that, in continuous time, x(t) fails to Granger cause y(t). That is, past x(t)'s are not useful in predicting future y(t)'s, given a continuous record on all past y(t)'s. Now suppose that x(t)'s and y(t)'s are only observed at integer values of t. In this case, a forecasting equation for future y(t)'s that only uses sampled past y(t)'s omits a massive amount of useful information. Missing are the observations on past y's between the integers. As long as there is some dynamic correlation between x(t) and y(t), past x's at the integers will be correlated with the missing past y's. For this reason, the past x's may serve as a useful proxy for the missing y's in forecasting future y's. In this case the apparent Granger causality going from x to y would be spurious in the sense that it is simply an artifact of temporal aggregation.

In order to gain some insight into the quantitative importance of these considerations, we investigated the Granger causality patterns between different measures of U.S. real output and money growth. Our results are based on estimated bivariate VARs which included twelve lags of each variable and a constant. These were estimated using data on six sample periods covering the period February 1952 through December 1985. Initially we measured output by the monthly Industrial Production (IP) Index constructed by the Federal Reserve Board. Money was measured by monthly data on M1 as published in the Federal Reserve Bulletin. Column 2 of Table 2.2 displays the significance level of the F-statistics testing the null hypothesis that output growth (the difference in the logarithm of IP) is not Granger caused by the growth rate of M1 (the difference in the logarithm of M1). Consistent with results in Eichenbaum and Singleton (1986), we found that in none of the six sample periods does the growth rate in M1 Granger cause the growth rate of IP at the 5 percent significance level. In five of the six sample periods we cannot reject, at even the 10 percent significance level, the null hypothesis that IP growth is not Granger caused by money growth.

Next, we examined the Granger causality patterns between the growth rate in quarterly real GNP and the quarterly growth rate of M1. Column 4 of Table 2.2 displays the significance level of the F statistics testing the null hypothesis that real quarterly GNP growth is not Granger caused by the quarterly growth rate of M1. Notice that the reported F statistics are all lower than the corresponding entries in Column 2. In fact, these numbers warrant rejecting the null hypothesis at the 7 percent level, although not at the 5 percent significance level. Overall, there is considerably more evidence that output is Granger caused by money when we use quarterly real GNP data than when we use monthly IP data as our measure of output.

How can we interpret the different results that we obtain using quarterly real GNP data and industrial production? One interpretation is that real GNP is simply a better indicator of real output than monthly industrial output. A different interpretation is that quarterly real GNP is a more temporally aggregated measure of real output than monthly industrial output. In light of Sims' results, the Granger causality pattern obtained with quarterly data could be interpreted as being spurious in the sense of reflecting the effects of temporal aggregation. In order to investigate the empirical plausibility of this second interpretation we constructed quarterly M1 and IP data by arithmetically averaging the monthly levels data. We then estimated a quarterly VAR(12) model using the quarterly growth rates for M1 and IP, and tested the null hypothesis that M1 growth fails to Granger cause IP growth. The significance levels of the test statistics for the six sample periods appear in column 3 of Table 2.2. Notice that the significance levels are lower than those in column 2 by a factor of 2 to 12, depending on the period. Moreover, in all periods, except the most recent, the significance levels have dropped enough so that the null hypothesis can be rejected at the five percent level. In the pre-1983 data, M1 growth appears to be useful in forecasting IP growth in the quarterly data only because it is proxying for missing data on lagged IP growth.

In our view these results provide support to the view that temporal aggregation contributes in a significant way to the role that money plays in forecasting quarterly real GNP. Of course in the absence of reliable monthly data on real GNP data we cannot draw definitive conclusions . Nevertheless our results do indicate the potential importance of temporal aggregation in generating spurious Granger causality patterns[2.8] .

Table 2.2

Significance[1] Levels of Granger Causality Tests of Null Hypothesis That Money Growth Fails to Granger Cause Output.[2]

	Industrial Production[4]		Real GNP[5]
Period[3]	Monthly	Quarterly	Quarterly
52 - 79	.220	.018	.067
61 - 79	.093	.023	.060
52 - 83	.327	.045	.004
61 - 83	.123	.039	.024
52 - 85	.406	.114	.012
61 - 85	.215	.107	.041

[1] Defined as the probability, under the null hypothesis, that the test statistic takes on a value greater than the computed value. When this quantity is small then the null hypothesis is unlikely.

[2] All results are based on a bivariate 12 lag VAR estimated by ordinary least squares.

[3] Signifies the period over which the estimation was carried out. Monthly (quarterly) results were obtained using data from the first month (quarter) in the first year to the last month (quarter) in the second year.

[4] Results for VAR on growth in industrial production and M1 growth.

[5] Results for VAR on real GNP growth and M1 growth.

3. Temporal Aggregation and Structural Parameters: The Stock Adjustment Model

Application of the stock adjustment model to the study of inventory behavior frequently produces implausibly low estimates of the speed of adjustment of actual to target inventories. For example, the parameter estimates reported by Feldstein and Auerbach (1976) imply that firms take almost 19 years to close ninety five percent of the gap between actual and desired inventories. Application of the stock adjustment model to other problems such as the demand for money also yields implausibly low speeds of adjustment.

A variety of interesting explanations for these anomalous results exist. Blinder (1986), Eichenbaum (1984), and McCallum (1984) explore different explanations for the slow estimated speed of adjustment of inventories. Goodfriend (1985) discusses this problem with respect to the demand for money. In this section we explore the possibility that estimated slow speeds of adjustment reflect temporal aggregation bias. Mundlak (1961) and Zellner (1968) showed theoretically that, if agents make decisions at intervals of time that are finer than the data sampling interval, then the econometrician could be led to underestimate speeds of adjustment. This is consistent with findings reported in Bryan (1967) who applied the stock adjustment model to bank demand for excess reserves. Bryan found that when the model was applied to weekly data, the estimated time to close ninety-five percent of the gap between desired and actual excess reserves was 5.2 weeks. When the model was applied to monthly aggregated data, the ninety-five percent closure time was estimated to be 28.7 months.

The empirical work discussed in this section is designed to shed light on whether temporal aggregation bias can account, in practice, for the slow speeds of adjustment typically found when the stock adjustment model is applied to inventories of finished goods. In subsection 3.A we formulate a continuous time equilibrium model of employment, inventories of finished goods and output. In subsection 3.B we discuss an estimation strategy which explicitly takes the temporal aggregation problem into account. Finally, in subsection 3.C we report our empirical results.

3.A A Continuous Time Model of Inventories, Output and Sales.

In this subsection we discuss a modified continuous time version of the model in Eichenbaum (1984). Our model is designed to nest, as a special case, the model considered by Blinder (1981,1986) and Blinder and Holtz-Eakin (1984). We take that model to be representative of an interesting class of inventory models. An important virtue of our model is that it provides an explicit equilibrium rationale for a continuous time version of the stock adjustment equation for inventories. An additional advantage of proceeding in terms of an equilibrium model is that we are able to make clear both the theoretical underpinnings and the weaknesses of an important class of inventory models which has appeared in the literature.

Consider a competitive representative household that ranks alternative streams of consumption and leisure using the utility function:[3.1]

$$(3.1)\quad E_t \int_0^{\infty} e^{-r\tau}\{u(t+\tau)s(t+\tau) - .5A(s(t+\tau))^2 - N(t+\tau)\}d\tau.$$

In (3.1),

t = the time unit, measured in months,

E_t = the linear least squares projection operator, conditional on the time t information set,

$s(t)$ = time t consumption of the single nondurable consumption good,

$N(t)$ = total work effort at time t,

$u(t)$ = a stochastic disturbance to the marginal utility of consumption at time t, and,

A,r = positive constants.

We now specify the technology for the production of new consumption goods and storing inventories of finished goods. Let $Q(t)$ denote the total output of new consumption goods at time t. The production function for $Q(t)$ is given by:

$$(3.2)\quad Q(t) = [(2/a)N(t)]^{\frac{1}{2}},$$

where a is a positive scalar. In order to accommodate two different types of costs associated with inventories that have been considered in the literature we suppose that total inventory costs, measured in units of labor, are given by:

$$(3.3) \quad C_I(t) = (b/2)[\overset{*}{s}(t) - cI(t)]^2 + v(t)I(t) + (e/2)I(t)^2,$$

where b,c and e are positive scalars, v(t) is a stochastic shock to marginal inventory holding costs and $\overset{*}{s}(t)$ denotes time t sales of the good. The last two terms in (3.3) correspond to the inventory holding cost function adopted by Blinder (1981,1986) and Blinder and Holtz-Eakin (1984), among others. This component of costs reflects the physical costs of storing inventories of finished goods. The first term in (3.3) reflects the idea that there are costs, denominated in units of labor, associated with allowing inventories to deviate from some fixed proportion of sales. Blanchard (1983,p.378) provides an extensive motivation of this component of inventory costs. Similar cost functions appear in Eichenbaum (1984), McCallum (1984) and Eckstein and Eichenbaum (1985).

The link between current production, inventories of finished goods and sales is given by,

$$(3.4) \quad Q(t) = \overset{*}{s}(t) + DI(t),$$

where D is the derivative operator, $Dx(t) = dx(t)/dt$.

It is well known that, in the absence of externalities or similar types of distortions, rational expectations competitive equilibria are Pareto optimal. Since our representative consumer economy has a unique Pareto optimal allocation, we could solve directly for the competitive equilibrium by considering the relevant social planning problem (see Lucas and Prescott (1971), Hansen and Sargent (1980b) and Eichenbaum, Hansen and Richard (1985)). On the other hand there are a variety of market structures which will support the Pareto optimal allocation. In the interest of preserving comparability with other papers in the inventory literature, we find it convenient to work with a particularly simple market structure that supports this allocation. As in Sargent (1979) we require only competitive spot markets for labor and the consumption good to support the Pareto optimal allocation.[3.2]

Suppose that the representative consumer chooses contingency plans for $s(t+\tau)$ and $N(t+\tau)$, $\tau \geq 0$, to maximize (3.1) subject to the sequence of budget constraints,

(3.5) $\quad P(t+\tau)s(t+\tau) = N(t+\tau) + \pi(t+\tau).$

In (3.5),
$P(t)$ = the price of the consumption good, denominated in labor units, and
$\pi(t)$ = lump sum dividend earnings of the household, denominated in labor units.

Solving the representative consumer's problem we obtain the following inverse demand function,

(3.6) $\quad P(t) = -As(t) + u(t).$

Given the very simple structure of relation (3.6) it is important to contrast our specification of the demand function with different specifications that have been adopted in the literature. In constructing empirical stock adjustment models, most analysts abstract from modelling demand. Instead, the analysis is conducted assuming a particular time series representation for an exogenous sales process (see for example Feldstein and Auerbach (1976) or Blanchard (1983)). Our model is consistent with this practice when A is very large. To see this, rewrite (3.6) as,

(3.6)´ $\quad s(t) = -(1/A)P(t) + \eta(t),$

where $\eta(t) = -(1/A)u(t)$. The assumptions we place on $u(t)$ below guarantee that $\eta(t)$ has a time series representation of the form $\gamma(D)\eta(t) = \upsilon(t)$, where $\upsilon(t)$ is continuous time white noise, uncorrelated with past values of $s(t)$ and $I(t)$. Also, $\gamma(D)$ is a finite ordered polynomial satisfying the root condition required for covariance stationarity. If A is very large ("infinite") then sales have the reduced form time series representation $\gamma(D)s(t) = \upsilon(t)$. This is the continuous time analogue of the assumption, made in many stock adjustment models, that sales are an exogenous stochastic process in the sense of not being Granger caused by the actions of the group of agents who make inventory decisions. (Our empirical results indicate that the assumption of one way Granger causality from sales to inventory stocks is reasonably consistent with the data.)

Other authors like Blinder (1986) and Eichenbaum (1984) begin their analysis by postulating the industry demand curve (3.6). Our analysis provides an equilibrium interpretation of this demand specification. In so doing we are forced to confront the strong assumptions implicit in (3.6). For example, we implement our model on nondurable manufacturing shipment and inventory data. This choice of data was dictated by the desire for our results to be comparable with those appearing in the relevant literature. Notice however that manufacturers' shipments do not enter directly as arguments into consumers' utility functions. Rather they represent sales from manufacturers to wholesalers and retailers who in turn sell them to households. Consequently, objective function (3.1) consolidates the wholesale, retail and household sectors. We know of no empirical justification for this assumption. By focussing on nondurable manufacturers, we place more faith than we care to on the stability of their relation to wholesalers and retailers. For example, shifts through time in the pattern of inventory holdings between manufacturer's and retailers and wholesalers would have effects on our empirical results that are hard to predict. At the same time they do not represent phonemena that we wish to model in this paper. In future research we plan to avoid this type of problem by consolidating data from the wholesale, retail and manufacturing sectors.

We assume that the representative firm seeks to maximize its expected real present value.The firm distributes all profits in the form of lump sum dividends to consumers. The firm's time t profits are equal to

$$(3.7) \qquad \pi(t) = P(t)\overset{*}{s}(t)-N(t)-C_I(t).$$

Substituting (3.2), (3.3) and (3.4) into (3.7) we obtain,

(3.8) $\pi(t) = P(t)\overset{*}{s}(t)-(a/2)[\overset{*}{s}(t)+DI(t)]^2-(b/2)[\overset{*}{s}(t)-cI(t)]^2-v(t)I(t)-(e/2)I(t)^2$.

The firm chooses contingency plans for $\overset{*}{s}(t+\tau)$ and $DI(t+\tau)$, $\tau \geq 0$, to maximize,

(3.9) $E_t \int_0^\infty e^{-r\tau}\pi(t+\tau)d\tau$

given I(t), the laws of motion of v(t) and u(t), (3.1) and beliefs about the law of motion for industry wide sales, $\overset{*}{s}(t)$.[3.3] In a rational expectations equilibrium these beliefs are self-fulfilling. Sargent (1979,p.375) describes a simple procedure for finding rational expectations equilibria in linear quadratic, discrete time models. The discussion in Hansen and Sargent (1980a) shows how to modify Sargent's solution procedure to accomodate our continuous time setup. Briefly, the procedure is as follows. Write,

(3.10) $F[I(t),DI(t),\overset{*}{s}(t),v(t),P(t),t] = e^{-rt}\pi(t)$,

so that (3.9) can be written as,

(3.11) $E_t\int_0^\infty F[I(t+\tau),DI(t+\tau),\overset{*}{s}(t+\tau),v(t+\tau),P(t+\tau),\tau]d\tau$,

by choice of $DI(t+\tau)$, $\overset{*}{s}(t+\tau)$, $\tau \geq 0$, subject to I(t) and the laws of motion of v(t) and P(t). Notice that the principle of certainty equivalence applies to this problem. Accordingly, we first solve a version of (3.11) in which future random variables are equated to their time t conditional expectation. Then we use a continuous time version of the Weiner-Kolmogorov forecasting formula to express the time t conditional expectation of time $t+\tau$ variables in terms of elements of agents' time t information set.

The variational methods discussed by Luenberger (1969) imply that firm's Euler equations for s(t) and I(t) are:

(3.12a) $\partial F/\partial \overset{*}{s}(t) = 0$ and,
(3.12b) $\partial F/\partial I(t) = D\,\{\partial F/\partial DI(t)\}$.

These imply respectively:

(3.13a) $P(t) - (a+b)\overset{*}{s}(t) - aDI(t) + bcI(t) = 0$, and,

(3.13b) $aD^2 I(t) - raDI(t) - (c^2 b+e)I(t) + aD\overset{*}{s}(t) + (cb-ra)\overset{*}{s}(t) = v(t)$.

In a rational expectations competitive equilibrium, P(t) must satisfy (3.6), with $s(t) = \overset{*}{s}(t)$. Substituting (3.6) into (3.13a) and replacing $\overset{*}{s}(t)$ by s(t) we obtain,

(3.14) $s(t) = -[a/(a+b+A)]DI(t) + [bc/(a+b+A)]I(t) + [1/(a+b+A)]u(t)$.

It is convenient to collapse (3.13b) and (3.14) into one differential equation in I(t). Substituting s(t) and Ds(t) from (3.14) into (3.13b) we obtain,

(3.15a) $(D-\lambda)[D-(r-\lambda)]I(t) = \frac{(a+b+A)}{a(b+A)} v(t) - \frac{1}{(b+A)}[(bc-ra)/a + D]\, u(t)$

where,

(3.15b) $\lambda = .5r + (k+.25r^2)^{\frac{1}{2}}$ and,

(3.15c) $k = [(a+b+A)/a(b+A)]\{(bc[c(a+A)+ra]/(a+b+A)) + e\}$.

Since $k > 0$, it follows from (3.15b) that $\lambda > 0$ is real. Moreover, it is easy to verify that $r-\lambda = .5r-[k+.25r^2]^{\frac{1}{2}}$.[3.4] Solving the stable root $(r-\lambda)$ backward and the unstable root λ forward in (3.15a) we obtain,[3.5]

$$DI(t) = (r-\lambda)I(t)\ \frac{a+b+A}{a(b+A)} \int_0^\infty e^{-\lambda\tau} v(t+\tau)d\tau +$$

$$+ \frac{1}{b+A}\int_0^\infty e^{-\lambda\tau}[(cb-ra)/a+D]u(t+\tau)d\tau,$$

(3.16)

$$= (r-\lambda)I(t) - \frac{a+b+A}{a(b+A)}\int_0^\infty e^{-\lambda\tau} v(t+\tau)d\tau$$

$$-\frac{1}{b+a}u(t) + \frac{1}{b+a}\left[\frac{bc}{a} - (r-\lambda)\right]\int_0^{\infty} e^{-\lambda\tau}u(t+\tau)d\tau,$$

where the second equality is obtained using integration by parts. Substituting (3.16) into (3.14), we obtain,

$$(3.17)\quad s(t) = \frac{bc-a(r-\lambda)}{a+b+A} I(t) + \frac{1}{b+A}\int_0^{\infty} e^{-\lambda\tau}v(t+\tau)d\tau + \frac{1}{b+A}u(t)$$

$$- \frac{1}{(b+A)(a+b+A)}\left[(bc/a) - (r-\lambda)\right]\int_0^{\infty} e^{-\lambda\tau}u(t+\tau)d\tau.$$

Equations (3.16) and (3.17) are the equilibrium laws of motion for inventory investment and consumption in the perfect foresight version of our model. Before allowing for uncertainty we discuss some qualitative features of this equilibrium.

First, suppose that the parameter b is equal to zero and there are no technology shocks. This is the model considered by Blinder (1981,1986) and Blinder and Holtz-Eakin (1984). The role of inventories in this version of the model is to smooth production in the sense that inventory investment is negatively related to current demand shocks and positively related to expected future demand shocks (see (3.16) and recall that $r-\lambda<0$). As Blinder (1986) points out, production smoothing, as defined here, does not necessarily imply that the variance of sales will exceed that of production. For example, if the serial correlation structure of u(t) were such that a jump in u(t) typically implies a large increase in u(t) in the future, then the current jump in u(t) could lead to an increase in inventory investment, as well as sales. We rule out these types of u(t) processes below. Consequently, production smoothing in our model implies that the variance of production is lower than the variance of sales when $b = v(t) = 0$.

Second, suppose that there are no preference shocks. Then, the role of inventories is to smooth sales. To see this, notice that inventory investment depends negatively on current and future shocks to the inventory holding cost function. The firm holds less inventories when the marginal cost of holding inventories increases. Suppose that inventory holding costs are viewed as general shocks to production costs. Firms will use inventories to smooth production costs, as opposed to production levels, over time in the face of stable demand for their product. For the kinds of production cost shocks that we consider in this paper, this implies that the variance of sales will be smaller than the variance of production.

A slightly different way of seeing these points is to remember that the competitive equilibrium solves the problem of a fictitious social planner/representative consumer. The representative consumer has a utility function which is locally concave in consumption so that, other things equal, he prefers a smooth consumption path. If preference shocks predominate we would expect sales/consumption to be volatile relative to production. On the other hand if technology shocks predominate, we would expect sales/consumption to be smooth relative to production. Blinder (1981,1986) and West (1986) document the fact that, at least for post World War II data, the variance of production exceeds the variance of sales/consumption. This suggests that the primary role of inventories is to smooth sales rather than production levels.

We now consider the equilibrium of the system in the uncertainty case. In order to derive explicit expressions for the equilibrium laws of motion of the system we parameterize the stochastic laws of motion of the shocks to preferences and technology. To this end we assume that u(t) and v(t) have the joint AR(1) structure,

(3.18a) $$u(t) = \epsilon_1(t)/(\beta+D) = \int_0^\infty e^{-\beta\tau}\epsilon_1(t-\tau)d\tau,$$ and

(3.18b) $$v(t) = \epsilon_2(t)/(\alpha+D) = \int_0^\infty e^{-\alpha\tau}\epsilon_2(t-\tau)d\tau\ ,$$

where α and β are positive scalars. The vector $\epsilon(t) = [\epsilon_1(t)\ \ \epsilon_2(t)]'$ is the continuous time linear least squares innovation in $[u(t)\ \ v(t)]'$, $E\epsilon(t)\epsilon(t-\tau)' = \delta(\tau)\tilde{V}$, where $\tilde{V}$ is a positive definite 2 x 2 symmetric matrix and $\delta(\tau)$ is the Dirac delta generalized function.

Given the above specification for the shocks it is obvious that, for $\tau \geq 0$,

(3.19a) $$E_t u(t+\tau) = \int_\tau^\infty e^{-\beta s}\epsilon_1(t+\tau-s)ds = \epsilon^{-\beta\tau}\int_0^\infty e^{-\beta s}\epsilon_1(t-s)ds = e^{-\beta\tau}u(t).$$

Similarly,

(3.19b) $E_t v(t+\tau) = e^{-\alpha\tau} v(t)$.

Simple substitution from (3.19) yields,

$$E_t \int_0^\infty e^{-\lambda\tau} u(t+\tau)d\tau = u(t)/(\beta+\lambda) \quad \text{and} \quad E_t \int_0^\infty e^{-\lambda\tau} v(t+\tau)d\tau = v(t)/(\alpha+\lambda).$$

Substituting these expressions into (3.16) and (3.17) we obtain the equilibrium laws of motion for s(t) and DI(t),

(3.20a) $$DI(t) = (r-\lambda)I(t) - \frac{a+b+A}{a(b+A)(\alpha+\lambda)} v(t) + \frac{(cb-ra)-a\beta}{a(b+A)(\beta+\lambda)} u(t)$$

(3.20b) $$s(t) = \frac{bc-a(r-\lambda)}{a+b+A} I(t) + \frac{v(t)}{(b+A)(\alpha+\lambda)} + \frac{a}{(b+A)(a+b+A)} \frac{[(bc-ra)-\beta a]}{a(\beta+\lambda)} + [\frac{1}{a+b+A}] u(t)$$

It is convenient to write the equilibrium laws of motion for I(t) and s(t) in the form of a continuous time moving average of $\epsilon_1(t)$ and $\epsilon_2(t)$. Substituting (3.18) into (3.20) and rearranging we obtain, in operator notation,

(3.21) $$\begin{bmatrix} I(t) \\ s(t) \end{bmatrix} = \theta(D)^{-1} \tilde{C}(D) \epsilon(t).$$

where

(3.22) $\theta(D) = (\alpha+D)(\beta+D)[D - (r-\lambda)]$,

(3.23) $\tilde{C}(D) = \tilde{C}_0 + \tilde{C}_1 L + \tilde{C}_2 L^2$,

$$\tilde{C}_0 = \begin{bmatrix} q_1\alpha & q_2\beta \\ \frac{\alpha(q_1bc-(r-\lambda))}{a+b+A} & \frac{q_2bc\beta}{a+b+A} \end{bmatrix}$$

$$\tilde{C}_1 = \begin{bmatrix} q_1 & q_2 \\ \frac{-aq_1(\alpha-\frac{bc}{a})+\alpha-(r-\lambda)}{a+b+A} & \frac{-aq_2(\beta-bc/a)}{a+b+A} \end{bmatrix}$$

$$\tilde{C}_2 = \begin{bmatrix} 0 & 0 \\ \frac{1-aq_1}{a+b+A} & \frac{-aq_2}{a+b+A} \end{bmatrix}$$

(3.24) $$q_1 = \frac{(cb-ra)-a\beta}{a(\lambda+\beta)(b+A)} \quad \text{and } q_2 = \frac{-(a+b+A)}{a(b+A)(\lambda+\alpha)}$$

We find it useful to write (3.21)´ as,

(3.21) $$\begin{bmatrix} I(t) \\ s(t) \end{bmatrix} = \theta(D)^{-1}C(D)e(t)$$

where $e(t) = \tilde{C}_0\epsilon(t)$, $C(D) = \tilde{C}(D)\tilde{C}_0^{-1}$, and $Ee(t)e(t)' = \delta(\tau)V = \delta(\tau)\tilde{C}_0\tilde{V}\tilde{C}_0'$. With this definition of C(D) and e(t), equations (3.21)-(3.24) summarize all of the restrictions that our model imposes on the continuous time Wold MAR of I(t) and s(t).

We conclude this section by showing that our model is consistent with a stock adjustment equation for inventories. Let I(t)* denote the aggregate level of inventories such that if I(t) = I(t)*, then actual inventory investment, DI(t), is equal to zero. I(t)* is taken to be the level of "desired" or "target" inventories. Relation (3.20a) implies that,

(3.25) $$I(t)^* = +\frac{a+b+A}{(r-\lambda)a(b+A)(\alpha+\lambda)}v(t) - \frac{(bc-ra) - a\beta}{a(r-\lambda)(b+A)(\beta+\lambda)}u(t)$$

Substituting (3.25) into (3.20a) we obtain a stock adjustment equation for inventory investment,

(3.26) $DI(t) = (\lambda - r)[I(t)^* - I(t)]$.

We require a measure of the "speed of adjustment" which can be compared with similar measures reported in the literature. In order to make this concept precise we imagine, counterfactually, that movements in $I(t)^*$ can be ignored over an interval $\tau \in (t,t+1)$, so that $I(\tau)^* = I(t)^*$ for $\tau \in (t,t+1)$. Then the solution to (3.26) is

(3.27) $I(t+\tau) - I(t)^* = e^{-(\lambda-r)\tau}[I(t) - I(t)^*]$.

Relation (3.27) gives rise to an interesting summary statistic regarding the speed of adjustment of actual to target inventories. In particular, the number of days required to close 95% of the gap between actual and target inventories is,

(3.28) $T^c = -30\log(1-.95)]/(\lambda-r)$,

where 30 is approximately the number of days in a month.

Given estimates of the structural parameters it is straightforward to calculate this statistic. In the next section we discuss a strategy for estimating the parameters of our model from discrete data. In addition we formulate a discrete time version of the model which is useful for estimating speeds of adjustment under the assumption that agents' decision intervals coincide with the data sampling interval.

3.B Estimation Issues.

In this subsection we discuss a strategy for estimating the continuous time model of subsection 3.A from discrete observations on inventories and sales. Since our estimator corresponds to the one discussed in Hansen and Sargent (1980a) we refer the reader to that paper for technical details. Christiano and Eichenbaum (1985) provide additional details for the model considered here. In this subsection we also display a discrete time version of our basic model and decsribe a method for estimating its parameters. By estimating both models we are able to derive an empirical measure of the effects of temporal aggregation on speed of adjustment estimates.

We now describe the procedure used to estimate the parameters of the continuous time model described in subsection 3.A. This procedure takes into account the fact that the inventory data are point-in-time, and measured at the beginning of the sampling interval, while sales are averages over the month.

Our estimation strategy involves maximizing an approximation of the Gaussian likelihood function of the data with respect to the unknown parameters, ζ, which we list explicitly in subsection 3.C. The approximation we use is the frequency domain approximation studied extensively in Hannan (1970). Hansen and Sargent (1980a) show how to use this approximation to estimate continuous time linear rational expectations models from discrete data records.

One way to describe our estimation strategy exploits the observation that estimation of a continuous time model actually is a special case of estimating a constrained discrete time model. Recall from the discussion of section 3.A that ζ implies a continuous time ARMA model, characterized by the polynomials $\theta(D)$ and $C(D)$ and a symmetric matrix, V (see (3.21)-(3.24)). This continuous time series representation implies a particular discrete time series representation for the sampled, averaged data. In Theorem 2, section 2.C, we characterized this discrete time representation by the polynomials $\theta^c(L)$ and $\bar{C}^c(L)$, and an innovation variance matrix, V^c. Given these objects it is possible to compute the spectral density of the data, $S_y(z;\zeta)$, which is one of the two ingredients of the spectral approximation to the likelihood function. It can be shown that $S_y(z;\zeta)$ is given by,

$$S_y(z;\zeta) = \tilde{C}^c(z)V^c\tilde{C}^c(z^{-1})'/\theta^c(z)\theta^c(z^{-1}), \text{ for } z=e^{-iw},\ w\in(-\pi,\pi).$$

The other ingredient of the spectral approximation to the likelihood function is the periodogram of the data. We denote the available data by $\{Y(t),t=1,2,\ldots T\}$. Here, $Y(t) \equiv (I(t),\bar{s}(t))'$, where $\bar{s}(t)$ denotes average sales:

$$(3.29)\quad \bar{s}(t) = \int_0^1 s(t+\tau)d\tau.$$

The periodogram of the data at frequency w_j, $I(w_j)$, is

$$I(w_j) = (1/T)Y(w_j)Y(w_j)^H,$$

where H denotes the Hermetian transpose and,

$$Y(w_j) = \sum_{t=1}^{T} Y(t)e^{-iw_jt}.$$

Here, $w_j = 2\pi j/T$, $j=1,2,\ldots,T$. Given these expressions for $S_y(z;\zeta)$ and $I(w_j)$ we can compute the spectral approximation to the likelihood function,

$$(3.30)\quad L_T(\zeta) = -T\log 2\pi - .5\sum_{j=1}^{T}\log\det[S(e^{-iw_j};\zeta)]$$
$$- .5\sum_{j=1}^{T}\text{trace}[S(e^{-iw_j};\zeta)^{-1}I(w_j)],$$

Since the likelihood function (3.30) is a known function of the data and the parameters of the model it can be maximized with respect to those parameters. We obtain an estimate of the variance-covariance matrix of the estimated coefficients by computing the negative of the inverse of the second derivative of L_T with respect to ζ, evaluated at the estimated values of ζ.

We now consider the problem of estimating a discrete time version of the model. Accordingly, we suppose that the representative consumer maximizes,

$$(3.31)\quad E_t\sum_{j=0}^{\infty}\phi^j\{u(t+j)s(t+j) - .5As(t+j)^2 - N(t+j)\},$$

subject to (3.5) by choice of linear contingency plans for s(t) and N(t). The parameter ϕ is a subjective discount rate that is between zero and one. As before the solution to the consumer's problem is given by the inverse demand function (3.6).

The representative competitive firm chooses linear contingency plans for $\overset{*}{s}(t)$ and I(t) to maximize,

$$(3.32)\quad E_t\sum_{j=0}^{\infty}\phi^j\{P(t+j)\overset{*}{s}(t+j) - (a/2)[\overset{*}{s}(t+j)+I(t+j)-I(t+j-1)]^2$$
$$-(b/2)[\overset{*}{s}(t+j)-cI(t+j)]^2 - v(t+j)I(t+j) - (e/2)I(t+j)^2\},$$

subject to I(t) given and the laws of motion of v(t) and P(t). We suppose that the shocks to technology and preferences have a discrete time AR(1) representation:

$$(3.33a)\quad u(t) = \mu u(t-1) + \epsilon_1(t), \quad \text{and}$$

$$(3.33b)\quad v(t) = \rho v(t-1) + \epsilon_2(t),$$

where $|\mu| < 1$ and $|\rho| < 1$. Also $\epsilon(t) = [\epsilon_1(t)\ \epsilon_2(t)]'$ is a vector white noise that satisfies,

$$(3.34)\quad E\epsilon(t)\epsilon(t-\tau)' = \Omega \qquad \tau \text{ equal to } 0,$$
$$= 0 \qquad \tau \text{ not equal to zero.}$$

The model summarized by (3.31)-(3.34) is the discrete time version of our continuous time model in that, essentially, it has been obtained by replacing the D operator by its "approximation", 1-L. An alternative would have been to specify the discrete time model so that the implied reduced form time series representation for inventories and sales would be an ARMA of the same order as that predicted by the continuous time model. In order to do this we would have to abandon the assumption that u(t) and v(t) have first order autoregressive representations or change other basic features of the discrete time model. This is an important point which we will return to in subsection 3.C.

Eichenbaum and Christiano (1985) show that the equilibrium laws of motion for inventories and sales are given by,

(3.35a) $I(t) = \psi I(t-1) + hu(t) + gv(t)$

(3.35b) $s(t) = -(a-bc)/(a+b+A)I(t) + a/(a+b+A)I(t-1) + [1/(a+b+A)]u(t).$

where

(3.35c) $h = \dfrac{-1}{a[b(c+1)+A]}\{\psi(a-bc) + [(a-bc)\psi-a]\psi\beta\mu/(1-\psi\beta\mu)\},$

$$g = \frac{-(a+b+A)\psi}{a[b(c+1)+A](1-\psi\beta\rho)},$$

$$\psi + 1/\psi\beta = \frac{-(a+b+A)}{\beta a[b(c+1)+A]}\left[\frac{\beta a^2 + (a-bc)^2}{a+b+A} - (a+bc^2 + e + \beta a)\right],$$

and $|\psi| < 1$.

The relevant measure of the speed of adjustment of inventories which can be compared to the measure which emerges from the continuous time model is,

(3.28)' $T^d = X[\log(.05)]/\log\psi,$

where X is the number of days in the data sampling interval.

It is convenient to write the equilibrium law of motion for s(t) and I(t) in the form of a moving average representation of the discrete time innovations to agents' information sets. Substituting (3.33) into (3.35) and rearranging we obtain,

$$(3.36)\quad \begin{bmatrix} I(t) \\ \\ s(t) \end{bmatrix} = \theta^d(L)^{-1}\bar{C}^d(L)\epsilon(t)$$

where,

$$(3.37)\quad \theta^d(L) = (1-\rho)(1-\mu L)(1-\psi L),$$

$$(3.38)\quad \bar{C}^d(L) = \bar{C}_0^d + \bar{C}_1^d L + \bar{C}_2^d L^2,$$

$$\bar{C}_0^d = \begin{bmatrix} h & g \\ \frac{1-(a-bc)h}{a+b+A} & \frac{g(bc-a)}{a+b+A} \end{bmatrix}$$

$$\bar{C}_1^d = \begin{bmatrix} -h\rho & -gu \\ \frac{(ah-\lambda)-\rho[1-(a-bc)h]}{a+b+A} & \frac{g[a-\mu(bc-a)]}{a+b+A} \end{bmatrix}$$

$$\bar{C}_2^d = \begin{bmatrix} 0 & 0 \\ \frac{-\rho(ah-\lambda)}{a+b+A} & \frac{-g\mu a}{a+b+A} \end{bmatrix}$$

Given these relations the free parameters of the discrete time model can be estimated by maximizing Hannan's spectral approximation to the likelihood function.

We are now in a position to demonstrate some of the possible sources of temporal aggregation bias in estimates of the speed of adjustment. Relations (3.21)-(3.24) and (3.36)-(3.38) summarize the restrictions on the continuous and discrete time Wold representation imposed by the continuous and discrete versions of the model, respectively. It can be shown that the continuous and discrete time models imply that I(t) and s(t) have continuous and discrete time VAR(2) representations, respectively. For example, to see this for the continuous time model, notice that (3.21)-(3.24) imply

(3.39) $\det C(D) = (\alpha+D)(\beta+D)[D-(r-\lambda]/(\lambda-r)a(b+A)$.

Premultiplying (3.22) by $C(D)^{-1} = C(D)^a/\det C(D)$ we obtain,

(3.40) $(\lambda-\alpha)a(b+A)C(D)^a Y(t) = e(t)$.

Here $C(D)^a$ denotes the adjoint matrix of C(D). Thus {Y(t)} is a pure VAR(2) in continuous time. However, Theorem 1 of section 2 implies that sampled and averaged {Y(t)} is a discrete time ARMA(2,2) process. One moving average term is due to sampling and the other is due to averaging. We choose not to focus upon this representation of the discrete data because its AR part requires stronger than usual restrictions to ensure identification (see Christiano and Eichenbaum(1985), pp.29-31). Instead we focus on an alternative reduced form representation for the data which emerges from the continuous time model,

(3.41) $\theta^c(L)Y(t) = [I + C_1{}^c + C_2{}^cL^2 + C_3{}^cL^3]e^c(t)$,

where $e^c(t)$ is the innovation in Y(t) which has covariance matrix V^c. Here $\det C^c(L) = \theta^c(L)\kappa(L)$, where $\kappa(L)$ is a second order polynomial in the lag operator L. The presence of $\kappa(L)$ is a symptom of the effects of sampling and of averaging s(t). Since $\det C^c(L)$ is not proportional to $\theta^c(L)$, the sampled representation is not VAR(2). As we indicated it is vector ARMA (2,2). Christiano and Eichenbaum (1985) discuss the mapping between the representations (3.40) and (3.41).

Of course the discrete time model remains a VAR(2). It is useful to write the reduced form of the discrete model in a manner that is analogous to (3.43). Define $e^d(t) = \bar{C}_0\epsilon(t)$ and $C^d(L) = \bar{C}^d(L)(\bar{C}_0{}^d)^{-1}$. Then (3.36) implies that the reduced form representation for Y(t) emerging from the discrete time model is

(3.42) $\theta^d(L)Y(t) = [I + C_1{}^dL + C_2{}^dL^2]e^d(t)$, where the first row of C_2^d is composed of zeros. We denote the covariance matrix of $e^d(t)$ by V^d.

Comparing (3.41) and (3.42) we see that the moving average component of the reduced form for the discrete model is of smaller order than that of the continuous time model. Again, this reflects the fact that the continuous time and discrete time models have different implications for measured data. Not surprisingly, estimation of the two models will yield different estimates of the underlying structural parameters and speeds of adjustment of actual to target inventories.

3.C Empirical Results

In this subsection we report empirical results obtained from estimating four different models. The continuous time model was estimated using monthly data. Three discrete models were estimated, one each using monthly, quarterly, and annual data. Our main results can be summarized as follows. First, the parameter estimates from the different models that we estimated are consistent with the Mundlak-Zellner hypothesis that temporal aggregation can account for slow speeds of adjustment in stock adjustment models. Secondly, we find that while the effects of temporal aggregation are substantial as we move from annual to quarterly to monthly specifications of the model, they are rather small when we move from the monthly to the continuous time specification. This second result is consistent with findings in Christiano (forthcoming) where the length of the timing interval in a rational expectations model is treated as a free parameter. Christiano (forthcoming) plots the maximized value of the likelihood function of an annual data record against various values of the model timing interval. As the interval is reduced from an annual to a quarterly specification the value of the likelihood function rises substantially. However, further decreases in the model timing interval result in smaller increases in the value of the likelihood function. This result is also consistent with findings in Christiano (1986) in which a continuous time model of hyperinflation is estimated using monthly data. When an analogous discrete time model is fit to the same data, the results are virtually indistinguishable from the continuous time results.

The 11 free parameters of our continuous time model are:

$$\Lambda^c = (r,a,b,c,e,A,\alpha,\beta,V_{11},V_{22},V_{12}).$$

Our discrete time model also has 11 free parameters:

$$\Lambda^d = (\phi,a,b,c,e,A,\rho,\mu,V^d_{11},V^d_{22},V^d_{12}).$$

Equation (3.40) implies that no more than 9 parameters of the continuous time can be identified. The same is true for the discrete time model. Consequently, we searched for a lower dimensional parameter set that was identified. We restricted our attention to sets that included $(\lambda\text{-}r)$ and ψ for the continuous and discrete time models respectively For present purposes, it does not concern us that we cannot identify all the elements of Λ^c and Λ^d, since our principle motivation is to identify the adjustment speeds implied by the two models. These are controlled by $(\lambda\text{-}r)$ and ψ in the continuous and discrete time cases, respectively. The parameter sets that we estimated are the following:

$$\zeta = [r,\alpha,\beta,\lambda\text{-}r,bc/a,(a+b+A)/a,V^c_{11},V^c_{22},V^c_{12}), \quad \text{and,}$$

$$\xi = [\phi,\rho,\mu,\psi,bc/a,(a+b+A)/a,\ V^d_{11},V^d_{22},V^d_{12}).$$

Christiano and Eichenbaum (1985) establish that ζ and ξ are identified.[3.6] In practice we fixed the discount rates r and ϕ, a priori, at values which imply a monthly discount rate of .997.[3.7]

Both models were estimated using seasonally adjusted monthly data on nondurable manufacturing shipments and finished goods inventories. The data correspond to those used by Blinder (1984). This data is published by the Bureau of Economic Analysis (BEA) except that Blinder has converted BEA's end-of-month inventory stocks to beginning-of-month figures. We constructed quarterly and annual data by taking arithmetic averages of the monthly data. The data cover the period February 1959 to April 1982 and are measured in millions of 1972 dollars. Shipments data are averages over the month. All data were demeaned and detrended using a second order polynomial function of time and seasonal dummies.[3.8]

Table 3.1 reports the results of estimating the continuous time model using monthly data.[3.9] We are particularly interested in the implications of these estimates for the speed of adjustment statistics. The point estimate for $\lambda\text{-}r$ is 5.29 with 90 percent confidence interval given by (1.83, 8.75). This implies that,

$$T^c = 17\ (10,49).$$

The ninety percent confidence interval is reported in parentheses. Thus the continuous time model implies that it takes 17 days to eliminate 95 percent of the gap between actual and desired inventories. This speed of adjustment seems plausible, especially in light of Feldstein and Auerbach's (1976) observation that even the largest swings in inventory stocks involve only a few days' worth of production.

We now turn to the results obtained with the discrete time models. Table 3.2, 3.3 and 3.4 report results obtained with monthly, quarterly and annual data, respectively. The point estimates of ψ obtained with monthly, quarterly and annual data are .14 (.036,.244), .28 (.070,.490) and .58 (.150,1.01), respectively. Ninety percent confidence intervals are reported in parentheses. The standard errors of the estimates of ψ increase with the degree to which the data are temporally aggregated. Presumably this reflects the smaller number of data points that are available for the more temporally aggregated data.

The implied speed of adjustment statistics are given by,

	Continuous	Monthly	Quarterly	Annual
Days to Close 95% of the Gap	17	46	212	1980
Condidence Interval	(10,49)	(27,63)	(101,378)	(577,∞)

(3.10)

The continuous time figures are repeated here for ease of comparison. The numbers in the last three columns of the first row correspond to T^d in (3.28)'. The number in the first column of row one corresponds to T^c in (3.28). Numbers in parentheses in the second row are 90 percent confidence intervals.

Notice that the number of days required to close 95 percent of the gap between actual and desired inventories (T^d) is more than twice as large with monthly data, more than twelve times as large with quarterly data, and more than one hundred and fifteen times as large with annual data, than the estimate obtained using the continuous time model. Evidently, the estimated speeds of adjustment are a monotonically decreasing function of the degree to which the data are temporally aggregated. We take this result to be supportive of the Mundlak-Zellner conjecture that temporal aggregation can account for slow speeds of adjustment in stock adjustment models. The estimated adjustment speeds are plausible for the continuous time and monthly models, but implausibly slow - in our view - in the quarterly and annual models.

An interesting feature of our results is that the estimated speed of adjustment increases in diminishing increments as the model timing interval is reduced. The increase is very large going from annual to quarterly data, but appears to have approximately converged at the monthly level. To see this, notice that the adjustment speed confidence intervals for the monthly and continuous time models overlap considerably. To investigate the conjecture that convergence has occured with the monthly specification, we compared the discrete time reduced forms of the monthly and continuous time models.

The reduced forms of the continuous and discrete time models are reported in the second columns of Tables 3.1 and 3.2 respectively. These are similar along a number of interesting dimensions. First, C_3^c is close to zero, while the third order term in $C^d(L)$ is exactly zero. Also, the 2,1 elements of C_1^c and C_2^c are small, and so compare well with the corresponding elements in $C^d(L)$. This feature of the reduced forms has the implication that sales fail to be Granger-caused by inventories.[3.11] One dimension along which the reduced forms differ concerns the first row of C_2^c, which does not appear to be close to zero. In contrast, the first row of C_2^d is identically equal to zero. Also, the variance of the second innovation error is three times larger in the continuous time model than in the discrete time model. Unfortunately, the importance of these differences and similarities is hard to judge, since we do not have the relevant distribution theory. Moreover, it is not clear that a direct comparison of the reduced form parameters is the most revealing one.

In our view, it is more interesting to compare the implications of the two reduced forms for both sets of structural parameters. We are particularly interested in the implications of the reduced form representation of the data emerging from the continuous (discrete) time model for the the structural parameters of the discrete (continuous) time model. Consider first the implications of the reported reduced forms for the structural parameters of the continuous time model. Since the continuous time model is identified the reduced form parameters in column 2 of Table 3.1 map uniquely into the parameter values reported in the first column of Table 3.1. It is less obvious how to deduce the implications of the reduced form emerging from the discrete time model for the structural parameters of the continuous time model. Since the reduced form of the discrete time model does not satisfy the cross equation restrictions implied by the continuous time model, there is in fact no set of continuous time structural parameters consistent with the discrete time model reduced form. In view of this, we decided that the most sensible thing to do was to compute the set of continuous time parameters that comes "closest" to reproducing the discrete time reduced form in Table 3.2.

A natural candidate for this set of parameters is the probability limit of the maximum likelihood estimator of the continuous time structural parameters calculated under the assumption that the data are generated by the estimated reduced form corresponding to the discrete time model.[3.12] If the discrete time model is true then the estimates of the continuous time model obtained using monthly data ought to be close to this probability limit. These probability limits are reported in the second of the two columns labeled "Plim" in Table 3.5. Numbers in parentheses are the estimated parameter values taken from column one of Table 3.2. We find some discrepancies. For example, the plim of α is .035, while its estimated value is .081. Other discrepancies which stand out are the results for bc/a, V_{22}, and V_{12}. Unfortunately, we cannot draw any definitive conclusions regarding the magnitude of these differences in the absence of the relevant distribution theory. Nevertheless it is interesting to note the similarity between the estimated value of λ-r and its reported probability limit. As noted earlier, the estimated value of λ-r implies that firms close 95 percent of the gap between actual and desired inventories in 17 days. The estimated probability limit of this number under the assumption that the data are generated by the discrete time monthly model is 19.5 days.

We now consider the implications of the two reduced form representations for the structural parameters of the discrete time model. In column 1 of Table 3.5 we report the probability limits of the structural parameters of the discrete time monthly model. These were calculated under the assumption that the data are generated by the continuous time model. If the continuous time model is true then the estimates of the structural parameters of the discrete time model obtained using the monthly data ought to be close to the corresponding probability limits reported in Table 3.5. In fact these appear to be quite close to each other. The principal discrepancy is that bc/a is larger than the value reported in Table 3.2. In addition V_{22}^{d} and V_{12}^{d} are somewhat different from the values reported in Table 3.2. As before we cannot draw any definitive conclusions from this exercise without the relevant distribution theory. Nevertheless, it is interesting to note how similar the estimate of ψ reported in Table 3.2 is to its plim in Table 3.5. In particular, inferences about the speed of adjustment of actual to target inventories are basically the same for the two values of ψ.

We conclude from the results in Table 3.5 that, when viewed from the point of view of their implications for the discrete time parameters, the reduced forms in Tables 3.1 and 3.2 are fairly similar. Some differences are apparent when examined from the point of view of certain structural parameters of the continuous time model.

A third way to compare the two reduced form representations is to compare their log likelihood values. The difference between the log likelihood value of the discrete time monthly and continuous time models is equal to 25.36. In this sense the discrete time monthly model "fits" the data better than the continuous time model. On the other hand, the likelihood ratio statistic obtained when either of the two models is compared with an unrestricted reduced form ARMA(3.3) model indicates rejection of both structural models at essentially the same level. The log likelihood value of the unrestricted ARMA(3,3) model is 3307.5 which is significantly greater than the log likelihood values associated with both the continuous and discrete time monthly models (see Tables 3.1 and 3.2.).

Overall, we conclude that the monthly discrete time and continuous time models appear to be fairly similar when examined from the perspective of the reduced form time series representations that they imply for the monthly data. Next, we report some diagnostic tests on the underlying statistical adequacy of the two structural models.

The validity of the formulas used to compute the confidence intervals around our speed of adjustment estimates requires that the underlying models be correctly specified. Unfortunately, we found evidence against this hypothesis. As we indicated, a likelihood ratio test rejects both models against an unrestricted ARMA(3,3) alternative. We also computed the multivariate Box-Pierce statistics proposed by Li and McCleod (1981) to test for serial correlation in the fitted residuals from the continuous time and monthly discrete time models. These statistics were computed at lags 12 and 24 and are denoted by BP(12) and BP(24), respectively. Under the null hypothesis that the underlying disturbances are white noise, BP(k) is drawn from a chi-square distribution with 4 x k - n degrees of freedom, where n is the number of free parameters.[3.13] In our case, n=9. The Box Pierce statistics for the continuous time model are BP(12) = 162 and BP(24) = 278. For the discrete time model, they are BP(12) = 386 and BP(24) = 602. These statistics indicate a substantial departure from white noise in the fitted residuals. Because the likelihood ratio statistic and Box-Pierce statistics supply evidence against our models the speed of adjustment confidence intervals that we reported above must be interpreted with caution.

To what extent are our results sensitive to the way in which we specified our discrete time model? As we indicated in subsection 3.B there are at least two ways to choose a discrete time analogue to the continuous time model of subsection 3.A. Our procedure was to specify the shocks in the discrete time model to have the same representation as the point-in-time sampled representation as the continuous time shocks. Since our continuous time shocks are AR(1), this implies an AR(1) representation for the shocks in the discrete time model. We adopted this specification of the discrete time model because it matches well with what is commonly done in the literature.[3.14] An alternative would have been to specify the shocks in the discrete time model so as to produce a reduced form for that model with AR and MA orders identical to those implied by the continuous time model.

This can be accomplished by adding a first order moving average term to the shocks in the discrete time model. We conjecture that the effect of these moving average terms would be to raise the estimated speed of adjustment implied by the discrete time model. This conjecture is based on the belief that the additonal MA terms would take over some of the burden borne by the AR parameters- one of which controls the speed of adjustment - for accommodating the serial correlation in the data. This would be consistent with results in Telser (1967). As yet, we have not formally investigated this conjecture. However, it is important to note that these comments illustrate the observations made in subsection 2.B, where we argued that that the temporal aggregation effects of shrinking the model timing interval can have the same effect on the reduced form implications of a model as allowing for more serial correlation in the unobserved shock terms.

We conclude this subsection by reiterating the main objectives of our empirical exercise. These were (i) to show that slow speeds of adjustment obtained with the stock adjustment model could be accounted for by temporal aggregation bias, and (ii) to show that structural inferences can, in practice, be substantively affected by different assumptions about the frequency with which agents make economic decisions. In our view these objectives have been accomplished.

Table 3.1

Continuous Time Model

Monthly Data

Structural Parameters*

	Estimate
α	.081 (.021)
β	.082 (.021)
λ-r	5.29 (2.10)
bc/a	610.9 (9120. 5)
a/(a+b+A)	0.00 (.001)

$$V \begin{bmatrix} \underset{(12046.2)}{13244.5} & \underset{(4854.8)}{-507.3} \\ & \underset{(25150.5)}{28310.7} \end{bmatrix}$$

$\mathcal{L}$** = -3352.33

Reduced Form Parameters

$\theta_1^c = -1.85$

$\theta_2^c = .851$

$\theta_3^c = -.004$

$$C_1^c = \begin{bmatrix} -.772 & -.035 \\ -.032 & -.698 \end{bmatrix}$$

$$C_2^c = \begin{bmatrix} -.104 & .009 \\ .088 & -.243 \end{bmatrix}$$

$$C_3^c = \begin{bmatrix} -.002 & .003 \\ -.001 & .002 \end{bmatrix}$$

$$V^c = \begin{bmatrix} 24852.1 & 12459.9 \\ & 187924.0 \end{bmatrix}$$

*Standard errors are displayed in parentheses.

**Value of the log likelihood function.

Table 3.2

Discrete Time Model

Monthly Data

Structural Parameters*

Parameter	Estimate
μ	.910 (.027)
ρ	.960 (.021)
ψ	.140 (.063)
bc/a	1.00 (1.17)
a/(a+b+A)	0.00 (.001)

$$V^d = \begin{bmatrix} \begin{matrix} 24808.7 \\ (2110.4) \end{matrix} & \begin{matrix} 7594.0 \\ (3781.3) \end{matrix} \\ & \begin{matrix} 54792.8 \\ (13156.5) \end{matrix} \end{bmatrix}$$

Reduced Form Parameters

$\theta_1^d = -2.01$

$\theta_2^d = 1.14$

$\theta_3^d = -.12$

$$C_1^d = \begin{bmatrix} -.910 & .008 \\ 0.00 & -1.10 \end{bmatrix}$$

$$C_2^d = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & .130 \end{bmatrix}$$

$\mathcal{L}$ = -3326.97**

*Standard errors are displayed in parentheses.

**Value of the log likelihood function.

Table 3.3

Discrete Time Model

Quarterly Data

Structural Parameters*

μ	.824	(.077)
ρ	.854	(.063)
ψ	.283	(.132)
bc/a	.078	(.602)
a/(a+b+A)	0.00	(.001)

$$V^d = \begin{bmatrix} \begin{matrix} 65530.8 \\ (9731.9) \end{matrix} & \begin{matrix} 8337.6 \\ (14396.0) \end{matrix} \\ & \begin{matrix} 276318.4 \\ 41021.8 \end{matrix} \end{bmatrix}$$

$\mathcal{L}^{**} = -1161.52$

Reduced Form Parameters

$\theta_1^d = -1.96$

$\theta_2^d = 1.18$

$\theta_3^d = -.20$

$$C_1^d = \begin{bmatrix} -.824 & -.007 \\ 0 & -1.14 \end{bmatrix}$$

$$C_2^d = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & .242 \end{bmatrix}$$

*Standard errors are displayed in parentheses.

**Value of the log likelihood function.

Table 3.4

Discrete Time Model

Annual Data

Structural Parameters*		Reduced Form Parameters
μ	.139 (.224)	$\theta_1^d = -1.31$
ρ	.584 (.256)	$\theta_2^d = .500$
ψ	.584 (.257)	$\theta_3^d = -.050$
bc/a	.998 (.525)	$C_1^d = \begin{bmatrix} -.139 & -.038 \\ .206 & -1.17 \end{bmatrix}$
a/(a+b+A)	.021 (.396)	

$$V^d = \begin{bmatrix} \begin{matrix}133765.1 \\ (42042.6)\end{matrix} & \begin{matrix}-42203.5 \\ (60428.8)\end{matrix} \\ & \begin{matrix}468030.5 \\ (146721.4)\end{matrix} \end{bmatrix} \qquad C_2^d = \begin{bmatrix} 0.00 & 0.00 \\ -.029 & .333 \end{bmatrix}$$

$\mathcal{I}$**

*Standard errors are in parentheses.

**Value of the log likelihood function.

Table 3.5

Probability Limits

Discrete[1] Parameter	Plim[3]	Continuous[2] Parameter	Plim[3]
ρ	.940 (.960)	α	.035 (.081)
μ	.938 (.910)	β	.164 (.082)
ψ	.116 (.140)	λ-r	4.60 (5.30)
bc/a	51.45 (1.00)	bc/a	.879 (611.1)
a/(a+b+A)	0.00 (0.00)	a/(a+b+A)	0.00 (0.00)
V^d_{11}	24951.7 (24808.7)	V_{11}	19013.6 (13244.5)
V^d_{22}	200570.0 (54792.8)	V_{22}	8276.7 (28310.7)
V^d_{12}	11701.0 (7594.0)	V_{12}	2661.6 (-507.3)

[1] Probability limit of parameters of monthly discrete time model, assuming data are generated by reduced form in column 2, Table 3.1.

[2] Probability limit of parameters of continuous time model, assuming data are generated by reduced form in column 2, Table 3.2.

[3] Numbers in parentheses are parameter estimates obtained from the data.

4. Concluding Remarks

This paper has investigated the impact of temporal aggregation bias on structural inference in macroeconomics. We have argued, by way of two empirical examples, that this source of bias should not be dismissed as a quantitatively unimportant theoretical curiosum. Our empirical examples indicate that temporal aggregation bias can be quantitatively important in the sense of significantly distorting inference.

Nowhere did we argue in favor of a particular decision interval as being the most natural or correct one for the purposes of modeling macroeconomic phenomena. In our view this is an entirely open question which in all likelihood cannot be resolved on the basis of the aggregate time series data alone. However we do not see any compelling reason for the standard practice of assuming that the interval of time between agents' decisions is equal to the data sampling interval. This practice might be defended on the grounds that it is empirically inocuous. In fact our results indicate that there is little reason to expect that empirical results are robust to different assumptions regarding the frequency with which agents make decisions.

Macroeconomists often have access to different data sets, corresponding to different sampling intervals. It is not our view that tests of economic models ought always to be conducted with the data set corresponding to the finest sampling interval. This is because there may be systematic differences in the degree of measurement error associated with data collected at different intervals of time. However, the specification of agents' decision intervals and which data should be used in implementing a given model, are, in principle, separate issues. It is not logically inconsistent to believe, for example, that agents make decisions on a monthly basis and still insist on using quarterly data. The quarterly data may simply be more reliably collected. However it is logically inconsistent, under these circumstances, to use quarterly data without taking into account the misallignment of agents' decision intervals and the data sampling interval. This inconsistency is even more pronounced when the quarterly data are not sampled on a point in time basis.

Economists have long understood the need for robustness checks of empirical results with respect to different data sets. One conclusion from this study is that more attention should be devoted to robustness checks using data sampled at different intervals of time. More generally, we hope that macroeconomists will begin to deal explicitly with the problem of temporally aggregated data. Fortunately, the technical apparatus for dealing with temporal aggregation problems exists, at least for linear models.

Footnotes

[1.1] See Garber (1977) for an example of a model in which the decision interval is endogenous. In our view, whether timing decisions should be endogenized is, to some extent, an empirical question. Christiano (forthcoming) develops a technique for estimating fixed timing intervals in economic models whose reduced form are linear in the variables. That technique could be applied to several data sets, say drawn from different countries or diffferent regimes. If the estimated timing interval varied in some systematic way across the data sets, then it might be desirable to modify the model by making the legnth of the timing interval a function of the other parameters. Of course, at the most general level, an endogenous timing interval would not be of fixed legnth. Instead it would be state dependant. A limitation of Christiano's analysis is that it cannot accomodate state dependant timing intervals.

[1.2] The reduced form approach is represented by the work of Working (1960), Telser (1967), Sims (1971b), Geweke (1978), Hansen and Sargent (1984) and Marcet (1985). Examples of the structural approach are Mundlak (1961), Zellner (1968), Sims (1971a), Zellner and Monmarquette (1971) Engle and Liu (1972), Hansen and Sargent (1980a) and Christiano (1984,1985).

[2.1] Although this section omits citations to rigorous presentations of the material, these are included in Appendix A.

[2.2] A linearly indeterministic process is one for which the mean and any other perfectly linearly predictable component, eg., a trend term, have been removed. See Sargent [1979,Chapter XI, sect. 11] for further discussion.

[2.3] Note that we depart slightly from the usual convention, according to which a vector ARMA model denotes a representation in which both the autoregressive and moving average parts are vectors. We refer to this kind of representation as a VARMA model. In section 3 we refer to VAR models, by which we mean a VARMA model with zero order moving average component.

[2.4] Given the definition of the continuous time lag operator, the definition of

L implies that $L = e^{-D}$.

[2.5] The argument is formalized as follows. Define $y_t = (1-\lambda_1^2)(1-\lambda_2^2)z_t$ and let $S_y(e^{-i\omega})$ denote the spectral density of y_t at frequency $\omega \in (-\pi,\pi)$. Since $y_t = \epsilon_t + (\lambda_1+\lambda_2)\epsilon_{t-1/2} + \lambda_1\lambda_2\epsilon_{t-1}$, $S_y(e^{-i\omega}) = c(0) + c(1)(e^{-i\omega} + e^{+i\omega})$, where $c(k) = Ey_t y_{t-k}$, for integer values of k, and $c(k) = 0$ for $k > 1$. The discussion in Sargent [1979, Chapter XI, section 13] applies so that unique scalars $|d| \leq 1$ and $\sigma^2 \geq 0$ can be found with the property that $S_y(e^{-i\omega}) = |1 + de^{-i\omega}|^2 \sigma^2$. Also, Sargent [1979, p.241] shows that $S_y(e^{-i\omega}) = |(1-\lambda_1^2 e^{i\omega})(1-\lambda_2^2 e^{-i\omega}|^2 S_z(e^{-i\omega})$, where S_z is the spectral density of $\{z(t), t = 0,\pm1,\pm2,\ldots\}$. Then, since by hypothesis $|\lambda_i|^2 < 1$, $i = 1,2$, we have that $S_z(e^{-i\omega}) = |(1-\lambda_1 e^{i\omega})(1-\lambda_2 e^{-i\omega})|^{-2} |1+de^{-i\omega}|^2 \sigma^2$. But the object on the right hand side of the equality is the time series representation for a process with AR component $(1-\lambda_1^2 L)(1-\lambda_2^2 L)$, MA component $1+dL$, and innovation variance σ^2. This establishes that $\{z_t, t = 0,\pm1,\pm2,\ldots\}$ has an ARMA(2,1) representation.

[2.6] The integrals were approximated by taking daily averages over the month.

[2.7] For other examples of cases were the random walk hypothesis may have been innapropriately rejected as a consequence of spurious correlation induced by data averaging, see Working (1960,ftn.1) and Cowles (1960).

[2.8] There is one dimension along which the preceding results are not at all robust. We redid the calculations reported in Table 2.2 using the levels of the logs of the data. VAR(12)'s with and without a quadratic trend were computed. The results are strikingly different from those reported in Table 2. 2. First, money significantly improves forecasts of output whether a trend is included or not. For example, using the 1952 to 1983 period, the significance level of the test statistic for the null hypothesis that money does not help predict output is .0007 when a trend is excluded, while it is .00006 when a trend is included in the VAR(12). Second, the significance level of the test statistic is smaller when quarterly averages of money and output are used. In this sense, money seems to be less important in predicting output when time aggregated data are used. These results are puzzling to us. We are currently working to develop an explanation for these results using Monte Carlo methods.

3.1 The fact that we specify utility to be linear in leisure warrants some discussion because it appears to be inconsistent with findings in two recent studies. Our specification implies that leisure in different periods are perfect substitutes from the point of view of the representative consumer. MaCurdy (1981) and Altonji (1986) argue, on the basis of panel data, that leisure in different periods are imperfect substitutes from the point of view of private agents. Rogerson (1984) and Hansen (1985) describe conditions under which the assumption that the representative consumer's utility function is linear in leisure is consistent with any degree of intertemporal substitutability at the level of private agents.

3.2 It is of interest to contrast our model with the equilibrium model in Sargent [1979, chapter XV]. In that model, the representative agent's utility function is linear in consumption and quadratic in leisure. As a result, the interest rate on risk free securities, denominated in units of the consumption good, is constant. In our model, the representative agent's utility function is quadratic in consumption, with the result that the interest rate on risk free securities, denominated in units of the consumption good, is time varying and stochastic. This feature of our model is attractive in view of the apparent non-constancy of real interest rates in the U.S. In order to remain within the linear-quadratic framework, we specify utility to be linear in leisure. This implies that the interest rate on risk free securities, denominated in units of leisure, is constant.

3.3 To avoid proliferating notation we do not formally distinguish between variables chosen by individual households and firms and their economy wide counterparts. Nevertheless the disticntion between them plays an important role in the model. By assumption agents are perfectly competitive and view economy wide variables, such as P(t) and economy wide sales and inventories, parameterically.

3.4 To see that $\lambda > 0$ consider $f(k) = .5r - [k+2.5r^2]^{1/2}$ and note that $f(0) = 0$ and $f'(k) < 0$ for $k \geq 0$.

3.5 See Hansen and Sargent (1980a) who show that this procedure yields the unique optimal solution to the social planning problem which the competitive equilibrium solves.

3.6 Specifically, Christiano and Eichenbaum (1985) show that ζ and ξ are locally identified. In addition, we show that, given any admissable ζ, then there are at least 5 other values of ζ which are observationally equivalent, i.e., yield an identical value for the likelihood function. We constructed an algorithm to find these ζ's in order to determine whether any of them is admissable in the sense of satisfying the non-negativity conditions imposed by the model. Generally, we find that one other ζ is admissable in this sense. This value of ζ is obtained by exchanging the values of α and $(\lambda\text{-}r)$ and suitably adjusting r. As we point out later, our continuous time parameter estimates imply $\alpha = .082$ and $(\lambda\text{-}r) = 5.29$ with $r = .003$. This parameterization implies a relatively rapid speed of adjustment of actual to desired inventories. An alternative parameterization which yields the same value of the likelihood function is one in which $\alpha = 5.29$ and $(\lambda\text{-}r) = .082$. This implies that the speed of adjustment is very slow and relatively little serial correlation in the inventory holding cost shock. This parameterization can be ruled out as being implausible since it requires the discount rate to be $r\times100 = 62{,}112$ percent. We experimented with numerous parameterizations, and always found that if we placed a reasonable upper bound on r, then global identification obtained. We found the same result regarding ξ.

3.7 Our results were insensitive to the different values of r and β that we considered.

3.8 This time trend can be rationalized as follows. Suppose that u(t) and v(t) are the sum of a covariance stationary component, as given by equation (3.18) and a linear function of time and seasonal dummies. Then the equilibrium laws of motion will have two components. The first component will be the law of motion given in the text. The second component will be a deterministic function of time and seasonal dummies. There are no restrictions across the two compnents. These claims are established in Christiano and Eichenbaum (1985). There are alternative ways to generate trend growth in inventories and sales. For example, the equilibrium laws of moton for s(t) and I(t) will inherit any unit roots in the VAR for u(t) and v(t). The fact that we choose to work with deterministic time trends does not necessarily reflect the view that this is the only reasonable model of trend growth for our variables. Instead it reflects the fact that almost the entire empirical literature that we wish to address assumes the existence of deterministic time trends.

3.9 In models where the timing interval is finer than the data sampling interval, estimates of the AR and MA parameters can be sensitive to the scale in which the data are measured. This contrasts with the case in which the timing interval coincides with the data sampling interval. In the latter case, multiplying the data by a constant scalar affects only the innovation variances but not the AR and MA parameters. To check that our continuous time speed of adjustment estimate is robust to a change of scale, we divided the data by 100 and re-estimated the model parameters. The results were virtually unchanged.

3.10 The upper bound of the ninety percent confidence interval for ψ in the annual model is 1.01. This implies that firms never reach their target inventory level. This is why the reported upper bound of the ninety percent confidence interval for T^d in the annual model is ∞.

3.11 We noted in section 3.A that this assumption is frequently made in the inventory literature.

3.12 These were computed by maximizing the frequency domain approximation to Gaussian likelihood function in which the periodogram was replaced by the spectral density function implied by the reduced form parameters in Table 3.2. The justification for calling the resulting numbers probability limits is given in Christiano (1984) where this technique is applied in another context.

3.13 Li and McCleod (1981) derive the distribution for their test statistic under the assumption that the model being estimated is an unconstrained vector ARMA with independent, identically distributed disturbances. They show that BP(k) has an asymptotic chi-square distribution with $m^2k-\ell$ degrees of freedom, where m is the number of equations in the vector ARMA model and ℓ is the number of AR and MA parameters. We assume that the appropriate modification regarding the number of degrees of freedom, in our problem, is obtained by replacing ℓ by n.

3.14 See for example, Blinder (1984), Eichenbaum (1984), Maccini and Rossana (1984) and the references in McCallum (1984).

Appendix A: A Primer on Continuous Time Models.

In this appendix we present a very informal discussion of certain properties of the class of continuous time models utilized in this paper. The presentation assumes familiarity with discrete time models of time series data. Our presentation makes heavy use of analogies between discrete time and continuous time models. The proofs for Theorems 1 and 2 of the paper are contained in subsection A.3 of this appendix.

A.1 The Continuous Time Wold Representation

In discrete time, it is common to write a time series representation for a covariance stationary, linearly indeterministic process, z(t), as an infinite ordered moving average (MA(∞)) of disturbances. The disturbances in this representation are the errors in forecasting z(t) one step ahead using a linear function of past z(t)'s. Because they are serially uncorrelated, the disturbances are often referred to as "white noise". The fact that the assumption of an MA(∞) model involves no loss of generality is guaranteed by Wold's theorem in discrete time (see Sargent[1979,p.257].) There is a continuous time version of this theorem (Rozanov[1967,p.118-119].) According to it, a covariance stationary, linearly indeterministic continuous time process can be written as an integral of current and past disturbances as follows:

(A.1a) $$z(t) = \int_{-\infty}^{\infty} f(\tau)\epsilon(t-\tau)d\tau$$

where,

(A.1b) $$E\epsilon(t)\epsilon(t-k)' = \delta(k)V,$$

Here, V is positive definite symmetric matrix which we refer to as the "variance" of $\epsilon(t)$. Also, δ is the Dirac delta function (see Hannan[1970,pp.514-516]) which is defined by the property that

$$\int_{-\infty}^{+\infty} h(\tau)\delta(\tau)d\tau = h(0)$$

for any function h that is continuous at zero. Loosely speaking, δ can be thought of as a function that is nonzero only when k = 0. Consequently, according to (A.1b), $\epsilon(t)$ is a serially uncorrelated process, and so we call it a "continuous time white noise". One sense in which the analogy between a discrete time MA(∞) model and (A.1a) is strained is that a continuous time white noise is considerably more difficult to analyze rigorously than its discrete time counterpart. This is because the continuous time white noise does not "exist" in the sense that a discrete time white noise does. The difference lies in the fact that a discrete time white noise can be simulated, say by repeatedly tossing a coin, or rolling a die. By contrast, it is not possible to simulate realizations from a continuous time white noise process. For this reason, a white noise process is said not to be "realizable". On the other hand, a weighted integral of a white noise, eg., (A.1a), is realizable. Although a rigorous understanding of continuous time white noise is mathematically demanding, it is sufficient, for the purposes of this paper, to rely on analogies with the discrete time case. A rigorous treatment of continuous time white noise processes can be found in Hannan(1970, section I.6) and Gel'fand and Vilenkin(1964). (See Sargent[1982] and Astrom [1970] for introductory treatments).

Applying (A.1b) and the definition of the Dirac delta function, it is easy to confirm that $c_z(k) \equiv Ez(t)z(t-k)'$ is

$$\text{(A.1c)} \qquad c_z(k) = E\int_{-\infty}^{\infty} f(\tau)\epsilon(t-\tau)d\tau \int_{-\infty}^{\infty}\epsilon(t-k-\nu)'f(\nu)'d\nu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\epsilon(\tau)\delta(k+\nu-\tau)Vf(\nu)d\nu$$

$$= \int_{-\infty}^{\infty} f(\tau)Vf(\tau-k)'d\tau < \infty,$$

for any real value of k. The presence of the inequality in (A.1c) reflects our assumption of covariance stationary which coincides with the requirement that f be a square integrable function of τ.

A final property satisfied by $\epsilon(t)$ in (A.1a) is that the error in forecasting $z(t+k)$ using a linear combination of $z(t-s)$, $s \geq 0$ is

$$(A.1d) \quad z(t+k) - E[z(t+k \mid z(t-s) : s \geq 0] = \int_0^k f(\tau)\epsilon(t+k-\tau)d\tau$$

for any $k > 0$. Because of the obvious analogy with the disturbance term in the discrete time MA(∞), property (A.1d) leads us to call $\epsilon(t)$ the "innovation" in $z(t)$.

A.2 Continuous Time Models in Operator Notation.

In the discrete time context it is often convenient to write the MA(∞) representation of a stochastic process in operator notation. This is also the case in continuous time, where the shift operator is $e^{\tau D}x(t) \equiv x(t+\tau)$, for any real value of τ. Here, $Dx(t) \equiv dx(t)/dt$, so that D is the time derivative operator. (In discrete time, the common notation for the lag operator is $Lx(t) \equiv x(t-1)$, so that $L = e^{-D}$.) Intuitively, we can think of the rationale for this notation as follows. Suppose $x(t)$ were infinitely differentiable. Then

$$e^{\tau D}x(t) = x(t) + \tau Dx(t) + \frac{1}{2!}\tau^2D^2x(t) + \frac{1}{3!}\tau^3D^3x(t) + \ldots$$

Here, we have simply written $e^{\tau D}$ as a series expansion. Notice, however, that the expression to the right of the equality is $x(t+\tau)$ expressed as a Taylor series expansion about $x(t)$.

Substituting the shift operator into (A.1a), we obtain

$$(A.2a) \quad z(t) = \int_0^\infty f(\tau)e^{-\tau D}d\tau\epsilon(t) = F(D)\epsilon(t),$$

where,

$$(A.2b) \quad F(D) = \int_0^\infty f(\tau)e^{-\tau D}d\tau.$$

It makes no substantive difference whether we parameterize the continuous time model at the level of f, or at the level of F, since given one it always possible to recover the other. (More precisely, the F polynomial corresponding to f via (A.2b) is unique. Also, there is only one f function satisfying (A.1) that corresponds to a given F polynomial.) In general, it is more convenient to parameterize the model at the level of F(D). We parameterize F by specifying it to be a rational polynomial in D. In doing so, we lose some generality, since Wold's theorem says only that F corresponds to some f satisfying (A.1) and (A.2b). Specifically we assume,

(A.3) $F(D) = C(D)/\theta(D)$,

were C(D) is a q-th ordered, nxn matrix polynomial in D, and $\theta(D)$ is a p-th ordered scalar polynomial in D.

The requirement that the f function corresponding to F satisfy (A.1c), (A.1d), and that z be realizable implies the following three sets of restrictions on θ and C:

(i)	$\theta(s) = 0$ implies $Re(s) < 0$	*covariance stationarity of* $\{z(t)\}$.
(ii)	$\det C(s) = 0$ implies $Re(s) < 0$	*condition* (*A.1d*)).
(iii)	$p \leq q-1$	*realizability of* $\{z(t)\}$.

Here, Re(s) denotes the real part of the complex variable s. The first restriction is required by covariance stationarity of z, and the second by the requirement that ϵ be the innovation in z. Restrictions (i) and (ii) are among the reasons why there is such a close analogy between continuous time and discrete time ARMA models. Recall that in discrete time, covariance stationarity and the requirement that the time series disturbance be an innovation imply that the roots of the autoregressive part and of the determinant of the moving average part lie outside the unit circle. Suppose, for the moment, that we think of the lag operator, L, as a complex variable. Then the restrictions just described are the following. If L is a zero of either the autoregressive part or the determinant of the moving average part of the discrete time representation, then $|L| > 1$. To see the analogy with the corresponding restrictions on the continuous time model, recall the link that $L = e^{-D}$, and notice that $|L| > 1$ and $re(D) < 0$ are equivalent conditions. (Here, $|\cdot|$ denotes the absolute value operator.)

Restriction (iii) does not have a counterpart in discrete time models. The need for it arises because of the fact that a continuous time white noise, unlike its discrete time counterpart, is not realizable. Notice that (iii) rules out p=q=0, in which case z is not realizable since it identically equals ϵ.

We now present three examples which are designed to further motivate the three restrictions which we impose on θ and C.

Example 1

Suppose n=1 and

(A.4a) $\theta(D) = (\beta + D),\ C(D) = 1$,

then,

(A.4b) $F(D) = 1/(\beta + D)$

(A.4c) $f(\tau) = e^{-\tau\beta} \quad \tau\geq 0$

$0 \quad \tau<0$

It is easy to verify that (A.4b) and (A.4c) satisfy (A.2b). Also, note from (A.4c) that unless $\beta > 0$, $f(\tau)$ will not converge to zero as $\tau\rightarrow\infty$ and (A.1c) will fail. This translates into the proposition that covariance stationarity restricts the zeros of $\theta(D)$ to be negative. This is consistent with restriction (i). (Here again, we lapse into referring to the operator D as a variable. This does not lead one astray).

The following example illustrates the role of restriction (iii).

Example 2

Suppose n=1 and

$\theta(D) = \beta + D,\ C(D) = \gamma D + 1,$

so that (iii) holds if, and only if, $\gamma = 0$. Then,

$$F(D) = \frac{\gamma D+1}{D+\beta} = \gamma + \frac{1-\gamma\beta}{D+\beta}$$

and,

$$z(t) = F(D)\epsilon(t) = \gamma\epsilon(t) + \frac{1-\gamma\beta}{D+\beta}\epsilon(t)$$

$$= \gamma\epsilon(t) + (1 - \gamma\beta)\int_0^\infty e^{-\beta\tau}\epsilon(t-\tau)\, d\tau.$$

Evidently, in this case z(t) is the sum of $\gamma\epsilon(t)$ - which is realizable if, and only if, $\gamma=0$ - and a second term which is realizable. Consequently, z(t) is realizable if, and only if, $\gamma=0$. This result coincides with restriction (iii).

Finding the f function corresponding to an arbitrary rational F was trivial in example 1 because θ had only one root. It is of interest to note that if the roots of θ are distinct, then finding the f function corresponding to a given F can be converted into a problem as simple as the one in example 1 by application of the partial fractions expansion formula. Example 3 illustrates this for the case p=2, q=1.

<u>Example 3.</u>

Suppose n=1 and

$\theta(D) = (D - \lambda_1)(D - \lambda_2)$ $\qquad$ λ_1-λ_2 n.e. 0, $Re(\lambda_j) < 0$ j=1,2,

$C(D) = D - b$ $\qquad$ $b < 0$,

In this case, restrictions (i), (ii) and (iii) are satisfied, implying that z(t) is realizable and covariance stationary, and that the disturbance ϵ in $\theta(D)z(t)=C(D)\epsilon(t)$ is the innovation in z(t). Application of the partial fractions expansion (see Sargent[1979,pp.188-89]) yields

$$F(D) = \frac{C(D)}{\theta(D)} = \frac{1}{\lambda_1-\lambda_2}\left[(\lambda_1-b)\frac{1}{D-\lambda_1} + (b-\lambda_2)\frac{1}{D-\lambda_2} \right]$$

Then, applying the result in example 1 twice, we obtain

$$f(\tau) = \begin{cases} \frac{1}{\lambda_1-\lambda_1}\left[(\lambda_1-b)e^{\lambda_1\tau} + (b-\lambda_2)e^{\lambda_2\tau} \right] & \tau \geq 0 \\ 0 & \tau < 0. \end{cases}$$

Evidently, (A.1c) is satisfied because $Re(\lambda_i) < 0$, i=1,2, as restriction (i) implies. If $b > 0$, then z(t) is still covariance stationary, but ϵ is not its innovation. For a heuristic explanation of this, see Sargent(1982).

A.3 Proof of Theorems 1 and 2 In Section 2.

The proofs essentially follow the strategy taken in Example 1 of section 2. We begin by developing some notation and presenting a useful lemma. Define the scalar polynomial in D:

(A.5) $$\theta(D) = (D - \lambda_1)(D - \lambda_2) \cdots (D - \lambda_p),$$

where $Re(\lambda_i) < 0$ for all i and the λ_i's are distinct. Also, define the n x n matrix polynomial in D,

(A.6) $$C(D) = C_0 + C_1D + \ldots + C_qD^q,$$

with $q \leq p-1$, and $\det C(s) = 0$ if, and only if, $Re(s) < 0$. Finally, let

(A.7) $$\theta^c(\zeta) = 1 + \theta_1{}^c\zeta + \theta_2{}^c\zeta^2 + \ldots + \theta_p{}^c\zeta^p$$

$$= (1 - \mu_1\zeta)(1 - \mu_2\zeta) \cdots (1 - \mu_p\zeta),$$

where $\mu_i = e^{\lambda_i}$, i=1,...,p, and ζ is a complex variable. In this case,

$$\text{(A.8)} \qquad F(D) = \frac{C(D)}{\theta(D)} = \sum_{j=1}^{p} \frac{W_j}{D - \lambda_j},$$

where

$$W_j = \frac{C(\lambda_j)}{\prod_{\substack{k=1 \\ k.ne.j}}^{p} (\lambda_j - \lambda_k)}.$$

Here, the standard partial fractions expansion formula has been applied, element by element, to F(D). From (A.8) it follows that z(t) has the law of motion given by

$$\text{(A.10)} \qquad z(t) = \sum_{j=1}^{p} W_j \int_0^{\infty} e^{\lambda_j \tau} \epsilon(t-\tau)\, d\tau.$$

The following Lemma is used in the proofs of Theorems 1 and 2:

Lemma 1.

If

(i) $\{z(t)\}$ is generated by the structure (A.5) - (A.10)

then,

(ii) $\theta^c(L)z(t) = \eta(t)$

(iii) $\eta(t) = \int_0^p f(\tau)\epsilon(t - \tau)d\tau$

(iv) $f(\tau) = \sum_{j=1}^{p} W_j(\sum_{k=0}^{s} \theta^c_k e^{-k\lambda_j})\, e^{\lambda_j \tau}$ $\quad \tau \in (s,s+1)$, s=0,1,...,p-1.

Proof

The proof consists of applying $\theta^c(L)$ to the right side of (A.10) and showing that the result is $\eta(t)$ in (iii) - (iv).

$$\theta^c(L)z(t) = \theta^c(L)\sum_{j=1}^{p} W_j \int_0^\infty e^{\lambda_j\tau}\epsilon(t-\tau)\,d\tau$$

$$= \sum_{\ell=1}^{p} \theta^c_\ell \sum_{j=1}^{p} W_j \int_0^\infty e^{\lambda_j\tau}\epsilon(t-\tau-\ell)\,d\tau.$$

Notice that

$$\int_0^\infty e^{\lambda_j\tau}\epsilon(t-\tau-\ell)\,d\tau = \int_0^{p-\ell} e^{\lambda_j\tau}\epsilon(t-\tau-\ell)\,d\tau + \int_{p-\ell}^{\infty} e^{\lambda_j\tau}\epsilon(t-\tau-\ell)\,d\tau$$

$$= \int_0^{p-\ell} e^{\lambda_j\tau}\epsilon(t-\tau-\ell)\,d\tau + \int_0^\infty e^{\lambda_j\tau} e^{\lambda_j(p-\ell)}\epsilon(t-\tau-p)\,d\tau$$

Substituting,

$$\theta^c(L)z(t) = \psi_t + \sum_{j=1}^{p} W_j \int_0^\infty e^{\lambda_j\tau}\,\theta^c(e^{-\lambda_j})\,e^{\lambda_j p}\,\epsilon(t-\tau-p)\,d\tau = \psi(t)$$

since by construction $\theta^c(e^{-\lambda_j}) = 0$, i=1,...,p. Now,

$$\psi_t = \sum_{\ell=0}^{p} \theta^c_\ell \sum_{j=1}^{p} W_j \int_0^{p-\ell} e^{\lambda_j\tau}\epsilon(t-\ell-\tau)\,d\tau$$

$$= \sum_{j=1}^{p} W_j \{ \theta_0^c \,[\int_0^1 e^{\lambda_j\tau}\epsilon(t-\tau)d\tau + \int_1^2 e^{\lambda_j\tau}\epsilon(t-\tau)d\tau$$

$$+ \int_2^3 e^{\lambda_j\tau}\epsilon(t-\tau)d\tau + \ldots + \int_{p-1}^{p} e^{\lambda_j\tau}\epsilon(t-\tau)\,d\tau\,]$$

$$+ \theta_1^c\, e^{-\lambda_j}[\int_1^2 e^{\lambda_1\tau}\epsilon(t-\tau)d\tau + \int_2^3 e^{\lambda_j\tau}\epsilon(t-\tau)d\tau + \ldots + \int_{p-1}^{p} e^{\lambda_j\tau}\epsilon(t-\tau)\,d\tau]$$

$$+ \theta_2^c\, e^{-2\lambda_j}[\int_2^3 e^{\lambda_j\tau}\epsilon(t-\tau)\,d\tau + \ldots + \int_{p-1}^{p} e^{\lambda_j\tau}\epsilon(t-\tau)\,d\tau]$$

$$+ \ldots + \theta^c_{p-1}\, e^{-\lambda_j(p-1)} \int_{p-1}^{p} e^{\lambda_j\tau}\epsilon(t-\tau)d\tau \,\}$$

Collecting terms in $\int_s^{s+1} e^{\lambda_j\tau}\epsilon(t-\tau)d\tau$, s = 0,1,...,p-1, yields $\psi_t = \eta(t)$ in (iii) and (iv).

Q.E.D.

Proof of Theorem 1

Let $S_z(i\omega)$ denote the spectral density at frequency $\omega \epsilon (-\infty,\infty)$ of the continuous time process, $z(t)$. Absolute integrability of the moving average function in (A.10) guarantees that $S_z(i\omega) = F(i\omega)VF(-i\omega)'$, where F is defined in (A.8) (see eg., Papoulis[1962,page 27].) Evidently $S_z(i\omega) > 0$, meaning that S_z is positive semidefinite for all $\omega \epsilon (-\infty,\infty)$ and its determinant is positive at all but possibly a finite number of points. Let $S^d_z(e^{-i\omega})$ denote the spectral density of z_t at frequency $\omega \epsilon (-\pi,\pi)$. According to Hannan [1970,p.45], $S_z{}^d(e^{-i\omega}) = \sum_{k=-\infty}^{\infty} S_z(i[\omega+\pi k])$. The fact that $S_z > 0$ therefore implies $S^d{}_z > 0$. Define $y_t = \theta^c(L)z_t$ for integer t. According to Hannan[1970,Theorem 9, p.58], the spectral density of $\{y_t\}$, denoted S_y, is given by $S_y(e^{-i\omega}) = |\theta^c(e^{-i\omega})|^2 S^d{}_z(e^{-i\omega})$, for $\omega \epsilon (-\pi,\pi)$. Because of the restrictions on the λ's, $|\theta^c(e^{-i\omega})|^2$ n.e. 0 for all $\omega \epsilon (\pi,\pi)$. Hence, $S_y > 0$. Since the conditions of Lemma 1 apply, $y_t = \eta(t)$ and $c(k) = Ey_t y_{t-k}' = 0$, $k \geq p-1$. Hannan's [1979,p.66] Theorem 10' then guarantees the existence of a unique set, $A_1, \ldots, A_{p-1}$, W, for which the zeroes of $\det A(\zeta)$ lie on, or outside the unit circle and W is positive definite, with the property $S_y(e^{-i\omega}) = A(e^{i\omega})WA(e^{i\omega})'$. Here, $A(e^{-i\omega}) = I + A_1 e^{-i\omega} + \ldots + A_{p-1}e^{-i\omega(p-1)}$. Therefore, we conclude that

$$S_z{}^d(e^{-i\omega}) = |\theta^c(e^{-i\omega})|^2 \, A(e^{i\omega})WA(e^{i\omega})'.$$

But the expression on the right of the equality is an ARMA(p,p-1) representation for a process with scalar autoregressive part $\theta^c(L)$, matrix moving average part $A(L)$, and innovation variance W.

Q.E.D.

Proof of Theorem 2

he proof is a trivial modification on the proof to Theorem 1, so the details are ommitted. The important thing to note is that $\int_0^1 \theta^C(L)z(t-\tau)d\tau = \theta^C(L)\bar{z}(t) = \int_0^1 \eta(t-\tau)d\tau = \bar{y}_t$, say. Hence, $\bar{c}(k) = E\bar{y}_t\bar{y}_{t-k}$ is not necessarily zero for $k = p$, although $\bar{c}(k) = 0$ for $k > p$.

Q.E.D

References

Altonji, J.G. (1986) "Intertemporal Substitution in Labor Supply: Evidence From Micro Data", manuscript.

Astrom, K.J., (1970) Introduction to Stochastic Control, Academic Press.

Blanchard, O.J. (1983) "The Production and Inventory Behavior of the American Automobile Industry," Journal of Political Economy, 91:365-400.

Blinder, A.S. (1981) "Inventories and the Structure of Macro Models", American Economic Review, 71:11-16.

Blinder, A.S. (1986) "Can the Production Smoothing Behavior Model of Inventory Behavior Be Saved", manuscript.

Blinder, A.S. and Holtz-Eakin, D. (1984) "Inventory Fluctuations in the United States Since 1929", manuscript.

Box, G.E.P and Jenkins, G.M.J. (1976) Time Series Analysis: Forecasting and Control, Holden-Day.

Bryan, W.R. (1967) "Bank Adjustments to Monetary Policy: Alternative Estimates of the Lag," American Economic Review, 57:855-864.

Christiano, L.J. (1984) "The Effects of Aggregation over Time on Tests of the Representative Agent Model of Consumption", manuscript.

Christiano, L.J. (1985a) "A Critique of Conventional Treatments of the Model Timing Interval in Applied Econometrics", manuscript.

Christiano, L.J. (1985b) "Temporal Aggregation and Government Policy Evaluation", manuscript.

Christiano, L.J. (1986) "A Continuous Time Model of Cagan's Model of Hyperinflation Under Rational Expectations", manuscript.

Christiano, L.J. (forthcoming) "A Method for Estimating the Timing Interval in a Linear Econometric Model, with an Application to Taylor's Model of Staggered Contracts", Journal of Economic Dynamics and Control.

Christiano, L.J. and Eichenbaum, M.S. (1985) "A Continuous Time, General Equilibrium, Inventory-Sales Model", manuscript.

Cowles, A. (1960) "A Revision of Previous Conclusions Regarding Stock Price Behavior", Econometrica, 28:909-915.

Eckstein, Z. and Eichenbaum, M.S. (1985), "Quantity-Constrained Equilibria in Regulated Markets: The U.S. Petroleum Industry, 1947- 1972", pps. 41-69, T.J. Sargent, ed., Energy, Foresight and Strategy, Resources For the Future, The Johns Hopkins University Press.

Eichenbaum, M.S. (1984) "Rational Expectations and the Smoothing Properties of Inventories of Finished Goods," Journal of Monetary Economics, 14:71-96.

Eichenbaum, M.S., Hansen, L.P. and Richard, S. "The Dynamic Equilibrium Pricing of Durable Goods", manuscript.

Eichenbaum, M.S. and Singleton, K.J. (1986) "Do Real Business Cycle Theories Explain Post War U.S. Time Series Data", Working Paper No. 1932, National Bureau of Economic Research.

Engle, Robert F. and Ta-Chung Liu (1972) "Effects of Aggregation Over Time on Dynamic Characteristics of an Econometric Model", in Econometric Models of Cyclical Behavior, vol. II, Proceedings of NBER Conference on Research in Income and Wealth, edited by B. Hickman, Columbia University Press, New York.

Feldstein, M.S. and Auerbach, A. (1976) "Inventory Behavior in Durable Manufacturing: The Target Adjustment Model," Brookings Papers on Economic Activity, 2:351-408.

Garber, P.M. (1977) Costly Decision and the Demand for Money, unpublished PH.d dissertation, University of Chicago.

Geweke, J.B. (1978) "Temporal Aggregation in the Multivariate Regression Model", Econometrica, 46:643-662.

Gel'fand, I.M. and Vilenkin, N.Ya (1964) Generalized Functions, Vol.4, Applications to Harmonic Analysis, translated by A.Feinstein, N.Y., Academic Press.

Goodfriend, M. (1985) "Reinterpreting Money Demand Regressions", (eds.) K. Brunner and A.H. Meltzer, Understanding Monetary Regimes, Carnegie-Rochester Conference on Public Policy, Vol. 22:207-242, Amsterdam, North Holland.

Hannan, E.J. (1970) Multiple Time Series Wiley Press.

Hansen, G. (1985) "Indivisible Labor and the Business Cycle", Journal of Monetary Economics, 16:309-327.

Hansen, L.P. and Sargent, T.J. (1980a) "Methods for Estimating Continuous

Time Rational Expectations Models from Discrete Data," Staff Report 59, Federal Reserve Bank of Minneapolis.

Hansen, L.P. and Sargent, T.J. (1980b), "Formulating and Estimating Dynamic Rational Expectations Models", Journal of Economic Dynamics and Control, 2:351-408.

Hansen, L.P. and Sargent, L.P. (1981) "Formulating and Estimating Continuous Time Rational Expectations Models from Discrete Data", manuscript.

Hansen, L.P. and Sargent, T.J. (1983) "Aggregation over Time and the Inverse Optimal Predictor Problem for Adaptive Expectations in Continuous Time", International Economic Review, 24:1-20.

Hansen, L.P. and Sargent, T.J. (1984) "Two Difficulties in Interpreting Vector Autoregressions," Working Paper 227, Federal Reserve Bank of Minneapolis.

Li, W.K. and McLeod, A.I. (1981) "Distribution of the Residual Autocorrelations in Multivariate ARMA Time Series Models", Journal of the Royal Statistical Society, (Series B), 43:231-239.

Lucas R.E. Jr. and Prescott, E.C. (1971) "Investment Under Uncertainty", Econometrica, 39:154-163.

Lucas, R.E. Jr. and Sargent, T.J. (1981), Introductory Essay to Rational Expectations and Econometric Practice, Minneapolis: University of Minnesota Press.

Luenberger, D.G. (1969) Optimization by Vector Space Methods, Wiley, New York.

Maccini, L.J. and Rossana, R.J. (1984) "Joint Production, Quasi-Fixed Factors of Production, and Investment in Finished Goods Inventories", Journal of Money Credit and Banking.

MaCurdy, T.E. (1981) "An Empirical Model of Labor Supply in a Life-Cycle Setting", Journal of Political Economy, 89:1059-1085.

Marcet, A. (1985) "Temporal Aggregation of Economic Time Series", manuscript.

McCallum, B.T. (1984) "Inventory Fluctuations and Macroeconomic Analysis: A Comment", manuscript.

Mundlak, Y. (1961) "Aggregation Over Time in Distributed Lag Models", International Economic Review, 2:154-163.

Papoulis, A. (1962) The Fourier Integral and Its Applications, McGraw Hill.

Phillips, A.W. (1959) "The Estimation of Parameters in Systems of Stochastic Differential Equations", Biometrika, 59, 657-665.

Rogerson, R. (1984) "Indivisible Labor, Lotteries and Equilibrium", manuscript

Rozanov, Yu. A. (1963) Stationary Random Processes, Holden-Day.

Sargent, T.J. (1979) Macroeconomic Theory, Academic Press.

Sargent, T.J. (1982) "Preliminary Introduction to Continuous Time Stochastic Processes", manuscript.

Sims, C.A. (1971a) "Approximate Specification in Distributed Lag Models", manuscript.

Sims, C.A. (1971b) "Discrete Approximations to Continuous Time Lag Distributions in Econometrics", Econometrica, 38:545-564.

Sims, C.A. (1980) "Macroeconomics and Reality", Econometrica, 48:1-48.

Taylor, J.B. (1980) "Output and Price Stability", Journal of Economics and Control, 2:109-132.

Telser, L.G. (1967) "Discrete Samples and Moving Sums in Stationary Stochastic Processes", Journal of the American Statistical Association 62, 484-99.

Tiao, G.C. (1972) "Asymptotic Behaviour of Temporal Aggregates of Time Series", Biometrika, 59:525-530.

West, K.D. (1986) "A Variance Bounds Test of the Linear-Quadratic Inventory Model", Journal of Political Economy, 94:374-401.

Working, H. (1960) "Note on the Correlation of First Differences of Averages in a Random Chain", Econometrica 28, 916-918.

Zellner, A. (1968) "Note on Effect of Temporal Aggregation on Estimation of A Stock Adjustment Equation", manuscript.

Zellner, A. and Montmarquette, C. (1971) "A Study of Some Aspects of Temporal Aggregation Problems in Econometric Analyses", Review of Economics and Statistics, 63:335-342.