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RATIONAL EXPECTATIONS MODELS WITH A CONTINUUM OF CONVERGENT SOLUTIONS

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ABSTRACT

This paper examines five examples of rational expectations models with a continuum of convergent solutions and demonstrates serious difficulties in the economic interpretation of these solutions. The five examples are (1) a model of optimal capital accumulation with a negative rate of time preference, (2) Taylor's (1977) linear rational expectations model of macroeconomic equilibrium; (3) Calvo's (1984) model of contract setting and price dynamics; (4) Obstfeld's (1984) equilibrium model of monetary dynamics with individual optimizing agents; and (5) Calvo's (1978) life-cycle model of savings and asset valuation. In every case, when these models yield a continuum of convergent infinite horizon solutions, these solutions fail to exhibit economically appropriate, forward looking dependence of the endogenous variables on the paths of the exogenous forcing variables--a difficulty that does not arise under the circumstances where these models yield unique convergent infinite horizon solutions. Further, the three models that have natural finite horizon versions, either lack finite horizon solutions or have solutions that do not converge to any of the infinite horizon solutions. Again, this difficulty arises only under the circumstances where these models have a continuum of infinite horizon solutions.

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1. Introduction

In recent years, Taylor (1977), Calvo (1978, 1984) and Obstfeld (1984), among others, have presented examples of rational expectations models with a continuum of convergent solutions. This situation arises when the dynamic systems associated with these models have steady states at which the number of stable roots is greater than the number of backward looking dynamic processes.[1] This paper demonstrates, through detailed consideration of five examples, serious difficulties that arise in the economic interpretations of solutions to such models.[2]

The first example is a model of optimal intertemporal consumption choice and capital accumulation with a negative rate of time preference. This model has a steady state toward which there converge a continuum of rational expectations paths. None of these paths, however, has any economic significance, and each must be constructed as the backward looking solution of what ought to be a forward looking economics relationship.

The second example is Taylor's (1977) model of macroeconomic equilibrium. This model has a continuum of convergent, rational expectations solutions when the effect of real money balances on aggregate supply is strong and positive. None of these solutions, however, has economically appropriate, forward looking expression for the price level as a function of the future behavior of the money supply.[3]

The third example is Calvo's (1984) model of price dynamics. In this model, a continuum of rational expectations solutions converge to a steady state equilibrium that is unstable in the Walrasian sense, while only a single solution converges to steady state equilibria that are stable in the usual Walrasian sense. The present analysis demonstrates that all of the solutions that converge to the Walrasian unstable equilibrium lack an economically appropriate dependence of the path of prices on exogenous factors that are shifting the excess demand function over time.

The fourth example is based on Obstfeld's (1984) equilibrium model of monetary and price level dynamics in which individual households maximize the lifetime utility from commodity consumption and services of real money balances. The infinite horizon version of this model yields a continuum of convergent rational expectations solutions when the elasticity of the marginal utility of consumption with respect to real money balances, η, is less than minus one. The finite horizon version of Obstfeld's model, however, does not have any solution when η is less than minus one, under the standard terminal condition that requires the utility value of final money balances to be zero. When this terminal condition is modified by a capital levy that fixes a positive level of final real money balances, the finite horizon version of Obstfeld's model sometimes has a solution, but solution does not converge to any of the continuum of solutions for the infinite version of Obstfeld's model. Moreover, all of the infinite horizon solutions when η is < -1, fail to exhibit economically appropriate forward looking relationships between real money balances and the price level and the future behavior of the nominal money supply.

The fifth example is Calvo's (1978) life-cycle model of consumption and savings behavior in which the young generation acquires land to finance consumption in old age. The infinite horizon version of this model yields a continuum of convergent solutions when consumption in youth responds sufficiently positively to an increase in the price of consumption in youth relative to consumption in old age. When this condition is met, however, the finite horizon solution of Calvo's model does not converge to any of the continuum of infinite horizon solutions. Moreover, as in the other examples, none of the continuum of convergent infinite horizon solutions exhibits a forward looking relationship between the relative price of consumption in successive periods and the future behavior of the rental on land.

Based on the analysis of these five examples, the paper concludes with some general observations on rational expectations models with a continuum of convergent solutions.

2. An Optimal Capital Accumulation Problem

Consider an optimal capital accumulation problem in which the objective is to maximize the discounted utility of consumption;

$$(1) \qquad \text{Maximize} \quad \int_0^T U(C(t)) \cdot \exp(-\rho t)\, dt$$

where $U(C)$ is a standard concave utility function, T is the time horizon, and ρ is the pure rate of time preference which is assumed to be <u>negative.</u>[4] The initial capital stock $K(0)$ is given, and rate rate of change of the capital stock, $\dot{K} = dK/dt$, is the excess of output (net of depreciation) over consumption;

$$(2) \qquad \dot{K} = F(K) - C$$

where $F(K)$ is the production function illustrated in the top panel of figure 1.

To determine the solution of this optimization problem, define the current value Hamiltonian,

$$(3) \qquad H = U(C) + q(F(K) - C)$$

where q is the shadow price of a unit of capital. The necessary conditions for a solution require that for $t < T$,

$$(4) \qquad \partial H/\partial C = U'(C) - q = 0$$

(5) $\dot{K} = \partial H/\partial q = F(K) - C$

(6) $\dot{q} = \rho q - \partial H/\partial K = \rho \cdot q - qF'(K)$

(7) $\lambda > 0.$

In addition, the boundary condition,

(8) $q(T)K(T) = 0.$

should be satisfied at time T.

For $t < T$, the necessary conditions for optimal behavior imply a dynamic system governing the behavior of K and q. Solving (4) for $C = N(q) = U'^{-1}(q)$, with $N'(q) < 0$, the differential equation governing K can be written as

(9) $\dot{K} = F(K) - N(q),$

The differential equation governing q can be written as

(10) $\dot{q} = q(\rho - F'(K)).$

The differential equation system (9) and (10) represents a rational expectations (or perfect foresight) model of the determination of K and q. Equation (9) should be thought of as a backward looking dynamic process since the current capital stock depends on past net investment, not on future net investment. In contrast, equation (10) should be thought of as a forward looking dynamic process since the shadow value of a unit of capital (which measures the current utility value of additions to the capital stock) ought to reflect the present value of the future increments to utility of consumption made possible by a current increment to capital.

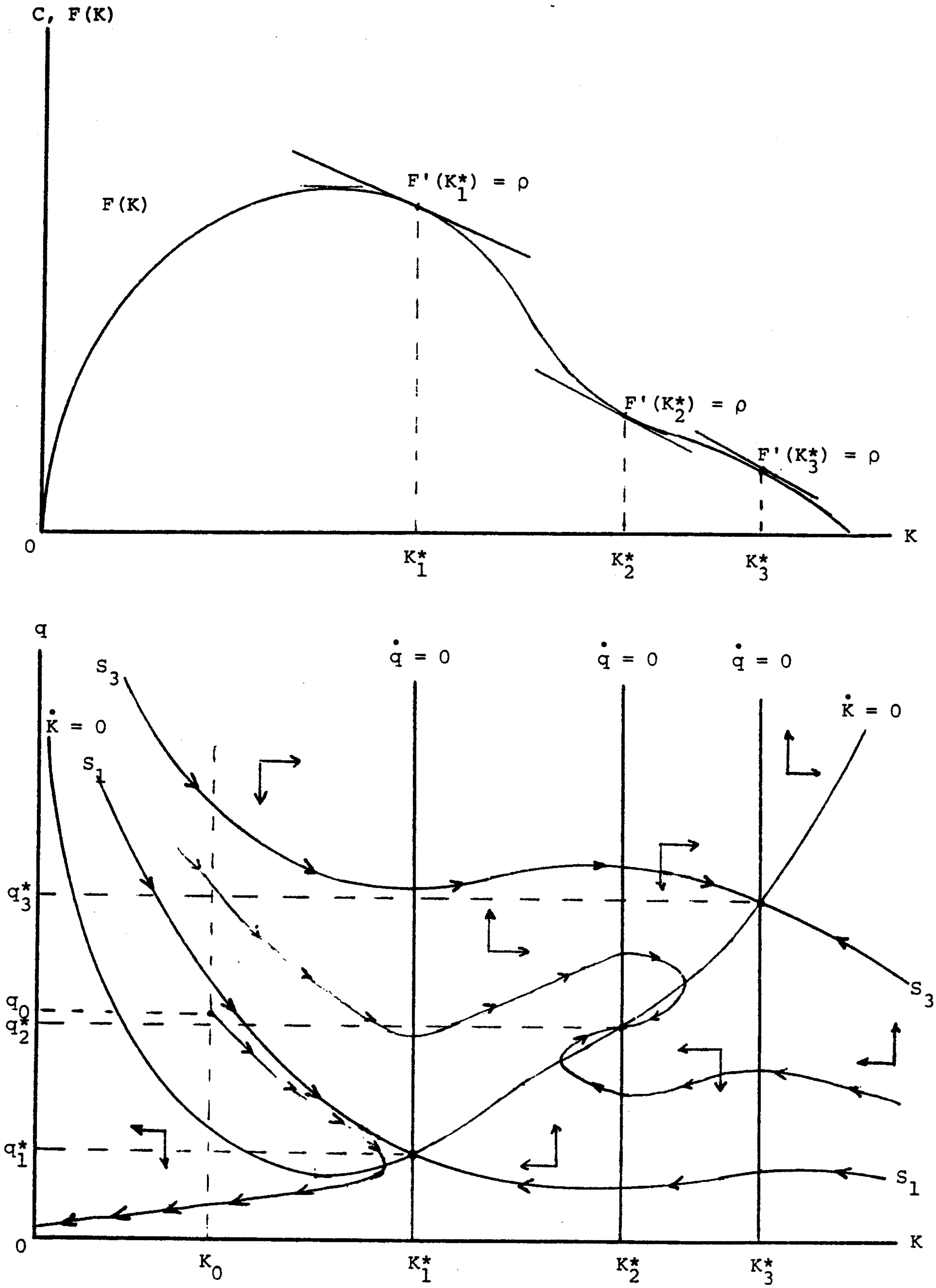

Fig.--1: The Production Function and Phase Diagram for the Optimal Consumption and Capital Accumulation Problem.

The behavior of K and q implied by dynamic system (9) and (10) is illustrated in the phase diagram in the lower panel of figure 1. In this diagram, there are three steady state positions, (K_1^*, q_1^*), (K_2^*, q_2^*) and (K_3^*, q_3^*), occuring at the three values of K at which

(11) $F'(K) = \rho < 0.$

Along the vertical lines passing through these three steady state points, $\dot{q} = 0$. The sign of $\dot{q}$ at other points indicated by the direction of the vertical arrows. The curve along which $\dot{K} = 0$ is determined by the combinations of K and q for which $F(K) = N(q)$. Above the $\dot{K} = 0$ curve, $\dot{K} > 0$, and below this curve, $\dot{K} < 0$, as indicated by the horizontal arrows in the diagram.

The characteristic roots of the dynamic system (9) and (10), at the three steady state positions, are given by

(12) $\lambda = (1/2)[\rho \pm \sqrt{\rho^2 - 4 \cdot F''(K)}].$

Since ρ is < 0 and $F''(K)$ is < 0 at the two outer steady state positions, (K_1^*, q_1^*) and (K_3^*, q_3^*), one characteristic root is negative and the other is positive. Consistent with this pattern of roots, there are stable branches of the dynamic system, labeled S_1S_1 and S_3S_3, that converge to each of these outer steady state positions. At the middle steady state position (K_2^*, q_2^*), where $F''(K) > 0$, the characteristic roots of that dynamic system both have negative real parts, implying more stable roots (two) than backward looking processes (one). Consistent with this fact, a continuum of paths converge to the steady state position (K_2^*, q_2^*); specifically, all of the paths lying between the stable branches S_1S_1 S_3S_3. Thus, (K_2^*, q_2^*) is a steady state position in a rational expectations model with a continuum of converging paths. It remains to determine the economic significance of this steady state and of the paths converging to it.

Given the initial capital stock $K(0)$ and a choice of the initial value of the shadow price $q(0)$, the differential equations (9) and (10) completely determine the subsequent paths of $K(t)$ and $q(t)$. The appropriate choice of $q(0)$ that yields the solution of the optimization problem is the choice that leads to satisfaction of the boundary condition (7) at time T. Specifically, for the situation illustrated in figure 1, if $K(0) = K_0$, then q_0 is the correct choice of $q(0)$ provided that the time it takes to move along the path from (K_0, q_0) to the vertical axis is equal to T. A lower choice for the initial q would lead to higher comsumption and higher utility during an initial period, but would drive the capital stock and consumption level to zero before T. Since the discount rate is negative and the utility function is concave, this would not be optimal for a horizon of length T. Conversely, choice of a higher initial q would leave the economy with a positive capital stock at time T. This also would be suboptimal because utility could be increased by consuming the remaining capital stock during a brief interval before T.

For any initial capital stock, a longer time horizon T, implies a higher optimal initial value of q, but one that always remains below the value of determined by the ordinate of the stable branch S_1S_1 at the initial capital stock. Indeed, as T grows very large, the optimal initial choice of $q(0)$ converges to the value of q determined by the point on the stable branch S_1S_1 corresponding to $K(0)$. It follows that for very large T, the optimal path for K and q initially runs very close to the stable branch S_1S_1. Moreover, since the speeds of adjustment of K and q are both very low in the neighborhood of (K_1^*, q_1^*), K and q remain in the neighborhood of this steady state for a long time. Only as the time horizon T approaches, does the economy move away from this neighborhood and increase consumption by running down the capital stock. Thus, the stable branch S_1S_1 and the steady state position (K_1^*, q_1^*) are relevant to describing the optimal behavior of the economic system as the time horizon becomes very long.[5]

In contrast, the steady state position (K_2^*, q_2^*) and the continuum of paths converging to this steady state have no economic relevance, no matter how long the time horizon. For any initial $K(0)$, a choice of $q(0)$ that places the economy on one of the paths between S_1S_1 and S_3S_3 that lead to the steady state (K_2^*, q_2^*) necessarily leaves the economy with a positive capital stock at time T. The longer the time horizon, the closer the economy comes to the steady state (K_2^*, q_2^*), and the more time the economy spends near to this steady state. Such behavior is distinctly suboptimal since the level of consumption associated with this steady state is lower than the level of consumption that could be sustained with a smaller capital stock, and since the economy never enjoys the benefit of consuming its excessively large capital stock. Mathematically, the non-optimality of these paths is indicated by the failure of the second-order condition for concavity of the Hamiltonian with respect to K at (K_2^*, q_2^*). The demonstrable suboptimality of all of these paths indicates existence of an infinity of non-explosive solutions for a rational expectations model does not insure that any of these solutions is economically meaningful.

Further insight into the economic significance of alternative solutions of the dynamic system (9) and (10) comes from considering the forward and backward looking solutions of (10):

$$(13) \qquad q_f(t) = A \cdot \exp(\rho \cdot t) + \int_t^T q(s) \cdot F'(K(s)) \cdot \exp(-\rho \cdot (s - t))\, ds$$

$$(14) \qquad q_b(t) = B \cdot \exp(\rho \cdot t) - \int_0^t q(s) \cdot F'(K(s)) \cdot \exp(-\rho \cdot (s - t))\, ds$$

where A and B are constants. For a finite horizon, these two solutions are well defined and yield the same value of $q(t)$ when evaluated along the same path of $K(s)$, provided that A and B satisfy the condition,

$$(15) \qquad B = A + \int_0^T q(s) \cdot F'(K(s)) \cdot \exp(-\rho \cdot s)\, ds.$$

From an economic perspective, however, the forward looking solution is the appropriate solution because the utility value of a marginal unit of capital should reflect the future path, rather than the past path, of the marginal product of capital.

When the discount rate is negative $(\rho < 0)$, the infinite time horizon, optimal capital accumulation problem does not have a solution. Appropriate to this situation, the forward looking solution $q_f(t)$ given by (13) does not have a well defined, bounded value for the stable branch converging to either (K_1^*, q_1^*) or (K_3^*, q_3^*) or for any of the infinity of paths converging to (K_2^*, q_2^*). The only a bounded solution for $q(t)$ is the backward looking solution $q_b(t)$ given by (14). The necessity of using a backward looking solution for $q(t)$ for which there is no corresponding forward looking solution means that $q(t)$ cannot be interpreted as forward looking, even though the economics of the situation require that $q(t)$ should be forward looking. As we shall see, this difficulty in obtaining a forward looking solution for what ought to be a forward looking variable is a problem that frequently arises in rational expectations models exhibiting a continuum of convergent solutions.

3. Taylor's Macroeconomic Model

Taylor's (1977) linear rational expectations model of macroeconomic equilibrium provides a much discussed example of a continuum of convergent solutions. This model consists of an aggregate demand function, an aggregate supply function, a money market equilibrium condition and a goods market equilibrium condition which jointly imply the following reduced form equation for the price level;

$$E_{t-1}p_{t+1} = E_{t-1}p_t + \delta_1 p_t - \delta_1 m_t + u_t \tag{16}$$

where p_t is the logarithm of the price level at time t, E_{t-1} indicates the expectation conditional on information available at $t - 1$, m_t is the logarithm of the money supply, and u_t is a serially independent, normally distributed error with zero mean and finite variance. The critical reduced form parameter in this model, δ_1, is a combination of parameters of the various behavior functions. Taylor argues that δ_1 must be nonzero, but could be either positive or negative. Taylor considers only the case where the money supply is constant $(m_t = m)$, and writes the solution of the model as

$$p_t = m + k + \sum_{i=0}^{\infty} \pi_i u_{t-1} \tag{17}$$

where k is a constant that depends on the parameters of the model and where the π_i's satisfy

$$\pi_0 = -1/\delta_1 \tag{18}$$

$$\pi_{i+1} = (1 + \delta_1)\pi_i \quad \text{for } i = i, 2, \ldots. \tag{19}$$

with π_1 a free parameter. When δ_1 is positive, the model has a unique non-explosive solution obtained by setting $\pi_1 = 0$ and hence $\pi_i = 0$ for all $i > 1$. When δ_1 is negative and $|\delta_1| < 2$, however, Taylor concludes that the model has an infinity of non-explosive solutions, one for each choice of π_1.

The difficulty with this conclusion becomes apparent in the general case where the money supply follows a known path but is not necessarily constant. In this case, the solution of the model should be written as

$$(20) \qquad p_t = F(t) + k + \sum_{i=0}^{\infty} \pi_i u_{t-1}$$

where the π_i's satisfy (18) and (19) with π_1 a free parameter, and where

$$(21) \qquad F(t) = (\delta_1/(1 + \delta_1)) \sum_{j=0}^{\infty} (1/(1 + \delta_1))^j m_{t+j} \quad .$$

$F(t)$ expresses the forward looking relationship between the current price level and current and future levels of the money supply. This relationship arises in Taylor's model because the real interest rate that affects aggregate demand depends on the expected inflation rate, and the expected inflation rate ought to depend (under the assumption of rational expectations) on the expected future behavior of the money supply.

When δ_1 is positive, the unique non-explosive solution that is obtained by setting $\pi_1 = 0$ and hence $\pi_i = 0$ for $i > 1$. When δ_1 is negative, however, that we no longer find an infinity of non-explosive solutions, one for each choice of π_1. Instead, we find no non-explosive solutions because the forward looking discounted sum that defines $F(t)$ does not converge for a wide class of reasonable specifications of the behavior of m_{t+j}, (including m_{t+j} constant at m) when δ_1 is negative and $|\delta_1| < 2$.

This difficulty with the convergence of the sum defining $F(t)$ is concealed when m_{t+j} is constant because the solution of the model can also be written as

$$(22) \qquad p_t = B(t) + k + \sum_{i=0}^{\infty} \pi_i u_{t-i}$$

where

$$(23) \qquad B(t) = -\delta_1 \sum_{j=0}^{\infty} (1 + \delta_1)^j m_{t-j-1}.$$

Here, $B(t)$ is the backward looking solution for the component of p_t that depends on the path of m. When δ_1 is negative and $|\delta_1| < 2$, $B(t)$ converges for reasonable assumptions about the behavior of m_s. In particular, when $m_s = m$, $B(t) = m$ and we obtain exactly Taylor's result that there is an infinity of solutions for p_t, one for each choice of π_1. However, use of $B(t)$ in the solution (23) is contrary to the economics of Taylor's model because it makes the price level a function of past levels of the money supply, not of present and future levels of the money supply.[6]. If one insists that dependence of p_t on present and future m's is essential for an economically sensible solution of Taylor's model, then when $\delta_1 < 0$ and $|\delta_1| < 2$, none of the continuum of solutions of Taylor's model qualifies as economically sensible. It follows that to satisfy this criterion of economic sensibility, it is necessary to rule out the case where $\delta_1 < 0$ and $|\delta_1| < 2$.[7]

4. Calvo's Model of Price Dynamics

Another example of a rational expectations model with an infinity of stable solutions is Calvo's (1984) model of price dynamics. In Calvo's model, prices of individual products are fixed by contracts that expire with a constant probability δ per unit time, implying that the fraction of contracts in force at time zero that will still be in force at time t is $\delta\exp(-\delta t)$. The general price level at time t is an average of the prices of individual products in force at time t;

$$(24) \qquad P(t) = \int_{-\infty}^{t} \delta V(s)\exp(\delta(s - t))\, ds$$

where $V(s)$ is the price of an individual product whose contract was last set at time s. The price in a contract set at time t is assumed to reflect the general price level and state of excess aggregate demand expected over the length of the contract;

$$(25) \qquad V(t) = \int_{t}^{\infty} \delta[P(s) + \beta f(P(s), r(s))]\exp(-\delta(s - t))\, ds$$

where $f(P, r)$ measures excess aggregate demand as a function of P and the real interest rate r, and $\beta > 0$ measures the responsiveness of an individual price to excess aggregate demand.[8] The real interest rate is equal to a fixed nominal interest rate, i, minus the expected rate of change (right hand derivative) of P. Differentiation of (24) reveals that

$$(26) \qquad \dot{P} = \delta[V - P].$$

Setting $r = i - \delta[V - P]$ in (25) and differentiating the result with respect to t, it follows that

$$(27) \qquad \dot{V} = \delta[V - P - \beta f(P, i - \delta(V - P))].$$

The differential equation system (26) and (27) constitutes a rational expectations model that determines the time paths of P and V, given an initial value of P. The steady states of this model are common values of P and V at which $f(P, i) = 0$. Assuming that excess aggregate demand is a continuous function of P and is positive for all sufficiently low values of P and negative for all sufficiently high values of P, it follows that there must be at least one steady state. If there is only one steady state, then at this steady state f_P must be negative, and it is easily shown that model linearized at this steady state has one positive and one negative characteristic root. In this case, there is a unique stable solution of the model for any initial price level.

The peculiar case of continuum of stable solutions arises at even numbered equilibria where $f(P, i) = 0$ and $f_P > 0$ when the total number of equilibria where $f(P, i) = 0$ is odd and greater than one. The behavior of P and V in the case of three solutions of $f(P, i) = 0$ is illustrated in figure 2. The curves labeled S_1S_1 and S_3S_3 are the stable branches of the dynamic system converging to the steady state equilibria $(P_1, V_1 = P_1)$ and $(P_3, V_3 = P_3)$, respectively. At these two steady state equilibria, the usual condition for Walrasian stability, $f_P < 0$, is satisfied. All paths lying outside the region bounded by these two stable branches diverge. All paths lying between these two stable branches converge to the steady state equilibrium $(P_2, V_2 = P_2)$. Such convergence is consistent with the fact that the linearized dynamic system has two stable roots at a steady state where $f_P > 0$. Thus, we observe the very peculiar result that an equilibium that is unstable in the Walrasian sense is super stable in Calvo's model of price dynamics. For any initial P(0), a continuum of choice of V(0) imply paths of P and V converging to this Walrasian-unstable equilibrium, while for each initial P(0), only one choice of V(0) yields a path converging to either of the two Walrasian-stable equilibria.[9]

The difficulty with this conclusion of super stability of Walrasian unstable equilibria in Calvo's model is revealed by considering a more general excess aggregate demand function, $f(P + z, i)$, where z is an exogenous forcing variable that shifts the excess aggregate demand function (and the positions of the Walrasian equilibria) over time. With this modification, the linearized version of Calvo's model can be written as

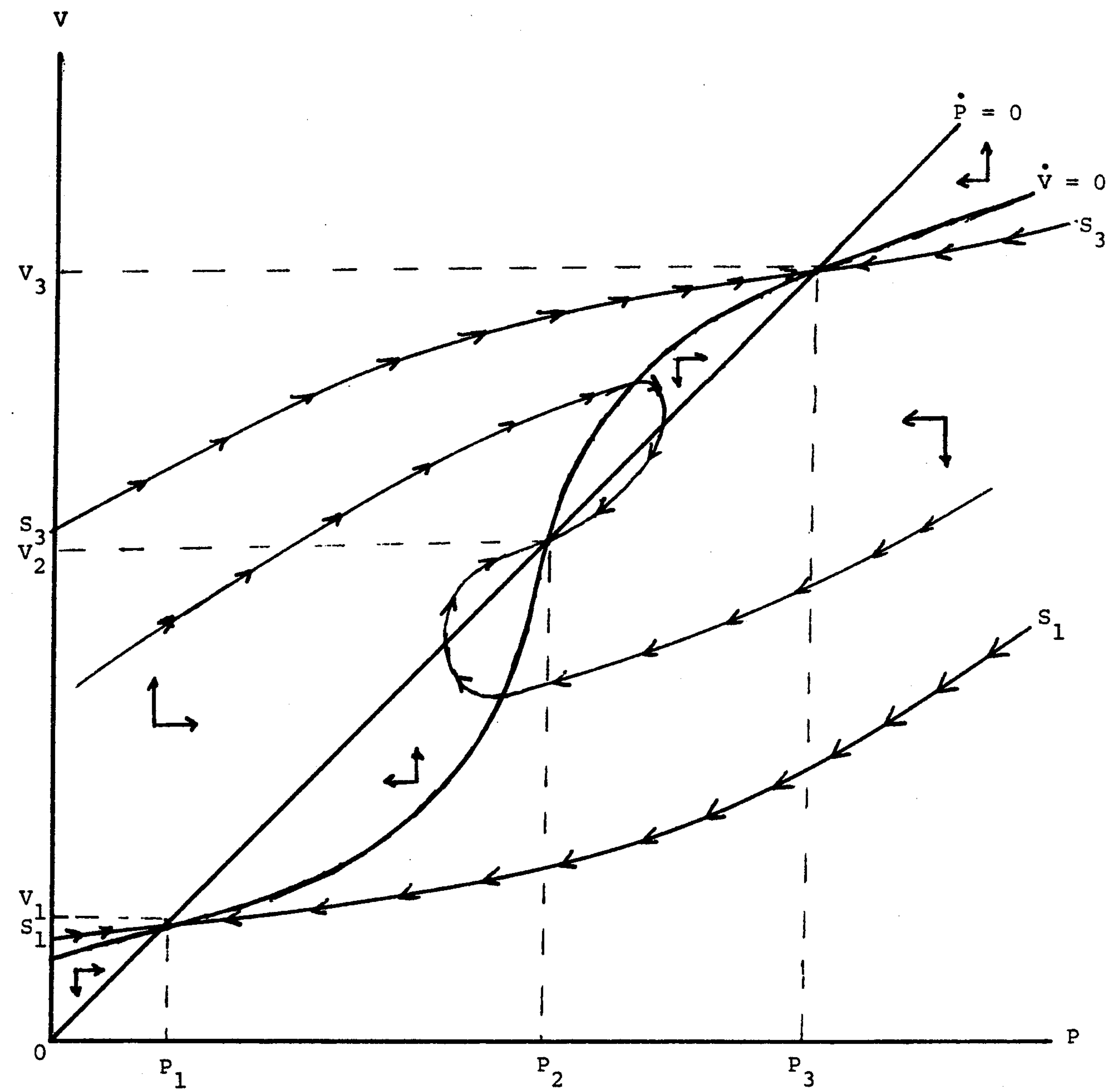

Fig.--2: The Dynamics of Calvo's Model of Price Adjustment.

$$(28)\quad \begin{bmatrix} -D-\delta & \delta \\ -\delta-\beta\delta f_p-\beta\delta^2 f_r & -D+\delta\beta\delta^2 f_r \end{bmatrix} \cdot \begin{bmatrix} P-\bar{P} \\ V-\bar{V} \end{bmatrix} = \begin{bmatrix} 0 \\ \beta\delta f_p z \end{bmatrix}$$

where D is the differential operator and $\bar{P} = \bar{V}$ is the common steady state value of P and V when $z = 0$. With $z = 0$, this system is exactly the linearized version of Calvo's model. The characteristic roots of this system are given by

$$(29)\quad \lambda_1 = (1/2)\cdot[\beta\delta^2 f_r - \sqrt{\beta^2\delta^4 f_r^{\,2} - 4\beta\delta^2 f_p}\,]$$

$$(30)\quad \lambda_2 = (1/2)\cdot[\beta\delta^2 f_r + \sqrt{\beta^2\delta^4 f_r^{\,2} - 4\beta\delta^2 f_p}$$

Since f_r is negative, a negative f_P implies that λ_1 is negative and λ_2 is positive; whereas a positive f_P implies that λ_1 and λ_2 are both negative or have negative real parts.

The general solution of (28) is given by

$$(31)\quad P(t) = A\cdot\exp(\lambda_1 t) + B\cdot\exp(\lambda_2 t) + a\cdot G(t) + b\cdot H(t) + \bar{P}$$

$$(32)\quad V(t) = (A(\lambda_1 + \delta)/\delta)\cdot\exp(\lambda_1 t) + (B(\lambda_2 + \delta)/\delta)\cdot\exp(\lambda_2 t) + c\cdot G(t) + d\cdot H(t) + \bar{V}$$

where A and B are arbitrary constants, where

$$(33)\quad a = b = -\beta\delta^2 f_p/(\lambda_1 - \lambda_2),\quad c = a(\lambda_1 + \delta)/\delta,\quad d = b(\lambda_2 + \delta)/\delta,$$

and where

$$(34)\quad G(t) = \int_0^t z(s)\cdot\exp(\lambda_1(t-s))ds$$

$$(35) \qquad H(t) = \begin{cases} H_f(t) = \int_t^{\infty} z(s)\cdot\exp(\lambda_2(t-s))ds & \text{when } f_P < 0 \\ H_b(t) = \int_0^{t} -z(s)\cdot\exp(\lambda_2(t-s))ds & \text{when } f_P > 0. \end{cases}$$

For a Walrasian-stable equilibrium where $f_P < 0$, the unique non-explosive solution consistent with a given initial $P(0)$ is obtained by setting $B = 0$ and $A = P(0) - b\cdot H_f(0)$. For a Walrasian-unstable equilibrium where $f_P > 0$, a non-explosive solution consistent with a given initial $P(0)$ is obtained by a continuum of choices of A and B subject to the constraint that $A + B = P(0)$. To obtain each of this continuum of solutions, however, it is essential to use the backward looking solution $H(t) = H_b(t)$. Thus, in each of this continuum of solutions, there is no relationship between $P(t)$ or $V(t)$ and the behavior of $z(s)$ for $s > t$.

It may be argued that use of an exclusively backward looking solution for $P(t)$ and $V(t)$ in the case where $f_P > 0$ is contrary to the economics of Calvo's model. In this model, prices of newly negotiated contracts are set with a view to what excess demand will be over the future life of the contract. It is reasonable to expect, therefore, that prices of newly negotiated contracts, represented by $V(t)$, will reflect the future behavior of the exogenous factor $z(s)$ that shifts the excess aggregate demand function. Further, since the general price level $P(t)$ is an average of prices of contracts negotiated up to time t, it is reasonable to expect that it too should reflect the expected evolution of $z(s)$ for $s > t$. If one accepts this argument, then failure of any of the continuum of solutions converging to a Walrasian-unstable equilibrium to exhibit forward looking dependence of

$V(t)$ and $P(t)$ on $z(s)$ for $s > t$ is reason for disqualifying these solutions as economically sensible solutions of Calvo's model.[10] This leaves the solutions converging to Walrasian-stable equilibria (one solution for each equilibrium and each initial value of P) as the only economically sensible solutions of Calvo's model.

5. Obstfeld's Model of Monetary Dynamics

Obstfeld (1984) presents an interesting example of a continuum of stable rational expectations solutions to a model in which households seek to maximize their lifetime utilities,

$$\text{(36)} \qquad \text{Max} \int_0^T u(c(t),\ m(t)) \cdot \exp(-\rho \cdot t)\, dt, \quad \rho > 0$$

by choice of their time paths of consumption, $c(t)$, and real money balances, $m(t)$, subject to the flow budget constraint,

$$\text{(37)} \qquad \dot{m} = y + v - c - \pi \cdot m,$$

where y is the fixed level of household income, v is the real flow of transfer payments from the government, and $\pi = \dot{P}/P$ is the inflation rate. From the perspective of individual households, initial nominal and real money balances, the fixed level of income, and the time paths of P, π and v are taken as given.[11] Optimal behavior by individual households requires choice time paths of c, m and the shadow price of consumption (or of real money balances), $\lambda > 0$, that satisfy the transition laws (37) and

$$\text{(38)} \qquad \dot{\lambda} = \lambda \cdot (\rho - \pi - x(c,\ m)), \quad x(c,\ m) \equiv u_m(c,\ m)/u_c(c,\ m);$$

the first order condition,

$$\text{(39)} \qquad u_c(c,\ m) = \lambda,$$

the initial condition

(40) $m(0) = M(0)/P(0)$

where $M(0)$ is the household's initial nominal money balance, and the terminal condition

(41a) $\exp(-\rho \cdot T) \cdot \lambda(T) \cdot m(T) = 0,$ for $T < \infty$

(41b) $\lim_{t \to \infty}(-\rho \cdot t) \cdot \lambda(t) \cdot m(t) = 0,$ for $T = \infty.$

From the perspective of the economy, equilibrium requires that consumption of the representative household equal a fixed level $\bar{c}$ that corresponds both to output of consumption goods (per household) and household income,

(42) $c = \bar{c} = y.$

Equilibrium also requires that the ratio of the (per household) nominal money supply to the price level equal the level of real money balances desired by households,

(43) $M/P = m.$

The government's budget constraint requires that real transfers (per household) equal the government's revenue (per household) from money creation;

(44) $v = \mu \cdot m,$

where $\mu = \dot{M}/M$ is the exogenously specified rate of monetary expansion. Differentiation of (43) with respect to time yields the result that

(45) $\pi = \mu - (\dot{m}/m)$

along any equilibrium path for the economy.

Combining (39) and (42), we obtian he relationship,

$$(46) \qquad \lambda = h(m) \equiv u_c(\bar{c}, m) \quad \text{or} \quad m = g(\lambda) \equiv h^{-1}(\lambda),$$

that must be satisfied along any equilibrium path for the economy. Differentiating this relationship and making use of (37) and (38), we obtain the differential equations that characterize the evolution of m and λ along any equilibrium path:

$$(47) \qquad \dot{m} = (1/(1 + \eta))\cdot[\rho + \mu - x(\bar{c}, m)]\cdot m$$

$$(48) \qquad \dot{\lambda} = (1/(1 + \gamma))\cdot[\rho + \mu - x(\bar{c}, g(\lambda))]\cdot\lambda$$

where

$$(49) \qquad \eta(m) = m\cdot h'/h \quad \text{and} \quad \gamma(\lambda) = \lambda\cdot g'/g = 1/\eta(g(\lambda)).$$

From these results it is apparent that the steady state values of m and λ for any constant rate of monetary expansion, $\bar{m}(\mu)$ and $\bar{\lambda}(\mu) = h(\bar{m}(\mu))$, are determined uniquely by the requirement that

$$(50) \qquad x(\bar{c}, m) = \rho + \mu.$$

Obstfeld focuses on the infinite horizon case, $T = \infty$. In this case, when the elasticity of the marginal utility of consumption with respect to m, $\eta(m) = m\cdot u_{cm}/u_c$, is > -1, there is a unique stable rational expectations equilibrium path for the economy--the path associated with m and λ remaining at their steady state values. This path is an equilibrium because all of the conditions (37) through (45), including the transversality condition (41b), are satisfied. It is the only stable equilibrium path because, as illustrated in panels A and B of figure 3, any initial choice of m and $\lambda = h(m)$ other than their steady state values implies a subsequent path of m and λ that diverges from the steady state.

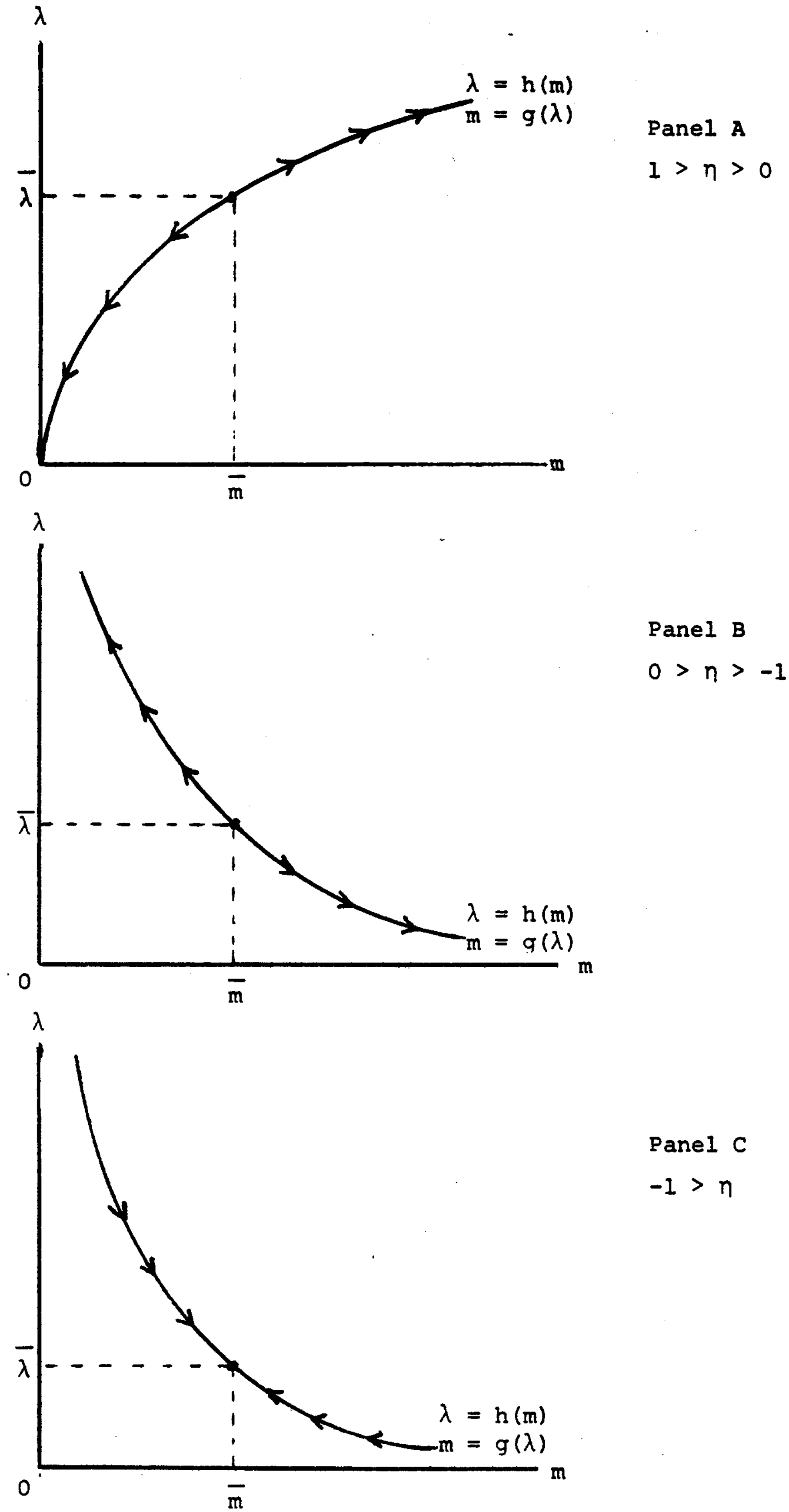

Fig.--3: Dynamics in Obstfeld's Model for Three Values of η.

When $\eta(m)$ is < -1, Obstfeld concludes that there is a continuum of stable rational expectations equilibrium paths. Each initial choice of m and $\lambda = h(m)$ implies, as illustrated in panel C of figure 3, a subsequent path of m and λ that converges to the steady state. Since each of these paths satisfies all of the conditions (37) through (45), including the transversality condition (41b), each is a rational expectations equilibrium path.

There is no doubt that when households solve their individual optimization problems taking as given the initial level of money balances and the paths of p and v associated with any of the solutions described in the preceding paragraph, they behave in a manner that sustains these solutions as equilibrium paths for the economy. There are, however, reasons for doubting the economic sensibility of Obstfeld's model and its solutions when η is < -1. These doubts arise first from considering solutions of the finite horizon version of Obstfeld's model and second from considering solutions of Obstfeld's model when the rate of monetary expansion is not necessarily constant.

The difference between the finite and infinite horizon versions of Obstfeld's model is in the terminal condition (41). All of the other conditions of optimal behavior by individual households and equilibrium in the economy reduce to the relationship (46) and the differential equations (47) and (48), and are the same in both versions. For a finite horizon, when $\eta > -1$, there is a unique rational expectations path for the economy.[12] This path is determined by the unique choice of $m(0) < \bar{m}$ and $\lambda(0) = g(m(0))$ that is consistent with $\lambda(T) \cdot m(T) = 0$ when m and λ subsequently evolve in accord with (47) and (48).[13] A longer time horizon T requires an initial $m(0)$ closer to $\bar{m}$ and an initial λ closer to $\bar{\lambda} = g(\bar{m})$. In the limit, as T grows large, the path of the economy lies most of the time in a near neighborhood of the steady state point $(\bar{m}, \bar{\lambda})$. Thus, when $\eta > -1$, the unique solution of the finite horizon version of Obstfeld's model converges to the unique solution of the infinite horizon version.

For a finite horizon, when $\eta < -1$, there is no rational expectations equilibrium path for the economy. The only eligible paths are those that start at some $m(0)$ and $\lambda(0) = h(m(0))$ and move along the path where $\lambda = h(m)$ as dictated by (47) and (48). But with $\eta < -1$, movement along the $\lambda = h(m)$ locus must be, as illustrated in panel C of figure 3, toward the steady state point where $m = \bar{m}$ and $\lambda = h(\bar{m})$. Such behavior does not satisfy the terminal condition (41a). This terminal condition is essential because with a finite horizon individual households recognize that the terminal utility value of real money balances is zero and hence wish to run down their real money balances as they approach the end of life.[14]

If the govermnent imposes a real capital levy, K, payable in money at date T, the terminal condition in the finite horizon version of Obstfeld's model is altered from (41a) to

(51) $$m(T) = K.$$

This altered terminal condition modifies only slightly the results when η is > -1. The starting point for the unique solution is the point on the $\lambda = h(m)$ such that when m evolves in accord with (47), $m(T) = K$. When $K = \bar{m}$, this requires that $m(0) = \bar{m}$ and the economy sits at the steady state from 0 to T. When $K \neq \bar{m}$, $m(0)$ lies between $\bar{m}$ and K and m moves away from $m(0)$ and $\bar{m}$ to reach K at T. As the time horizon becomes longer (for a fixed K), $m(0)$ moves closer to $\bar{m}$ and $m(t)$ remains longer in the neighborhood of $\bar{m}$. Thus when η is > -1, the unique solution for the finite horizon converges to the unique solution for the infinite horizon as the finite horizon becomes long.

When η is < -1 and the terminal condition (51) replaces (41a), solutions for the finite horizon version of Obstfeld's model sometimes exist. When $K = \bar{m}$, setting $m(0) = \bar{m}$ and holding $m(t)$ at $\bar{m}$ until T is always the solution. But, unlike the infinite horizon problem with the terminal condition (41) or (51), this is the only solution for the finite horizon problem. When $K \neq \bar{m}$, the situation is more complicated and depends on properties of the utility function. The case considered by Obstfeld where the utility function is given by

$$(52) \qquad u(c, m) = (c^{\alpha} m^{\beta})^{1-R}/(1 - R), \quad \alpha, \beta > 0, \quad \alpha + \beta < 1, \quad R < 1$$

with β and R chosen so that $\eta = \beta \cdot (1 - R)$ is less than -1, is especially interesting. With this utility funciton, if K is $> \bar{m}$, a unique solution to the finite horizon problem always exists. This solution is obtained by choosing $m(0) > K$, so that $m(t)$ evolving in accord with (47) falls to K at T. As the time horizon lenghtens, $m(0)$ must be set higher, tending toward a limiting value of infinity. For any horizon length, $m(t)$ never approaches nearer to $\bar{m}$ than K. Thus, the solution to the finite horizon problem for $K > \bar{m}$ does not converge to any of the continuum of solutions for the infinite horizon problem. When K is $< \bar{m}$, a unique solution for the finite horizon problem exists only when the horizon is not too long. When it exists, this solution is obtained by choosing $m(0) < K$ so that $m(t)$ evolving in accord with (47) rises to K at T. A solution fails to exist when the horizon is sufficiently long that the time it takes $m(t)$ to reach K from any positive $m(0)$ is less than T. Since this always happens for any $K < \bar{m}$, it is again clear that the solution to the finite horizon problem does not converge to any of the continuum of solutions of the infinite horizon problem when $\eta < -1$.

Next, consider Obstfeld's model when the rate of monetary expansion is not necessarily constant. The requirements for a rational expectations equilibrium solution are still summarized by (46), (47) and (48) and the boundary condition (41). When η is > -1, there is a unique solution for fairly general specifications of the time path of μ. This solution is obtained by choosing the unique initial values $m(0)$ and $\lambda(0) = h(m(0))$ that imply satisfaction of the terminal conditon (41) when m and λ subsequently evolve in accord with (47) and (48). The behavior of real money balances along this solution path is described by the forward looking solution of (47);

$$(53)\quad m(t) = \int_t^T ((x(s) - \mu(s))\cdot m(s)/(1 + \eta))\cdot \exp(R(t) - R(s))ds + A\cdot \exp(R(t))$$

where

$$(54)\qquad R(u) = \int_0^u (\rho/(1 + \eta))du$$

and where A is a constant whose value is determined by the terminal condition (41) (which requires that $A = 0$ in the infinite horizon case). This expression indicates that real money balances at time t depend in an economically appropriate fashion (in rational expectations model) on the future course of the rate of monetary expansion. Future monetary expansion ought to matter for current equilibrium real money balances because future monetary expansion influences the inflation rate and hence the costs that households perceive from holding money balances.

Further, when $\eta > -1$, in both the finite and infinte horizon cases the equilibrium level of real money balances at time t is independent of the initial nominal money supply $M(0)$ and the rate of monetary expansion between 0 and t.[15] This is economically sensible because in Obstfeld's model there is no reason why past behavior of the nominal money supply ought to affect current or future behavior of any real variable. Moreover, from these results, it follows that the equilibrium price level at time t, which is given by

$$(55) \qquad P(t) = M(t)/m(t),$$

depends in an econmically sensible manner on the past and future behavior of the nominal money supply: The initial money supply $M(0)$ and the rate of monetary expansion between 0 and t affect $P(t)$ by determining $M(t)$; while the future behavior of the rate of monetary expansion affects $P(t)$ by influencing $m(t)$ in the manner indicated by the integral on the right hand side of (53).

When η is < -1, we do not retain these economically sensible relationships between the exogenously specified path of the rate of monetary expansion and the endogenously determined equilibrium paths of real money balances and the price level. For the finite horizon with the terminal condition (41a), there is no solution for the equilibrium path of the economy and hence no implied relationship between the behavior of μ and that of m and P. With the terminal condition (51), when a finite horizon solution of $m(t)$ exists, it is given by (53); but as previously emphasized, this solution does not converge to any of the continuum of infinite horizon solutions. Further, in each of the continuum infinite horizon solutions, $m(t)$ cannot be related to the future behavior of μ in the manner indicated by (53) because the integral on the right hand side of (53) does not converge when η is < -1. Instead of using (53) as the solution of (47), it is necessary to use the backward looking solution,

$$(56) \qquad m(t) = \int_0^t ((\mu(s) - x(s))\cdot m(s)/(1 + \eta))\cdot \exp(R(t) - R(s))ds + B\cdot \exp(R(t))$$

where $B > 0$ is a free parameter that determines $m(0)$. In general, there is a continuum of choices of $B = m(0) > 0$ that yield equilibrium paths for the economy consistent with all of the conditions (37) through (45) including the transversality condition (41b). In none of these solutions, however, is the behavior of the real money supply or the price level related to the future course of the rate of monetary expansion.

Summarizing the results of this analysis, it is clear that Obstfeld's model exhibits strange behavior when η is < -1. In the infinite horizon case, this strangeness is manifest in a continuum of solutions in which the paths of real money balances and the price level are not related in an economically sensible, forward looking manner to the future behavior of the money supply. In the finite horizon case, this strangeness means either absence of any solution, or solutions that do not converge to any of the continuum of infinite horizon solutions as the length of the finite horizon becomes long.[16] In contrast, when η is > -1, Obstfeld's model always has a unique, economically sensible solution in both the finite and infinite horizon cases, and the finite horizon solution converges to the infinite horizon solution as the length of the finite horizon becomes long. The conclusion that seems warranted by these results is simply that Obstfeld's model lacks economically sensible solutions when η is < -1.

Given this conclusion, it is reasonable to enquire whether $\eta < -1$ can be excluded on some economically plausible argument. Standard concavity restrictions on the utility function $u(c, m)$ are not sufficient to insure that $\eta = m \cdot u_{cm}/u_c$ is > -1. However, the rationale for introducing m into the utility function is to represent the services of cash balances. Consistent with this rationale, it it might be argued that for any given level of m, a higher level of c ought to imply a higher marginal service yield from money balances since each real unit of these balances is being used more intensively. This would imply that u_{cm} and hence η would be positive, thereby ruling out the situation in which Obstfeld's model fails to yeild economically sensible solutions.

6. Calvo's Life-Cycle Model

Calvo (1977) provides an example of a continuum of stable rational expectations equilibrium paths in a life-cycle model of consumption and savings. In this model, $u(c_t, x_t)$ denotes the utility of the generation born at t as a function of their consumption in youth, c_t, in period t, and their consumption in age, x_t, in period $t + 1$. In youth, each generation earns wages $\bar{w}$ which it uses to finance current consumption and to save by purchasing claims to land from the old generation. Land is in fixed supply (at one unit) and earns a rental R_t. The old generation consumes this rental plus the price, q_t, at which it sells the land to the young generation. From the perspective of the generation born at t, therefore, the relative price of consumption in youth (in period t) in terms of consumption in old age (in perod $t + 1$) is given by

(57) $$v_t = (q_{t+1} + R_{t+1})/q_t.$$

With the wealth of each generation fixed at $\bar{w}$, the utility maximizing levels of c_t and x_t are determined as functions of this relative price (which is taken as given by individual members of the generation);

(58) $$c_t = C(v_t)$$

(59) $$x_t = v_t \cdot (\bar{w} - C(v_t)).$$

Equilibrium requires that total consumption by the young and old each period equal the available supply of consumption goods that period; specifically, in t we require that

$$C(v_t) + v_{t-1}\cdot(\bar{w} - C(v_{t-1})) = \bar{w} + R_t \tag{60}$$

Alternatively, using (57) and using the resource constraint and the budget constraint to establish that $q_t = v_{t-1}\cdot(\bar{w} - C(v_{t-1})) - R_t$, the equilibrium condition may be written as

$$S((q_{t+1} + R_{t+1})/q_t) = q_t \tag{61}$$

where $S(v_t) = \bar{w} - C(v_t)$ is the saving function. Calvo considers time paths of q that satisfy (61). For present purposes, however, it is more convenient to consider time paths of v that satisfy (60).[17]

Equation (60) is a first order difference equation in v_t. Under fairly general assumptions about the utility function, when R_t is constant at $\bar{R}$, there is a unique steady state solution of this difference equation, denoted by $\bar{v}$, and associated levels of consumption the young and old generations, denoted by $\bar{c} = C(\bar{v})$ and $\bar{x} = \bar{v}\cdot(\bar{w} - \bar{c}) = \bar{w} + \bar{R} - \bar{c}$. The difference equation is stable in the neighborhood of this steady state if and only if

$$-1 < \bar{v} - ((\bar{w} - \bar{c})/C'(\bar{v})) \equiv \theta < 1 \tag{62}$$

In the infinite horizon case, when this stability condition is violated, the model has a unique (stable) rational expectations solution, obtained by setting $v_t = \bar{v}$ for all t. In the infinite horizon case, when this stability condition is satisfied, a continuum of choices of initial values of v_1 (at least in the neighborhood of $\bar{v}$) imply subsequent equilibrium paths of v_t determined by (60) (with $R_t = \bar{R}$) that converge to $\bar{v}$. Hence, when the stability condition (62) is satisfied, Calvo's model has a continuum of (stable) rational expectations solutions.

As with Obstfeld's model, there is no doubt that Calvo's model has a continuum of convergent solutions when the stability condition (62) is satisfied. However, these solutions have peculiar properties that are exhibited by considering first solutions to the finite horizon version of Calvo's model and then solutions to the infinite horizon version when the rental on land is not constant.

To analyze the solutions of the finite horizon version of Calvo's model, it is useful to consider the diagrams in the four panels of figure 4. In each of these panels, the line with slope -1 connecting the point $c = \bar{w} + \bar{R}$ on the horizontal axis with the point $x = \bar{w} + \bar{R}$ on the vertical axis shows the combinations of c_t and x_{t-1} that are consistent with resource constraint, $c_t + x_{t-1} = \bar{w} + \bar{R}$. The curves labeled PCC in each panel are the price consumption curves which show the loci of consumption points $(c_t = C(v_t), x_t = v_t \cdot (\bar{w} - C(v_t)))$ chosen by each generation as a function of the relative price v_t. These price consumption curves are drawn on the assumption that the utility function is in the CES family.[18] In panel A, the elasticity of substitution is greater than one, implying a negatively sloped PCC. The stability condition (62) is necessarily violated in this case. In panel B, the elasticity of substitution is greater than one, implying a positively sloped PCC, but the distribution parameter is such that the slope at the steady steady point (where PCC intersects the resource constraint line) is greater than one. In this case, the stability condition (62) is also violated. In panel C, the elasticity of substitution is less than one and the distribution parameter is such that the slope of PCC at the steady state is less than one, implying that the stability condition is satisfied. In panel

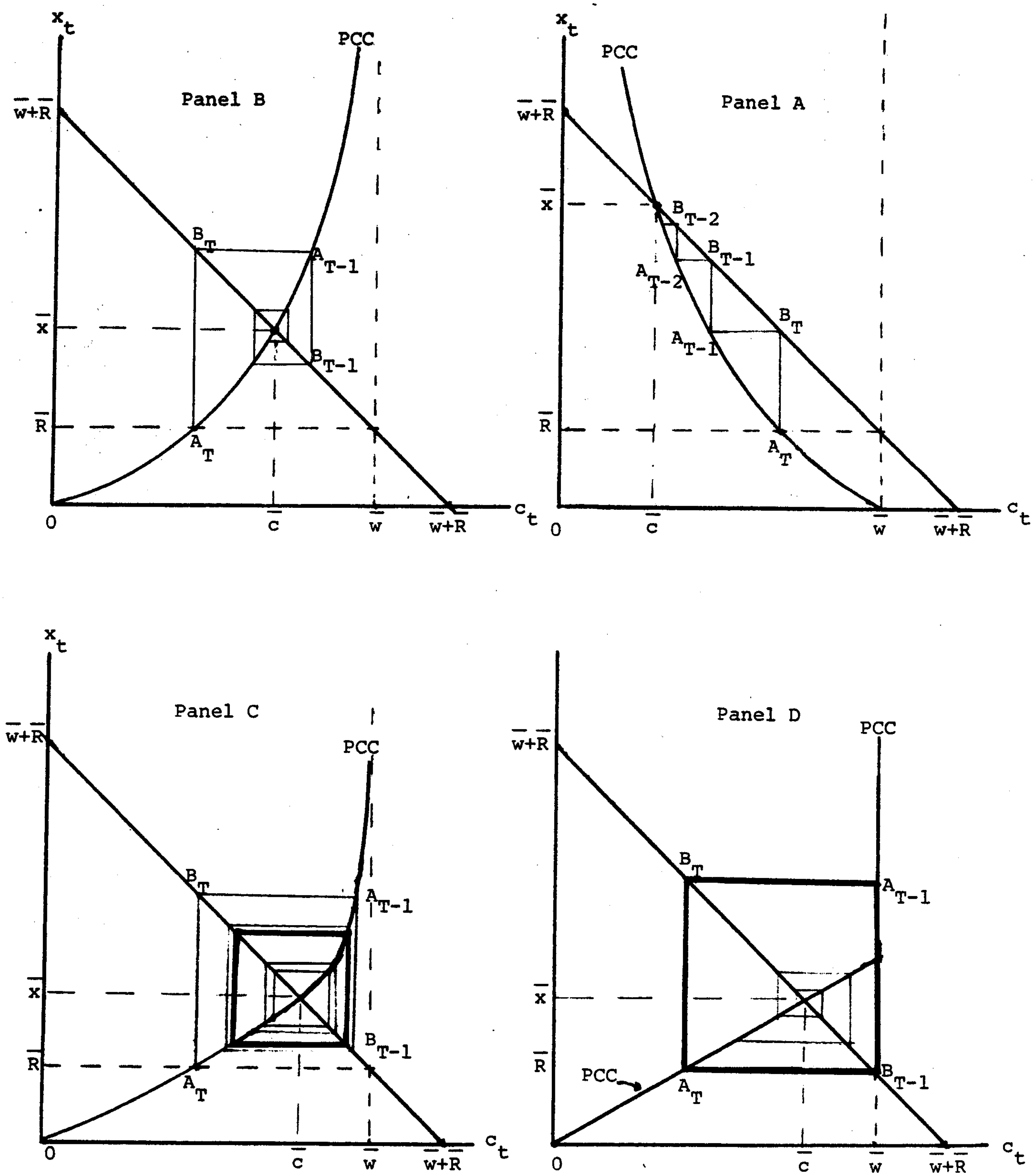

Fig.--4: The Dynamics of Calvo's Life-Cycle Model.

D, the elasticity of substitution is zero and the distribution parameter is such that the PCC proceeds along a ray through the origin with slope less than one until the vertical line $c = \bar{w}$ is reached and then proceeds upward along this vertical line. In this case, the stability condition is also satisfied.

In figure 4, the dynamic behavior of v_t, c_t and x_t implied by (60) may be described as follows. In any of the panels, a point $A_t = (c_t, x_t)$ on PCC represents a possible consumption point of the generation born at t. The associated value of v_t is the negative of the slope of the line connecting this point with the point $\bar{w}$ on the horizontal axis. Proceed horizontally (left or right) from A_t to a point B_{t+1} on the resource constraint line to determine the level of c_{t+1} consistent with A_t. Proceed vertically (up or down) from B_{t+1} to the point $A_{t+1} = (c_{t+1}, x_{t+1})$ on PCC that represents the consumption point of the generation born at $t + 1$ implied by A_t and (60). The negative of the slope of the line connecting A_{t+1} to the point $\bar{w}$ on the horizontal axis is v_{t+1}. Repeat this procedure starting with A_{t+1}, and so on, to traceout the path of c_{t+j}, x_{t+j}, and v_{t+j} implied by (60) given A_t. In panels A and B, this procedure implies a path that diverges from the steady state whenever A_t is not the steady state point. In contrast, in panels C and D, this procedure implies a path that converges to the steady state, provided that A_t is sufficiently near to the steady state.

To determine the solution of the finite horizon version of Calvo's model, assume provisionally that land continues to yield its rental $\bar{R}$ in $t + 1$ and that this determines the consumption of the last generation, born at T, in old age. (Modifications of this assumption will be considered later.) The consumption point $A_T = (c_T, x_T = \bar{R})$ of the last generation in each of the panels of figure 4 must be the point where the horizontal line $x = \bar{R}$

intersects PCC. To construct the solution that ends at this point, reverse the procedure of the preceding paragraph. From A_T move vertically to find B_T on the resource constraint line, then move horizontally to find A_{T-1} on PCC, then vertically to B_{T-1} on the resource constraint, and so on until the appropriate starting point $A_1 = (c_1, x_1)$ for the first generation is located. The unique rational expectations equilibrium path for the economy is the path that starts at A_1 and follows the procedure described in the preceding paragraph (reverse the procedure of this paragraph) to end up at A_T.

For the situations described in panels A and B of figure 4, the solution in the finite horizon case starts at a point A_1 that is nearer to the steady state point that A_T and moves outward (either directly or cyclically) away from the steady state to reach A_T. In these cases, as the length of the time horizon is lengthened, the initial point A_1 moves closer to the steady state, and the path of the economy remains longer in the neighborhood near the steady state. Thus, in these cases where the stability condition (62) is violated, the infinite horizon solution (remaining forever at the steady state) is the limiting result of the finite horizon solution.

For the situations described in panels C and D of figure 4, where the stability condition (62) is satisfied in the neighborhood of the steady state, the solution for the finite horizon does not converge to any of the continuum of solutions for the infinite horizon. Instead, in each of these cases the path of the converges to a limit cycle (shown by the rectangle drawn with heavy lines whose four corners lies on the resource constraint and the price consumption curve). In panel D, the terminal point A_T is the south-east corner of this rectangle. This point marks the consumption point for all generations that are an even number before T (i.e. $A_{T-j} = A_T$ for j even and less than T). The opposite corner of limit cycle rectangle is the consumption point for all generations that are an odd number before T (i.e., $A_{T-j} = A_{T-1}$ for all odd j less than T). In panel C, the terminal point A_T lies outside the rectangle describing the limit cycle. Constructing the solution for the finite horizon problem by moving backward from the point A_T (in the manner previously described), we obtain a path that converges toward (but remains outside) the rectangle that describes the limit cycle. Thus, for the situations described in panels C and D, where the stability condition (62) is satisfied, the limit cycle in these diagrams, rather than the steady state point, describes the path toward which the finite horizon solution of Calvo's model converges.

This conclusion is reinforced by considering modifications of the terminal condition determining A_T. Suppose that rent in $T + 1$, which determines x_T, differs from the rent $\bar{R}$ that land earns in every other period. The terminal point A_T now must lie at the intersection of the horizontal line where $x = R_{T+1}$ and PCC, and the equilibrium path is constructed by moving backward from this terminal point. If R_{T+1} happens to equal $\bar{x}$, the result for all four cases in figure 4 (and in any other case) is that the economy remains at its steady state point in the fintite horizon case. Any other value of R_{T+1} does not alter the main qualitative features of the results illustrated in panels A and B. In panels C and D, if the terminal point lies outside the limit cycle rectangle, the equilibrium path cycles on (in panel D) or just outside (in panel C) the limit cycle rectangle until shortly before the terminal date, and then cycles out to reach the terminal point. In these panels, if the terminal point lie inside the limit cycle rectangle, the equilibrium path cycles on (in panel D) or just inside (in panel C) the limit cycle rectangle until shortly before the terminal date and then cycles in to reach the terminal point. In the circumstances in panels C and D (except $R_{T+1} = \bar{x}$), as the time horizon becomes long, the path of the economy spends most of its time on or near the limit cycle rectangle. Thus, for the situations illustrated in panels C and D, where the stability condition (62) is satisfied at the steady state, the limit cycle, rather than the steady state, is the convergence path for the economy as the length of the horizon becomes long.

More generally, when the stability condition (62) is satisfied at a steady state position it need not be the case that solutions for the finite horizon version of Calvo's model converge to a limit cycle as the horizon becomes long. For example, it is possible that there could be multiple intersections of the price consumption curve with the resource constraint line implying multiple steady state equilibria. Solutions to the finite horizon model might then converge to one of the steady states at which the stability condition (62) is violated. What must be true in general is that solutions to the finite horizon model do not converge to a steady state at which the stability condition (62) is satisfied, except in the case where the terminal condition fixes this exact point as the ending point for the economy. The reason is that the dynamics of the model in the neighborhood of such a stable steady state cause the path of the economy to cycle in toward this steady state. Hence, it is not possible for a solution that must terminate at a point other than this steady state to start out too near this steady state or remain long in its neighborhood.

Another difficulty with Calvo's model when the stability condition (62) is satisfied arises from the relationship between the relative price of consumption in successive periods and the path of the rental on land. To exhibit this difficulty, consider the linear approximation of (60) in the neighborhood of the steady state;

(63) $$u_t - \theta \cdot u_{t-1} = z_t / C'(\bar{v})$$

where $u_s = v_s - \bar{v}$ measures the deviation of v_s from its steady state value and $z_s = R_s - \bar{R}$ measures the deviation of R_s from the constant level of R that determines $\bar{v}$. When $|\theta| > 1$, the stability condition (62) is violated and the unique solution for the equilibrium path of u_t (which determines the paths of v_t and c_t) must be written in the foward looking form

(64) $$u_t = -(1/\theta \cdot C'(\bar{v})) \cdot \sum_{j=0}^{\infty} (1/\theta)^j \cdot z_{t+j+1} + K \cdot \theta^t$$

where the constant K is set equal to zero. This result expresses the economically appropriate dependence of the relative price of consumption between t and t + 1 on the rental of land in t + 1 and beyond. This result is exploited (usually with a non-zero K) to construct the solution in the finite horizon case for any value of θ. However, in the infinite horizon case, when $|\theta| \leq 1$ and the stability condition (62) is satisfied, this forward looking expression cannot be used because the sum on the right hand side of (64) does not converge. Instead, it is necessary to use the backward looking solution of (63);

(65) $$u_t = C'(\bar{v}) \cdot \sum_{j=0}^{\infty} \theta^j \cdot z_{t-j} + L \cdot \theta^t.$$

With $|\theta| < 1$, this solution converges for a continuum of choices of the constant L, implying a continuum of solutions for the equilibrium path of the economy. However, none of these solutions exhibits forward looking dependence of the relative price of consumption between t and t + 1 on the rental of land in t + 1 and beyond.

In connection with this result, it should be noted that when $|\theta| < 1$ and the path of $v_t = \bar{v} + u_t$ is given by (65) (for any value of L), the price of land, q_t, can always be expressed as the present value of future rentals;[19]

$$(66) \qquad q_t = \sum_{j=1}^{\infty} R_{t+j} / \prod_{i=o}^{j-1} v_{t+1}$$

The difficulty, therefore, is not in obtaining an appropriate forward looking expression for the price of land, given the path of the relative price of consumption in successive periods. The difficulty is in obtaining an expression for the relative price of consumption that exhibits an appropriate forward looking dependence of future rentals on land.

In summary, when the stability condition (62) is violated both the finite and infinite time horizon versions of Calvo's life-cycle model have unique rational expectations solutions. In both versions, the relative price of consumption in successive periods depends in an economically appropriate manner on the future course of the rental on land. When this rental is constant, the solution for the finite horizon case converges to the solution for the infinite horizon case as the length of the finite horizon becomes long. When the stability condition (62) is satisfied, Calvo's model exhibits strange behavior. Specifically, the infinite time horizon version of this model exhibits a continuum of rational expectations solutions. The relative price of consumption in successive periods in these solutions, however, is linked to past rather than future rentals on land. A unique finite horizon solution that does not suffer from this deficiency can be constructed when the stability condition (62) is satisfied, but this solution has its own peculiarities (convergence to a limit cycle) and it does not converge to any of the continuum of infinite horizon solutions. It may be argued, therefore, that the assumptions that allow the stability condition (62) to be satisfied should be excluded on the grounds that Calvo's life-cycle model fails to exhibit economically sensible behavior when these assumptions are satisfied.[20]

7. Conclusions

Some of the results of the five examples of rational expectations models with a continuum of convergent solutions considered in this paper are specific to those examples. There are, however, three general lessons to be learned from these examples.

First, existence of a steady state equilibrium in a rational expectations model which is the convergence point for a continuum of solutions of that model does not imply that any of these solutions is economically meaningful. This is clearly demonstrated by the first example where the continuum of paths converging to the steady state (K_2^*, q_2^*) are demonstrably irrelevant to the solution of the dynamic optimization problem that gives rise to the dynamic system (9) and (10). This example indicates that caution is required in interpreting similar sets of solutions when they arise in rational expectations models whose behavioral functions are specified on an ad hoc basis.

Second, when a rational expectations model has a steady state which is the convergence point for a continuum of solutions, none of these solutions may exhibit an economically appropriate, forward looking dependence of the endogenous variables on future behavior of the exogenous variables. This difficulty was observed in all five examples considered in this paper. It is likely to be a difficulty in other models as well because the condition that gives rise to a continuum of convergent solutions (more stable roots than backward looking dynamic processes) automatically implies difficulty in constructing the forward looking components of the solutions for the endogenous variables (because there are fewer unstable roots than forward looking dyanamic processes).[21] When appropriate forward looking dependence of endogenous variables on exogenous variables is an essential feature of economiclly sensible solutions of a rational expectations model (as arguably it is in the five examples considered here), there may be grounds for rejecting a continuum of solutions converging to a steady state and for questioning the economic sensibility of the model under the circumstances that give rise to such a continuum of solutions.

Third, under circumstances that give rise to a continuum of solutions of the infinite horizon versions of rational expectations models, finite horizon solutions of these models may not exist, and when they do exist, they may not converge to any of the continuum of infinite horizon solutions. This may provide additional grounds for questioning the relevance of the continuum of infinite horizon solutions. These grounds may be strengthened if (as the case for two examples considered here) the circumstances that insure a unique infinite horizon solution also imply a unique finite horizon solution that converges to the infinite horizon solution as the horizon becomes long.

Finally, it should be emphasized that this paper has not demonstrated the impossibility of constructing rational expectations models with a continuum of convergent solutions. It can be argued that Taylor's macroeconomic model, Calvo's price dynamics model, Obstfeld's monetary dynamics model and Calvo's life-cycle model all exhibit such a continuum of solutions for some specifications of their parameter values or behavior functions. Each of these solutions satisfies all of the formal requirements in their respective models for a rational expectations solution of the model. Only by raising the additional considerations summarized in the two preceding paragraphs has it been possible to question the economic relevance of the continuum of convergent solutions of these models. Moreover, it is possible that a rational expectations model can be constructed which has a continuum of convergent solutions that meet the difficulties discussed in this paper.[21] Only by examining each such model on its own merits is it possible to consider the economic meaning and relevance of its solutions.

FOOTNOTES

1. The conditions for existence of a continuum of convergent solutions to linear rational expectations models are examined in Blanchard and Kahn (1980) and Buiter (1982). It is well known that these conditions apply locally in the neighborhoods of steady states for non-linear rational expectations models; see Burmeister (1980).

2. The issues examined in this paper are different from the issue of the justification for selecting the unique nonexplosive solution to a rational expectations model when other explosive solutions are available. This latter issue will not be addressed in this paper.

3. This deficiency also applies to the minimum variance solution suggested by Taylor and to the solution dependent on the minimal set of state variables suggested by McCallum (1983) as the economically relevant solution of Taylor's model.

4. The assumption of a negative rate of time preference and the strange shape of the production function illustrated at the top of figure 1 are necessary to create a dynamic system with a steady state at which there are two stable characteristic roots.

5. Since the steady state (K_3^*, C_3^*) lies below the tangent to the total product curve at K_1^* in the upper panel of figure 1, it can be shown that it is not optimal to choose paths of K and q that lie near the stable branch S_3S_3 and remain (for long time horizons) near the steady state (K_3^*, q_3^*) for a long time.

6. McCallum (1983) has shown that the indeterminacy arising from a continuum of solutions to Taylor's model when $\delta_1 < 0$ and $|\delta_1| < 2$ can be resolved by insisting on the solution that depends on the minimal set of state vari-

ables. This solution, however, suffers from the same difficulty of all other solutions that there is no forward looking dependence of the price level on the future behavior of the money supply.

7. In order for δ_1 to be negative in Taylor's model, real money balances must have a strong positive effect on aggregate supply. It might be that explicit consideration of the microeconomic foundations of Taylor's model would rule out this possibility. For example, if real balances affect aggregate supply because wealth affects the supply of labor, then normality of leisure in the household utility function would imply negative coefficient for real balances in the aggregate supply function. Introducing money as an argument in the production function, however, would provide a rationale for a positive effect of real balances on aggregate supply. Application of a generalized version of Samuelson's correspondence principle might be used to argue that this effect should not be so strong as to induce a negative value of δ_1.

8. Suppose that excess demand at time s for the product whose contract price $V(t)$ is set at t is given by $x(s) = \gamma\cdot(P(s) - V(t)) + \theta\cdot f(P(s), r(s))$, where γ measures the responsiveness of excess demand to deviations between the price of this product and the general price level and θ measures the effect of excess aggregate demand on excess demand for an individual product. Suppose further that the objective of the price setting process is to set each contract price so that expected excess demand for the product over the length of the contract is zero. It may then be shown that $V(t)$ should be set in accord with (25) with β set equal to θ/γ.

9. There is no obvious way in which McCallum's criterion of a solution that depends on a minimal set of state variables would select a unique solution from the continuum of solutions converging to a Walrasian unstable equi-

librium. If both characteristic roots are real (and negative) at such an equilibrium, there will be a continuum of directly convergent solutions along which it is possible to express V as a function of P and thereby reduce the set of state variables to just P. There is no obvious basis, however, for selecting one of these directly convergent solutions in preference to all others. When the two roots are a complex conjugate pair (with negative real parts), all solutions cycle in toward a Walrasian-unstable equilibrium, and the minimal set of state variables consists of P and V.

10. Insistence upon a solution that exhibits forward looking dependence of $V(t)$ and $P(t)$ on $z(s)$ for $s > t$ imposes a condition beyond the conditions expressed by the equations that constitute Calvo's model. The argument for imposing this additional condition is that it is implied by the economic content of Calvo's model. If one argues that this additional condition is not an essential part of the model, then he is left with a continuum of solutions converging to any Walrasian-unstable equilibrium.

11. The model examined in this section is a simplified version of Obstfeld's model. The simplifications do not alter the substance of the analysis.

12. In this discussion it is assumed that η does not change sign and that is always either greater than or less than -1. Peculiar things can happen when these assumptions are not satisfied.

13. The differential equation (47) controlling $\dot{m}$ can be rewritten as $\dot{m} = (1/(1 + \eta))\cdot[(\rho + \mu)\cdot m - u_m(\bar{c}, m)]$. Since $u_m(\bar{c}, m)$ rises as m declines, while $(\rho + \mu)\cdot m$ falls as m declines, it may be shown that $\dot{m} < -b$ for some positive b whenever $m < m(0) < \bar{m}$. It follows that m will be driven to zero (while λ is driven to zero or to infinity) in finite time starting from $m(0) < \bar{m}$.

14. In particular, if $m(0)$ is set equal to $\bar{m}$, then (47) and (48) imply that m and λ will remain at their steady state values between 0 and T. With m constant at $\bar{m}$, it follows that $\pi = \mu$ and $v = \mu \cdot \bar{m}$. Taking these time paths of π and v as given, however, it is easy to show that individual households will not want to hold their consumption at $\bar{c}$ and the real money balances at $\bar{m}$. Each individual household wants to consume in excess of $\bar{c}$ and run down its real money balance, especially as t approaches T.

15. Consider the finite horizon problem starting from t rather than from 0. This problem is identical to the finite horizon problem starting from 0 and running to $T - t$. In this shortened horizon problem, the initial nominal money supply (which is $M(t)$ in the horizon T problem) matters only for the initial price level and does not affect the behavior of any real variable (including the inflation rate).

16. With the terminal condition (51) and K set at $\bar{m}$, the solution of the finite horizon problem does converge to the solution of the infinite horizon solution with $m(0)$ set at $\bar{m}$. This inifinite horizon solution, however, still lacks an economically sensible, forward looking dependence of real money balances and the price level on the future behavior of the money supply.

17. Given the path of R_{t+1}, a solution for the path of q_t implies through (57) a solution for the path of v_t. To solve for the path of q_t given the path of v_t requires one additional condition. In the finite horizon case, where the generation born at T is the last generation, this condition is supplied by the requirement that $q_{T+1} = 0$. In the infinite horizon case, this additional condition is supplied by the requirement given in (66) that the price of land equal the present value of future rentals.

18. In the Cobb-Douglas case of unit elasticity of substitution (not illustrated), the price consumption curve is a vertical line at the level of c where the ratio $c/\bar{w}$ is equal to the share of current consumption in wealth. In the general CES case where the utility function is $[\delta \cdot c^{\nu} + (1 - \delta) \cdot x^{\nu}]^{1/\nu}$ with $0 < \delta < 1$ and $\nu < 1$, the price consumption curve is described by $x = (\delta/(1 - \delta))^{1/\nu} \cdot c \cdot ((\bar{w}/c) - 1)^{1/\nu}$ for $0 < c < \bar{w}$.

19. Calvo (1977) makes this point in the case where R_t is constant at $\bar{R}$.

20. Utility functions $u(c_t, x_t)$ which imply satisfaction of the stability condition (62) need not have any peculiar properties (inferior goods or convex indifference curves) that utility functions are not usually assumed to possess. The difficulty is that when a usually sensible utility function is embedded in Calvo's life-cycle model, it implies very peculiar behavior of the solutions of this model. Applying the same logic used to rule out utility functions with convex indifference curves in demand theory (because they always imply specialization in consumption), utility functions that imply satisfaction of the stability condition (62) can be ruled out for use in Calvo's model (because they imply peculiar behavior of the solutions of this model).

21. With fewer unstable roots than independent forward looking dynamic processes, the dimension of the manifold of forward looking components of solutions for endogenous variables will be smaller than the dimension called for by the economic structure of the model. For the examples considered in this paper, where there is only one independent forward looking dynamic process, absence of any unstable roots implies absence of any forward looking component in the solutions for the endogenous variables.

22. Consider the problem of finding the minimum distance path connecting two polar opposite points on the surface of the sphere in R^3. This problem has a

continuum of solutions since every great circle connecting the two poles is a minumum distance path. Another example is Kareken and Wallace's (1981) model of floating exchange rates. In this model there is a determinate demand for total real money balances, domestic plus foreign, but (with an exchange rate that is expected to remain constant) there is nothing that determines the division of this total between domestic and foreign money. Hence, corresponding to each initial exchange rate, there is a distinct equilibrium solution of the model, with a different division of total real money balances between domestic and foreign money. A third example is Obstfeld's money of section 5 when the rate of monetary expansion is made a function, $\mu(m)$, of the level of real money balances. If we define this function implicitly by the requirement that $x(\bar{c}, m) = \rho + \mu(m)$ over some range of values of m, then all values of m within this range will define steady state solutions of Obstfeld's model.

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