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OPTIMAL AND TIME-CONSISTENT POLICIES IN CONTINUOUS TIME RATIONAL EXPECTATIONS MODELS

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Abstract

In this note the method of Hamiltonian dynamics is used to characterize the time-consistent solution to the optimal control problem in a deterministic continuous time rational expectations model. A linear-quadratic example based on the work of Miller and Salmon is used for simplicity. To derive the time-consistent rational expectations (or subgame-perfect) solution we first characterize the optimal solution made familiar e.g. through the work of Calvo. The time-consistent solution is then obtained by modifying the optimal solution through the requirement that the co-state variables (shadow prices) of the non-predetermined variables be zero at each instant. Existing solution methods and computational algorithms can be used to obtain the behaviour of the system under optimal policy and under time-consistent policy.

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1. INTRODUCTION

In this note the method of Hamiltonian dynamics is used to characterise the time-consistent solution to the optimal control problem in a deterministic continuous time rational expectations model. A linear-quadratic example based on the work of Miller and Salmon [1982, 1983] is used for simplicity. To derive the time-consistent rational expectations (or subgame-perfect) solution we first characterise the optimal solution made familiar e.g. through the work of Calvo [1978]. The time-consistent solution is then obtained by modifying the optimal solution through the requirement that the co-state variables (shadow prices) of the non-predetermined variables be zero at each instant. Existing solution methods and computational algorithms can be used to obtain the behaviour of the system under optimal policy and under time-consistent policy.

2. THE MODEL AND THE OBJECTIVE FUNCTIONAL

Consider the non-stochastic continuous time linear rational expectations model of equation (1)

$$\begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} z(t) \qquad ..(1)$$

x_t is an n_1 vector of predetermined variables, ie state variables such as the economy-wide capital stock for which the boundary conditions take the form of initial conditions. y_t is a $n-n_1$ vector of non-predetermined variables for which the boundary conditions take the form of terminal or transversality conditions. Examples are asset prices whose current values are functions of their expected rates of change. u_t is a k vector of policy instruments or controls and z_t an r vector of exogenous variables. $A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ $B \equiv \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $F \equiv \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ are constant matrices partitioned conformably with x and y. E_t is the rational expectation operator conditional on the information set at time t. Since the model is non-stochastic, rational expectations of y(t) are forecasts based on the correct model, given in (1), conditional on subjective expectations of future values of u(t) and z(t) which are held with complete certainty. We make the standard assumptions that

$$E_t\, y(s) = y(s) \qquad s \leq t \qquad \ldots\ldots (2a)$$

and

$$E_t E_s y(\tau) = E_t y(\tau) \qquad t \leq s \leq \tau \qquad \ldots\ldots (2b)$$

The boundary conditions are given in (3a,b,c). The exogenous variables are assumed not to explode 'too fast'. The same convergence condition characterises the non-predeterminal variables

$$x(t_o) = \bar{x}(t_o) \qquad \text{...... (3a)}$$

$$\lim_{s\to\infty} e^{-\beta sI} E_t\, z(s) = 0 \quad \forall\beta>0\ \forall t\geq t_o \qquad \text{...... (3b)}$$

$$\lim_{s\to\infty} e^{-\beta sI} E_t\, y(s) = 0 \quad \forall\beta>0\ \forall t\geq t_o \qquad \text{...... (3c)}$$

The objective functional to be minimized is the familiar quadratic and given in (4)[1/]

$$\min_{\{u(t)\}} J(t_o) \equiv \min_{\{u(t)\}} E_{t_o} \int_{t_o}^{\infty} \left[\tfrac{1}{2} \left[x(t)^T y(t)^T u(t)^T z(t)^T\right] \Omega \begin{bmatrix} x(t) \\ y(t) \\ u(t) \\ z(t) \end{bmatrix} + \omega^T \begin{bmatrix} x(t) \\ y(t) \\ u(t) \\ z(t) \end{bmatrix} \right] e^{-\zeta(t-t_o)}\, dt \qquad \text{....(4)}$$

where

$$\Omega = \begin{bmatrix} \Omega_{xx} & \Omega_{xy} & \Omega_{xu} & \Omega_{xz} \\ \Omega_{xy}^T & \Omega_{yy} & \Omega_{yu} & \Omega_{yz} \\ \Omega_{xu}^T & \Omega_{yu}^T & \Omega_{uu} & \Omega_{uz} \\ \Omega_{xz}^T & \Omega_{yz}^T & \Omega_{uz}^T & \Omega_{zz} \end{bmatrix} \qquad \text{...... (5a)}$$

$$\omega^T = \left[\omega_x^T \quad \omega_y^T \quad \omega_u^T \quad \omega_z^T \right] \qquad \text{...... (5b)}$$

$$\zeta \geq 0 \qquad \text{...... (5c)}$$

Ω is a symmetric positive semi-definite matrix. Like the vector ω^T it is partitioned conformably with x,y,u and z

Ω_{uu} is a symmetric, positive definite matrix.

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1. m^T denotes the transpose of m.

3. OPTIMAL POLICIES

The natural interpretation of this optimal control problem is that of a non-co-operative Stackelberg leader-follower game. Equation (1) represents the 'reaction function' of the follower, who takes as given the current and anticipated future actions of the controller who is the leader.

To derive the optimal policy we define the Hamiltonian H

$$H(t) = \{\tfrac{1}{2}[\, x(t)^T\Omega_{xx}x(t)+2y(t)^T\Omega_{xy}x(t)+y(t)^T\Omega_{yy}y(t)+2x(t)^T\Omega_{xu}u(t)+2y(t)^T\Omega_{yu}u(t)$$

$$+\ 2x(t)^T\Omega_{xz}z(t)+2y(t)^T\Omega_{yz}z(t)+u(t)^T\Omega_{uu}u(t)+\ 2u^T(t)\Omega_{uz}z(t)+z(t)^T\Omega_{zz}z(t)]$$

$$+\ \omega_x^T\ x(t)+\omega_y^T\ y(t)\ +\ \omega_u^T\ u(t)\ +\ \omega_z^T\ z(t)\}e^{-\zeta(t-t_o)}$$

$$+\lambda_x(t)^T\,[\,A_{11}\ x(t)\ +\ A_{12}\ y(t)\ +\ B_1\ u(t)\ +\ F_1\ z(t)\,]$$

$$+\lambda_y(t)^T\,[\,A_{21}\ x(t)\ +\ A_{22}\ y(t)\ +\ B_2\ u(t)\ +\ F_2\ z(t)\,] \quad \ldots\ldots (6)$$

$\lambda_x(t)$ is the n_1 vector of co-state variables corresponding to the predetermined state variables x(t) while $\lambda_y(t)$ is the $n-n_1$ vector of co-state variables corresponding to the non-predetermined state variables y(t).

The first-order conditions for an optimum are given by the equations of motion (1) and (7a, b, c)

$$\frac{\partial H(t)}{\partial u(t)} = 0 \qquad \forall t \qquad \ldots\ldots (7a)$$

$$\frac{-\partial H(t)}{\partial x(t)} = \dot{\lambda}_x^T(t) \qquad \forall t \qquad \ldots\ldots (7b)$$

$$\frac{-\partial H(t)}{\partial y(t)} = \dot{\lambda}_y^T(t) \qquad \forall t \qquad \text{...... (7c)}$$

Defining the current value co-state variables (shadow prices)

$$\mu_x(t) \equiv e^{\zeta(t-t_o)I}\lambda_x(t) \qquad \text{...... (8a)}$$

$$\mu_y(t) \equiv e^{\zeta(t-t_o)I}\lambda_x(t) \qquad \text{...... (8b)}$$

we can solve (7a) for the optimum instrument values as in (9)

$$u(t) = -\Omega_{uu}^{-1}\Omega_{xu}^T\, x(t) - \Omega_{uu}^{-1}\, \Omega_{yu}^T\, y(t) - \Omega_{uu}^{-1}\, B_1^T\, \mu_x(t) - \Omega_{uu}^{-1}\, B_2^T\, \mu_y(t)$$
$$-\Omega_{uu}^{-1}\, \Omega_{uz}\, z(t) - \Omega_{uu}^{-1}\, \omega_u \qquad \text{...... (9)}$$

Substituting for u(t) from (9) into (1) and into (7b, c) the behaviour of the state variables and the co-state variables under optimal control is given in (10).

$$\begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \\ \dot{\mu}_x(t) \\ \dot{\mu}_y(t) \end{bmatrix} = \begin{bmatrix} A_{11} - B_1\Omega_{uu}^{-1}\Omega_{xu}^T & A_{12} - B_1\Omega_{uu}^{-1}\Omega_{yu}^T & -B_1\Omega_{uu}^{-1}B_1^T & -B_1\Omega_{uu}^{-1}B_2^T \\ A_{21} - B_2\Omega_{uu}^{-1}\Omega_{xu}^T & A_{22} - B_2\Omega_{uu}^{-1}\Omega_{yu}^T & -B_2\Omega_{uu}^{-1}B_1^T & -B_2\Omega_{uu}^{-1}B_2^T \\ -\left[\Omega_{xx} - \Omega_{xu}\Omega_{uu}^{-1}\Omega_{xu}^T\right] & -\left[\Omega_{xy}^T - \Omega_{xu}\Omega_{uu}^{-1}\Omega_{yu}^T\right] & -\left[A_{11}^T - \Omega_{xu}\Omega_{uu}^{-1}B_1^T - \zeta I_{n_1}\right] & -\left[A_{21}^T - \Omega_{xu}\Omega_{uu}^{-1}B_2^T\right] \\ -\left[\Omega_{xy} - \Omega_{yu}\Omega_{uu}^{-1}\Omega_{xu}^T\right] & -\left[\Omega_{yy} - \Omega_{yu}\Omega_{uu}^{-1}\Omega_{yu}^T\right] & -\left[A_{12}^T - \Omega_{yu}\Omega_{uu}^{-1}B_1^T\right] & -\left[A_{22}^T - \Omega_{yu}\Omega_{uu}^{-1}B_2^T - \zeta I_{n-n_1}\right] \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ \mu_x(t) \\ \mu_y(t) \end{bmatrix}$$

$$+ \begin{bmatrix} F_1 - B_1\Omega_{uu}^{-1}\Omega_{uz} \\ F_2 - B_2\Omega_{uu}\Omega_{uz} \\ -\left[\Omega_{xz} - \Omega_{xu}\Omega_{uu}^{-1}\Omega_{uz}\right] \\ -(\Omega_{yz} - \Omega_{yu}\Omega_{uu}^{-1}\Omega_{uz}) \end{bmatrix} z(t) + \begin{bmatrix} -B_1\Omega_{uu}^{-1}\omega_u \\ -B_2\Omega_{uu}^{-1}\omega_u \\ \Omega_{xu}\Omega_{uu}^{-1}\omega_u - \omega_x \\ \Omega_{yu}\Omega_{uu}^{-1}\omega_u - \omega_y \end{bmatrix} \qquad \text{[2/]} \qquad \text{...... (10)}$$

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2. I_k is the k x k identity matrix

The 2n boundary conditions take the form:

$$x(t_o) = \overline{x(t_o)} \qquad \text{...... (11a)}$$

$$\lim_{t\to\infty} e^{-\beta t} I E_s y(t) = 0 \qquad \forall\beta>0,\ \forall s \qquad \text{...... (11b)}$$

$$\lim_{t\to\infty} \mu_x(t)^T x(t) = 0 \qquad \text{...... (11c)}$$

$$\mu_y(t_o) = 0 \qquad \text{...... (11d)}$$

The crucial boundary condition is the one relating to the initial values of the co-state variables corresponding to the non-predetermined state variables given in (11d). Since $y(t_o)$ is free, it will be set optimally, ie, the values of the co-state variables μ_y at the initial date, t_o, which measure the marginal contribution of $y(t_o)$ to the objective functional will be zero. (See Bryson and Ho (1975, p.55-59), Calvo (1978)).

The dynamic system under optimal control, given in (10) therefore contains n predetermined variables (x, the predetermined state variables and μ_y the shadow prices of the non-predetermined state variables) and n non-predetermined variables (y, the non-predetermined state variables and μ_x, the shadow prices of the predetermined state variables). Following Miller and Salmon [1982, 1983], we rearrange (10) by grouping together the predetermined and non-predetermined variables and by subsuming the constant vector under the exogenous variables. Letting $\bar{z} \equiv \begin{bmatrix} z \\ 1 \end{bmatrix}$, we obtain

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\mu}_y(t) \\ \dot{\mu}_x(t) \\ E_t\dot{y}(t) \end{bmatrix} = \bar{A} \begin{bmatrix} x(t) \\ \mu_y(t) \\ \mu_x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \bar{z}(t) \qquad \text{[3/]} \qquad \ldots\ldots(12a)$$

and from (9)

$$u(t) = C \begin{bmatrix} x(t) \\ \mu_y(t) \\ \mu_x(t) \\ y(t) \end{bmatrix} + D\,\bar{z}(t) \qquad \ldots\ldots(12b)$$

If $\bar{A}$ can be diagonalised and if it has n stable and n unstable characteristic roots, a unique solution exists to (11, 12) given by (see Buiter (1983)).

$$\begin{bmatrix} \mu_x(t) \\ y(t) \end{bmatrix} = W_{21} W_{11}^{-1} \begin{bmatrix} x(t) \\ \mu_y(t) \end{bmatrix} - V_{22}^{-1} \int_t^{\infty} e^{\Lambda_2(t-\tau)} \left[V_{21}\bar{B}_1 + V_{22}\bar{B}_2 \right] E_t\bar{z}(\tau)\,d\tau \qquad \ldots\ldots(13a)$$

$$\begin{bmatrix} x(t) \\ \mu_y(t) \end{bmatrix} = W_{11} e^{\Lambda_1(t-t_o)} W_{11}^{-1} \begin{bmatrix} x(t_o) \\ \mu_y(t_o) \end{bmatrix} + \int_{t_o}^{t} W_{11} e^{\Lambda_1(t-s)} W_{11}^{-1}\bar{B}_1\bar{z}(s)\,ds$$

$$- \int_{t_o}^{t} W_{11} e^{\Lambda_1(t-s)} \{\Lambda_1 V_{12} V_{22}^{-1} + W_{11}^{-1} W_{12} \Lambda_2\} \int_s^{\infty} e^{\Lambda_2(s-\tau)} \left[V_{21}\bar{B}_1 + V_{22}\bar{B}_2 \right] E_s\bar{z}(\tau)\,d\tau\,ds \qquad \ldots\ldots(13b)$$

where

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3. The matrices $\bar{B}_1$ and $\bar{B}_2$ are each n x (r+1)

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \bar{A} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \qquad \ldots\ldots (14)$$

Λ_1 is a diagonal matrix whose diagonal elements are the n stable characteristic roots of $\bar{A}$ and Λ_2 is a diagonal matrix whose diagonal elements are the n unstable characteristic roots of $\bar{A}$.

$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$ is a matrix of linearly independent left eigenvectors of $\bar{A}$

and $\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$ its inverse.

4. TIME-CONSISTENT RATIONAL EXPECTATIONS SOLUTIONS

It is obvious from the boundary condition (11d) and the equations of motion under optimal control (10) that in general if the controller re-optimises at $t=t_1>t_o$ his optimal plan from time $t_1>t_o$ onwards will not be the continuation for $t\geq t_1$ of the optimal plan derived at time t_o, even if no new information about the exogenous variables has accrued between t_o and t_1. The optimal plan is not in general time consistent. (See Kydland and Prescott (1977)). The reason is that while $\mu_y=0$ at $t=t_o$, it will, in general, be different from zero for $t>t_o$, given the dynamics of equation (10). Reoptimizing at $t=t_1>t_o$, the controller will, taking $x(t_1)$ as given, be tempted to adopt a plan for $t\geq t_1$ that will set $\mu_y(t_1) = 0$.

Unless, under the optimal plan adopted at $t=t_o$, the value of μ_y at $t=t_1$ would have been equal to zero anyway, the re-optimization at $t=t_1$ would falsify the expectations held between t_o and t_1 by the agents represented in the model of equation (1). It is these expectations, as shown in the last term on the right-hand side of (13b) that brought the system to $x(t_1)$.[4/]

If the agents in the model anticipate that the controller will reoptimize at t_1, taking as given their past expectations of his future actions, they will expect $\mu_y(t_1) = 0$. If the controller can reoptimize at each and every instant, they will anticipate $\mu_y(t) = 0 \ \forall t\geq t_o$. The characterization of the time-consistent rational expectations solution (or the 'subgame perfect' solution) is then straightforward.

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4. Note that the "followers" whose behaviour is given by (1) form expectations not only of future values of $\bar{z}$ but also of future values of u.

Proposition

The time-consistent rational expectations solution is characterised by zero values at each instant of the co-state variables corresponding to the non-predetermined state variables, i.e. by $\mu_y(t) \equiv 0$, $t \geq t_o$. The optimality condition $-\frac{\partial H}{\partial y} = \dot{\lambda}_y^T$ no longer applies as the controller overrides it by choosing $\mu_y(t) = \frac{\partial J(t)}{\partial y(t)} = 0$ for all t .

The equations of mention under time-consistent control are therefore obtained by omitting the rows corresponding to $\dot{\mu}_y(t)$ and the columns corresponding to $\mu_y(t)$ in (10). The behaviour of the system under time-consistent control is given by (11a, b, c) and

$$\begin{bmatrix} \dot{x}(t) \\ E_t\dot{y}(t) \\ \dot{\mu}_x(t) \end{bmatrix} = \begin{bmatrix} A_{11}-B_1\Omega_{uu}^{-1}\Omega_{xu}^T & A_{12}-B_1\Omega_{uu}^{-1}\Omega_{yu}^T & -B_1\Omega_{uu}^{-1}B_1^T \\ A_{21}-B_2\Omega_{uu}^{-1}\Omega_{xu}^T & A_{22}-B_2\Omega_{uu}^{-1}\Omega_{yu}^T & -B_2\Omega_{uu}^{-1}B_1^T \\ -[\Omega_{xx}-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{xu}^T] & -[\Omega_{xy}^T-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{yu}^T] & -[A_{11}^T-\Omega_{xu}\Omega_{uu}^{-1}B_1^T-\zeta I_{n_1}] \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ \mu_x(t) \end{bmatrix}$$

$$+\begin{bmatrix} F_1-B_1\Omega_{uu}^{-1}\Omega_{uz} \\ F_2-B_2\Omega_{uu}^{-1}\Omega_{uz} \\ -[\Omega_{xz}-\Omega_{xu}\Omega_{uu}^{-1}\Omega_{uz}] \end{bmatrix} z(t) + \begin{bmatrix} -B_1\Omega_{uu}^{-1}\omega_u \\ -B_2\Omega_{uu}^{-1}\omega_u \\ \Omega_{xu}\Omega_{uu}^{-1}\omega_u-\omega_x \end{bmatrix} \quad \ldots\ldots (15a)$$

$$u_t=-\Omega_{uu}^{-1}\Omega_{xu}^T x(t)-\Omega_{uu}^{-1}\Omega_{yu}^T y(t)-\Omega_{uu}^{-1}B_1^T\mu_x(t)-\Omega_{uu}^{-1}\Omega_{uz}z(t)-\Omega_{uu}^{-1}\omega_u \quad \ldots\ldots (15b)$$

Note that while the time-consistent solution is a product of the realisation by the agents in the model (the followers) that the controller (the leader) will cheat if he has an incentive to do so (if $\mu_y(t)\neq 0$), there is no cheating along the time-consistent path because the incentive to cheat has been eliminated.

Obviously, the optimal policy will be time-consistent i.f.f. under the optimal policy $\mu_y(t) \equiv 0 \ t \geq t_o$. If this is not the case, precommitment is necessary for the controller to implement the optimal solution.

The behaviour of the system under time-consistent control can be solved for using the methods outlined for the case of optimal control, provided the state matrix in (15a) is diagonalizable and has n_1 stable roots and n unstable roots.

CONCLUSION

The proposition that the time-consistent rational expectations solution is characterised by zero shadow prices of the non-predetermined state variables at each moment is not confined to the linear-quadratic example analysed here. It applies to non-linear continuous time models with general objective functionals. The analysis can also be extended in a straightforward manner to discrete time models using the discrete time maximum principle. The example in this note was chosen because of the existence of simple analytical and computational solution methods for both optimal and time-consistent policies (Austin and Buiter (1982)).

I would like to thank Marcus Miller for many extended discussions about the subject matter of this note. A comprehensive analysis of linear-quadratic optimal control and differential games in rational expectations models can be found in Miller and Salmon [1982, 1983]. My interest in characterising time-consistent solutions for continuous time dynamic optimization problems was first stimulated by Driffill [1982].

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