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IDENTIFICATION IN DYNAMIC LINEAR MODELS WITH RATIONAL EXPECTATIONS

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ABSTRACT

This paper characterizes identification in dynamic linear models. It shows that identification restrictions are linear in the structural parameters and are therefore easy to use. Using these restrictions, it analyzes the role of exogenous variables in helping to achieve identification.

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Introduction

This paper derives a set of identification conditions for a class of models with rational expectations.

The model analyzed in the paper is a first-order linear model. The analysis applies therefore to the class of models that can be reduced to this form; as examples will show, this class is sufficiently large to be of interest. The analysis can be applied to ad hoc models, i.e., models not explicitly derived from optimization or to models derived explicitly from a linear quadratic optimization problem ([5], [7] for recent examples). In this last case however, the first-order linear model is simply the canonical difference system associated with the optimization problem [9] and has additional structure. This structure allows an alternative and sometimes more direct derivation of identification conditions than the one given in this paper. This derivation has been given by Chow [4].

The set of identification conditions derived in the paper is as follows. Define the first-order form of the system, in which expectations appear, as the "structural form." It is characterized by an information structure and a pair (A, Σ) where A is a matrix of parameters and Σ a variance covariance matrix of disturbances. This system can be solved to give an observable "reduced form," characterized by a pair (Π, Θ) where Π is a matrix of parameters, Θ a covariance matrix and where Π and Θ are identified. The identification problem is the derivation of the set of restrictions on (A, Σ) imposed by (Π, Θ). As in the standard simultaneous equation model, in the absence of prior restrictions on Σ, the set of restrictions on A depends on Π and not on Θ; it is this set of restrictions that is derived in the paper. As in the standard model, prior restrictions on Σ, if they yield additional restrictions,[1] yield restrictions that are often intractable.

The answer to this restricted identification problem turns out to be simple: although the mapping from A to Π is highly nonlinear, knowledge of Π imposes linear restrictions on the elements of A. Checking identification of one or a set of parameters is therefore usually feasible and straightforward.

The paper is organized as follows. Section I introduces the framework by using a simple example. Section II presents the general model. Sections III and IV characterize identification in various cases.

Section I. An Overview

It is easier to introduce the terminology, the framework and the issues involved by using a simple example:

The model

$$(1.1)\qquad \begin{bmatrix} z_t \\ E(q_{t+1}|\Omega_t) \end{bmatrix} = \begin{bmatrix} A(\alpha) \end{bmatrix} \begin{bmatrix} z_{t-1} \\ q_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{zt} \\ \varepsilon_{qt} \end{bmatrix} \; ; \; A(\alpha) \equiv \begin{bmatrix} a_{11}(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha) \end{bmatrix}$$

ε_{zt} and ε_{qt} are scalar white noise processes. Their covariance structure satisfies either A1a or A1b:

(A1a): $E(\varepsilon_{zt}\varepsilon_{qs}) = \sigma_{zq}$ if $t = s$, 0 otherwise

(A1b): $E(\varepsilon_{zt}\varepsilon_{qs}) = 0$ for all t, s

The information set Ω_t satisfies either A2a or A2b:

(A2a): $\Omega_t = \{\varepsilon_{zt}, \varepsilon_{zt-1}, \cdots ; \varepsilon_{qt}, \varepsilon_{qt-1}, \cdots\}$

(A2b): $\Omega_t = \{\quad \varepsilon_{zt-1}, \cdots ; \varepsilon_{qt}, \varepsilon_{qt-1}, \cdots\}$

Equation (1.1) gives the structural form of the model. Even if the system is not derived from a control problem, it is convenient to use the control terminology. Thus we refer to z_t as the state variable, to q_t as the costate variable and to the first equation in (1.1), which does not involve expectations, as the transition equation for z_t.[2]

We refer to the elements of A as the structural parameters. The set of parameters of interest may however not be A but a set (α) of "deep" parameters (this expression is due to Sargent [7]). We do not specify the mapping from (α) to A; as a result we can only characterize identification of A although the ultimate goal is identification of (α).

To show the importance of covariance and information assumptions for identification, we allow for alternative assumptions. The cross correlation between disturbances may or may not be zero. The realization of ε_{zt} may or may not be part of the information at time t.

Many examples in the economic literature--or at least linear approximations to them--fit this model. It could be a model of money and growth, with z_t being the capital stock and q_t being the price level. It could be a model of an asset market, with z_t being the stock and q_t being the price of this asset; the first equation would be an accumulation equation, the second an arbitrage relation.

Suppose we have observations on q_t and z_t. What are the restrictions imposed on A and ultimately on (α)? The first step is to solve (1.1) to obtain an observable reduced form, with identified parameters.

<u>The solution</u>

The solution to (1.1) gives the costate variable q_t as a function of variables in the information set at time t. It is well known ([1] for example and references therein) that the solution to (1.1) is not unique. If however, in this case, A has eigenvalues with absolute value on each side of unity, there is a unique stationary solution for q_t and z_t. We shall assume that the eigenvalue condition is satisfied; it would necessarily be satisfied if (1.1) followed from an optimization problem [9]. We choose the stationary solution and refer to it as "the" solution.[3]

In this simple case, the method of undetermined coefficients can be used and gives:

$$(1.2)\quad \begin{bmatrix} 1 & -a_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_t \\ q_t \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ \pi_1 & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ q_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{zt} \\ \pi_2 \varepsilon_{qt} + \pi_3 E(\varepsilon_{zt} | \Omega_t) \end{bmatrix}$$

where

$$(1.3)\quad \begin{aligned} &\pi_1 \text{ such that } a_{12}\pi_1^2 + (a_{11}-a_{22})\pi_1 - a_{21} = 0\ ;\ |\pi_1| < 1 \\ &\pi_2 = (\pi_1 a_{12} - a_{22})^{-1} \\ &\pi_3 = -(\pi_1 a_{12} - a_{22})^{-1}\pi_1 \end{aligned}$$

Let us define the reduced form as the form in which matrix parameters--in this case a_{11}, a_{12}, π_1--and the covariance matrix are identified. Inspection of (1.2) shows it to be a reduced form only if the covariance matrix is diagonal, as in this case (1.2) is a recursive system. This, in turn, depends on the assumptions about the structural covariance matrix and the information set.

<u>The recursive case</u>

Suppose that ε_{zt} and ε_{qt} are uncorrelated and that ε_{zt} is not in Ω_t, so that A1b and A2b hold. In this case, the system (1.2) is recursive and therefore is the reduced form. π_1 and the parameters of the transition equation, a_{11} and a_{12}, are identified. Equation (1.3) imposes, given $\pi_1 \neq 0$, one additional <u>linear</u> restriction on the elements of A. There are no further restrictions on A from the reduced form covariance matrix. Thus, in the absence of prior restrictions, A is underidentified but, depending on the mapping from (α) to A, A and (α) may be identified.

The simultaneous case

If either assumption A1b or assumption A2b does not hold, the covariance matrix in (1.2) is not diagonal, (1.2) is not recursive and thus is not the reduced form. Suppose for example that the structural covariance matrix is unrestricted and that ε_{zt} is known at time t, so that A1a and A2a hold. In this case, $E(\varepsilon_{zt}|\Omega_t) = \varepsilon_{zt}$. The reduced form is obtained by eliminating q_t from the first equation to get:

$$(1.4) \qquad \begin{bmatrix} z_t \\ q_t \end{bmatrix} = \begin{bmatrix} \phi_1 & 0 \\ \pi_1 & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ q_{t-1} \end{bmatrix} + \begin{bmatrix} a_{12}\pi_2\varepsilon_{qt} + (a_{12}\pi_3 + 1)\varepsilon_{zt} \\ \pi_2\varepsilon_{qt} + \pi_3\varepsilon_{zt} \end{bmatrix}$$

$$(1.5) \qquad \phi_1 \equiv a_{11} + a_{12}\pi_1$$

In this case, ϕ_1 and π_1 are identified, imposing if $\pi_1 \neq 0$ the two linear restrictions on A given by (1.3) and (1.5). There are no further restrictions from the reduced form covariance matrix. Identification of A and (α) depends again on the specific mapping of (α) to A.

This leaves us with two pairs of assumptions to consider. If ε_{zt} is correlated with ε_{qt} but its realization is unknown, so that A1a and A2b hold, identification restrictions on A are the same as above, namely (1.3) and (1.5). There are no further restrictions from the reduced form covariance matrix.[4] If however ε_{zt} and ε_{qt} are uncorrelated but are both known at time t, so that A1b and A2a hold, there is in addition to (1.3) and (1.5) a restriction on A from the reduced form covariance matrix. Let this matrix be:

$$\Theta \equiv \begin{bmatrix} \theta_z^2 & \theta_{qz} \\ \theta_{qz} & \theta_q^2 \end{bmatrix}$$

Then the additional restriction is:

$$(\theta_{qz} - a_{12}\theta_z^2)/(\theta_q^2 - a_{12}\theta_{qz}) = (1 - a_{12}\pi_3)\pi_3 \; ; \; \pi_3 = -(\pi_1 a_{12} - a_{22})^{-1}\pi_1$$

It is highly nonlinear. Thus, restrictions on the covariance of disturbances whose realizations are known at time t lead to restrictions on A from Θ; such restrictions are hard to use and this case will not be analyzed further.

Unobservable costate variables

We have assumed until now that both z and q are observable. It is sometimes the case however that not all costate variables are observable. This arises in three circumstances. First, the time series on some costate variables may simply be unavailable; there is little that can be said in general in this case ([2] gives an example of a market in which even in the absence of a time series for prices the "deep" parameters are identified up to a scalar). The second case arises in the control problem where costate variables are unobservable; in this case however control variables, which are linearly related to them but possibly less in number, are observable. As mentioned in the introduction, there is a more direct approach to identification in control problems and we shall not treat this case explicitly. The third case, which we shall study, arises when higher order systems are reduced to the required first-order form.

The rest of the paper is as follows. Section II gives the structural form and solution of the general first-order model. Section III analyzes identification in the recursive and the simultaneous cases. Section IV analyzes identification when some costate variables are unobservable.

Section II. The Model and Its Solution

The structural model is:

$$
(2.1)\quad \begin{bmatrix} \Psi_t \\ (k\times 1) \\ X_t \\ (n\times 1) \\ EP_{t+1} \\ (m\times 1) \end{bmatrix} = \begin{bmatrix} A(\alpha) \end{bmatrix} \begin{bmatrix} \Psi_{t-1} \\ (k\times 1) \\ X_{t-1} \\ (n\times 1) \\ P_t \\ (m\times 1) \end{bmatrix} + \begin{bmatrix} \varepsilon_{\psi t} \\ (k\times 1) \\ \varepsilon_{xt} \\ (n\times 1) \\ \varepsilon_{pt} \\ (m\times 1) \end{bmatrix}
$$

$$
A(\alpha) \equiv \left[\begin{array}{c|c} \underset{(k+n)\times(k+n)}{A_{11}(\alpha)} & \underset{(k+n)\times m}{A_{12}(\alpha)} \\ \hline \underset{m\times(k+n)}{A_{21}(\alpha)} & \underset{m\times m}{A_{22}(\alpha)} \end{array}\right] \equiv \left[\begin{array}{cc|c} \underset{(k\times k)}{A_{11}^{\psi}(\alpha)} & \underset{(k\times n)}{0} & \underset{(k\times m)}{0} \\ \underset{(n\times k)}{A_{11}^{x\psi}(\alpha)} & \underset{(n\times n)}{A_{11}^{x}(\alpha)} & \underset{(n\times m)}{A_{12}^{x}(\alpha)} \\ \hline \underset{(m\times k)}{A_{21}^{\psi}(\alpha)} & \underset{(m\times n)}{A_{21}^{x}(\alpha)} & \underset{(m\times m)}{A_{22}(\alpha)} \end{array}\right]
$$

X_{t-1}, Ψ_{t-1} predetermined with respect to $\varepsilon_{\psi t}$, ε_{xt}, ε_{pt}. $\varepsilon_{\Psi t}$, ε_{xt}, ε_{pt} are vector white noise processes. Their covariance matrix satisfies either A3a or A3b:

$$
\text{(A3a):}\quad E\left(\begin{bmatrix} \varepsilon_{\psi t} \\ \varepsilon_{xt} \\ \varepsilon_{pt} \end{bmatrix} \begin{bmatrix} \varepsilon_{\psi s} \\ \varepsilon_{xs} \\ \varepsilon_{ps} \end{bmatrix}' \right) = \Sigma \text{ if } s = t,\ 0 \text{ otherwise}
$$

A(3b): $$E\left(\begin{bmatrix}\varepsilon_{\psi t}\\ \varepsilon_{xt}\\ \varepsilon_{pt}\end{bmatrix}\begin{bmatrix}\varepsilon_{\psi s}\\ \varepsilon_{xs}\\ \varepsilon_{ps}\end{bmatrix}'\right) = \begin{bmatrix}\Sigma_{\psi x} & 0\\ 0 & \Sigma_p\end{bmatrix} \text{ if } s = t, \text{ 0 otherwise}$$

In (2.1), $EP_{t+1} \equiv E(P_{t+1}|\Omega_t)$. The information set Ω_t satisfies either A4a or A4b:

(A4a): $$\Omega_t = \{\varepsilon_{\psi t}, \varepsilon_{\psi t-1}, \cdots; \varepsilon_{xt}, \varepsilon_{xt-1}, \cdots; \varepsilon_{pt}, \varepsilon_{pt-1}, \cdots\}$$

(A4b): $$\Omega_t = \{\quad \varepsilon_{\psi t-1}, \cdots; \quad \varepsilon_{xt-1}, \cdots; \varepsilon_{pt}, \varepsilon_{pt-1}, \cdots\}$$

This model is a straightforward extension of the example of the previous section. There are (k+n) transition equations in (k+n) state variables. It is now useful to distinguish between state exogenous variables, Ψ_t, and state endogenous variables, X_t. Corresponding blocks of zeros in the matrix A are associated with the state exogenous variables. There are m costate variables. A is the matrix of structural parameters, depending on a set (α) of deep parameters. Assumptions on covariance and information are extensions of those of Section I.

The solution

The solution to this model has been derived in [1] to which the reader is referred for proof and details.[5] Let Π be the eigenvector matrix and J be the eigenvalue--or Jordan if need be--matrix associated with A. Order J by increasing absolute value of the eigenvalues and partition Π and J conformably to the partition of A so that:

$$(2.2)\qquad \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$$

A stationary solution for P_t exists and is unique if and only if the diagonal elements of J_2 are outside the unit circle. As the focus of this paper is neither on existence nor on uniqueness, this condition will be assumed to hold and we shall choose this solution, referring to it as "the" solution.

The solution gives (Ψ, X, P) as linear functions of variables in Ω. The equations for Ψ_t, X_t are simply the transition equations for Ψ_t and X_t, i.e., the first (k+n) lines of (2.1). The equations for P_t are given by:

$$(2.3)\qquad P_t = -\Pi_{22}^{-1}\Pi_{21}\begin{bmatrix} \Psi_{t-1} \\ X_{t-1} \end{bmatrix} - \Pi_{22}^{-1}J_2^{-1}\left(\Pi_{21}E\left(\begin{bmatrix} \varepsilon_{\psi t} \\ \varepsilon_{xt} \end{bmatrix} | \Omega_t\right) + \Pi_{22}\varepsilon_{pt}\right)$$

This characterization of the solution is the most convenient to study identification. As it does not however use any information on the structure of A and does not exploit the difference between Ψ_t and X_t, it is not the most efficient way of computing the solution. The particular structure of A allows in this case recursive computation first of the coefficients on X_{t-1} (feedback), then of the coefficients on Ψ_{t-1} (feedforward) (see [1] and [5]). Furthermore, if A derives from a control problem an alternative computation of (2.3) is the use of Ricatti equations (the solution above is simply an algebraic solution to a Ricatti equation in this case).

Whether the matrix parameters and the covariance matrix of the system composed of the transition equations and equation (2.3) are identified, i.e., whether the system is the reduced form of (2.1), depends again on whether this system is recursive. This in turn depends on the specific assumptions about Σ and Ω. We now turn to identification under the different sets of assumptions.

Section III. Observable State and Costate Variables

The recursive case

Suppose that $\varepsilon_{\psi t}$ and ε_{xt} are uncorrelated with ε_{pt} and that their realizations are not known at time t, so that A3b and A4b hold. In this case, the solution derived above is the reduced form. Repeating it for convenience:

$$
\begin{aligned}
\Psi_t &= A^{x}_{11}\Psi_{t-1} && + \varepsilon_{\psi t} \\
X_t &= A^{x\psi}_{11}\Psi_{t-1} + A^{x}_{11}X_{t-1} + A^{x}_{12}P_t && + \varepsilon_{xt} \qquad (3.1) \\
P_t &= (-\Pi^{-1}_{22}\Pi_{21})\begin{bmatrix}\Psi_{t-1} \\ X_{t-1}\end{bmatrix} && - \Pi^{-1}_{22}J^{-1}_{2}\Pi_{22}\varepsilon_{pt}
\end{aligned}
$$

The system being recursive, the matrices in the transition equations are identified. What additional restrictions are imposed on A by $(-\Pi^{-1}_{22}\Pi_{21})$ which is also identified? (There are no further restrictions on A from the reduced form covariance matrix.)

Expand the last m rows of (2.2). This gives:

$$\Pi_{21}A_{11} + \Pi_{22}A_{21} = J_2\Pi_{21}$$

$$\Pi_{21}A_{12} + \Pi_{22}A_{22} = J_2\Pi_{22}$$

Premultiplying both equations by Π^{-1}_{22} gives:

$$\Pi^{-1}_{22}\Pi_{21}A_{11} + A_{21} = (\Pi^{-1}_{22}J_2\Pi_{22})\Pi^{-1}_{22}\Pi_{21}$$

$$\Pi^{-1}_{22}\Pi_{21}A_{12} + A_{22} = (\Pi^{-1}_{22}J_2\Pi_{22})$$

This implies:

$$\Pi_{22}^{-1}\Pi_{21}A_{11} + A_{21} = (\Pi_{22}^{-1}\Pi_{21})A_{12}(\Pi_{22}^{-1}\Pi_{21}) + A_{22}(\Pi_{22}^{-1}\Pi_{21}) \tag{3.2}$$

Partition $(\Pi_{22}^{-1}\Pi_{21})$ so that:

$$\Pi_{22}^{-1}\Pi_{21} = [\underset{(m\times k)}{\Phi_\psi} \quad \underset{(m\times n)}{\Phi_x}]$$

Equation (3.2) can be decomposed in two sets of identification restrictions:

$$\Phi_x A_{11}^x \qquad + A_{21}^x = \Phi_x A_{12}^x \Phi_x + A_{22}\Phi_x \tag{3.3}$$

$$\Phi_\psi A_{11}^\psi + \Phi_x A_{11}^{\psi x} + A_{21}^\psi = \Phi_x A_{12}^x \Phi_\psi + A_{22}\Phi_\psi \tag{3.4}$$

The first set (3.3) imposes <u>linear</u> restrictions across (A_{12}^x, A_{11}^x, A_{21}^x, A_{22}). The restrictions follow from knowledge of Φ_x only and therefore are the only restrictions imposed on A if there are no exogenous variables. They are (m×n) in number. Given that A_{11}^x, A_{12}^x are identified, there are m×(n+m) elements of A_{21}^x, A_{22} to be identified in (3.3).

The second set (3.4) also imposes <u>linear</u> restrictions on (A_{11}^ψ, $A_{11}^{\psi x}$, A_{12}^x, A_{21}^ψ, A_{22}). These restrictions are present only if there are exogenous state variables Ψ and are (m×k) in number. Given that A_{11}^ψ, $A_{11}^{\psi x}$, A_{12}^x are identified, there are m×(m+k) elements of A_{21}^ψ, A_{22} to be identified in (3.4).

Simply counting the total number of restrictions on A gives (m×m) less restrictions than nonidentically zero elements of A. This number is independent of the number of state variables and thus, in this sense, exogenous variables do not help identification.

Exogenous variables are however often associated with prior restrictions on A. A frequent case is the following: Suppose that the transition equations for Ψ_t are the quasi-first-order form of a $\tilde{k}$th order process of p variables $\tilde{\Psi}_t$, with $\tilde{k}p = k$. This by itself imposes restrictions on A_{11}^{ψ} but not on A_{21}^{ψ}. Suppose furthermore that only current values of $\tilde{\Psi}_{t-1}$ affect X_t and P_t. In this case, A_{21}^{ψ} is subject to the following additional prior restrictions:

$$(3.5) \qquad A_{21}^{\psi} = [\underset{(m\times\tilde{k})}{A_{21}^{\tilde{\psi}}} \;\vdots\; \underset{(m\times(k-\tilde{k}))}{0}]$$

With these additional prior restrictions on A_{21}^{ψ}, equation (3.4) imposes $(m\times k)$ restrictions while adding only $(m\times\tilde{k})$ nonzero elements of A. The total number of restrictions minus the number of nonzero elements of A is equal to $m\times(\tilde{k}(p-1)-m)$, which for $\tilde{k}$ or p large enough is positive. Another often encountered case is that of a subset of exogenous variables which enter the model only because they help predict future values of the exogenous variables affecting X_t and P_t; the analysis of this case parallels the one above. This makes precise the sense in which a high order process for--or a large number of--exogenous variables help identification [7].

Equations (3.3) and (3.4), together with the matrices identified in the transition equations, characterize all the restrictions on A and thus on (α). Local identifiability if the system of restrictions is nonlinear in (α), and global identifiability if the system is linear in (α), can be checked using standard rank conditions on the Jacobian (see Rothenberg [6, Section V]. Note that, in order to check identification of a specific (α^0), the reduced form parameters must be derived as a function of (α^0) before the rank condition can be checked. If however the purpose is only to check necessary conditions for identification of (α^0), the derivation of the reduced form parameters from (α^0) is not required; in this case, the system of restrictions (3.3) and (3.4) on the elements of A can simply be assumed to be of full rank.

The simultaneous case

If either assumption A3b or A4b does not hold, conditional expectations of ε_{xt} and $\varepsilon_{\psi t}$ are not identically zero. The system composed of transition equations and equation (2.3) is thus not the reduced form, as the costate variables P_t are in general correlated with the disturbances ε_{xt} in the transition equations for the state variables X_t. The reduced form is then obtained by eliminating P_t from the transition equations for X_t.

Suppose that the structural covariance matrix is unrestricted and that ε_{xt} and $\varepsilon_{\psi t}$ are known at time t, so that assumptions A3a and A4a hold. The reduced form is then:

$$
\begin{aligned}
\Psi_t &= A_{11}^{\psi}\Psi_{t-1} + \varepsilon_{\psi t} \\
X_t &= \Gamma_{\psi}\Psi_{t-1} + \Gamma_x X_{t-1} + \left(\varepsilon_{xt} - A_{12}^{x}\Pi_{22}^{-1}J_2^{-1}\left(\Pi_{21}\begin{bmatrix}\varepsilon_{\psi t}\\ \varepsilon_{xt}\end{bmatrix} + \Pi_{22}\varepsilon_{pt}\right)\right) \\
P_t &= \Phi_{\psi}\Psi_{t-1} - \Phi_x X_{t-1} + \left(- \Pi_{22}^{-1}J_2^{-1}\left(\Pi_{21}\begin{bmatrix}\varepsilon_{\psi t}\\ \varepsilon_{xt}\end{bmatrix} + \Pi_{22}\varepsilon_{pt}\right)\right)
\end{aligned}
\tag{3.6}
$$

(3.7) $\Gamma_\psi \equiv A_{11}^{x\psi} - A_{12}^{x}\Phi_\psi$

(3.8) $\Gamma_x \equiv A_{11}^{x} - A_{12}^{x}\Phi_x$

A_{11}^{ψ}, Γ_ψ, Γ_x, Φ_ψ, Φ_x are identified. Thus (3.3), (3.4), (3.7), (3.8) impose $(m+n)\times(k+n)+(k\times k)$ restrictions on the $(m+n)\times(k+n+m)+(k\times k)$ nonidentically zero elements of A. There are therefore $(m+n)\times m$ less restrictions than nonzero elements; this number is independent of the number of exogenous variables and is increasing with the number of endogenous state and costate variables. Prior restrictions associated with exogenous variables may again yield identification. There are no further restrictions on A from the reduced form covariance matrix.

If ε_{xt} and $\varepsilon_{\psi t}$ are not known at time t, but there are no restrictions on the structural covariance matrix, i.e., if assumptions A3a and A4b hold, the restrictions on A are exactly the same as above. If however A3b and A4a hold, so that there are restrictions on the structural covariance matrix and ε_{xt}, $\varepsilon_{\psi t}$ are known at time t, there is in addition to the above restrictions, a set of restrictions on A from the reduced form covariance matrix; these restrictions have no simple form and such prior restrictions on the covariance structure of disturbances whose realizations are in Ω_t at time t are not considered further.

We now illustrate these results with an example similar to a model by Taylor [8]:

<u>Example</u>. The scalar random variable χ_t satisfies:

$$E(\chi_{t+1}|\Omega_t) + a\chi_t + b\chi_{t-1} + c\psi_t + \varepsilon_t = 0 \quad ; b \neq 0$$

$$\psi_t = a_1\psi_{t-1} + \cdots + a_k\psi_{t-k} + \eta_t$$

$$\Omega_t = \{\eta_t, \eta_{t-1}, \cdots; \varepsilon_t, \varepsilon_{t-1}\cdots\}$$

ε_t and η_t are white noise processes; their covariance structure satisfies:

$$E(\varepsilon_t\eta_s) = 0 \ \ \forall t,s \quad \text{or} \quad E(\varepsilon_t\eta_s) = \sigma \quad \text{if } t = s, \text{ 0 otherwise.}$$

Suppose first that ε_t and η_t are uncorrelated, so that ψ is strictly exogenous with respect to ε. The above model can be rewritten as:

$$(3.9) \qquad \begin{bmatrix} \psi_{t+1} \\ \psi_t \\ \vdots \\ \psi_{t-k+2} \\ \chi_t \\ \hline E(\chi_{t+1}|\Omega_t) \end{bmatrix} = \left[\begin{array}{cccc|c} a_1 & \cdots\cdots & a_k & 0 & 0 \\ & & 0 & | & | \\ & I_{(k-1)\times(k-1)} & | & | & | \\ & & 0 & 0 & 0 \\ 0 & \text{———} & 0 & 0 & 1 \\ \hline -c & 0 \text{———} & 0 & -b & -a \end{array}\right] \begin{bmatrix} \psi_t \\ \psi_{t-1} \\ \vdots \\ \psi_{t-k+1} \\ \chi_{t-1} \\ \hline \chi_t \end{bmatrix} + \begin{bmatrix} \eta_{t+1} \\ 0 \\ | \\ | \\ 0 \\ \hline -\varepsilon_t \end{bmatrix}$$

This satisfies the assumptions of the recursive case of the general model. Note that χ_{t-1} is a state variable while χ_t is a costate variable. Note also that the transition equation for χ_t is an identity and involves no unknown parameters or disturbances. Using equation (3.1), the reduced form for χ_t is:

$$\chi_t = -\phi_x \chi_{t-1} - \phi_{\psi 1}\psi_t - \cdots - \phi_{\psi k}\psi_{t-k+1} + \phi_\varepsilon \varepsilon_t$$

Equations (3.3) and (3.4) impose the following restrictions on (a,b,c):

(3.10) $$\phi_x^2 - a\phi_x + b = 0$$

(3.11) $$\begin{aligned} \phi_{\psi 1}(\phi_x - a) &= \phi_{\psi 1}a_1 + \phi_{\psi 2} - c \\ \phi_{\psi 2}(\phi_x - a) &= \phi_{\psi 1}a_2 + \phi_{\psi 3} \\ &\vdots \\ \phi_{\psi k-1}(\phi_x - a) &= \phi_{\psi 1}a_{k-1} + \phi_{\psi k} \\ \phi_{\psi k}(\phi_x - a) &= \phi_{\psi 1}a_k \end{aligned}$$

Therefore only if Ψ_t follows a second or higher order autoregressive process may (a,b,c) be identified.

Suppose now that Ψ is not strictly exogenous with respect to ε. (3.9) does not satisfy the assumptions of the general model as its first-order autocovariance matrix is not identically zero. The model can however be rewritten as:

$$(3.12)\quad \begin{bmatrix} \Psi_t \\ \Psi_{t-1} \\ \vdots \\ \Psi_{t-k+1} \\ \chi_t \\ \hline E(\chi_{t+1}|\Omega_t) \end{bmatrix} = \left[\begin{array}{cccc|c} a_1 & \cdots & a_k & 0 & 0 \\ & & 0 & | & | \\ & I_{(k-1)\times(k-1)} & | & | & | \\ & & 0 & 0 & 0 \\ 0 & \text{———} & 0 & 0 & 1 \\ \hline -ca_1 & \cdots & -ca_k & -b & -a \end{array}\right] \begin{bmatrix} \Psi_{t-1} \\ \Psi_{t-2} \\ \\ \Psi_{t-k} \\ \chi_{t-1} \\ \hline \chi_t \end{bmatrix} + \begin{bmatrix} \eta_t \\ 0 \\ \\ \\ 0 \\ \\ -\varepsilon_t - c\eta_t \end{bmatrix}$$

This satisfies the assumptions of the simultaneous case of the general model. The reduced form for χ_t is, using (3.5):

$$\chi_t = -\Phi_x \chi_{t-1} - \Phi_{\psi 1}\Psi_{t-1} - \cdots - \Phi_{\psi k}\Psi_{t-k} + \Phi_\varepsilon \varepsilon_t + \Phi_\eta \eta_t$$

Because, in this example, the transition equation for the state variable χ_t involves no unknown parameters, all the elements of the first (k+1) lines of A are still identified. Equations (3.3) and (3.4) impose the following restrictions on (a,b,c):

$$(3.13)\quad \Phi_x^2 - a\Phi_x + b = 0$$

$$(3.14)\quad \begin{aligned} \Phi_{\psi 1}(\Phi_x - a) &= \Phi_{\psi 1}a_1 + \Phi_{\psi 2} - ca_1 \\ &\ \ \vdots \\ \Phi_{\psi k-1}(\Phi_x - a) &= \Phi_{\psi 1}a_{k-1} + \Phi_{\psi k} - ca_{k-1} \\ \Phi_{\psi k}(\Phi_x - a) &= \Phi_{\psi 1}a_k - ca_k \end{aligned}$$

The form of the restrictions is different from above but their implication is the same as above: only if Ψ_t follows a second or higher order process may a,b,c be identified. The similarity of results in this case follows from the absence of unknown parameters in the transition equation for χ_t.

Section IV. Unobservable Costate Variables

If the system (2.1) is derived from a higher order system, some of the costate variables are themselves expectations and not observable. To analyze this case we take as the initial model not the form (2.1) but the original higher order model. The system we consider is:

$$\sum_{i=1}^{m} B_i E(y_{t+i}|\Omega_t) + B_o y_t + \sum_{i=1}^{n} B_{-i} y_{t-i} + D\Psi_t + \varepsilon_{yt} = 0$$

(4.1)

$$\Psi_t = B\Psi_{t-1} + \varepsilon_{\psi t}$$

y_t and Ψ_t are vectors of p and k random variables respectively; ε_{yt} and $\varepsilon_{\psi t}$ are vector white noise processes and their covariance structure satisfies:

$$E(\varepsilon_{yt}\varepsilon_{\psi s}) = 0 \qquad \forall t, s$$

The information set is given by

$$\Omega_t = \{\varepsilon_{yt}, \varepsilon_{yt-1}, \cdots; \varepsilon_{\psi t}, \varepsilon_{\psi t-1}, \cdots\}$$

B_m is of full rank.

The assumption that B_m is invertible is not restrictive. B_m will not be invertible if for example expectations of a subset of y_t do not appear in (4.1); in this case however, the initial system can be reduced to a lower order system in more variables satisfying the assumptions of (4.1).[6]

Note also that Ψ_t in (4.1) could itself be given by:

$$\Psi_t = \sum_{i=1}^{m'} H_i E(Z_{t+i}|\Omega_t) + H_o Z_t + \sum_{i=1}^{n'} H_{-i} Z_{t-i} \; ;$$

$$Z_t = HZ_{t-1} + \varepsilon_{zt}$$

$$\Omega_t = \{\varepsilon_{yt}, \varepsilon_{yt-1}, \cdots; \varepsilon_{zt}, \varepsilon_{zt-1}, \cdots\}$$

This would be reduced to the above form by solving for future expectations, $E(Z_{t+i}|\Omega_t) = H^i Z_t$ and defining lagged values of Z as state variables.

Note finally that Ψ is assumed strictly exogenous with respect to ε_y. This can easily be relaxed as in the example of the previous section.

The system (4.1) can be rewritten in quasi-first-order form as follows:

$$(4.2) \quad \left[\begin{array}{c} \Psi_{t+1} \\ \hline y_{t-n+1} \\ \vdots \\ y_t \\ \hline Ey_{t+1} \\ \vdots \\ Ey_{t+m} \end{array}\right] = \left[\begin{array}{c|cc|cc} B & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{(n-1)p} & 0 & 0 \\ 0 & 0 & 0 & I_p & 0 \\ \hline 0 & 0 & 0 & 0 & I_{(n-1)p} \\ -B_m^{-1}D & -B_m^{-1}B_{-n} & -B_m^{-1}B_{-1} & -B_m^{-1}B_o & -B_m^{-1}B_{m-1} \end{array}\right] \left[\begin{array}{c} \Psi_t \\ \hline y_{t-n} \\ \vdots \\ y_{t-1} \\ \hline y_t \\ \vdots \\ Ey_{t+m-1} \end{array}\right] + \left[\begin{array}{c} \varepsilon_{\psi t+1} \\ 0 \\ | \\ 0 \\ -B_m^{-1}\varepsilon_{yt} \end{array}\right]$$

This satisfies the assumptions of the recursive case of the general model. Note that only y_t in the vector of costate variables is observable.

The reduced form gives $y_t, \cdots, Ey_{t+m-1}$ as functions of $\Psi_t, y_{t-n}, \cdots, y_{t-1}$. Partition the reduced form matrix $\Pi_{22}^{-1}\Pi_{21}$ as in Section II to get:

$$[\ \Phi_{\psi} \ \vdots \ \Phi_{x}\ ] \equiv \left[\begin{array}{c|ccc} \Phi_{\psi 1} & \Phi_{n1} & \cdots & \Phi_{11} \\ \vdots & \vdots & & \vdots \\ \Phi_{\psi m} & \Phi_{nm} & & \Phi_{1m} \end{array}\right]$$
$$\underset{mp\times k}{} \quad \underset{mp\times np}{}$$

The identification restrictions are still given by equations (3.3) and (3.4). Equation (3.3) gives here:

$$(4.3)\quad \begin{bmatrix} \Phi_{11}\Phi_{n1}+\Phi_{n2} & \cdots \Phi_{11}\Phi_{11}+\Phi_{12} \\ \vdots & \vdots \\ \Phi_{1m-1}\Phi_{n1}+\Phi_{nm} & \cdots \Phi_{1m-1}\Phi_{11}+\Phi_{1m} \\ \Phi_{1m}\Phi_{n1}+\sum_{i=0}^{m-1} B_m^{-1}B_i\Phi_{ni+1} & \cdots \Phi_{1m}\Phi_{11}+\sum_{i=0}^{m-1} B_m^{-1}B_i\Phi_{1i+1} \end{bmatrix} = \begin{bmatrix} 0 & \Phi_{n1} & \cdots & \Phi_{21} \\ \vdots & \vdots & & \vdots \\ 0 & \Phi_{nm-1} & \cdots & \Phi_{2m-1} \\ B_m^{-1}B_{-n} & \Phi_{nm}+B_m^{-1}B_{-n+1} & \cdots & \Phi_{2m}+B_m^{-1}B_{-1} \end{bmatrix}$$

As only y_t is observable, only $(\Phi_{i1})_{i=1,\cdots,n}$ is directly identified. We first show however that the restrictions above imply that all $(\Phi_{ij})_{i,j}$ are identified. Consider the first p lines of equation (4.3). They give n matrix equalities:

$$(4.4)\quad \begin{aligned} \Phi_{11}\Phi_{n1} &+ \Phi_{n2} &= 0 \\ \Phi_{11}\Phi_{n-1,1} &+ \Phi_{n-1,2} &= \Phi_{n1} \\ \vdots \quad & \quad \vdots & \vdots \\ \Phi_{11}\Phi_{11} &+ \Phi_{12} &= \Phi_{21} \end{aligned}$$

Given $(\Phi_{i1})_{i=1,\cdots,n}$, these equalities determine recursively $(\Phi_{i2})_{i=1,\cdots,n}$. By the same argument, the first (m-1)×p lines of equation (4.3) determine all $(\Phi_{ij})_{i=1,\cdots,n;j=1,\cdots,m}$. This reflects the special structure of A in this case. The last p lines impose restrictions on $(B_i)_{i=+m,\cdots,-n}$:

$$
\begin{aligned}
B_m\Phi_{1m}\Phi_{n1} &+ B_o\Phi_{n1} + \cdots + B_{m-1}\Phi_{nm} = B_{-n} \\
B_m\Phi_{1m}\Phi_{n-1,1} &+ B_o\Phi_{n-1,1} + \cdots + B_{m-1}\Phi_{n-1,m} = B_{-n+1} + B_m\Phi_{nm} \\
&\vdots \\
B_m\Phi_{1m}\Phi_{11} &+ B_o\Phi_{11} + \cdots + B_{m-1}\Phi_{1m} = B_{-1} + B_m\Phi_{2m}
\end{aligned}
$$

If a set $(B_i)_i$ satisfies (4.4), then for any nonsingular matrix C, $(CB_i)_i$ also satisfies (4.4): a normalization such as $B_o = I$ is required. Once the system is normalized, the system (4.4) imposes np^2 restrictions on the $(n+m)\times p^2$ elements of $(B_i)_i$.

Equation (3.4), which gives the restrictions imposed by the presence of Ψ, is here:

$$
(4.5) \qquad
\begin{aligned}
\Phi_{11}\Phi_{\Psi 1} &+ \Phi_{\Psi 2} = \Phi_{\Psi 1}B \\
&\vdots \\
\Phi_{1m-1}\Phi_{\Psi 1} &+ \Phi_{\Psi m} = \Phi_{\Psi m-1}B \\
B_m\Phi_{1m}\Phi_{\Psi 1} &+ B_o\Phi_{\Psi 1} + \cdots + B_{m-1}\Phi_{\Psi m} = B_m\Phi_{\Psi m} + D
\end{aligned}
$$

Again, only $\Phi_{\Psi 1}$ and B are directly identified but the first (m-1)×p lines of equation (4.5) determine, given B, $\Phi_{\Psi 1}$ and $(\Phi_{1j})_{j=1,\cdots,m}$ from above, $(\Phi_{\Psi i})_{i=2,\cdots,m}$. The last lines impose pk restrictions but also add pk elements of D. In the absence of additional prior restrictions on D, there are no additional restrictions on $(B_i)_i$ from the presence of exogenous variables. As in the previous section, prior restrictions are however likely. Suppose for example that Ψ_t is the quasi-first-order form of a kth order univariate process $\tilde{\Psi}_t$ and that only current $\tilde{\Psi}_t$ affects y_t. In this case, after normalization, the number of linear restrictions imposed by (4.4) and (4.5) minus the number of nonidentically zero parameters of $(B_i)_i$ and D is $p\times(k-1)-mp^2$. Thus, if $k \geq mp+1$, the model is, subject to the rank condition, identified.

Conclusion

This paper has extended the study of identification in rational expectations model, considered by Wallis for the static model [10] and Chow for the dynamic control problem [4], to the first-order dynamic linear model. The identification restrictions are linear in the structural parameters and thus easy to use.

Footnotes

1. Some prior restrictions, such as disturbances being identically zero if some of the equations are identities, or block diagonality under some assumptions about information, do not yield additional restrictions and will be considered in the paper.

2. z_t and q_t appear with different time indices in the right-hand-side vector. By defining $y_t \equiv z_{t-1}$ and substituting y for z_{-1} on each side, each vector would have variables with the same time index; the specification used in (1.1) is often more natural.

3. In doing so, we implicitly use a transversality condition. This has implications for identification. Chow [3] has recently argued that imposing the transversality condition may not be warranted and that identification should be studied without imposing it.

4. Identification of the structural covariance matrix requires, as there is a conditional expectation in the reduced form, a specification of the distribution of disturbances. If they are joint normal, so that $E(\varepsilon_{zt}|\Omega_t) = E(\varepsilon_{zt}|\varepsilon_{qt}) = \sigma_{zq}\sigma_q^{-2}\varepsilon_{qt}$, the structural covariance matrix is identified if A is identified.

5. The model considered in [1] is more general than the one considered here.

6. Because of the many cases and the notational complexity, the method to achieve this is cumbersome to describe. The following example shows how to proceed. Consider the system in two scalar variables y_{1t} and y_{2t}:

$$\begin{bmatrix} b^2_{11} & 0 \\ b^2_{12} & 0 \end{bmatrix} \begin{bmatrix} E(y_{1t+2}|\Omega_t) \\ E(y_{2t+2}|\Omega_t) \end{bmatrix} + \begin{bmatrix} b^2_{11} & b^2_{12} \\ b^1_{21} & b^1_{22} \end{bmatrix} \begin{bmatrix} E(y_{1t+1}|\Omega_t) \\ E(y_{2t+1}|\Omega_t) \end{bmatrix} + \begin{bmatrix} b^o_{11} & b^o_{12} \\ b^o_{21} & b^o_{22} \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

Note that the parameters on $E(y_{2t+2}|\Omega_t)$ are equal to zero. Define

$$p_t \equiv E(y_{1t+1}|\Omega_t) \Rightarrow E(p_{t+1}|\Omega_t) = E(E(y_{1t+2}|\Omega_{t+1})|\Omega_t))$$
$$= E(y_{1t+2}|\Omega_t).$$

Rewrite the system as:

$$\begin{bmatrix} b^1_{11} & b^1_{12} & b^2_{11} \\ b^1_{21} & b^1_{22} & b^2_{12} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} E(y_{1t+1}|\Omega_t) \\ E(y_{2t+1}|\Omega_t) \\ E(p_{t+1}|\Omega_t) \end{bmatrix} + \begin{bmatrix} b^o_{11} & b^o_{12} & 0 \\ b^o_{21} & b^o_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \\ p_t \end{bmatrix} = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ 0 \end{bmatrix}$$

If the first matrix is nonsingular, this system satisfies (4.1).

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