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POLICY EVALUATION AND DESIGN FOR CONTINUOUS TIME LINEAR RATIONAL EXPECTATIONS MODELS: SOME RECENT DEVELOPMENTS

Willem H. Buiter

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ABSTRACT

The paper surveys some recent developments in policy evaluation and design in continuous time linear rational expectations models. Much recent work in macroeconomics and open economy macroeconomics fits into this category. First the continuous time analogue is reviewed of the discrete time solution method of Blanchard and Kahn. Some problems associated with this solution method are then discussed, including non-uniqueness and zero roots. Optimal (but in general time-inconsistent) and time-consistent (but in general suboptimal) solutions are derived to the general linear-quadratic optimal control problem, based on work by Calvo, Driffill, Miller and Salmon and the author. A numerical example is solved, involving optimal and time-consistent anti-inflationary policy design in a contract model.

Willem H. Buiter
London School of Economics
Houghton Street
London WC2A 2AE
England

01-405-7686 ext 437

Policy evaluation and design for continuous time linear rational expectations models: some recent developments

1. Introduction

The first systematic introduction to economic dynamics came for me, as for many of my contemporaries, through William Baumol's lucid and "user-friendly" book Economic Dynamics (Baumol [1970]).[1] It seems appropriate, therefore, to survey in this volume honouring William Baumol's contributions to economics, some of the recent developments in modelling dynamic macroeconomic systems. All these developments bear the hallmark of the rational expectations revolution which has swept macroeconomics and international finance since the early Seventies. Only models represented by systems of first order linear differential equations with constant coefficients are considered. The reason for limiting the discussion to linear systems will be obvious to those who have attempted to analyse even very simple non-linear rational expectations models. The restriction to continuous time systems reflects the existence of many excellent survey articles on general discrete time systems (e.g. Whiteman [1983], Blanchard [1983] and McCallum [1983]). Continuous time rational expectations models, by contrast, appear extensively in the literature in one, two or occasionally three dimensions, but have not been the subject of

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1. The first edition of this book appeared as early as 1951.

systematic surveys to anything like the same extent. (Exceptions are Dixit [1980], Buiter [1981a, 1982] and Currie and Levine [1982].)

Section II of the paper summarises the continuous time analogue of the discrete time solution method of Blanchard and Kahn [1980], as developed in Buiter [1982]. Section III considers some problems that are associated (or may appear to be associated) with this solution method. Section IV contains the solution to the general linear-quadratic optimal control problem in continuous time rational expectations models. It builds on work by Calvo [1978], Driffill [1982], Miller and Salmon [1982, 1983] and Buiter [1983].

Both optimal (but in general time-inconsistent) and time-consistent (but in general sub-optimal) solutions are derived in a uniform framework. A numerical example, involving optimal and time-consistent anti-inflationary policy design in a contract model (using an algorithm developed by Austin and Buiter [1982]), serves as an illustration of the general approach in Section V.

II. Solving continuous time linear rational expectations models

Consider the continuous time linear rational expectations model given in (1).

$$(1) \quad \begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + Bz(t)$$

with boundary conditions

(2a) $F_1 x(t_o) + F_2 y(t_o) = f$; F_1 is $n_1 \times n_1$ and of full rank.

(2b) The solution is restricted to lie on the stable manifold.

x is an n_1 vector of predetermined state variables, y an $n - n_1$ vector of non-predetermined state variables and z a k-vector of exogenous or forcing variables. A, B, F_1 and F_2 are known constant matrices; f is a known vector of constants. E is the expectation operator and $\Omega(t)$ the information set conditioning expectations formed at time t. For any vector w , $E_t w(s) \equiv E(w(s)|\Omega(t))$ and $\dot{w}(t) \equiv \lim_{s \downarrow t} \left(\frac{w(s) - w(t)}{s - t} \right)$. The information set $\Omega(t)$ contains all current and past values of x, y and z and the true structure of the model given in (1) and (2a, b).

Formally, we assume :

(A1) $E_t w(s) = w(s) \qquad s \leq t$

(A2) $\Omega(t) \geq \Omega(s) \qquad t > s$

We shall make use of the "law of iterated projections", i.e.

(3) $$E(E(w(s)|\Omega(t_o))|\Omega(t_1)) = \begin{cases} E(w(s)|\Omega(t_1)) & t_1 \leq t_o \\ E(w(s)|\Omega(t_o)) & t_o \leq t_1 \end{cases}$$

Assumption (A1) combines "perfect hindsight" ($s < t$) and "weak consistency" ($s = t$) (see Turnovsky and Burmeister [1977]). Assumption (A2) means that memory doesn't decay. Condition (3) is a basic property of conditional expectations, if (A2) holds.

For ordinary n-dimensional first order linear differential equation systems, a unique solution exists if there are n linearly independent boundary conditions.[2/] For the n_1 predetermined variables x, the boundary conditions take the form of n_1 linear restrictions at the initial date t_o. For many applications these linear restrictions will take the form of n_1 initial values, i.e.

$$\text{(2a')} \quad x(t_O) = \bar{x}(t_O)$$

In Buiter and Miller [1982, 1983a] a more general form of the boundary conditions for the predetermined variables such as (2a) was necessary.

The meaning of the boundary condition (2b) will become apparent below. A sufficient condition for ruling out the explosive growth of the expectation, held at time t, of future values of z, is that $E_t z(s)$ is a bounded function of s on $[t, +\infty)$ and continuous almost everywhere.

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2. Note that, through the presence of the conditional expectations operator E_t, equation (1) strictly speaking represents a partial differential equation system. The solution chosen here is the "minimal state" solution (see McCallum [1983]) involving only "fundamentals". Rational expectations and weak consistency ensure that the additional degrees of freedom introduced through the presence of the expectation operator are actually very limited.

The solution for x and y is restricted to be a continuous function of time when there is no change in current expectations of future values of the forcing variables, i.e. x(t) and y(t) are continuous functions of t as long as $E_t z(s)$, $s > t$, doesn't vary with t.

This rules out anticipated future discrete jumps in y.[3/] The economic rationale for this restriction appears sound: an infinite instantaneous rate of capital gain cannot be anticipated in models with reasonably rich opportunities for intertemporal arbitrage and speculation.

There is no formal recognition of uncertainty in the model. The expectations are to be interpreted as single-valued or point expectations, i.e. expectations held with complete subjective certainty. It will be clear, however, that the results obtained for (1) are applicable to the stochastic linear differential equation system given in (4), provided there are no measurement errors in the observation of the state vector $[x\ y]^T$.[4/]

$$(4) \qquad \begin{bmatrix} dx(t) \\ E_t\, dy(t) \end{bmatrix} = A \begin{bmatrix} x(t)dt \\ y(t)dt \end{bmatrix} + B\ z(t)dt + dv$$

The continuous time vector process v(t) is a stationary zero mean stochastic process with independent increments. Examples are Wiener

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3. and, if $F_2 \not\equiv 0$, in x.

4. m^T denotes the transpose of m.

$$(6) \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}; \quad V^{-1} \equiv W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

Λ_1 is an $n_1 \times n_1$ diagonal matrix containing the stable roots of A and Λ_2 an $(n - n_1) \times (n - n_1)$ diagonal matrix containing the unstable roots of A.

We also define

$$(7) \quad \begin{bmatrix} p \\ q \end{bmatrix} = V \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = W \begin{bmatrix} p \\ q \end{bmatrix}$$

p is an n_1 vector and q an $n - n_1$ vector.

Taking expectations conditional on E_t on both sides of (4) and using (5), (6) and (7) we obtain

$$(8) \quad E_t \dot{q}(t) = \Lambda_2 E_t q(t) + D E_t z(t)$$

where

$$(9) \quad D \equiv V_{21} B_1 + V_{22} B_2$$

From the law of iterated projections given in (3) it follows that, for $t \leq s$

$$E_t \dot{q}(s) = \Lambda_2 E_t q(s) + D E_t z(s)$$

Treating this as a differential equation in s, conditional on E_t, we can write the solution for $E_t q(s)$ in "forward-looking" form as

$$(10) \quad E_t q(s) = e^{\Lambda_2 s} K_2 - \int_s^{\infty} e^{\Lambda_2 (s-\tau)} D E_t z(\tau) d\tau \qquad s \geq t$$

K_2 is an $n - n_1$ vector of arbitrary constants. Since Λ_2 contains unstable roots only, boundary condition (2b), that the solution should be convergent, compells us to choose K_2 as follows:

$$(2b') \quad K_2 = 0$$

Given (2b') we evaluate (10) at $t = s$. From the weak consistency assumption (A1) it then follows that

$$(10') \quad q(t) = - \int_t^{\infty} e^{\Lambda_2 (t-\tau)} D E_t z(\tau) d\tau$$

From equations (6) and (7) we know that $q = V_{21} x + V_{22} y$. If V_{22} is invertible, the solution for the non-predetermined variables can therefore be written as

(11) $$y(t) = - V_{22}^{-1} V_{21} x(t) - V_{22}^{-1} \int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau) d\tau$$

An equivalent expression, provided W_{11} has an inverse, is

(11') $$y(t) = W_{21} W_{11}^{-1} x(t) - V_{22}^{-1} \int_t^{\infty} e^{\Lambda_2(t-\tau)} D E_t z(\tau) d\tau$$

Substituting (11) or (11') into the equations of motion for $\dot{x}$ given in (1) and choosing the backward-looking solution for $x(t)$ we find that the predetermined variables are given by (12) or (12')

(12) $$x(t) = W_{11} e^{\Lambda_1(t-t_0)} W_{11}^{-1} x(t_0) + \int_{t_0}^{t} W_{11} e^{\Lambda_1(t-s)} W_{11}^{-1} B_1 z(s) ds - \int_{t_0}^{t} W_{11} e^{\Lambda_1(t-s)} W_{11}^{-1} A_{12} V_{22}^{-1} \int_s^{\infty} e^{\Lambda_2(s-\tau)} D E_s z(\tau) d\tau ds$$

(12') $$x(t) = W_{11} e^{\Lambda_1(t-t_0)} W_{11}^{-1} x(t_0) + \int_{t_0}^{t} W_{11} e^{\Lambda_1(t-s)} W_{11}^{-1} B_1 z(s) ds - \int_{t_0}^{t} W_{11} e^{\Lambda_1(t-s)} \{\Lambda_1 V_{12} V_{22}^{-1} + W_{11}^{-1} W_{12} \Lambda_2\} \int_s^{\infty} e^{\Lambda_2(s-\tau)} D E_s z(\tau) d\tau ds$$

Boundary condition (2a) can be written as

(13) $$x(t_0) = - F_1^{-1} F_2 y(t_0) + F_1^{-1} f$$

The initial value for x at $t = t_0$ is solved for from (13) and (11) or (11') with y evaluated at $t = t_0$.

Thus the non-predetermined variables can be expressed as a function of the current predetermined variables and of current expectations of future values of the forcing variables. The predetermined variables at time t depend in a non-explosive manner on their initial values at t_0, on the <u>actual</u> values of the forcing variables between t_0 and t and on the expectations, formed at each instant between t_0 and t, of the future values of the forcing variables.

It is clear that, if the process governing the forcing variables z can be expressed by a system of simultaneous first order linear differential equations $\dot{z} = Lz$ [5/], then the x vector can be augmented to include z; the solution of this augmented homogeneous system only involves the first terms on the r.h.s. of (11) (or (11')) and (12) (or 12')). For many purposes, and especially for optimal policy design, it is however very informative to keep the explicit dependence of x and y on actual and anticipated future values of z.

III. Three Problems

Three issues arise in connection with the solution method outlined in Section II. They are 1) the rather minor problem of ensuring that the eigenvalues are "assigned to" the proper state variables where such an unambiguous assignment is dictated by the structure of the model; 2) the existence of solutions other than the minimal state solution involving

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5. Or, in the stochastic case, $dz = Lzdt + dw$ where w is a stationary, zero mean stochastic process with independent increments and independent of the state vector $[x\ y]^T$.

only fundamentals and 3) the problem of zero eigenvalues or eigenvalues with zero real parts.

III. 1 The right root in the right place

Consider the simple two-variable homogeneous system given in equation (14).

$$(14) \qquad \begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Let $\alpha_{12} = 0$. The eigenvalues are $\lambda_1 = \alpha_{11}$ and $\lambda_2 = \alpha_{22}$ $x(t)$ is predetermined, with $x(t_0) = \bar{x}(t_0)$ and $y(t)$ is non-predetermined.

The solution is given by :

$$x(t) = e^{\lambda_1(t-t_0)}\,\bar{x}(t_0) = e^{\alpha_{11}(t-t_0)}\,\bar{x}(t_0)$$

$$y(t) = \frac{\alpha_{21}}{\lambda_1-\alpha_{22}}\,e^{\lambda_1(t-t_0)}\,\bar{x}(t_0) + Ke^{\lambda_2 t} = \frac{\alpha_{21}}{\lambda_1-\alpha_{22}}\,x(t) + Ke^{\lambda_2(t)}$$

$$= \frac{\alpha_{21}}{\alpha_{11}-\alpha_{22}}\,x(t) + Ke^{\alpha_{22}t}$$

K is to be determined by a boundary condition for y.

Let $\alpha_{11} > 0$ and $\alpha_{22} < 0$. Clearly we have the right number of stable and unstable eigenvalues (one of each) but unfortunately the unstable root is unambiguously attached to the predetermined variable. Also, since $\alpha_{22} < 0$, we cannot use the convergence criterion to set $K = 0$. This problem will of course be revealed if the system is solved correctly. The purpose of pointing it out here is merely to remind the reader that equality between the number of stable eigenvalues and the number of predetermined state variables and between the number of unstable eigenvalues and the number of non-predetermined variables is not strictly sufficient for the applicability of the solution methods of Section II.

III. 2 Sunspots and other forms of non-uniqueness

The solution for the non-predetermined variables given in (11) in terms of the current values of the predetermined variables and the current and anticipated future values of the forcing variables is what McCallum has called the "minimal state" solution (McCallum [1983]). It involves only the fundamentals (i.e. the forcing variables actually appearing in the equations of the model) and a minimal representation of the state variables.

A simple scalar example will illustrate the wealth of alternative solutions that satisfy the equations of motion of these rational expectations models.

(15) $$E_t \dot{y}(t) = \alpha\, y(t) + \beta\, z(t) \qquad \alpha > 0.$$

The minimal state solution for the non-predetermined variable y is

(16) $$y(t) = -\beta \int_t^{\infty} e^{-\alpha(s-t)} E_t\, z(s)\,ds.$$

It is easily checked that any variable $u(t)$ can be added to this solution, provided $u(t)$ satisfies the homogeneous equation of (15) i.e. provided

(17) $$E_t\, \dot{u}(t) = \alpha\, u(t)$$

For instance, $u(t) = y(t_0)e^{\alpha(t-t_0)}$ satisfies (15) as would $u(t) = z(t_0)e^{\alpha(t-t_0)}$. $u(t)$, however, need not involve y or z and could involve processes that are completely extraneous to the model under consideration (see e.g. Buiter [1981b]). It is easily checked that $u(t)$ can be written as :

$$u(t) = \lim_{\tau\to\infty} E_t\, e^{-\alpha(\tau-t)}\, y(\tau).$$

The extraneous element in the solution of (15) is generally ruled out on the grounds that unless $u(t) = 0$ for all t, an explosive process will be added to the behaviour of the system and that this would cause the system to violate (implicit) physical boundaries or other plausible constraints in finite time. Boundary condition (2b) is the expression of this view.

The same kind of nuisance process cannot be added to the solution of a boundary value problem involving an ordinary differential equation for a predetermined variable such as x in equation (18) because it would violate the initial condition.

(18a) $\dot{x}(t) = \gamma x(t) + \delta z(t)$

(18b) $x(t_0) = \bar{x}(t_0)$

The minimal state solution for x, given the initial boundary condition is

(19) $$x(t) = e^{\gamma(t-t_0)}\bar{x}(t_0) + \delta\int_{t_0}^{t} e^{\gamma(t-s)} z(s)ds$$

We cannot add to this solution any non-zero term $u(t)$, because although any $u(t)$ satisfying the homogeneous equation $\dot{u}(t) = \gamma u(t)$ would satisfy the equation of motion (18a), it would violate the condition $x(t_0) = \bar{x}(t_0)$ unless $u(t_0)$, and therefore $u(t)$, $t \geq t_0$, is equal to zero. The reason for the non-uniqueness in the solution for (15) and its absence in (18) is therefore not that, as was pointed out by Shiller [1978], (15) is a partial differential equation involving time in two ways: calendar time and the expectations or forecast horizon. At each instant t, a boundary condition must therefore be given for $\lim_{\tau\to\infty} e^{-\alpha(\tau-t)} E_t\, y(\tau)$. These boundary conditions cannot, however, be set completely independently of each other, as reflected in the constraint that $u(t)$ must satisfy (17). Without the expectation operator in (15) we would have to select a single boundary condition to determine $u(t) = \lim_{\tau\to\infty} e^{-\alpha(\tau-t)} y(\tau)$. It is the lack of compelling economic economic arguments for choosing $u(t) = 0$ that is the fundamental reason for the indeterminacy, not the presence of the expectation operator.

In terms of the general model of Section II, we can add to the fundamental solution for the canonical forward-looking variables q, given in (10'), any $n - n_1$ vector process u (deterministic or stochastic) which satisfies the homogeneous system $E_t \dot{u}(t) = \Lambda_2 u(t)$. Through $q = V_{21}x + V_{22}y$ this non-uniqueness of q can be translated into non-uniqueness for y and x.

In what follows, the analysis will be restricted to the minimal state solution, for convenience rather than out of a deep conviction that any properly specified macroeconomic model would generate the right set of boundary conditions to puncture any extraneous bubbles at their inception.

III. 3 Zero roots and the hysteresis phenomenon

There is nothing in the analysis thus far to rule out zero roots in Λ_1, the set of eigenvalues governing the behaviour of the homogeneous solution for x.[6/] From (12) it can be seen that a zero root in Λ_1 means that for one or more of the predetermined variables,[7/] the influence of the initial conditions does not wear off, even asymptotically, and that the contribution of the exogenous variables is similarly undamped. The model will exhibit hysteresis: if the forcing variables become constant after some point in time and if the

6. Multiple zero roots will complicate the solution method somewhat, but the Jordan canonical form representation of the system can always be used even when A cannot be diagonalized as in (5).

7. and through them possibly also for one or more of the non-predetermined variables (see equation (11).

system converges to a stationary or steady state equilibrium, the stationary equilibrium values of one or more of the state variables will be functions of the initial conditions and of the values of the exogenous variables along the adjustment path to the stationary equilibrium; the steady state conditions alone do not suffice to determine unique steady state values for x and y (see e.g. Buiter and Gersovitz [1981] and Buiter and Miller [1983b]). A general algebraic treatment of the case where Λ_1 contains a zero root can be found in Giavazzi and Wyplosz [1983]. The main points can be brought out quite simply with the example given below, which also has some intrinsic economic interest. We also use this example to consider the case where a zero root is contained in Λ_2, i.e. where the non-predetermined variables (or q) are governed by a zero root.

The example is a contract model of the inflation-unemployment trade-off due to Marcus Miller. This is discussed in Buiter and Miller [1983b]. The basic version is represented in equations (20)-(22).

(20) $$\dot{p}(t) = \psi(y(t) - \bar{y}(t)) + \pi(t) \qquad \psi > 0$$

(21) $$\pi(t) = \pi(t_0)e^{-\zeta_1(t-t_0)} + \zeta_1 \int_{t_0}^{t} c(s)e^{-\zeta_1(t-s)} ds \qquad \zeta_1 > 0$$

(22) $$c(t) = \zeta_2 \int_{t}^{\infty} E_t \dot{p}(\tau)e^{-\zeta_2(\tau-t)} d\tau \qquad \zeta_2 > 0$$

p is the logarithm of the general price level, π the "core" rate of inflation, c the current rate of contract inflation, y actual output and $\bar{y}$ the exogenous natural level of output.

Equation (20) is the familiar core inflation-augmented Phillips curve. Core inflation, in (21), is a backward-looking exponentially declining moving average of past contract inflation. Current contract inflation in (22) is a forward-looking exponentially declining moving average of future expected inflation. Both the price level, p, and core inflation, π, are treated as predetermined. Current contract inflation, c, however, is non-predetermined and can move discontinuously at a point in time in response to "news". The model can be viewed as a modification of Calvo's [1983] continuous time contract model of the inflation process. Calvo specified the current general price level as a backward-looking function of past contract prices, and the current contract price level as a forward-looking function of expected future general price levels and excess demands. Inertia or sluggishness therefore characterizes only the price level in Calvo's model, not both the price level and the core rate of inflation as in equations (20 - 22).

We can represent the model in state-space form as in equations (23a, b), treating the output gap $y - \bar{y}$ as exogenous.

(23a) $$\begin{bmatrix} \dot{\pi}(t) \\ E_t \dot{c}(t) \end{bmatrix} = \begin{bmatrix} -\zeta_1 & \zeta_1 \\ -\zeta_2 & \zeta_2 \end{bmatrix} \begin{bmatrix} \pi(t) \\ c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\zeta_2 \psi \end{bmatrix} \left[y(t) - \bar{y}(t) \right]$$

(23b) $$\dot{p}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \pi(t) \\ c(t) \end{bmatrix} + \psi \left[y(t) - \bar{y}(t) \right]$$

The two characteristic roots of the state equation system (23a) are $\lambda_1 = 0$ and $\lambda_2 = \zeta_2 - \zeta_1$. The solutions for π, c and $\dot{p}$ are therefore given by[7/]

(24a) $$\pi(t) = \pi(t_0) + \zeta_1 \zeta_2 \psi \int_{t_0}^{t} \int_{s}^{\infty} e^{(\zeta_2-\zeta_1)(s-\tau)} E_s(y(\tau) - \bar{y}(\tau) d\tau ds$$

(24b) $$c(t) = \pi(t) + \zeta_2 \psi \int_{t}^{\infty} e^{(\zeta_2-\zeta_1)(t-\tau)} E_t(y(\tau) - \bar{y}(\tau)) d\tau$$

(24c) $$\dot{p}(t) = \pi(t) + \psi(y(t) - \bar{y}(t))$$

The fact that $\lambda_1 = 0$ creates no problems whatsoever. Core inflation $\pi(t)$ can be reduced below its initial value $\pi(t_0)$ only through past expectations (formed between t_0 and t) of future recessions (negative values of $E_s(y(\tau) - \bar{y}(\tau))$). Current contract inflation c(t) differs from current core inflation $\pi(t)$ if the "present value" of currently anticipated future booms or recessions differs from zero. Note that a sustained and sustainable reduction (e.g. a steady state reduction) in inflation $\dot{p}(t)$ requires an equal reduction in core inflation $\pi(t)$.

Note that, in terms of the solution method of Section II, $V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $W = \begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix}$. There are two linearly independent eigenvectors even though at least one and possibly both eigenvalues are zero.

Consider an aggregate demand policy which keeps constant the output gap after some time $t_1 \geq t$ at $y(t_1) - \bar{y}(t_1)$. The only value of this permanent output gap for which a stationary equilibrium exists is of course zero. In that case the steady state conditions of (23b) only give us $\pi = c = \dot{p}$. The common stationary equilibrium value of core inflation, contract inflation and actual inflation cannot be determined from the steady state conditions alone. It is, from (24a), a function of the initial value of π, and of the entire sequence of expectations of future values of the output gap. The rank deficiency of the state matrix in (23a) produces this "hysteresis". If the zero output gap for $t \geq t_1$ has been anticipated correctly from t_0 onwards, i.e. if $E_s(y(\tau) - \bar{y}(\tau)) = 0$, $\tau \geq t_1$; $s \geq t_0$ then

(25) $$\lim_{t\to\infty} \pi(t) = \lim_{t\to\infty} c(t) = \lim_{t\to\infty} \dot{p}(t) = \pi(t_0) + \zeta_1 \zeta_2 \psi \int_{t_0}^{t_1} \int_{s}^{t_1} e^{(\zeta_2 - \zeta_1)(s-\tau)} E_s(y(\tau) - \bar{y}(\tau)) d\tau ds$$

It will be apparent from equations (24a, b, c) that even if $\lambda_2 = \zeta_2 - \zeta_1 = 0$ (if there is no discounting of expected future inflation in the contract inflation equation) the model is still well-behaved, i.e. $c(t)$ is finite, if the undiscounted expected cumulative net output gap $\int_t^\infty E_t(y(\tau) - \bar{y}(\tau))d\tau$ is finite. If we again make the stronger assumption that the output gap expected after some time t_1 is zero, then this is sufficient (but not necessary) for π_t to remain bounded for all time with its steady-state value given by (25) with $\zeta_1 = \zeta_2$. A zero root in Λ_2 therefore merely puts tighter constraints on the permissible forcing processes to ensure bounded values for the non-predetermined variables; it doesn't invalidate the general solution procedure of Section II.

IV Optimal and time-consistent policy design

In this section we consider the optimal control of the model given in (1). The vector of forcing variables is divided into two components, u and z. u is an ℓ vector of policy instruments and z a k vector of exogenous variables. The model is rewritten in (26a, b, c, d). For simplicity the boundary conditions for the predetermined variables are assumed to take the form of n_1 initial values at t_0. Without significant loss of generality the non-explosiveness condition for the exogenous variables and the convergence condition for the non-predetermined variables given in (2b) are expressed as (26c) and (26d) respectively.

$$\text{(26a)} \quad \begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + B\,u(t) + F\,z(t)$$

$$\text{(26b)} \quad x(t_0) = \bar{x}(t_0)$$

$$\text{(26c)} \quad \lim_{s\to\infty} e^{-\beta sI} E_t\, z(s) = 0 \quad \forall \beta > 0 \quad \forall t \geq t_0$$

$$\text{(26d)} \quad \lim_{s\to\infty} e^{-\beta sI} E_t\, y(s) = 0 \quad \forall \beta > 0 \quad \forall t \geq t_0$$

The objective functional to be minimized is the familiar quadratic given in (27)

$$\text{(27)}\quad \min_{\{u(t)\}} J(t_O) \equiv \min_{\{u(t)\}} E_{t_O} \int_{t_O}^{\infty} [\tfrac{1}{2}[x(t)^T y(t)^T u(t)^T z(t)^T]\Omega \begin{bmatrix} x(t) \\ y(t) \\ u(t) \\ z(t) \end{bmatrix} + \omega^T \begin{bmatrix} x(t) \\ y(t) \\ u(t) \\ z(t) \end{bmatrix}] e^{-\zeta(t-t_O)} dt$$

where

$$\Omega = \begin{bmatrix} \Omega_{xx} & \Omega_{xy} & \Omega_{xu} & \Omega_{xz} \\ \Omega_{xy}^T & \Omega_{yy} & \Omega_{yu} & \Omega_{yz} \\ \Omega_{xu}^T & \Omega_{yu}^T & \Omega_{uu} & \Omega_{uz} \\ \Omega_{xz}^T & \Omega_{yz}^T & \Omega_{uz}^T & \Omega_{zz} \end{bmatrix}$$

$$\omega^T = [\ \omega_x^T \quad \omega_y^T \quad \omega_u^T \quad \omega_z^T\ ]$$

$\zeta \geq 0$ is the discount rate.

Ω is a symmetric positive semi-definite matrix. Like the vector ω^T it is partitioned conformably with x,y,u and z

Ω_{uu} is a symmetric, positive definite matrix.

A, B and F are also partitioned conformably with x and y.

The objective function (27) is sufficiently general to include the case where the state equation (26a) is supplemented by an output equation $v(t) = G_1 \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + G_2\, u(t) + G_3\, z(t)$, and the integrand in the objective functional is specified in terms of the output vector as $v^T(t)\, \tilde{\Omega}\, v(t) + \tilde{\omega}^T v(t)$.

Optimal Policies

The natural interpretation of this optimal control problem is that of a non-co-operative Stackelberg leader-follower game. Equation (26a) represents the 'reaction function' of the follower (the economic system) who takes as given the current and anticipated future actions of the controller, who is the leader.

To derive the optimal policy we define the Hamiltonian H

$$
\begin{aligned}
(28)\quad H(t) = \{\tfrac{1}{2}[& x(t)^T\Omega_{xx}x(t)+2y(t)^T\Omega_{xy}x(t)+y(t)^T\Omega_{yy}y(t)+2x(t)^T\Omega_{xu}u(t)+2y(t)^T\Omega_{yu}u(t) \\
&+2x(t)^T\Omega_{xz}z(t)+2y(t)^T\Omega_{yz}z(t)+u(t)^T\Omega_{uu}u(t)+2u^T(t)\Omega_{uz}z(t)+z(t)^T\Omega_{zz}z(t)] \\
&+\omega_x^T x(t)+\omega_y^T y(t)+\omega_u^T u(t)+\omega_z^T z(t)\}e^{-\zeta(t-t_0)} \\
&+\lambda_x^T(t)[A_{11}x(t)+A_{12}y(t)+B_1 u(t)+F_1 z(t)] \\
&+\lambda_y^T(t)[A_{21}x(t)+A_{22}y(t)+B_2 u(t)+F_2 z(t)]
\end{aligned}
$$

$\lambda_x(t)$ is the n_1 vector of co-state variables corresponding to the predetermined state variables $x(t)$ while $\lambda_y(t)$ is the $n-n_1$ vector of co-state variables corresponding to the non-predetermined state variables $y(t)$.

The first-order conditions for an optimum are given by the equations of motion (26a) and (29a, b, c)

(29a) $$\frac{\partial H(t)}{\partial u(t)} = 0 \qquad \forall t$$

(29b) $$\frac{-\partial H(t)}{\partial x(t)} = E_t\, \dot{\lambda}_x^T(t) \qquad \forall t$$

(29c) $$\frac{-\partial H(t)}{\partial y(t)} = \dot{\lambda}_y^T(t) \qquad \forall t$$

Defining the current value co-state variables (shadow prices)

(30a) $$\mu_x(t) \equiv e^{\zeta(t-t_0)I} \lambda_x(t).$$

(30b) $$\mu_y(t) \equiv e^{\zeta(t-t_0)I} \lambda_y(t)$$

we can solve (29a) for the optimum instrument values as in (31).

(31) $$u(t) = -\Omega_{uu}^{-1}\Omega_{xu}^T\, x(t) - \Omega_{uu}^{-1}\, \Omega_{yu}^T\, y(t) - \Omega_{uu}^{-1}\, B_1^T\, \mu_x(t) - \Omega_{uu}^{-1}\, B_2^T\, \mu_y(t) - \Omega_{uu}^{-1}\, \Omega_{uz}\, z(t) - \Omega_{uu}^{-1}\, \omega_u$$

Substituting for u(t) from (31) into (26a) and into (29b, c) the behaviour of the state variables and the co-state variables under optimal control is given in (32).

(32)

$$\begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \\ E_t\dot{\mu}_x(t) \\ \dot{\mu}_y(t) \end{bmatrix} =$$

$$\begin{bmatrix} A_{11}-B_1\Omega_{uu}^{-1}\Omega_{xu}^T & A_{12}-B_1\Omega_{uu}^{-1}\Omega_{yu}^T & -B_1\Omega_{uu}^{-1}B_1^T & -B_1\Omega_{uu}^{-1}B_2^T \\ A_{21}-B_2\Omega_{uu}^{-1}\Omega_{xu}^T & A_{22}-B_2\Omega_{uu}^{-1}\Omega_{yu}^T & -B_2\Omega_{uu}^{-1}B_1^T & -B_2\Omega_{uu}^{-1}B_2^T \\ -[\Omega_{xx}-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{xu}^T] & -[\Omega_{xy}^T-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{yu}^T] & -[A_{11}^T-\Omega_{xu}\Omega_{uu}B_1^T-\zeta I_{n_1}] & -[A_{21}^T-\Omega_{xu}\Omega_{uu}^{-1}B_2^T] \\ -[\Omega_{xy}-\Omega_{yu}\Omega_{uu}^{-1}\Omega_{xu}^T] & -[\Omega_{yy}-\Omega_{yu}\Omega_{uu}^{-1}\Omega_{yu}^T] & -[A_{12}^T-\Omega_{yu}\Omega_{uu}^{-1}B_1^T] & -[A_{22}^T-\Omega_{yu}\Omega_{uu}^{-1}B_2^T-\zeta I_{n-n_1}] \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ \mu_x(t) \\ \mu_y(t) \end{bmatrix}$$

$$+ \begin{bmatrix} F_1 - B_1\Omega_{uu}^{-1}\Omega_{uz} \\ F_2 - B_2\Omega_{uu}^{-1}\Omega_{uz} \\ -[\Omega_{xz} \;\; \Omega_{xu}\Omega_{uu}^{-1}\Omega_{uz}] \\ -(\Omega_{yz}-\Omega_{yu}\Omega_{uu}^{-1}\Omega_{uz}) \end{bmatrix} z(t) + \begin{bmatrix} -B_1\Omega_{uu}^{-1}\omega_u \\ -B_2\Omega_{uu}^{-1}\omega_u \\ \Omega_{xu}\Omega_{uu}^{-1}\omega_u-\omega_x \\ \Omega_{yu}\Omega_{uu}^{-1}\omega_u-\omega_y \end{bmatrix}$$ [8]

The 2n boundary conditions for the economy under the optimal policy take the form:

(33a) $$x(t_0) = \overline{x(t_0)}$$

(33b) $$\lim_{t\to\infty} e^{-\beta(t-s)I} E_s\, y(t) = 0 \qquad \beta > 0\,, \quad s \geq t_0$$

---

8. I_k is the $k \times k$ identity matrix.

(33c) $\lim_{t\to\infty} e^{-\zeta(t-s)} E_s \mu_x^T(t)\ x(t) = 0$

(33d) $\mu_y(t_0) = 0$

The crucial boundary condition is the one relating to the initial values of the co-state variables corresponding to the non-predetermined state variables given in (33d). Since $y(t_0)$ is free, it will be set optimally, i.e., the values of the co-state variables μ_y at the initial date, t_0, which measure the marginal contribution of $y(t_0)$ to the objective functional, will be zero. (See Bryson and Ho [1985, p. 55-59], Calvo [1978].) [9/]

The dynamic system under optimal control, given in (32) therefore contains n predetermined variables (x, the predetermined state variables and μ_y, the shadow prices of the non-predetermined state variables) and n non-predetermined variables (y, the non-predetermined state variables and μ_x, the shadow prices of the predetermined state variables). Following Miller and Salmon [1982, 1983], we rearrange (32) by grouping together the predetermined and non-predetermined variables and by subsuming the constant vector (the last term on the r.h.s. of (32)) under the exogenous variables. Letting $\bar{z} \equiv \begin{bmatrix} z \\ 1 \end{bmatrix}$, we obtain

9. For (33d) to hold, a controllability condition for y must be satisfied: there must exist at t_o a path of expected future policy $\{E_{t_o} u(s)\ ;\ s \geq t_o\}$ such that $y(t_o)$ can be set at the value required to make $\mu_y(t_o)$ equal to zero. See Bryson and Ho [1975; p.58, p.164 and Appendix B, pp. 455-457].

$$(34)\quad \begin{bmatrix} \dot{x}(t) \\ \dot{\mu}_y(t) \\ E_t \dot{\mu}_x(t) \\ E_t \dot{y}(t) \end{bmatrix} = \bar{A} \begin{bmatrix} x(t) \\ \mu_y(t) \\ \mu_x(t) \\ y(t) \end{bmatrix} + \bar{B}\,\bar{z}(t)$$

and from (31)

$$(35)\quad u(t) = C \begin{bmatrix} x(t) \\ \mu_y(t) \\ \mu_x(t) \\ y(t) \end{bmatrix} + D\,\bar{z}(t)$$

If $\bar{A}$ can be diagonalised and if it has n stable and n unstable characteristic roots the solution method of Section II can be applied to (34) and the optimal policy as well as the behaviour of the economy under optimal policy can be computed easily.

Time-Consistent Rational Expectations Solutions

It is obvious from the boundary condition (33d) and the equations of motion under optimal control (32) that in general if the controller re-optimises at $t = t_1 > t_0$ his optimal plan from time $t_1 > t_0$ onwards will not be the continuation for $t \geq t_1$ of the optimal plan derived at time t_0, even if no new information about the exogenous variables has accrued between t_0 and t_1. The optimal plan is not in general time consistent. (See Kydland and Prescott [1977]). The reason is that while

$\mu_y = 0$ at $t = t_0$, it will, in general, be different from zero for $t > t_0$, given the dynamics of equation (32). Reoptimizing at $t = t_1 > t_0$, the controller will, taking $x(t_1)$ as given, be tempted to adopt a plan for $t \geq t_1$ that will set $\mu_y(t_1) = 0$.

Unless, under the optimal plan adopted at $t = t_0$, the value of μ_y at $t = t_1$ would have been equal to zero anyway, the re-optimization at $t = t_1$ would falsify the expectations held between t_0 and t_1 by the agents represented in the model of equation (26a). It is these expectations that will have brought the system to $x(t_1)$ [10/] in the first place. Past expectations of future policy actions would have been used as an additional policy instrument, unconstrained by the requirement that they be equal to actual, realized policy actions (except for unforeseen exogenous shocks).

If the agents in the model anticipate that the controller will reoptimize at t_1, taking as given their past expectations of his future actions, embodied in $x(t_1)$, they will expect $\mu_y(t_1) = 0$. If the controller can reoptimize at each and every instant, they will

---

10. Note that the "followers" whose behaviour is given by (26) form expectations not only of future values of $\bar{z}$ but also of future values of u. This is clear from equation (12), if we interpret z as containing both policy instruments and variables exogenous to the system and to the controller. The followers (the agents forming expectations in (26)) take an open-loop view (in stochastic models an "innovation-contingent" open-loop view [see Buiter (1981b)]) of future policy.

anticipate $\mu_y(t) = 0 \; \forall t \geq t_0$. The characterization of a time-consistent rational expectations solution is then straightforward.

A time-consistent rational expectations solution is characterised by zero values at each instant of the co-state variables corresponding to the non-predetermined state variables, i.e. by $\mu_y(t) \equiv 0$, $t \geq t_0$. The optimality condition $-\frac{\partial H}{\partial y} = \dot{\lambda}_y^T$ no longer applies as the controller is effectively forced to treat $y(t)$ as exogenous rather than as driven by the equations of motion of the system.

The equations of motion under time-consistent control are therefore obtained by omitting the rows corresponding to $\dot{\mu}_y(t)$ and the columns corresponding to $\mu_y(t)$ in (32). The behaviour of the system under time-consistent control is given by (33a, b, c) and

$$(36) \quad \begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \\ E_t\dot{\mu}_x(t) \end{bmatrix} = \begin{bmatrix} A_{11}-B_1\Omega_{uu}^{-1}\Omega_{xu}^T & A_{12}-B_1\Omega_{uu}^{-1}\Omega_{yu}^T & -B_1\Omega_{uu}^{-1}B_1^T \\ A_{21}-B_2\Omega_{uu}^{-1}\Omega_{xu}^T & A_{22}-B_2\Omega_{uu}^{-1}\Omega_{yu}^T & -B_2\Omega_{uu}^{-1}B_1^T \\ -[\Omega_{xx}-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{xu}^T] & -[\Omega_{xy}^T-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{yu}^T] & -[A_{11}^T-\Omega_{xu}\Omega_{uu}^{-1}B_1^T-\zeta I_{n_1}] \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ \mu_x(t) \end{bmatrix}$$

$$+ \begin{bmatrix} F_1 - B_1\Omega_{uu}^{-1}\Omega_{uz} \\ F_2 - B_2\Omega_{uu}^{-1}\Omega_{uz} \\ -[\Omega_{xz}-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{uz}] \end{bmatrix} z(t) + \begin{bmatrix} -B_1\Omega_{uu}^{-1}\omega_u \\ -B_2\Omega_{uu}^{-1}\omega_u \\ \Omega_{xu}\Omega_{uu}^{-1}\omega_u - \omega_x \end{bmatrix}$$

$$(37) \quad u_t = -\Omega_{uu}^{-1}\Omega_{xu}^T x(t) - \Omega_{uu}^{-1}\Omega_{yu}^T y(t) - \Omega_{uu}^{-1}B_1^T\mu_x(t) - \Omega_{uu}^{-1}\Omega_{uz} z(t) - \Omega_{uu}^{-1}\omega_u$$

Note that while the time-consistent solution is a product of the realisation (ex ante) by the agents in the model (the followers) that the controller (the leader) will cheat if he has an incentive to do so (if $\mu_y(t) \neq 0$), there is no cheating (ex post) along the time-consistent path because the incentive to cheat has been eliminated; the leader has lost his leadership.

Obviously, the optimal policy will be time-consistent i.f.f. under the optimal policy, $\mu_y(t) \equiv 0$, $t \geq t_0$. If this is not the case, precommitment is necessary for the controller to implement the optimal solution.

Two comments on this time-consistent solution are pertinent. First, the "loss of leadership" solution characterised in (36) and (37) doesn't solve the time-inconsistency problem associated with optimal policy in rational expectations models. It is merely an alternative solution that may be relevant when precommitment is impossible. Miller and Salmon [1982, 1983] have shown that the time-consistent solution is equivalent to the open-loop Nash equilibrium in a two-player linear-quadratic differential game. This sheds further light on the "loss of leadership" interpretation of the time-consistent solution.

Second, the analysis of Section IV brings out the incompleteness of the standard specification of the optimal control problem. As pointed out by Reinganum and Stokey [1981], the period of commitment is a crucial parameter of the optimization problem. Treating the period of commitment as exogenous, we can interpret the optimal policy as the equilibrium policy when credibility is complete and the period over which the leader can make binding commitments is infinite. The time-consistent solution represents the other extreme when the period of commitment has shrunk to zero and no credible announcements of future policy actions are possible at all. Clearly, one could plausibly think of intermediate cases in which the period of commitment is positive but finite. Even more interesting would be an endogenous determination of the period of commitment or a theory of precommitment. Reputation effects, threats and sanctions, voluntary or self-imposed constraints on future freedom of action etc. all would come into play. We are unfortunately still far removed from such a positive theory of constitutions.

Section V : An example of optimal and time-consistent policies: anti-inflationary policy in a contract model

As an example of optimal and time-consistent policy design we shall consider anti-inflationary policy in the model given by equations (20)- (22), whose state-space representation is in (23a, b). The level of demand y is treated as the control variable and the objective functional is given in (38).

$$(38) \quad \min_{\{y(s)\}} E_t \int_t^{\infty} \frac{1}{2} \left[(y(s) - y^*)^2 + \gamma (\dot{p}(s))^2 \right] e^{-\rho(s-t)} ds$$

$\gamma, \rho \geq 0$.

Deviations of output from its target level y^* are penalized, as are deviations of the inflation rate from zero. y^* need not equal the natural level of output.

The equations of motion for the state variables π, c and their current value co-state variables μ_π and μ_c and the optimal path of demand are given in equations (39a, b).

$$(39a) \quad \begin{bmatrix} \dot{\pi}(t) \\ \dot{\mu}_c(t) \\ E_t \dot{\mu}_\pi(t) \\ E_t \dot{c}(t) \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & 0 & \zeta_1 \\ 0 & (\rho - \zeta_2) & -\zeta_1 & 0 \\ \frac{-\gamma}{1+\gamma\psi^2} & \frac{\zeta_2}{1+\gamma\psi^2} & \rho+\zeta_1 & 0 \\ \frac{-\zeta_2}{1+\gamma\psi^2} & \frac{-(\zeta_2\psi)^2}{1+\gamma\psi^2} & 0 & \zeta_2 \end{bmatrix} \begin{bmatrix} \pi(t) \\ \mu_c(t) \\ \mu_\pi(t) \\ c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{-\gamma\psi}{1+\gamma\psi^2}(y^* - \bar{y}) \\ \frac{-\zeta_2\psi}{1+\gamma\psi^2}(y^* - \bar{y}) \end{bmatrix}.$$

$$(39b) \quad y(t) = \frac{-\gamma\psi}{1+\gamma\psi^2} \pi(t) + \frac{\zeta_2\psi}{1+\gamma\psi^2} \mu_c(t) + \bar{y} + \frac{y^* - \bar{y}}{1+\gamma\psi^2}$$

with

$\pi(t_0) = \overline{\pi(t_0)}$

and

$\mu_c(t_0) = 0$

Note from (39b) that the optimal policy does not feed back directly from μ_π, the shadow price of core inflation. In (39a), $\dot{\pi}$ and $E_t\dot{c}$ similarly don't feed back from μ_π directly, but only indirectly through the effect of μ_π or $\dot{\mu}_c$. This is a reflection of our assumption that core inflation is simply an exponentially deciding moving average of past contract inflation. The shadow price of contract inflation therefore contains the relevant information about the shadow price of core inflation. The optimal policy has the sensible property that if we start off with the "bliss" rate of core inflation $(\pi = 0)$ and if the target and natural levels of output coincide, then demand will be kept at the natural level and full employment with zero inflation endures. Cet. par. a higher value of y^* relative to $\bar{y}$ means a higher optimal level of demand; also a higher inherited value of core inflation implies a lower optimal level of demand. If $\mu_c > 0$, i.e. if current core inflation makes a positive marginal contribution to the minimized value of the loss function, then current demand is high relative to its long run value.

The steady state equilibrium under optimal policy is characterised by:

$$\pi = c = \overset{\circ}{p} = \left[\frac{\rho^2 + (\zeta_1 - \zeta_2)\rho}{\gamma\psi(\rho+\zeta_1)(\rho-\zeta_2))} \right] (y^* - \bar{y})$$

$$\mu_c = \frac{\zeta_1}{\psi(\rho+\zeta_1)(\rho-\zeta_2)} (y^* - \bar{y})$$

$$\mu_\pi = \frac{1}{\psi(\rho+\zeta_1)} (y^* - \bar{y})$$

$$y = \bar{y}$$

Thus the system under optimal policy always converges towards the natural level of output. If the target level of output coincides with the natural level of output steady state inflation will be zero.[11/] For there to be a unique convergent saddlepoint equilibrium the model should possess two stable and two unstable roots; the state matrix in (39a) should therefore have a positive determinant. This requires $\zeta_2 > \rho$. The shadow price of core inflation is always positive (negative) in long-run equilibrium if $y^* > \bar{y}$ $(y^* < \bar{y})$; the determinant condition for saddlepoint stability implies that the opposite holds for the shadow price of contract inflation. This reflects the backward-looking nature of π and the forward-looking nature of c.

In the numerical example given below, $\rho^2 + (\zeta_1 - \zeta_2)\rho < 0$,

---

11. If there is no discounting, the steady state value of the loss function is unbounded (unless $\bar{y} = y^*$) and no solution exists.

so the long-run rate of inflation will be positive (negative) if $y^* > \bar{y}$ $(y^* < \bar{y})$. I have not been able to establish whether or not this is a necessary condition for saddlepoint stability in general.

Following Section IV, a time-consistent policy is obtained by deleting the rows and columns corresponding to μ_c in (39a, b). The behaviour of the state variables, the remaining co-state variable and the policy instrument y is given in equations (40a, b):

$$\text{(40a)} \quad \begin{bmatrix} \dot{\pi}(t) \\ E_t \dot{c}(t) \end{bmatrix} = \begin{bmatrix} -\zeta_1 & \zeta_1 \\ \frac{-\zeta_2}{1+\gamma\psi^2} & \zeta_2 \end{bmatrix} \begin{bmatrix} \pi(t) \\ c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\zeta_2\psi}{1+\gamma\psi^2} (y^* - \bar{y}) \end{bmatrix}$$

$$\text{(40b)} \quad y(t) = -\frac{\gamma\psi}{1+\gamma\psi^2}\pi(t) + \bar{y} + \frac{1}{1+\gamma\psi^2}(y^* - \bar{y})$$

$$\pi(t_0) = \overline{\pi(t_0)} .$$

The shadow price of core-inflation, μ_π is determined recursively, given the solution for (40a), by (41) but policy has become completely "backward-looking" and y no longer "feeds back" from any shadow price.

$$\text{(41)} \quad E_t\dot{\mu}_\pi(t) = -\frac{\gamma}{1+\gamma\psi^2}\pi(t) + (\rho + \zeta_1)\mu_\pi - \frac{\gamma\psi}{1+\gamma\psi^2}(y^* - \bar{y}).$$

The steady state conditions are

$$\pi = c = \dot{p} = \frac{1}{\gamma\psi}(y^* - \bar{y})$$

$$\mu_\pi = \frac{1}{\psi(\rho+\zeta_1)}(y^* - \bar{y})$$

$$y = \bar{y}$$

Again $y^* = \bar{y}$ implies zero inflation in the long run. Long-run inflation will be positive (negative) if $y^* > \bar{y}$ $(y^* < \bar{y})$. Note that if $\rho^2 + (\zeta_1 - \zeta_2)\rho < 0$, inflation will be higher in the long run under the time-consistent policy than under the optimal policy, if $y^* > \bar{y}$.

Figures 1 and 2 depict the behaviour of some of the variables of interest under optimal and time-consistent policy for the following values of the parameters : $\zeta_1 = .5$; $\zeta_2 = .6$; $\psi = .5$; $\gamma = 1$ and $\rho = .03$. In Figure 1, $y^* = .02$ and $\bar{y} = 0$. In Figure 2 both y^* and $\bar{y}$ equal zero. The initial rate of core inflation at $t = 0$ is ten per cent.

In Figure 1 under optimal policy, inflation remains slightly above zero (at .028 per cent) even in the long run. There is a fairly sharp initial recession. The shadow price of contract inflation starts at zero but becomes sharply negative and converges to a negative long-run value. The shadow price of core inflation (not drawn) jumps to .19 at t = 0 and converges to .075. The negative values of μ_c under the optimal policy signal the time inconsistency problem. The time-consistent policy has a smaller recession throughout. The authorities cannot credibly announce a path of deep recession, as they would be tempted not to have a fierce recession once the announcement effect of that recession has succeeded in bringing down core inflation. When the target level of output exceeds the natural level, the cost of not having credibility is a long-run rate of inflation which in the numerical example is four per cent - well above the optimal long-run inflation rate.

In Figure 2, $y^* = \bar{y}$ and the long-run conflict between output target and output constraint is absent. Both time-consistent and optimal policies yield zero long-run inflation. Inflation is, however, brought down more rapidly under the optimal policy. This is reflected in a deeper initial recession under the optimal policy; after period 9, however, the recession is slightly more severe under the time-consistent policy.

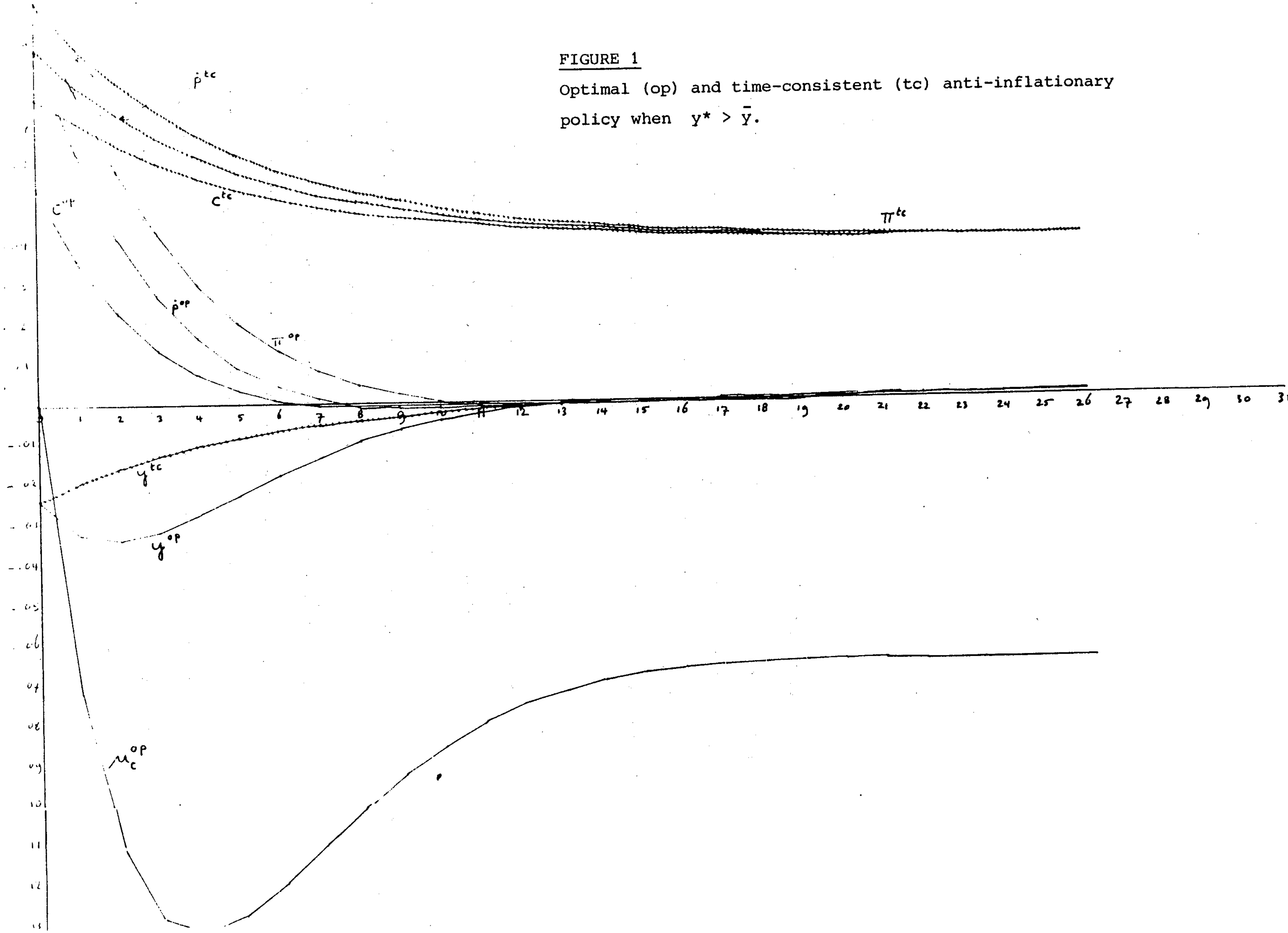

FIGURE 1

Optimal (op) and time-consistent (tc) anti-inflationary policy when $y^* > \bar{y}$.

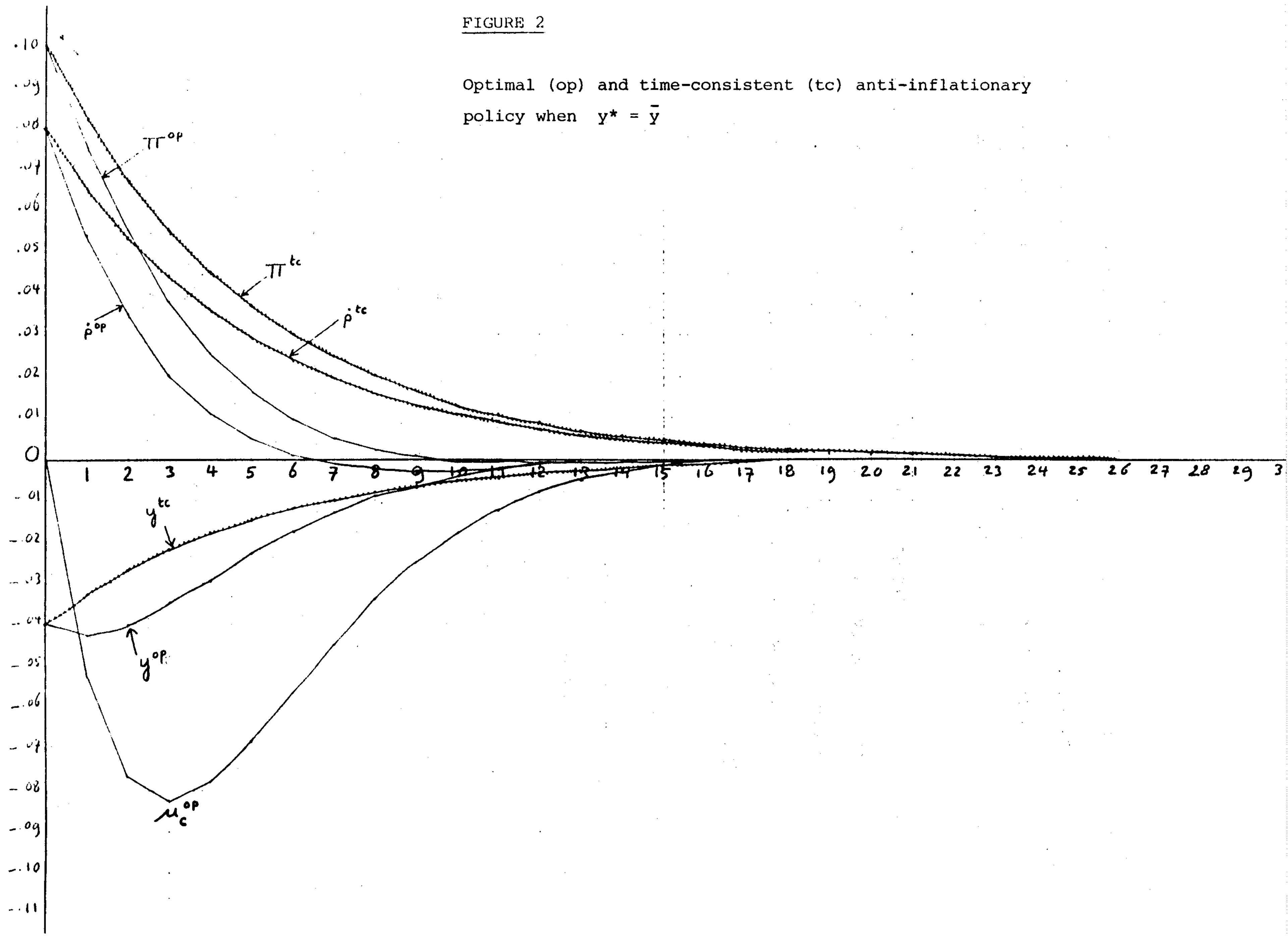

FIGURE 2

Optimal (op) and time-consistent (tc) anti-inflationary policy when $y^* = \bar{y}$

Conclusion

The solution method discussed in this paper can be used to study the behaviour of continuous time linear rational expectations models under exogenous policy, under ad-hoc linear policy feedback rules, under optimal policy and under time-consistent policy. The consequences of any combination and sequence of anticipated or unanticipated, current or future and permanent or transitory shocks can be evaluated. The great virtue of the method is its analytical simplicity and computational efficiency, even for fairly large dynamic systems. As was indicated in Section II, the explicit consideration of uncertainty in the form of additive white noise is, because of certainty equivalence, a very simple matter. A more general specification of uncertainty (e.g. random parameters) very soon leads to awsome complications. Deterministic non-linear systems can be tackled on a "try-it-and-see-if-it-works" basis with a wide variety of existing non-linear two point boundary value problem solution algorithms. E.g. successful applications of the technique of "multiple shooting" in economics can be found e.g. in the work of Bruno and Sachs [1982] (see also Lipton, Poterba, Sachs and Summers [1982]).

Decentralized, non-cooperative policy design in continuous time linear rational expectations models has been pioneered by Miller and Salmon [1982, 1983] using a linear-quadratic differential game approach. As with the "single player" optimal control problem of

Section IV, the behaviour of the system under various kinds of decentralized control can be reduced to the standard format of equation (1). It appears safe to predict continued growth in the range of applications of these methods in the fields of macroeconomics and international finance.

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