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METHODS OF SOLUTION AND SIMULATION FOR DYNAMIC RATIONAL EXPECTATIONS MODELS

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Abstract

Many methods have been proposed for the solution and simulation of medium or large size models under the assumption of rational expectations. The purpose of this paper is to present these methods, and to show how and where each can be applied.

The methods fall into two groups. Methods in the first can be used to solve for perfect foresight paths in non-linear models. Methods in the second can be used in linear models, to solve either for paths or processes followed by endogenous variables. All the methods described here have been used in empirical applications and computer algorithms are available for most.

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The introduction of the hypothesis of rational expectations has raised a host of new conceptual and technical issues. This paper is concerned with a subset of the technical issues, namely those associated with the solution and simulation of models under rational expectations. Its purpose is to introduce the principal methods of solution to potential builders and users of such models. The paper is in the nature of a survey but is not encyclopedic: some methods are excluded because their properties are not yet fully understood; some may not be included simply because of ignorance on my part. All the methods described have been used in empirical economic applications.

Solving a dynamic RE (rational expectations) model is usually more difficult than solving a standard dynamic model. A standard model has as many initial conditions as endogenous variables and can be solved recursively forward using the equations of motion. A typical RE model, however, lacks initial conditions for some of the endogenous variables. Additional conditions therefore have to be used; they are usually terminal or "transversality" conditions. Whether and where the use of such transversality conditions is justified and whether their use leads to the existence of a unique solution has been the subject of much research and debate. Although this debate is not the focus of the paper, it needs to be briefly presented so that the status of these additional conditions is understood. This is done in Section I. The rest of the paper takes as given that these transversality conditions can be applied.

Solving a dynamic RE model then becomes a problem of finding a solution which satisfies the equations of motion, initial conditions for some of the variables and transversality conditions for the others. The combination of uncertainty and non-linearity makes this problem difficult

if not impossible to solve in all but a few cases. Two approaches have therefore been followed. The first has been to remove uncertainty and consider perfect foresight paths; the problem is then a standard two point boundary value problem for which numerical methods have been developed. The second approach has been to remove non-linearity. Various methods have been developed; some apply to general linear models, others exploit the specific structure of some models, such as the equivalence of the model to the set of necessary conditions of an optimization problem. Section II describes these methods.

Section I. The Role of Transversality Conditions

In dynamic RE models, i.e. in models in which both current values and expectations of future values of endogenous variables appear, some endogenous variables do not have natural initial conditions. Thus, in the absence of other conditions or restrictions on these variables, these models admit an infinity of solutions. Finding "the" solution then requires the use of additional conditions; the additional conditions usually imposed are in the nature of transversality conditions.

The nature of the problem and the role of the transversality conditions can be made more precise by considering the following simple example:

$$(1) \qquad p_t = a\, E\,(p_{t+1}|\Omega_t) + bz_t \;;\quad a < 1 \;.$$

$E\,(\cdot|\Omega_t)$ denotes the expectation conditional on the information set Ω_t, which contains current and lagged values of z_t. Assume further that z_t does not explode[1], namely:

$$\lim_{i\to\infty} \beta^i\, E(z_{t+i}|\Omega_t) = 0 \quad \forall \beta < 1 \;,\; \forall t$$

This example is often used for discussions of issues in RE models and susceptible of many interpretations. For example, arbitrage between the return on a share and a constant interest rate r gives such a relation with p_t as the price of a share, z_t as the dividend and $a = (1+r)^{-1}$. Other interpretations are as the reduced form of Cagan's money model (Sargent-Wallace (1973) and many others) or the reduced form of Samuelson's overlapping generation model with money

A solution to (1) is:

$$p_t^* = b \sum_{i=0}^{\infty} a^i E(z_{t+i}|\Omega_t)$$

Under the above share price interpretation, this solution gives the price of a share as the expected present discounted value of dividends. By extension, this solution has sometimes been referred to as the "market fundamental" solution. It is not, however, the only one. The set of solutions is given by:

$$(2) \qquad p_t = p_t^* + \eta_t \quad ; \quad E(\eta_{t+1}|\Omega_t) = a^{-1}\eta_t$$

Thus a solution may differ from p_t^* by a term η_t which satisfies the second part of (2). Various authors have shown that, if η_t is not identically zero, p_t may then depend on extraneous variables, i.e. on variables not affecting p_t^* (Taylor (1977), Shiller (1978)), p_t may be subject to "bubbles", "crashes" (Flood and Garber (1980), Blanchard and Watson (1982)), or that p_t may depend directly on past values of z_t, beyond the effects of such past values on the expectations of future z. (Taylor (1977), Blanchard (1979)).

Note that, as $a^{-1} > 1$, if η_t is not equal to zero the expected value of η_{t+i} in (2) goes to infinity as i increases. Thus, adding the condition that p_t does not explode, for example that $\lim_{i\to\infty} \beta^i E(p_{t+i}|\Omega_t) = 0$ $\forall t$, $\forall \beta < 1$ is enough to leave only one solution, p_t^*. Imposing this transversality condition leads to the choice of a unique solution;[2] this raises two questions:

The first is whether imposing such a condition is justified. In some cases, in particular if the model is derived from an optimization problem, transversality conditions are part of the characterization of the solution. In other cases, the implications of the explosion

of an endogenous variable are inconsistent with some assumption of the model. (In the well known example of Hahn (1966), some price becomes negative in finite time if any solution other than the fundamental solution is chosen). In the remaining cases, however, the transversality condition has to be imposed by the economist himself. There seems to be general agreement among macroeconomists that, unless the focus is on bubble-type phenomena, imposition of a transversality condition is a reasonable way to proceed (some have expressed reservations; see Gourieroux, Laffont and Monfort (1982) for example). This still leaves open the exact specification of such a condition: various possibilities exist, such as a non explosion condition as above, stationarity of p if z is stationary, boundedness in mean of p... (See Gourieroux et al (1982)). In our case all three coincide to give p^* as the unique solution. A different criterion has been suggested by McCallum (1981); in effect, it requires that p depend only on variables which affect expectations of z and leads here also to the choice of p^*.

The second question is whether the use of a transversality condition always yields a unique solution. This is the case in the above example as long as $a < 1$: The effect of expected future p on current p has to be less than unity, a plausible restriction. In more general models, the conditions under which transversality conditions yield a unique solution are extensions of this simple condition. Heuristically, they require that the effects of the past and the expected future on the current variables not be too large. More precisely, they require the models to have a saddle point structure; the exact condition will be given in the next section. This condition is plausible and is satisfied in most existing theoretical and empirical models. Nevertheless, examples of theoretical and

estimated empirical models in which it fails have also been given. What should be done in such models is unclear.

In what follows, we whall assume that "non explosion" transversality conditions can be imposed. The use of a different type of condition might require minor changes in the results and methods presented below.

Section II. Methods of Solution

Dynamic models which combine uncertainty, non-linearity and two types of boundary conditions — initial conditions for some variables, transversality conditions for the others — are usually difficult, if not impossible, to solve even numerically. Only ingenious restrictions on functional forms and distributions of the exogenous variables allow the derivation of a solution. If such restrictions are not imposed, either uncertainty or non-linearity must be abandoned for the model to be solved. We first consider methods to solve non linear models under certainty, then methods to solve linear models under uncertainty.

A. Non Linear Models

Consider the following model:

$$(3) \qquad f(x_{t+1}, p_{t+1}, x_t, p_t, z_t) = 0 \quad , \quad t = 0,\ldots,\infty$$

x_t is nxl , p_t is mxl , z_t is kxl $\quad f: R^{2(n+m)+k} \longrightarrow R^{n+m}$

$$\lim_{t\to\infty} \beta^t z_t = 0 \quad \forall \beta < 1$$

$$x_0 = \bar{x}_0 \; ; \quad \lim_{t\to\infty} \beta^t p_t = 0 \quad \forall \beta < 1$$

z_t is a vector of exogenous variables, x_t and p_t are vectors of endogenous variables. x_t is predetermined at any time t and subject to initial conditions at t=0. p_t is not predetermined and thus subject instead to transversality conditions; as explained in the previous section, the specific form of this condition may differ from model to model. An example of such a model would be a model of money and growth, with x_t being capital, p_t being the price level and z_t being nominal money.

The solution is a sequence (x_t, p_t) $t=0,\ldots,\infty$ which satisfies (3), for a given sequence (z_t) $t=0,\ldots,\infty$. The implicit assumption is that agents know at time t=0 the complete sequence of future z or, equivalently, hold at time t=0 expectations of future z with subjective certainty. In this last case, if at time $\bar{t} > 0$, there is an unanticipated change in the remaining sequence $(z_t)_{t=\bar{t},\ldots,\infty}$, a new solution must be derived for $\bar{t},\ldots,\infty$. Computation of such "perfect foresight" paths has proved useful in understanding the dynamic effects of anticipated and unanticipated changes in policy or other exogenous variables (for example Sachs (1982), Summers (1981)); whether these paths are reliable guides to the case of uncertainty is, especially if non-linearities are essential, an open question.

A numerical method which is well adapted to solve systems with a saddle point structure and two types of boundary conditions is the method of "parallel" or "multiple" shooting (see Keller (1968), Section 2-4; a simple description and economic examples are given in Lipton, Poterba, Sachs and Summers (1982)). Two modifications must be made to the initial model (3) before the method can be used. The first is an additional assumption on z, namely that z converges over time to some value $\bar{z}$. The second is the replacement of the infinite time transversality condition by a finite time condition; a natural condition is that by time T where T is large, p_T has converged to the steady state value $\bar{p}$ associated with z. Thus, the method is used to solve the modified problem:

$$(3') \quad f(x_{t+1}, p_{t+1}, x_t, p_t, z_t) = 0 \quad , \quad t = 0,\ldots,T$$

$$z_T = \bar{z} \quad ; \quad \bar{x}, \bar{p} \mid f(\bar{x}, \bar{p}, \bar{x}, \bar{p}, \bar{z}) = 0$$

$$x_0 = \bar{x}_0 \quad ; \quad p_T = \bar{p}$$

A simple method, or method of "shooting" would be to choose an initial guess for p_0, solve forward — "shoot" — using the equations of motion in (3') to derive the implied value of p_T and then revise the initial guess according to the difference between p_T and $\bar{p}$. Formally, let $p_T = H(p_0)$ be the relation implied by (3') when solved forward from 0 to T, where the effects of $\bar{x}_0$ and the sequence of z are subsumed in the H function. Then starting with the initial guess p_0^1, we could use Newton's method and iterate the following system until approximate convergence:

$$(4) \qquad p_0^j - p_0^{j-1} = [H_p]^{-1}(\bar{p} - p_T^{j-1}) \; ; \; p_T^{j-1} = H(p_0^{j-1}).$$

The Jacobian H_p may be computed at p_0^1 or at each iteration at p_0^{j-1}. This method, however, does not work well in saddle point systems: small deviations of initial guesses lead to very large deviations of p_T from $\bar{p}$; H_p may have very large elements; in the example of the previous section, $H_p = a^{-T}$, $a < 1$. As a result, overflow problems usually prevent computation and iteration of (4).

The method of multiple shooting modifies the above method to treat such cases. It divides the period [0,T] into I intervals. It guesses not only p_0 but also intermediate values for x and p at the beginning of each interval i, i = 2,...,I. Using the equations of motion, it solves forward for each interval — "multiple shoots" — to derive intermediate values and terminal values for x and p. Differences between initial guesses and implied values of intermediate x and p, and between the implied terminal value of p and $\bar{p}$ are then used to revise initial guesses. Formally, denote values of x and p at the beginning of interval i by x_{i-1} and p_{i-1} and define the following vectors:

$$\hat{\psi} \equiv \begin{bmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \hat{x}_1 \\ \vdots \\ \hat{p}_{I-1} \\ \hat{x}_{I-1} \end{bmatrix} \quad \psi \equiv \begin{bmatrix} p_1 \\ x_1 \\ \vdots \\ p_{I-1} \\ x_{I-1} \\ p_T \end{bmatrix} \quad \overline{\psi} \equiv \begin{bmatrix} \hat{p}_1 \\ \hat{x}_1 \\ \vdots \\ \hat{p}_{I-1} \\ \hat{x}_{I-1} \\ \overline{p} \end{bmatrix} \quad M \equiv \begin{bmatrix} p_1 - \hat{p}_1 \\ x_1 - \hat{x}_1 \\ \vdots \\ p_{I-1} - \hat{p}_{I-1} \\ x_{I-1} - \hat{x}_{I-1} \\ p_T - p \end{bmatrix}$$

$\hat{\psi}$ is the vector of "guesses", ψ the vector implied by $\hat{\psi}$ and the equations of motion. M is the vector of "misses". ψ is a function of $\hat{\psi}$ and so is M. A solution $\hat{\psi}$ is such that $M(\hat{\psi}) = 0$. This suggests iterating the following system until approximate convergence:

$$\hat{\psi}^j = \hat{\psi}^{j-1} - [M_\psi]^{-1} M(\hat{\psi}^{j-1})$$

where M_ψ is the Jacobian of $M(\hat{\psi})$, evaluated either at $\hat{\psi}^1$ or at each evaluation at $\hat{\psi}^{j-1}$. As this method now solves (3') forward over shorter intervals, the problem of explosion is reduced; the number of intervals must be chosen so as to eliminate explosion and overflow problems. The method implies the inversion of a large Jacobian matrix (of rank $m + (I-1)(m+n)$) but this matrix has a band diagonal structure which facilitates inversion. An algorithm corresponding to this method has been written by Lipton et al (1982).

When a numerical method such as multiple shooting is used, the uniqueness of the solution or the saddle-point structure of the system cannot be checked. Two partial checks are, however, feasible and desirable.

The first is a check of local saddle point structure around steady state, by linearizing the system around its steady state values; the precise condition on the Jacobian is given in the next subsection. The second is a check of the sensitivity of p_0 to changes in either $\bar{p}$ or T. If the system has a saddle point structure, p_0 should be approximately insensitive to such changes.

B. General Linear Models: No Lagged Expectations

The approach above eliminates uncertainty and maintains non-linearity. The alternative is to either specify models as linear — or else to linearize non-linear models around their steady state — and to maintain uncertainty. Consider the following model:

$$(5) \qquad \begin{bmatrix} x_{t+1} \\ E(p_{t+1}|\Omega_t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} z_t \; ; \; t = 0,\ldots,\infty$$

x_t is nxl, p_t is mxl, z_t is kxl

x_t, p_t, z_t and lagged values ε Ω_t

$$\lim_{i\to\infty} \beta^i E(z_{t+i}|\Omega_t) = 0 \quad \forall \beta < 1 \; , \; \forall t$$

$$x_0 = \bar{x}_0 \; ; \; \lim_{i\to\infty} \beta^i E(p_{t+i}|\Omega_t) = 0 \quad \forall \beta < 1 \; , \; \forall t.$$

Definitions of z_t, p_t, x_t are as before. $E(\cdot|\Omega_t)$ denotes an expectation conditional on the information set Ω_t. The certainty transversality conditions for exogenous variables z_t and endogenous variables p_t are replaced by transversality conditions on their conditional expectation.

Models with either variables lagged more than once or expectations

of variables more than one period ahead are easily reduced to the above form by the introduction of auxiliary variables (Blanchard and Kahn (1980)). However, models with lagged expectations of past, current and future endogenous variables usually cannot be reduced to form (5). We shall consider them in the next subsection and show how the method used to solve (5) can be extended to such models.

The condition for existence of a unique solution to (5) has been derived by Blanchard and Kahn (1980): The number of eigenvalues of the matrix A outside the unit circle must be equal to m, the number of non-predetermined variables. If this condition holds, the solution may be characterized explicitly:

Let J be the diagonal matrix to which A is similar, with diagonal elements, which are the eigenvalues of A, ordered by increasing absolute value.[3] Let C be the matrix of eigenvectors and partition J and C conformably to A:

$$\begin{bmatrix} A_{11} & A_{12} \\ (n\times n) & (n\times m) \\ A_{21} & A_{22} \\ (m\times n) & (m\times m) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ (n\times n) & (n\times m) \\ C_{21} & C_{22} \\ (m\times n) & (m\times m) \end{bmatrix}^{-1} \begin{bmatrix} J_1 & 0 \\ (n\times n) & (n\times m) \\ 0 & J_2 \\ (m\times n) & (m\times m) \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ (n\times n) & (n\times m) \\ C_{21} & C_{22} \\ (m\times n) & (m\times m) \end{bmatrix}$$

Then:

$$(6) \qquad x_t = A_{11} x_{t-1} + A_{12} p_{t-1} + \gamma_1 z_{t-1}$$

$$(7) \qquad p_t = - C_{22}^{-1} C_{21} x_t - C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-(i+1)} (C_{21}\gamma_1 + C_{22}\gamma_2) E(z_{t+i} | \Omega_t)$$

p_t can be expressed as a function of the predetermined variables x_t and the sequence of expected future z.

Equations (6) and (7) may be used to compute the sequence (x_t,p_t) $t = 0,\ldots,\infty$ associated with a given sequence (z_t) $t = 0,\ldots,\infty$. They may alternatively be used to characterize the stochastic process of x_t and p_t for a given stochastic process z_t. Suppose for example that z_t is a subvector of $\bar{z}_t$, which follows a first order vector autoregressive process; $\bar{z}_t$ would typically include lagged values of z_t and current and lagged values of variables which help predict z_t. Assume, further, that $\bar{z}_t \in \Omega_t$. Thus:

$$(8) \qquad \underset{kx1}{z_t} = \underset{kxk\ kx(\ell-k)}{[\,I \quad 0\,]}\ \bar{z}_t \ ;\ \bar{z}_{t+1} = B\bar{z}_t + \varepsilon_{t+1} \ ;\ E(\varepsilon_{t+1}|\Omega_t) = 0 \qquad \underset{\ell x1}{}$$

This implies:

$$(9) \qquad E(z_{t+i}|\Omega_t) = [I\ 0]\ B^i\ \bar{z}_t$$

Replacing (9) in (7) gives p_t as a function of x_t and $\bar{z}_t$:

$$(10) \qquad p_t = -C_{22}^{-1}C_{21}\,x_t - C_{22}^{-1}\sum_{i=0}^{\infty} J_2^{-(i+1)}(C_{21}\gamma_1+C_{22}\gamma_2)[I\ 0]\ B^i\ \bar{z}_t$$

The problem of computing the second term in (10) is similar to the problem of computation of autocovariance matrices in linear systems (Chow (1975), Chapter 3). It is notationally tedious but straightforward. Assume that B is similar to a diagonal matrix Λ, so that $B = D^{-1}\Lambda\, D$. Define $F \equiv J_2^{-1}(C_{21}\gamma_1+C_{22}\gamma_2)[I\ 0]D^{-1}$. The sum on the right-hand side of (10) becomes:

$$\Big(\sum_{i=0}^{\infty} J_2^{-i} F\ \Lambda^i\Big)D\bar{z}_t$$

The only remaining problem is thus the evaluation of the sum in parentheses.

J_2 is a diagonal matrix, with typical element j_θ ; $\theta = 1,...m$. Λ is a diagonal matrix, with typical element λ_σ; $\sigma = 1,...,\ell$. F is an $(m \times \ell)$ matrix with typical element $f_{\theta\sigma}$. Then $(\sum_{i=0}^{\infty} J_2^{-i} F \Lambda^i)$ has typical element $f_{\theta\sigma}(1-j_\theta^{-1}\lambda_\sigma)^{-1}$. Collecting terms gives an expression for p_t as a function of x_t and $\bar{z}_t$. This, together with (6) and (8) characterizes the joint process of $(x_t, p_t, \bar{z}_t)$.

An alternative approach to the derivation of the process of x_t and p_t is to rewrite (5) treating $\bar{z}_t$ as state variables, so that (5) becomes:

$$(11) \quad \begin{bmatrix} \bar{z}_{t+1} \\ x_{t+1} \\ E(p_{t+1}|\Omega_t) \end{bmatrix} = \left[\begin{array}{c:cc} B & 0 & 0 \\ \hdashline \gamma_1 & A_{11} & A_{12} \\ \gamma_2 & A_{21} & A_{22} \end{array}\right] \begin{bmatrix} \bar{z}_t \\ x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ 0 \end{bmatrix}$$

Defining as before the eigenvalue matrix $\tilde{J}$ and the matrix of eigenvectors $\tilde{C}$ associated with the matrix in (11) and applying the same method as before, gives:

$$p_t = -\tilde{C}_{22}^{-1} \tilde{C}_{21} \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix}$$

This approach gives directly p_t as a function of the state variables, $\bar{z}_t$ and x_t. It, however, implies the computation of eigenvalues and eigenvectors of a much larger matrix and is likely to be more expensive than the approach described before.

An algorithm which computes (6) and (7) for systems of form (5) and a given sequence of z has been written by Buiter and Dunn (1982).

C General Linear Models: Lagged Expectations

As mentioned above, models which include lagged expectations of either past, current or future endogenous variables usually cannot be written in form (5). Such models must be solved in two steps. The method described above can be used for the first; the second step varies from model to model and is harder to describe in general. To avoid notational complexities, we shall show how this can be done in a special case and indicate how the method extends to the general model with lagged expectations. Consider:

$$(12) \qquad \begin{bmatrix} x_{t+1} \\ E(p_{t+1}|\Omega_{t-1}) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} z_t \; ; \; t = 0,\ldots,\infty$$

The model differs from (5) in that $E(p_{t+1}|\Omega_{t-1})$ replaces $E(p_{t+1}|\Omega_t)$. All definitions and other assumptions are the same as in (5).

The first step is to solve for $E(p_t|\Omega_{t-1})$. It uses the fact that (12) implies:

$$E\left(\begin{bmatrix} x_{t+1} \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} z_t \quad |\Omega_{t-1} \right)$$

The same method of proof as used to derive (6) and (7) in Blanchard and Kahn can be used to solve the system inside the expectation operator. Thus, the condition for existence of a unique solution is the same saddle point condition on A, and in a way analogous to (7):

$$(13) \qquad E(p_t|\Omega_{t-1}) = -C_{22}^{-1}C_{21}E(x_t|\Omega_{t-1}) - C_{22}^{-1}\sum_{i=0}^{\infty} J_2^{-(i+1)}(C_{21}\gamma_1+C_{22}\gamma_2)E(z_{t+i}|\Omega_{t-1})$$

$$= -C_{22}^{-1}C_{21}x_t - C_{22}^{-1}\sum_{i=0}^{\infty} J_2^{-(i+1)}(C_{21}\gamma_1+C_{22}\gamma_2)E(z_{t+i}|\Omega_{t-1})$$

The second step is to solve for p_t itself. In this example, $E(p_{t+1}|\Omega_{t-1})$, derived by leading (13) and taking expectations conditional on Ω_{t-1}, is replaced in (12). This gives p_t.

The general linear model may include expectations lagged more than once. The first step is the same as above: Let the furthest lagged expectation of an endogenous variable in the model be $E(\cdot|\Omega_{t-\theta})$. Then by taking expectations on both sides conditional on $\Omega_{t-\theta}$ the model implies:

$$E\left(\begin{bmatrix} x_{t+1} \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} z_t \quad |\Omega_{t-\theta} \right)$$

The first order form inside the expectation operator is not restrictive in this case: variables lagged and led more than once can be eliminated, using auxiliary variables. Applying again the same method allows the derivation of $E(p_t|\Omega_{t-\theta})$ as a function of $E(x_t|\Omega_{t-\theta})$ and the sequence $E(z_{t+i}|\Omega_{t-\theta})$, $i = 0,\ldots,\infty$.

The second step replaces the values of $E(p_t|\Omega_{t-\theta})$ (and $E(x_t|\Omega_{t-\theta})$) so obtained in the original model to solve for p_t. In models with complicated lagged expectation structures, this step may be tedious and appears difficult to describe in general. This suggests, as a modeling strategy, that the lagged expectation structure of a model should be as simple as possible. This is often feasible, as lagged expectations are likely to affect the current equilibrium through other variables which can be introduced explicitly and treated as endogenous variables.

D. Linear Models With Specific Lag Structures

In some models, the predetermined variables x_t are simply lagged values of p_t. These models can be written as:

$$(14) \quad E(A_{-q}p_{t-q}+\ldots+A_0p_t+\ldots+A_\ell p_{t+\ell}|\Omega_t) = Bz_t \; ; \; t = 0,\ldots,\infty$$

p_t is mxl, z_t is kxl ; A_i, $i=-q,\ldots,\ell$ is mxm, B is mxk.

$$\lim_{i\to\infty} \beta^i E(z_{t+i}|\Omega_t) = 0 \quad \forall\beta < 1 \; \forall t$$

Although such models can be rewritten in form (5) and solved by the method described in subsection B above, (Blanchard and Kahn (1980)), they can also be solved by factorization methods which use their specific structure and may therefore be cheaper.

The condition for existence of a unique solution for models of form (14) has been derived by Whiteman (1981), assuming A_ℓ invertible. Consider the roots of $|y^\ell(A_{-q}y^q+\ldots+A_\ell y^{-\ell})| = 0$ where y is a scalar variable. There exists a unique solution if ℓm of these roots are inside the unit circle. (If (14) is rewritten in the quasi first order form (5), this condition would be stated as a condition on the eigenvalues of A, the matrix of the first order system. The condition would be that ℓm of its eigenvalues be outside the unit circle).

If this condition is satisfied, the solution can be derived in two steps. (This follows closely Whiteman (1981) and Hansen and Sargent (1981)). The first step is the factorization of the matrix polynomial $(A_{-q}L^q+\ldots+A_\ell L^{-\ell})$ where L is a lag operator. Under the above condition, the polynomial can be factorized as:

$$(C_0+\ldots+C_qL^q)\;(D_0+\ldots+D_\ell L^{-\ell})$$

where all roots of $|C_0+\ldots+C_qy^q| = 0$ are on or outside the unit circle and all roots of $|y^\ell(D_0+\ldots+D_\ell y^{-\ell})|$ are inside the unit circle. This factorization is easy if p and therefore the A_i's, are scalars (see for example

Taylor (1980), or Chapter 9 in Sargent (1979)). It is more difficult if the A_i's are matrices. A special case which is easier to solve is $\ell = q$, $A_i = A_{-i}$, $i = 1,\ldots,\ell$; in this case the problem is the same as the problem of factorization of a spectral matrix for a vector moving average process; methods developed for spectral analysis can then be used (see Whittle (1963), Chapter 9). Hansen and Sargent (1981) have shown how these methods can be extended to the case $\ell = q$, $A_i = \Theta^i A_{-i}$, $i = 1,\ldots,\ell$.

With this factorization, (14) can be rewritten as:

$$(15) \quad (C_0+\ldots+C_qL^q)p_t = E((D_0+\ldots+D_\ell L^{-\ell})^{-1}Bz_t|\Omega_t)$$

The second step requires the expansion of the inverse of the matrix polynomial in partial matrix fractions:

$$(D_0+\ldots+D_\ell L^{-\ell})^{-1} = \sum_{j=1}^{\ell m} M_j(1-y_jL^{-1})^{-1}$$

where M_j's are mxm matrices of rank one, and y_j's are the roots of $|y^\ell(D_0+\ldots+D_\ell y^{-\ell})| = 0$. The method is an extension of the method of expansion of inverse scalar polynomials (see examples in Sargent (1979), Chapter 9) and is described in Hansen and Sargent (1981). Equation (15) can then be rewritten as:

$$(C_0+\ldots+C_qL^q)p_t = E(\sum_{j=1}^{\ell m} M_j(1-y_jL^{-1})^{-1}Bz_t|\Omega_t) \quad \text{or}$$

$$C_0p_t+\ldots+C_qp_{t-q} = \sum_{j=1}^{\ell m} M_j \sum_{i=0}^{\infty} y_j^i BE(z_{t+i}|\Omega_t)$$

The final step is the derivation of the path of p given a path of z, or of the process for p given a process for z. The methods described in (B) above can be used. Alternatively, Wiener-Kolmogorov prediction formulas

can be used (Hansen and Sargent (1980) describe their use in this context).

The method of factorization appears very convenient in the scalar case and in the case where the lag polynomial is symmetric or quasi-symmetric, less convenient otherwise. To my knowledge, no algorithm for solving models of form (14) with a given process for z is yet available.

E. Linear Models as Solutions to an Optimization Problem

The model given by (5) may be the set of optimality conditions in an optimization problem. This may be the case even if the model is derived without reference to optimization: there may exist an artificial optimization problem to which it corresponds (this is for example the case for the model in Taylor (1980) as shown by Quah (1981)). In such cases, the system of equations has a special structure not exploited by the method of subsection B. We first show how linear quadratic dynamic optimization problems lead to models in form (5) and then how they can be solved directly, by the method of Ricatti equations.

Consider the following optimization problem. At any time t , $t = 0,\ldots,\infty$, maximize the following function with respect to x and v:

$$(17)\quad E\Big(\frac{1}{2}\sum_{i=0}^{\infty}\beta^i\Big(\begin{bmatrix}x_{t+i}\\ z_{t+i}\end{bmatrix}'\begin{bmatrix}R_{11} & R_{12}\\ R_{12}' & R_{22}\end{bmatrix}\begin{bmatrix}x_{t+i}\\ z_{t+i}\end{bmatrix} + v_{t+i}'Q\,v_{t+i}\Big) \quad |\Omega_t\Big)\ ;\ \beta < 1$$

subject to:

$$(18)\quad \begin{bmatrix}x_{t+1}\\ z_{t+1}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}\begin{bmatrix}x_t\\ z_t\end{bmatrix} + \begin{bmatrix}B_1\\ 0\end{bmatrix}v_t + \begin{bmatrix}0\\ \varepsilon_{zt+1}\end{bmatrix}\ ;\ t = 0,\ldots,\infty$$

Q symmetric negative definite, R symmetric negative semidefinite

x_t is nxl, z_t is kxl, v_t is ℓxl, $\ell \leq n$

$x_0 = \bar{x}_0$; $z_0 = \bar{z}_0$; x_t, z_t, v_t and lagged values $\varepsilon\ \Omega_t$

ε_{zt} white noise, $E(\varepsilon_{zt+1}|\Omega_t) = 0$

This specification distinguishes between two types of state variables, x and z. z depends neither on x nor on the control variables v ; x depends on both z and v. (Specification of this optimization problem differs from author to author, in particular with respect to the time index of control variables in the transition equations, or the presence of controls in the objective function. See Chow (1975)).

Let p_t be the nxl vector of lagrange multipliers — costate variables — associated with the first n equations in (18). Deriving first order conditions and rearranging gives:

$$(19)\quad \begin{bmatrix} x_{t+1} \\ E(p_{t+1}|\Omega_t) \end{bmatrix} = \begin{bmatrix} A_{11} & B_1 Q^{-1} B_1' \\ A_{11}'^{-1} R_{11} A_{11} & A_{11}'^{-1}(\beta^{-1} I + R_{11} B_1 Q^{-1} B_1') \end{bmatrix} \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} A_{12} \\ A_{11}'^{-1}(R_{11}A_{12} + R_{12}A_{22}) \end{bmatrix} z_t$$

$$\lim_{i\to\infty} \beta^i\, E(p_{t+i}|\Omega_t) = 0 \quad , \quad \forall t = 0,\ldots,\infty \ .$$

The system in (p_t, x_t) is in the form of (5) and can therefore be solved by the method described in (B).[4] Note that the transversality condition is slightly different, as β is now the same as in the objective function (17). Once p_t is obtained, v_t follows from:

$$(20)\quad v_t = Q^{-1} B_1'\, p_t$$

This is the approach followed by Vaughan (1970) (Vaughan does not distinguish between the state variables x and z and works with the canonical system of state variables x, z and associated costate variables — lagrange multipliers —).

The alternative method of solution which is likely to be cheaper is the method of Ricatti equations (see for example Chow [1975]), exploiting the difference between the two types of state variables x_t and z_t (as suggested in Hansen and Sargent (1981)). The solution to the above optimization problem may also be stated as:

$$(21) \quad v_t = - [F_1 \quad F_2] \begin{bmatrix} x_t \\ z_t \end{bmatrix}$$

$$(22) \quad F_1 = \beta(Q+\beta B_1' P_{11} B_1)^{-1} B_1' P_{11} A_{11}$$

$$(23) \quad F_2 = \beta(Q+\beta B_1' P_{11} B_1)^{-1} (B_1' P_{11} A_{12} + B_1' P_{12} A_{22})$$

where P_{11} and P_{12} are themselves given by:

$$(24) \quad P_{11} = \beta A_{11}' P_{11} A_{11} + R_{11} - \beta^2 A_{11}' P_{11} B_1 (Q+\beta B_1' P_{11} B_1)^{-1} B_1' P_{11} A_{11}$$

$$(25) \quad P_{12} = \beta A_{11}' (P_{11} A_{12} + P_{12} A_{22}) + R_{12} - \beta^2 A_{11}' P_{11} B_1 (Q+\beta B_1' P_{11} B_1)^{-1} B_1' (P_{11} A_{12} + P_{12} A_{22})$$

Note that (24) and (25) give P_{11} and P_{12} only in implicit form. P_{11} may, however, be found by iterating the following matrix equation, starting with $P_{11} = 0$.

$$P_{11}^k = \beta A_{11}' P_{11}^{k-1} A_{11} + R_{11} - \beta^2 A_{11}' P_{11}^{k-1} B_1 (Q+\beta B_1' P_{11}^{k-1} B_1)^{-1} B_1' P_{11}^{k-1} A_{11}$$

When approximate convergence is achieved, P_{11} and F_1 can be computed.

(Conditions for iteration to converge are stated in Hansen and Sargent (1981) and references therein). The solution P_{11} can then be used in (25) and the same iterative method can be used to derive P_{12}. F_2 can then be computed. This method requires only matrix addition, multiplication and inversion of relatively small matrices as opposed to the computation of eigenvalues or roots in the methods presented in subsections B-D. It has a long history of use in control and economics and many algorithms are available.

Conclusion

This paper has described various methods of solution for dynamic RE models. The descriptions are necessarily sketchy and the potential user is advised to go to the original sources to learn about the various conditions, caveats and special cases associated with each method.

The issue does not seem anymore to be how to solve dynamic RE models but at what cost. Issues of cost are important as, for example, estimation of dynamic RE models typically implies solving models many times for different values of the parameters. Some comparisons have been made (Hansen and Sargent (1981)) but much remains to be done.

Footnotes

1. What is needed is in fact the weaker assumption that z_t does not explode too fast, namely that:

$$\lim_{i \to \infty} a^i E(z_{t+i} | \Omega_t) = 0 \quad \forall t$$

This assumption is needed for p_t^* below to be finite.

2. If the assumption of footnote 1 is used for z, then the required transversality condition is instead:

$$\lim_{i \to \infty} a^i E(p_{t+i} | \Omega_t) = 0 \quad \forall t$$

3. More generally, J is the Jordan matrix to which A is similar.

4. If the transition equations for x in (18) included a disturbance term ε_{xt+1}, this term would appear in the transition equations for x in (19). As $E(\varepsilon_{xt+1} | \Omega_t) = 0$, the solution for p_t would be the same as in the absence of such a disturbance term and still be given by (7).

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