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STOCHASTIC CAPITAL THEORY
I. COMPARATIVE STATICS

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ABSTRACT

Introductory lectures on capital theory often begin by analyzing the following problem: I have a tree which will be worth X(t) if cut down at time t. If the discount rate is r, when should the tree be cut down? What is the present value of such a tree? The answers to these questions are straightforward. Since at time t a tree which I plan to cut down at time T is worth $e^{rt}e^{-rT}X(T)$, I should choose the cutting date T* to maximize $e^{-rT}X(T)$; at $t < T^*$ a tree is worth $e^{rt}e^{-rT^*}X(T^*)$. In this paper we analyze how the answers to these questions of timing and evaluation change when the tree's growth is stochastic rather than deterministic. Suppose a tree will be worth $X(t,\omega)$ if cut down at time t when $X(t,\omega)$ is a stochastic process. When should it be cut down? What is its present value?

We study these questions for trees which grow according to both discrete and continuous stochastic processes. The approach to continuous time stochastic processes contrasts with much of the finance literature in two respects. First, we obtain sharp comparative statics results without restricting ourselves to particular stochastic specifications. Second, while the option pricing literature seems to imply that increases in variance always increase value, we show that an increase in the variance of a tree's growth has ambiguous effects on its value.

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STOCHASTIC CAPITAL THEORY

I. Comparative Statics

by

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1. Introduction.

Introductory lectures on capital theory often begin by analyzing the following problem: I have a tree which will be worth X(t) if cut down at time t. If the discount rate is r, when should the tree be cut down? What is the present value of such a tree? The answers to these questions are straightforward. Since at time t a tree which I plan to cut down at time T is worth $e^{rt}e^{-rT}X(T)$, I should choose the cutting date T* to maximize $e^{-rT}X(T)$; at $t < T^*$ a tree is worth $e^{rt}e^{-rT^*}X(T^*)$. On this simple foundation most of capital theory can be built. It is our purpose in this paper to analyze how the answers to these questions of timing and evaluation change when the tree's growth is stochastic rather than deterministic. Suppose a tree will be worth $X(t,\omega)$ if cut down at time t when X(t,) is a stochastic process. When should it be cut down? What is its present value? We ask these questions not because we are intrinsically interested in optimal policies for managing forests (although we would gladly accept research support from the Sierra Club or the Weyerhauser Corporation), but because we believe that, as in the certainty case, one can use such an analysis to answer many other questions of valuation and timing.

In this paper we show how to pose and analyze the basic problem of stochastic capital theory. The sequel to this paper will apply the techniques developed here to several problems of economic interest. Our main concern here is with the effect of uncertainty on the value of an asset whose instantaneous market value evolves stochastically but which may be marketed at a time and under conditions determined by its owner. We will use the standard metaphors of Austrian capital theory in this paper and call the asset a tree; the time at which an owner sells a tree is the tree's cutting time.

In the next section we analyze a tree whose growth follows a discrete time stochastic process and obtain three main results. First, for processes which are strictly increasing, as uncertainty increases so does the value of the tree; however, the size at which it is cut down is not affected by uncertainty. Secondly, for processes which can decrease, the reverse may be the case. Third, the results for discrete time processes, particularly those which can decrease, are not as sharp as we would like them to be because they are confounded by round off problems. Trees which follow a discrete time process necessarily grow by leaps and bounds. They cannot be cut down and harvested at an exact size. These problems of overshooting and undershooting make it difficult to obtain the kind of strong qualitative results we are looking for. Thus we turn in Section 3 to an analysis of trees whose growth path are continuous, i.e., they are governed by diffusion processes.

Diffusion processes can decrease as well as increase. Thus, to analyze the value of a tree whose growth is governed by a diffusion process, it is necessary to specify how it behaves when it reaches some lower boundary. In this paper we focus on absorbing boundaries. When the tree reaches a fixed size Q its growth stops and it is sold for Q. We find this boundary condition the most economically appealing -- an Appendix briefly treats other boundary conditions. The results of our analysis of the comparative statics of trees whose growth is described by a stationary diffusion process are as follows. The tree's growth path is completely described by an instantaneous variance $\sigma^2(x)$ and an instantaneous mean $b(x)$ where x is the current size of the tree. The other parameters of the problem are Q, the level of the absorbing barrier and r, the discount rate. Suppose for simplicity that the expected growth rate, $b(x)$ is a decreasing function of x. Then the optimal rule is to cut down the tree the first time it reaches a size y^*. Under these conditions y^* is always greater than $\bar{y}$, the solution to $b(y)/y = r$. If its growth were certain, the tree would be cut down at $\bar{y}$; uncertainty increases the cutting size of the tree.

Let $w(x)$ be the expected discounted value of a tree of size x which will be cut down when it first reaches y^* (unless it hits Q before it reaches y^*, in which case it will be cut down at Q). Both y^* and $w(x)$ are functions of the parameters of the problem -- $\sigma^2(x)$, $b(x)$, Q and r. Any parametric change which increases $w(x)$ at a point x increases $w(x)$ for all x. Such a change also increases y^*. Thus all changes are either good, in which case they increase value uniformly and they increase the optimal cutting size, or bad, in which case they decrease value uniformly and cause the tree to be cut down when smaller. Increases in the growth rate are good. Increases in the discount rate and the absorbing barrier are bad. Whether increases in variance, $\sigma^2(x)$, are good or bad depend on where they take place. If the value function w(x) is convex near x, a local increase in variance at x is good, it increases value and cutting size. If the value function is concave near x, then an increase in variance is bad. Near y*, w(x) is convex but there always is a region below y* where w(x) is concave. In many cases this region will include Q, so that the interval (Q,y*) is broken up into two regions. There is a Z such that for $x \varepsilon (Q,Z)$, w(x) is concave and increases in variance are bad. For $x \varepsilon (Z,y^*)$ increases in variance are good. A sufficient (but not necessary) condition for this to happen is that b(x) be decreasing. The ambiguous effects of variance increases in value stand in sharp contrast to the option pricing literature. Increases in the variance of an asset increase the value of most financial options (like calls) written on that asset.

2. Discrete time.

This section contains observations about trees which grow according to discrete time stochastic processes. We believe these examples and observations provide considerable intuition about the way in which uncertainty affects value and cutting time. They also indicate why we have chosen to analyze continuous processes at greater length in succeeding sections of this paper.

In all the examples of this section we will assume that the tree's size follows a discrete time Markov process which we write as: $X_1, X_2, \ldots, X_t$. The tree's owner is trying to pick a stopping time τ to maximize the expected present discounted value of the tree. That is, he seeks a stopping time which will maximize $EX_\tau e^{-r\tau}$.

Example 1.

Let

(1) $$X_{t+1} = X_t + \varepsilon_t$$

The ε_t are independent, identically distributed (i.i.d.) random variables with

$$\Pr\{\varepsilon_t = 2\} = \Pr\{\varepsilon_t = 0\} = 1/2.$$

Let $r = .1$ so $e^{-r} = .905$. Suppose I have a tree of size 8 and I plan to cut it down the first time it is size 10. Then $V_{10}(8)$, its present value, must satisfy the equation

$$V_{10}(8) = .905[(1/2)V_{10}(8) + (1/2)10]$$

or

(2) $$V_{10}(8) = .826.$$

If I have a tree of size 10 and plan to cut it down at size 12, then its present value, $V_{12}(10)$, satisfies

$$V_{12}(10) = .905[(1/2)V_{12}(10) + (1/2)12]$$

so that

(3) $$V_{12}(10) = 9.91 < 10.$$

A comparison of (2) and (3) suggests that an optimal policy is to cut down the tree when it first reaches size 10. (That this is so is a consequence of Proposition 2 below). It is interesting to note that this is the rule which would be followed if the tree grew constantly at a speed equal to its mean. In this case we would have in the notation of the previous section $X'(t) = 1$, so the optimal cutoff size X^* satisfies $\frac{X^{*\prime}}{X^*} = r = .1$ or $X^* = 10$ and a tree of size $X < 10$ would be worth $W(X) = 10\, e^{-.1(10-X)}$. Thus

(4) $$W(8) = 8.19.$$

Comparing (4) and (2) we see that uncertainty increases the value of the tree. Given the form of the optimal stopping rule -- harvest the tree when it first reaches the optimal cutting size X^* -- this is quite reasonable. The tree's expected present discounted value is then just $EX^*\, e^{-r\tau_{X^*}}$ where τ_{X^*} is the first time the tree reaches the size X^*. The only uncertain quantity in this expression is τ_{X^*} and our valuation function is the expected value of a <u>convex</u> function of this random variable. Since uncertainty increases the expected value of convex functions it is not surprising that uncertainty should increase the value of the tree. The next example shows that this intuition is not always correct.

<u>Example 2</u>.

Consider a tree which grows according to the same rules as the tree in Example 1. Its present size is 9.5. As its owner, I can either sell it now or let it grow. Since the tree grows two units at a time, if I elect to let it grow, I must wait until it is at least 11.5 units tall. It is easy to calculate that

$$V_{11.5}(9.5) = 9.501$$

so that it is worthwhile to keep the tree. Since $V_{13.5}(11.5) = 11.15 < 11.5$ it seems the best policy is to keep the tree until it reaches 11.5. On the other hand, if growth of the tree is certain, then

$$(5) \qquad W(9.5) = 10e^{-.05} = 9.512 > V_{9.5}(11.5)$$

In this case uncertainty decreases the value of the tree. The explanation for this anomaly is straightforward. As we will prove shortly, if the tree grows according to

$$X_{t+1} = X_t + \varepsilon_t$$

where the ε_t are non-negative independent, identically distributed (i.i.d.) random variables with $E\varepsilon_t = \mu$, an optimal policy is to cut down the tree the first time it reaches or exceeds $\overline{X}$ where $\overline{X}$ is the solution to

$$(6) \qquad \overline{X} = \beta(\overline{X} + \mu)$$

or

$$(7) \qquad \overline{X} = \left[\frac{\beta}{(1- \ )}\right] \mu$$

and $\beta = e^{-r}$. In our case $\mu = 1$ and $\beta = e^{-1} = .905$, so

$$\overline{X} = 9.508.$$

I would like to cut down the tree the first time it exceeds 9.508. Ideally, I would cut it down when its size reached $\overline{X} = 9.508$. Since the tree grows in steps of 2, I cannot cut it down exactly at $\overline{X}$. If it is currently of a size 9.5 I can only plan to cut it down at size $9.5 < \overline{X}$ or at 11.5 which overshoots the optimal harvesting size. It is not so much uncertainty per se which is responsible for the inequality (5) as the fact that the uncertain process is discrete and the tree cannot be cut down at its optimal size.

To see that this is correct, consider a tree now worth X. Next period it will be $Y = X + \Delta$ with probability P and X with probability (1-p). The discounted value of my expected returns if I realize Y the first time the tree reaches Y is

$$V = YEe^{-rN_Y}$$

where the random variable N_Y is the number of periods I have to wait until the tree reaches Y. Since $EN_Y = \frac{1}{p}$, Jensen's inequality implies $Ee^{-rN_Y} > e^{-r/p}$. But if the tree grew at the rate $\mu = p\Delta$ per period for certain, I would wait exactly 1/p periods for it to grow to Y and its value would be $Ye^{-r/p}$. The uncertain tree is worth more -- if it is to be cut down precisely at Y.

In the simple examples we have given so far, we have suggested that uncertainty does not change the form of the optimal stopping rule. If

$$X_{t+1} = X_t + \varepsilon_t \tag{1}$$

and the ε_t are independent, identically distributed (i.i.d.) random variables with $E\varepsilon_t = \mu$ and $\beta = e^{-r}$ is the discount factor, then we have shown that the optimal stopping rule is to cut down the tree the first time it reaches or exceeds a height of $\overline{X}$ when $\overline{X}$ is the solution to

$$\overline{X} = \beta(\overline{X} + \mu). \tag{6}$$

In Proposition 1, we use a simple heuristic argument (for which we are grateful to Herbert Scarf) to show that this is so if ε_t is strictly positive (and thus the tree's size is an increasing process). A more rigorous proof is also given in Proposition 2 below. Let X be the optimal cutting time for the tree.

<u>Proposition 1</u>. If (1) holds and if

(8) $$P\{X_{t+1} \geq X_t\} = 1$$

then $\bar{X} = X$.

<u>Proof</u>: Let V(X) be the value of having a tree of size X assuming it will be cut down when it reaches the optimal size. Then if $\hat{X}$ is the cutting size it must be that

(9) $$V(X) = X \qquad \text{for } X \geq \hat{X}$$

and

(10) $$V(X) > X \qquad \text{for } X < \hat{X}$$

Also at X, the tree owner must be indifferent between cutting the tree down now and letting it grow for a period. That is, $\hat{X}$ must satisfy

(11) $$\hat{X} = \beta E[\max(\hat{X}+\varepsilon), V(\hat{X}+\varepsilon)].$$

However, since ε is non-negative, $V(\hat{X}+\varepsilon) = \hat{X} + \varepsilon$ and (11) becomes $\hat{X} = \beta E[\hat{X}+\varepsilon] = \beta(\hat{X} + \mu)$ which is the same as (6). Since the solution to (6) is unique, $\hat{X} = \bar{X}$.

If the tree may shrink as well as grow, the conclusion of Proposition 1 may not hold. The next example shows that when ε may be negative, uncertainty can increase the optimal cutting size of the tree.

Example 3.

Suppose again that $X_{t+1} = X_t + \varepsilon_t$ but now assume that ε has a density function f(.) with support on [-1,+1]. then the optimal cutting time $\hat{X}$ exceeds $\overline{X}$. To see this, note that in this case (11) becomes

$$\hat{X} = \beta \left[\int_1^0 V(\hat{X}+\varepsilon)f(\varepsilon)d\varepsilon + \int_0^1 (\hat{X}+\varepsilon)f(\varepsilon)d\varepsilon \right]$$

$$> \beta \left[\int_1^0 (\hat{X}+\varepsilon)f(\varepsilon)d\varepsilon + \int_0^1 (\hat{X}+\varepsilon)f(\varepsilon)d\varepsilon \right]$$

$$= \beta(\hat{X} + \mu).$$

Thus, $\hat{X} > \frac{\mu\beta}{1-\beta} = \frac{\mu}{r} = \overline{X}.$

The mere fact that decreasing processes can go down leads to another reason why processes which can decrease are different from processes which grow certainly or processes which are uncertain but increasing. A process which can decrease can go to a region in which it is stuck or from which it can escape only with difficulty. Uncertainty makes this unfavorable prospect likely and thus can decrease the value of the tree even when there is no overshooting problem. The next example illustrates this point.

Example 4.

Suppose that we have a tree of size .5 which grows according to (1); suppose further that $\beta = (1.1)^{-1}$ so that $r = .0953$. The average growth rate of the tree is $E(\varepsilon_t) = \mu = 1$. If the tree grows at this rate for certain then it will be held until its size is r^{-1} and its present value is

(12) $$W = r^{-1} e^{-r(r^{-1} - .5)} = 4.048$$

Suppose instead that with probability .1 the tree grows by 10 and with probability .9 it stays the same height. Then Proposition 1 states that it is optimal to hold onto the tree until its size exceeds 10, which in this case means waiting until it grows. The value of the tree if this policy is followed is V, the solution to

$$V = (1.1)^{-1}(.9V + .1 \times 10.5)$$

Thus

(13) $$V = 5.25.$$

Suppose that the tree can shrink as well as decline, that the probability of an increase of 10 is 1/7 and of a decrease of .5 is 6/7. Then the average gain is again 1. Suppose further that if the tree's value declines to 0 it is worthless. (This is meant to catch in as strong a way as possible the notion that a decline in value may be disastrous -- more realistic examples with this property appear in Section 3 below. Then, the value of a tree is

(14) $$Z = (1.1)^{-1}(1/7)10.5 = 1.36.$$

Comparing these results we see that

$$V > W > Z$$

so that one kind of uncertainty increases the value of the tree while another kind decreases it.

These examples give rise to something of a dilemma for the researcher who wants to build simple but general models to analyze the effect of uncertain growth on such Austrian assets as trees and wine. Comparison of Examples 1 and 2 suggests that continuous models will yield the most easily interpretable results. For if the asset's value or the tree's size is continuous, then it can be stoppped at any point. Problems of overshooting -- which are responsible for the anomalous results of Example 2 -- will be avoided. On the other hand, Examples 3 and 4 suggest that strictly increasing processes behave very differently from processes which can decrease. Unfortunately, processes with continuous sample paths cannot be both genuinely stochastic and increasing. The only non-deterministic processes with continuous sample paths are diffusions and such processes behave locally like Brownian motion. They can go up and they can go down. It is impossible to use continuous models to examine increasing processes.[1] We have resolved this problem by focussing largely on continuous (and thus possibly decreasing) processes.

We conclude this section with two observations about increasing discrete processes. We first prove a general version of Proposition 1 which shows that if X_t is increasing, and satisfies some regularity conditions (which are equivalent to the second-order conditions in the non-stochastic case), then uncertainty does not affect the optimal stopping size.

<u>Proposition 2</u>. Let X_t be a stochastic process such that

$$P\{X_{t+1} \geq X_t\} = 1$$

and let $\overline{X}$ satisfy

$$E[\beta X_{t+1} - X_t \mid X_t \geq \overline{X}] \leq 0$$

$$E[\beta X_{t+1} - X_t \mid X_t < \overline{X}] \geq 0.$$

Let $Y_t = \beta^t X_t$ and $\mathcal{M}$ be the set of Markov times for X_t.

Define

$$W(X) = \sup_{\tau \in \mathcal{M}} E_X Y_\tau .$$

Then $W(X) = E_X Y_{\tau(\overline{X}}$ where $\tau(\overline{X}) = \inf \{t | X_t \geq \overline{X}\}$.

<u>Proof</u>: Clearly, if $t < \tau(\overline{X})$, $E[Y_{t+1} \mid X_t] > Y_t$ so it cannot be optimal to stop if $t < \tau(\overline{X})$. Conversely, if $t \geq \tau(\tilde{X})$ we will show that Y_t is a super-martingale. Then the optimal stopping theorem [Chow, Robbins and Siegmund, 2, p. 21] states that if σ is any stopping time such that $\sigma \geq t$, $E[Y_\sigma | t \geq \tau(\tilde{X})] \leq Y_t$. We must show that for all s

$$E[Y_{t+s+1} - Y_{t+s} | X_t > \overline{X}] \leq 0.$$

However, since $Y_{t+s+1} - Y_{t+s} = \beta^{t+s}(\beta X_{t+s+1} - X_{t+s})$, it suffices to observe that

$$E[\beta X_{t+s+1} - X_{t+s} | X_t > \overline{X}] = E\left[E[\beta X_{t+s+1} - X_{t+s} | X_{t+s}] | X_t > \overline{X}\right]$$

$$= E\left[E[\beta X_{t+s+1} - X_{t+s} | X_{t+s} > \overline{X}] | X_t > \overline{X}\right] \geq 0.$$

Proposition 2 states that the introduction of uncertainty does not change the cutting size of trees which increase. They will be cut down when the first reach height $\overline{X}$, whether the increments in their size are certain or stochastic. What is uncertain is $\tau(\overline{X})$, the number of periods it will take to reach $\overline{X}$. The expected present discounted value of such a tree is $\overline{X}\,E\,\beta^{\tau(\overline{X})}$. Since β^{τ} is a convex function of τ, Jensen's inequality implies that uncertainty should increase the expected value of trees. If overshooting problems are absent, this intuition is correct, as is demonstrated in Proposition 3.

To avoid overshooting, we analyze a stationary process which at each step either grows a fixed amount (which can vary from step to step), or stays constant. Let $\{Y_n\}$ n = 0, ..., be an increasing sequence and consider a Markov process on the points $\{Y_n\}$ defined as follows:

$$\Pr\{X_{t+1} = Y_{n+1} \mid X_t = Y_n\} = P_n$$

$$\Pr\{X_{t+1} = Y_n \mid X_t = Y_n\} = 1 - P_n$$

Let $T(K,N) = \inf(t|X_t = Y_N|X_0 = Y_K)$; then T(K,N) is a random variable with

$$ET(K,N) = \sum_{i=K}^{N} P_i^{-1}$$

<u>Proposition 3</u>. The expected value of a tree which grows according to this process and will be cut down when it first reaches Y_N is greater than the present value of the corresponding certain continuous process which will be cut down at Y_N.

Proof: The corresponding certain continuous process is one which grows at a rate equal to the mean of the uncertain process,

$$\mu_N = P_N(Y_{N+1} - Y_N)$$

when it is between Y_N and Y_{N+1}. It may be defined by

$$Z(0) = Y_0$$

$$Z'(t) = \mu_0 \quad \text{for } 0 < t < P_0^{-1}$$

$$Z'(t) = \mu_n \quad \text{for } \sum_{i=1}^{n-1} p_i^{-1} < t < \sum_{i=1}^{n} p_i^{-1}$$

$$Z\left(\sum_{i=0}^{n-1} p_i^{-1}\right) = Y_n$$

Thus, if $Z(t_1) = Y_K$ and $Z(t_2) = Y_N$, $t_2 - t_1 = \sum_{i=0}^{N-1} p_i^{-1} - \sum_{i=0}^{K-1} p_i^{-1} = \sum_{i=K}^{N-1} p_i^{-1}$ $= T(K,N)$. The value of a tree of size Y_K which will be cut down at size Y_N is

$$V = Y_N \, Ee^{-rT(K,N)}$$

if it grows according to the uncertain process. If it grows according to the certain process, its value is

$$W = Y_N \, e^{-rT(K,N)}$$

and Jensen's inequality implies $V > W$.

It is hard to know whether this observation is a trivial tautology or an important insight. The discrete process is a very special case and not in itself of much interest. However, to generalize it we must either deal with overshooting problems or with processes which can decrease.[2]

It is our view that the former problems defy useful analysis while, as we see in the next section, an analysis of diffusion processes suggests that the conclusion that uncertainty increases the value and cutting time of trees is quite robust and general. The important exceptions to this rule come about because trees can decrease.

3. Diffusion Processes.

In this section we analyze the value of a tree whose size (or market value if cut down and sold) at t evolves according to

$$x_t = x + \int_0^t \sigma(x_s)dW_s + \int_0^t b(x_s)ds \tag{1}$$

where W_s is a Weiner process. A common shorthand notation for (1) is

$$dx = a(x)dW + b(x)dt \tag{2}$$

where

$$a(x) = \frac{1}{2}\sigma^2(x). \tag{3}$$

The meaning of equations (1) and (2) is roughly that x_t behaves locally like Brownian Motion with instantaneous drift b(x) and instantaneous variance $\sigma(x)$. Thus, if $\Delta x = x_{t+\Delta t} - x_t$

$$E\Delta x \simeq b(x)\Delta t \tag{4}$$

$$E(\Delta x)^2 \simeq \sigma^2(x)\Delta t \tag{5}$$

Many texts have explained in detail the meaning of equations like (1) and (2). Two excellent examples are Arnold [1] and Karlin and Taylor [4].

The techniques we use for analyzing value and optimal cutting time of a tree are largely taken from Krylov [5].

a. Remarks on generality.

The most general continuous time stochastic processes which have sample paths continuous with probability one can be written in the form (1) or (2) with the requirement that the coefficients a( ) and b( ) satisfy some regularity conditions. See Karlin and Taylor [4, Section 15.1] and Wentzell [9, Chapter 11] for a discussion of the necessary regularity conditions. We make two significant restrictions by assuming that a( ) and b( ) are functions of the tree's current size alone. First, we assume that the process which affects the tree's growth is stationary; growth is not a function of time. Second we assume that the only factor which determines the tree's growth is its size. This is quite a severe restriction. One might naturally suppose that the value of a tree was a function of its monetary value $M_t = P_t x_t$ where P_t, the price of lumber in board feet, also follows a diffusion process. While we believe that the techniques discussed below -- which are, as noted above, mainly taken from Krylov [5] -- can be used to solve these problems, we are not optimistic that any general qualitative results -- like those of Propositions 5 through 9 -- can be obtained except in special cases.

We will impose the mild regularity conditions that a(x) and b(x) are bounded and satisfy a Lipschitz condition. We will also insist that the process be genuinely stochastic so that a(x) is bounded away from 0 for all x.

b. Heuristics and Boundary Conditions.

To begin our analysis we <u>assume</u> that the optimal stopping rule is of the form: harvest the tree when it first reaches height y. Then the tree's value is

$$H(x,y) = E[y\, e^{-rT_y} | x(0) = x] \tag{6}$$

where

$$T_y = \inf\{t | x_t = y\}.$$

Let us suppress y for a minute and consider the value of the tree as a function of the tree's current size alone. The valuation functional, w(x) must satisfy the linear second-order differential equation

$$L[w] = 0$$

where

$$L[u](x) = ru(x) - b(x)u'(x) - a(x)u''(x). \tag{7}$$

That this is so follows from general results stated in Krylov [5] or Shiryayev [7]. We give a heuristic argument on which the more rigorous proofs are based. Suppose $x_t << y$. Since x_t is continuous, if Δt is sufficiently small we may be sure that $x_{t+\Delta t} < y$ and the tree will not be cut down before $t+\Delta t$. Thus

$$w(x) = e^{-r\Delta t} E[w(x_{t+\Delta t}) | x_t = x] \tag{8}$$

Use (3), (4) and (5) to expand the right-hand side of (8) in a Taylor series so that

$$w(x) = e^{-rt}E[w(x) + w'(x)(\ x) + \frac{1}{2}w''(x)(\Delta x)^2 + \dots]$$

$$= (1 - r\Delta t + \dots)\ (w(x) + w'(x)b(x)\Delta t + w''(x)a(x)\Delta t + \dots).$$

Rearrange and discard all terms of order $(\Delta t)^2$ and higher to get (8).

Another condition which w( ) must satisfy is that

(9) $$w(y) = y$$

which is an immediate consequence of (6). Another consequence of (6) is that <u>if y is chosen optimally</u>

(10) $$w'(y) = 1.$$

Rigorous proofs of the necessity of (10) -- known as the smooth pasting condition -- can be found in Krylov [5] and Shiryayev [7]. We give a heuristic argument which holds only for the special case where the function H(x,y) of (6) can be written in the form

(11) $$w(x) = H(x,y) = f(x)\ g(y).$$

(We will observe shortly that (11) holds for some interesting special cases.) In this case it follows from (9) that

$$w(y) = f(y)g(y) = y$$

so that $g(y) = \frac{y}{f(y)}$ and

(12) $$w(x) = f(x)\frac{y}{f(y)}$$

and

(13) $$w'(x) = f'(x)\frac{y}{f(y)}.$$

However, if y is chosen optimally, y must maximize $\frac{y}{f(y)}$ so that

(14) $$f(y) - f'(y)y = 0.$$

Using (14) to evaluate (13) at y we obtain (10).

We have thus argued that the valuation functional w must satisfy a second-order differential equation $L[w] = 0$ and the two boundary conditions (9) and (10). This might at first sight seem to determine w completely as the two boundary conditions determine a unique solution to a second-order differential equation. However, this reasoning is incorrect since (9) and (10) are meant to determine the optimal stopping time y while for <u>every y</u> there is a solution to the differential equation $L[u] = 0$ satisfying $u'(y) = y$, $u'(y) = 1$.[3] Obviously all y cannot be optimal cutting sizes so the problem must be rethought. The most straightforward approach is to look for another boundary condition. We stated that a distinctive aspect of diffusion processes was that they could go down as well as up. A natural question to ask is whether there is some obvious lower bound to the size of the tree and if there is, what the value of the tree should be if it ever reaches that size.

One possible answer to this question is to suppose that there is no natural lower bound to the size of the tree. Then we must be prepared to evaluate trees of any size; $w(x)$ must make sense for arbitrarily large negative x. This requirement turns out to provide the needed missing boundary condition. To see this most easily, consider the constant coefficients case where $a(x) = a$ and $b(x) = b$. Then the requirement that $L[w] = 0$ implies that

(15) $$rw - bw' - aw'' = 0$$

or

(16) $$w(x) = A_1 e^{\lambda x} + A_2 e^{\mu x}$$

where λ and μ are roots of the characteristic equation of (15). Thus

(17a) $$\lambda = \frac{-b + (b^2 - 4ar)}{2a} > 0$$

(17b) $$\mu = \frac{-b - (b^2 - 4ar)}{2a} > 0 .$$

It follows from (6) that for all x, $w(x) \leq y$ so that if we are going to consider arbitrarily large negative values of x we must require that w(x) remain bounded as $x \to -\infty$. From (16) and (17) it follows then that $A_2 = 0$ and w(x) is of the form

(18) $$w(x) = A_1 e^{\lambda x}.$$

Note that (18) is of the form (11) with the constant A_1 playing the role of g(y). We can repeat the analysis given above observing that if the tree is cut down at size y, we must have $A_1 = ye^{-\lambda y}$, so that $w(x) = ye^{-\lambda(y-x)}$. If y is chosen optimally it maximizes $ye^{-\lambda y}$ so that $y = \lambda^{-1}$ and

(19) $$w(x) = \lambda^{-1} e^{-\lambda x - 1}.$$

Note that since $x > \lambda^{-1}$, w(x), as given by (19), is a decreasing function of λ.

If the coefficients a(x) and b(x) are not constant, the same results hold. It follows from general results of Hartman [7: Chapter

XI, Section 6] that if a(x) and b(x) satisfy the conditions set forth in Section 3a above, then there is a solution w(x) to L[w] = 0 which satisfies

$$\lim_{x \to -\infty} |w(x)| < \infty \tag{20}$$

Furthermore, if w(x) is a solution to L[w] = 0 satisfying (20), so is Aw(x) for any constant A. Once again these solutions satisfy (11). The optimal boundary y is determined in the same way as in the constant coefficient case.

While consistent, this analysis is obviously incomplete. On both economic and biological grounds, it makes sense to suppose sometimes that a lower bound to tree size exists, that there is some value Q below which the tree cannot sink. We must ask what determines this boundary and what the value of the tree is if it ever hits this boundary. The most straightforward procedure is to assume that when the tree becomes of size Q, it dies and is worth nothing. In this case we add to the conditions (9) and (10) the condition

$$w(Q) = 0. \tag{21}$$

Note that if w(x) is a solution of L[w] = 0 which satisfies (21), so is Aw(x). Although this may have biological appeal, a little thought will make clear that (21) is suspect on economic grounds. In the first place, since we are measuring tree size in market value rather than board feet, (21) simply makes no sense unless $Q \geq 0$. However, if [Q > 0], it is not clear that the optimal policy is simply to let the tree die if it reaches Q. It would seem better to sell it at size $Q + \varepsilon$

and get $Q + \varepsilon$ where $\varepsilon > 0$ is small. Since ε can be arbitrarily small, we can approximate arbitrarily closely a tree which has boundary condition

$$(22) \qquad w(Q) = Q$$

instead of (21); a tree which grows according to this boundary condition is more valuable than one with boundary condition (21).

If we consider not trees but stocks bought on margin, the condition (22) has appeal. Suppose I buy one share of stock on margin. When I buy it, its price is x and I pay $P < x$, borrowing $x - P$ from my broker. If the price gets so low that the amount I have borrowed is greater than or equal to γ times the current value of the stock, my broker will sell me out at the current market price. Thus I will get sold out at price $Q = \frac{(x-p)}{\gamma}$. When a sale is made at price S at time T (either voluntarily by me or involuntarily by my broker) I receive S and pay off my loan. If I can borrow at my discount rate r, then I must pay back a sum of $(x-P)e^{rT}$ which has a present value of $(x-P)$. Thus the value of the transaction to me will be

$$ye^{-rT} - (x-P)$$

if I sell voluntarily and

$$Qe^{-rT} - (x-P)$$

if I am sold out. Since the term $(x-P)$ is a constant independent of anything I do, this case corresponds to the boundary condition (22).

Unfortunately, while (22) is appealing on economic grounds, it does not have the appealing homogeneity property which the previous conditions we have discussed had. That is, it is not true that if w(x) is a solution to L[w] = 0 which satisfies (22), so is Aw(x). Equation (11) does not hold. However, powerful techniques exist which allow us to analyze processes which satisfy (22).

We close this subsection by mentioning one boundary condition which is homogeneous. Probabilists who analyze diffusion processes often consider reflecting barriers. This corresponds to a tree which when it hits a size Q simply bounces off Q and starts again. As can be proved by a Taylor series argument [Cox and Miller, 2] if there is a reflecting barrier at Q, then

$$w'(Q) = 0. \tag{23}$$

It is difficult to imagine cases where (23) makes sense where (1) represents the value of a tree or other assets. However, it is not hard to imagine processes which could be modeled by diffusion processes for which the boundary condition (23) is appropriate. In models of invention (23) corresponds to "Oh well, back to the drawing board." (This case might also be modeled by a process which stuck on the boundary for a random period of time before it bounced back.)

c. Comparative Statics

In this section we will analyze how the tree's value and cutting size change as we change the parameters of the problem.

i. Constant coefficients; no absorbing barrier. To illustrate the kind of results we seek it will be well to begin with the constant coefficient case which is bounded at $-\infty$. As we saw the optimal cutting size is $y = \lambda^{-1}$ and the value of a tree of size x is $w(x) = \lambda^{-1}e^{\lambda x-1}$ where λ is given by (17a). Since w(x) is a decreasing function of λ, the parameters of the problem (a, b, and r) affect value and cutting time through their effect on λ. It is straightforward to calculate that $\frac{d\lambda}{dr} > 0$; $\frac{d\lambda}{db} < 0$ so that increases in the discount rate decrease value and cutting time while increases in the growth rate b have the opposite effect. Similarly, $\frac{d\lambda}{da} > 0$ so that increased variance increases both cutting time and value. This result is consistent with the observations we made in the preceding section that barring overshooting problems and the possibility of falling into a disaster, uncertainty should increase value and cutting time because e^{-rt} is a convex function of t.

In the Appendix we analyze the effects of parameter changes on trees with boundary conditions (20) and (23) (reflecting barriers and bounded at $-\infty$). Our analysis is not confined to the constant coefficient case and we consider the effects of local parameter changes as described in the next section. In general, the conclusions described above hold. Increases in the interest rate, and decreases in the growth rate decrease value and cutting time. Often, but not always, increases in variance increase value and cutting time.

ii. Absorbing barriers. In this section we present a complete analysis of the absorbing barrier case. That is, we consider a process whose growth is controlled by

$$(1) \qquad x_t = x_0 + \int_0^t \sigma(x_s)dW_s + \int_0^t b(x_s)ds$$

on the interval $I = [Q,Z]$. Let $a(x) = \frac{1}{2}[\sigma(x)]^2$ and $\gamma > 0$ be the first exit time of x_t from I. Let $\mathcal{J}$ be the set of stopping times for x_t and define

$$w(x) = \sup_{\tau\varepsilon\mathcal{J}} E_x x_{(\gamma\wedge\tau)} e^{-r(\gamma\wedge\tau)} \tag{24}$$

where

$$a\wedge b = \mathrm{Min}(a,b)$$

and

$$E_x f(x_t) = E[f(x_t)|x_0 = x].$$

The meaning of (24) is that when the process is stopped according to τ or when it exits from I the tree owner gets the value of the tree at that time. If the tree exits from I at Q, he gets Q -- in accord with boundary condition (22); if it exits at Z, he gets Z and if it is stopped at τ he gets x_τ. The upper boundary Z is added to the statement of the problem because this is the way in which Krylov's results are stated. It has no significance as we can make Z arbitrarily large. We give conditions below which guarantee that it is never optimal to stop but the examples are pathological.

In this section, we analyze the effects of parameter changes -- that is, changes in the functions a( ), b( ) and the numbers Q and r on the valuation function w(x) and the optimal stopping rule. Though the analysis is somewhat intricate, the results -- which are summarized in the introduction -- are rather clean.

Krylov [4: Section 1.5] shows that if $a(x)$, $a(x)^{-1}$ and $b(x)$ are bounded and satisfy a Lipschitz condition on I, then we may divide I into two disjoint regions:

(a) A closed set $\mathcal{S}$ called the stopping region on which w(x) = x.

(b) An open set $\mathcal{C}$ on which w(x) > x.

Furthermore

(c) L[w] = 0 for $x \in \mathcal{C}$

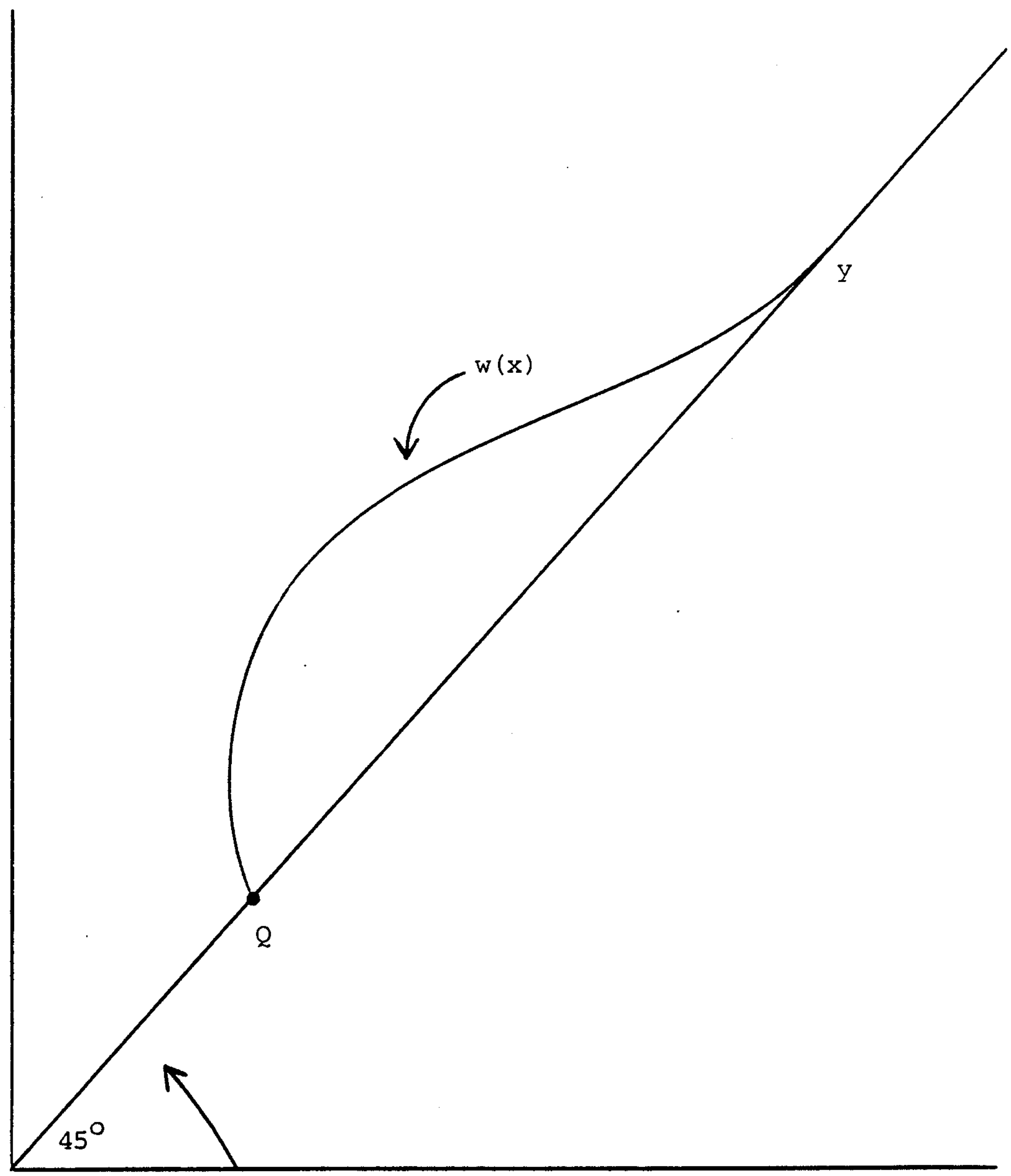
(d) $w'(x) = 1$ for $x \in \mathcal{S} \cap (Q,Z)$.

(e) The optimal stopping rule is to stop on the first entry into the region $\mathcal{S}$. (If $x_0 \in \mathcal{S}$ the tree should be cut down immediately).

These conditions are sufficient as well as necessary. If there is a function $\tilde{w}$ and regions $\mathcal{C}$ and $\mathcal{S}$ satisfying (a), (b), (c), and (d), and if $\tilde{w}$ is absolutely continuous on I, then $\tilde{w} = w$ and (e) describe the optimal stopping rule.

These conditions are best understood graphically. In Figure 1 we have drawn the function w(x). The continuation region $\mathcal{C}$ corresponds to the areas where w(x) is above the 45° line. On the stopping region $\mathcal{S}$, w(x) = x and thus coincides with the 45° line. There are two boundaries of $\mathcal{C}$ in Figure 1. The first is at Q and there it is not true that $w'(x) = x$. We will call such a situation a <u>coerced</u> boundary. There is a <u>free</u> boundary at y and at that point, $w'(y) = y$ (in accordance with condition (d)). While other possible arrangements of continuation regions and stopping regions are possible, we will show below the condition that b(x)/x is decreasing is sufficient to insure that Figure 1 describes the problems. Some other examples of

Figure 1



Coerced Boundary at Q, Free Boundary at y possible boundaries and continuation regions are shown in Figures 2 through 5. Figure 2 corresponds to two free boundaries (the boundary at Q is not a binding constraint), Figure 3 to a disconnected continuation region, Figure 4 to a tree which will never be cut down and Figure 5 to a tree which is always cut down immediately.

We begin our analysis of the optimal stopping time with a result which implies that in general trees are kept until they are at least as big under uncertainty as they are under certainty.

Proposition 4. Consider the region

$$\mathcal{A} = \{x \in I;\ b(x)/x > r\}$$

then

(a) $\mathcal{A} \subset \mathcal{C}$

(b) Each interval of $\mathcal{C}$ contains an interval of $\mathcal{A}$. In particular if $\mathcal{A}$ consists of a single interval, $\mathcal{C}$ consists of a single interval.

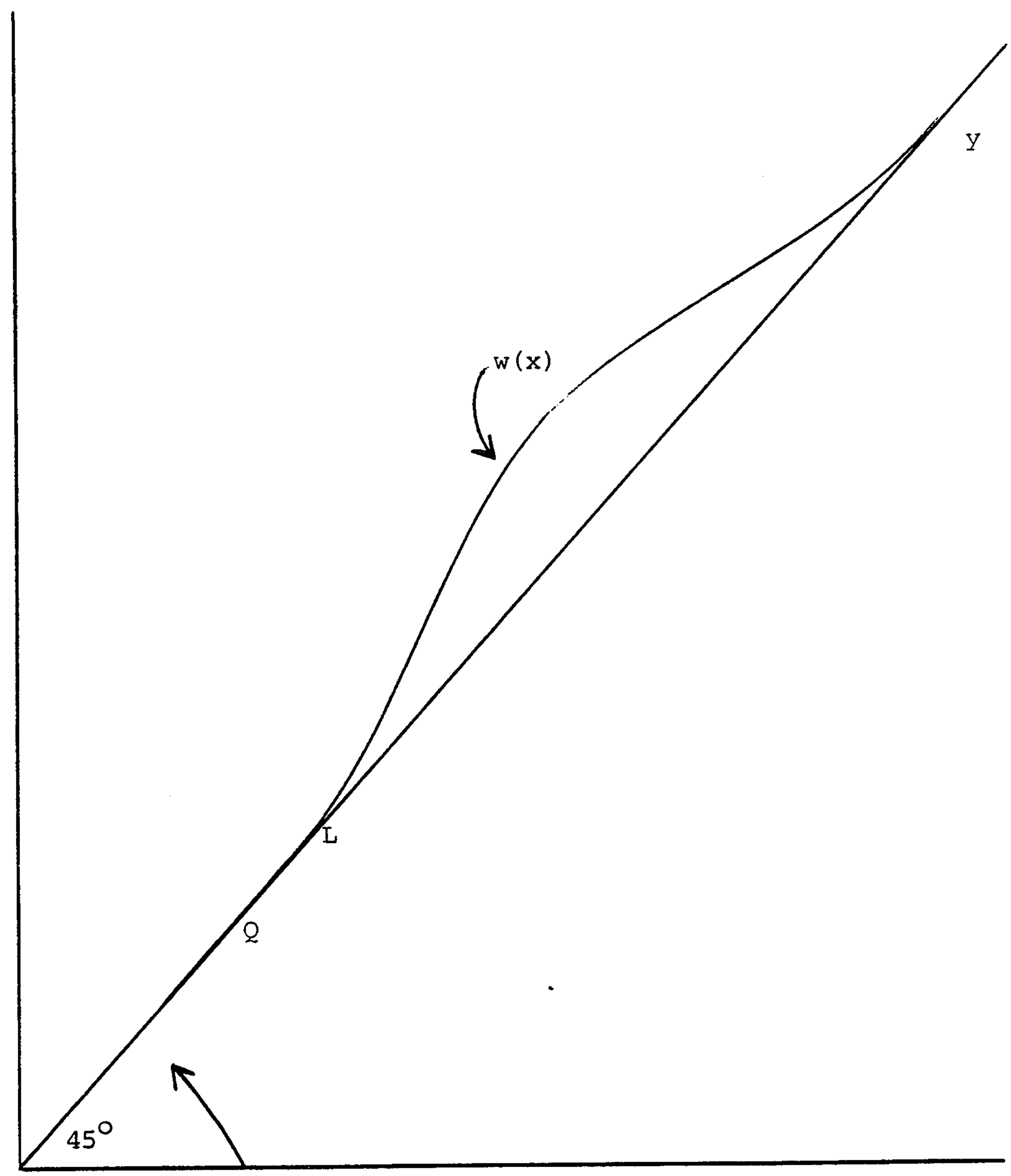
Proof: This result is due to Miroshnichenko [6] and it is only necessary to adapt his proof to our special case. To prove (a) consider a point $x \in \mathcal{S}$. Then

(25) $$w(x) = \sup_{\tau \in \mathcal{J}} E_x\, e^{-r(\gamma \wedge \tau)} x_{\gamma \wedge \tau} = x$$

thus, for any $t > 0$, $E_x e^{-r(t \wedge \gamma)} x_{\gamma \wedge t} \leq x$. But, applying Ito's lemma to $h(x,t) = e^{-rt}x$ we have, for any stopping time τ

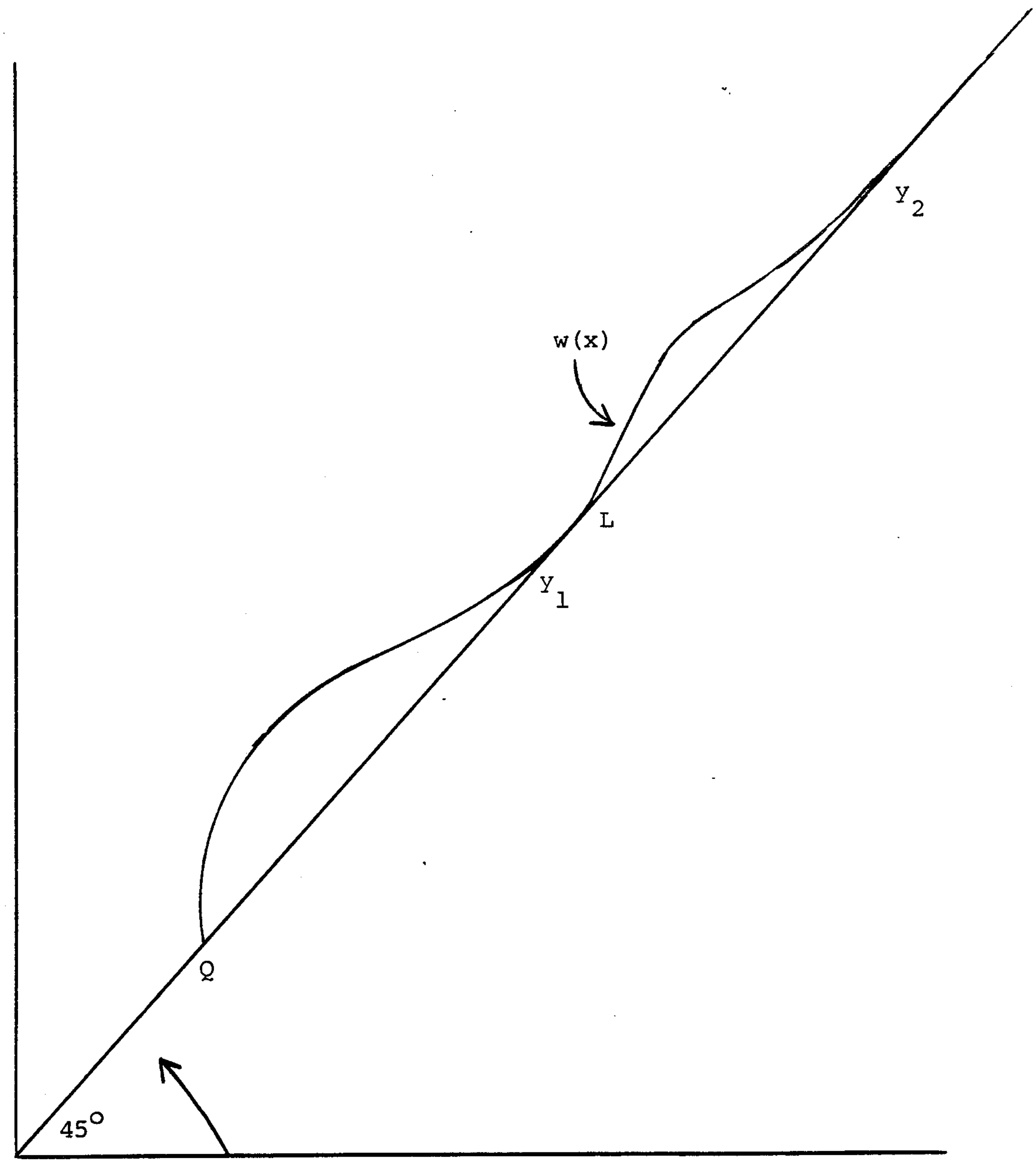
$$h(x_\tau, \tau) = h(x,0) + \int_0^\tau dh$$

Figure 2



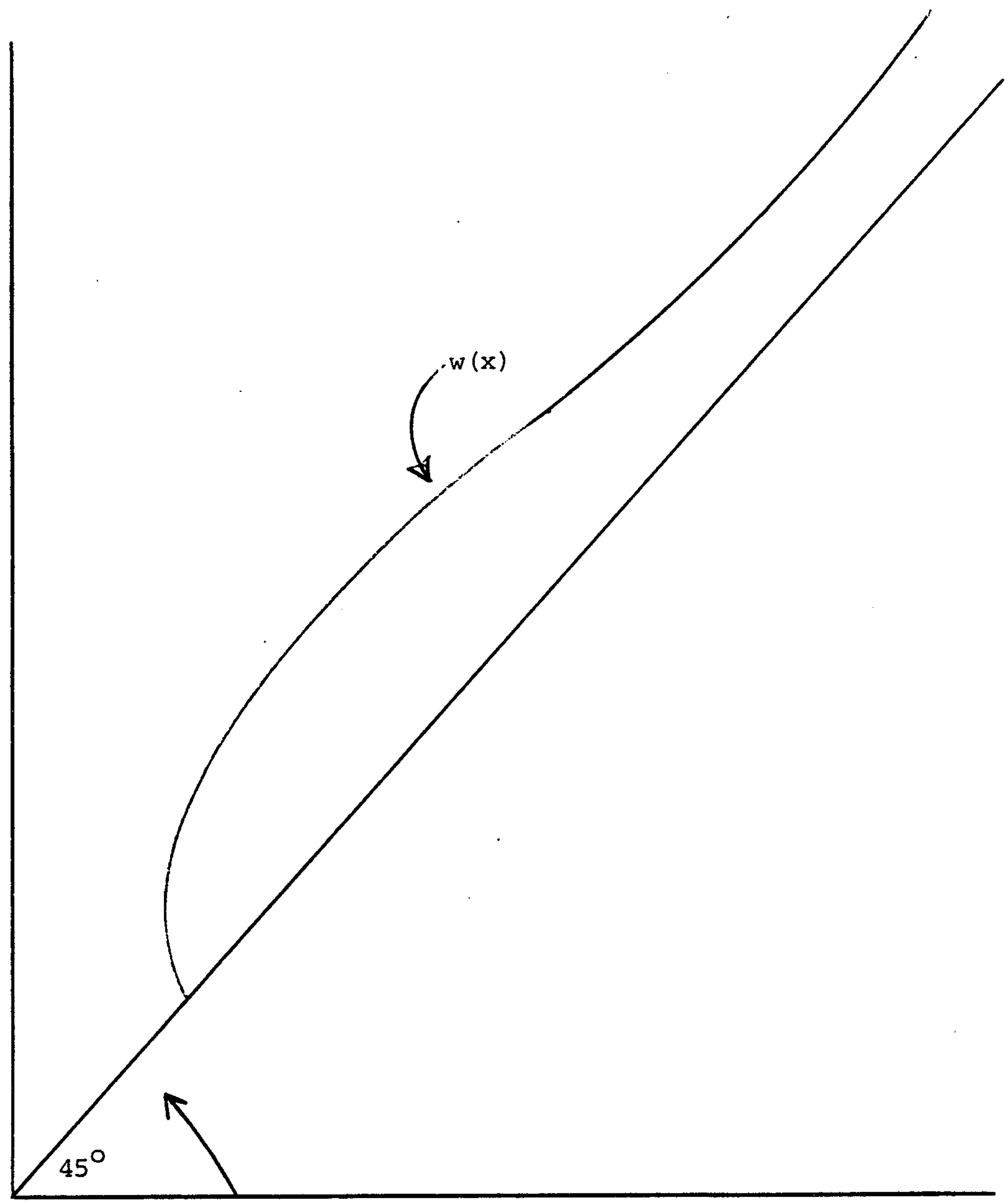
Free Boundaries at L and y

Figure 3



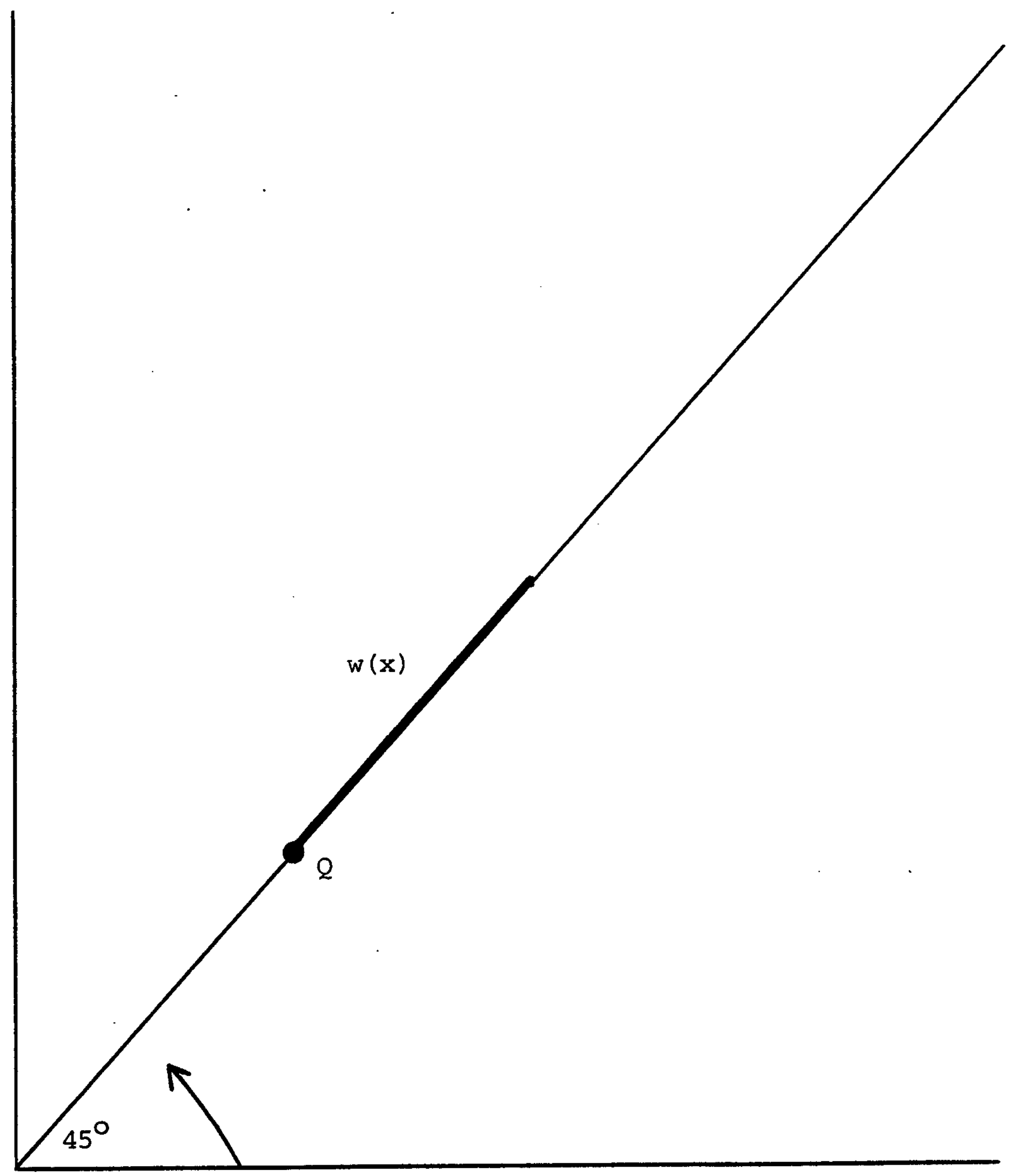
Disconnected Continuation Region

Figure 4



Unbounded Cutting Size

Figure 5



w(x) coincides with the 45° line.

$$= x + \int_0^\tau dh$$

$$= x + \int_0^\tau [a(x_s)h_x + b(x_s)h_x - rh]ds + \int_0^\tau a(x_s)h_x dw_s$$

Taking expectations and observing that $h_{xx} = 0$ while the second integral is -- by the definition of the Ito integral -- a martingale, we have

$$E_x h(x_\tau, \tau) = x + E\int_0^\tau e^{-rs}[b(x_s) - rx_s]ds$$

By (25) we have $0 \geq E_x \int_0^\tau e^{-rs}[b(x_s) - rx_s)]ds$. Since b is continuous, if $\frac{b(x)}{x} > r$, then we could choose $\hat{\tau}$ as the first t such that $\frac{b(x_t)}{x_t} = r$ and $\tau = \gamma \wedge \hat{\tau}$. But then we would have

$$E_x \int_0^\tau e^{-rs}[b(x_s) - rx_s] \; ds < 0$$

a contradiction. Thus we must have that $\frac{b(x)}{x} \leq r$.

Miroshnichenko's proof of part (b) applies without change.

Since b(x) is the average rate of change of the tree, b(x)/x is its (expected) growth rate; if b(x)/x > r, the expected growth rate of the tree is greater than the interest rate. The rule for cutting down the tree under certainty is to let it grow as long as the growth rate exceeds the interest rate. Proposition 4 generalizes and strengthens this rule. We note some obvious consequences of Proposition 4.

Corollary 1. (a) If $b(x)/x > r$ for all x I, then $\mathcal{C} = (Q,Z)$, and $\hat{\mathcal{C}} = \{(Q),(Z)\}$. (b) If $b(x)/x \leq r$ for all x I, then $\mathcal{S} = I$.

Proof: Obvious.

Note that (a) corresponds to Figure 1 and (b) to Figure 5. The next result gives necessary conditions for Figure 1 to describe the stopping region.

Corollary 2. If

(26) $b(x)/x$ is decreasing and $b(Q)/Q > r$

then the continuation region consists of a single interval. Furthermore, there is a coerced boundary of $\mathcal{C}$ at Q.

Proof: Obvious.

Our approach to comparative statics will consist of changing the parameters a(x) and b(x) in a small interval $J \subset I$ and leaving them unchanged outside of J. Specifically, we increase both instantaneous mean b(x) and instantaneous variance a(x) and ask whether this will increase the value of the tree. To analyze this question we use Krylov's [4] discussion of the optimal control of diffusion processes. Krylov shows that if a diffusion evolves according to

$$x_t = x + \int_0^t \sigma(x_s,\alpha_s)\, dW_s + \int_0^t b(x_s,\alpha_s)\,ds$$

where α_s is a control variable, then on the continuation region the value function w(c) satisfies

(27) $$a(x,\alpha^*(x))w''(x\ ) + b(x,\alpha^*(x))w'(x) = w(x) = 0$$

where $\alpha^*(x)$ is chosen to maximize the left hand side of (27). This implies that if increases in α correspond to increases in variance [$a(x,\alpha)$ is increasing in while $b(x,\alpha)$ is independent of α] α will be set as high as possible if the value function is convex near x, if $w''(x) < 0$, then α will be set as low as possible. Since choosing to set α high means voluntarily accepting more variance, whether or not local increases in variance increase or decrease value depends on whether the value function is convex or concave. Similarly, increases in mean will increase value if the valuation function is increasing -- as we will see in Proposition 6 below. We now show that this heuristic argument is correct.

Consider an optimal stopping problem on $I = [Q,Z]$ with parameters $a(0,x)$ and $b(0,x)$. Let $w_0(x)$ be the value function for this problem. Suppose, for simplicity, that $\mathcal{C}_0 = (L_0,y_0)$, the continuation region for this problem, is an interval. Let $J = (c,d)$ be an interval with $L_0 < c < d < y_0$ and $\gamma > 0$, such that

$$w'_0(x) > \gamma > 0, \quad w''_0(x) > \gamma > 0 \quad \text{for } x \in J.$$

Let $a(x,\alpha)$ and $b(x,\alpha)$ be smooth functions on $I\ x[0,1]$ satisfying

$$a(x,\alpha) = a(x,0) \quad \text{for} \quad x \notin J$$
$$b(x,\alpha) = b(x,0) \quad \text{for} \quad x \notin J$$

while for $x\varepsilon J$, $a(x,\alpha)$ and $b(x,\alpha)$ are strictly increasing in α. Let $w_\alpha(x)$ be the payoff function and $\mathcal{C}_\alpha = (L_\alpha,y_\alpha)$ the interval of the continuation region containing J, for the problem with parameters $a(x,\alpha)$

and $b(x,\alpha)$. Since $\mathcal{C}_0$ is an interval, Proposition 4 implies $\mathcal{C}_\alpha$ is also an interval and $\mathcal{C}_0 \subset \mathcal{C}_\alpha$ for all α.

One can show (the details are available from the authors) by adapting the arguments Krylov [5] used to establish his lemmas 1.4.6 and 1.5.2 that the functions $a(x,\alpha)$ and $b(x,\alpha)$ can be chosen so that

(28) $$w'_\alpha(x) > 0; \; w''_\alpha(x) > 0 \quad \text{for} \quad \alpha \varepsilon [0,1], x \varepsilon J .$$

We shall suppose that the perturbations $a(x,\alpha)$, $b(x,\alpha)$ have been chosen sufficiently smoothly that

$$w'_\alpha(x) > 0; \quad w''_\alpha(x) > 0 \quad \text{for } x \varepsilon J.$$

Proposition 5. Under these conditions

(i) $w_1(x) > w_0(x)$ for $x \varepsilon \mathcal{C}_0$.

(ii) $\mathcal{C}_1 \supset \mathcal{C}_0$

(iii) If L_0 is a free boundary, $L_1 < L_0$; if y_0 is a free boundary, $y_1 > y_0$.

Proof. Consider the optimal stopping and control problem: Choose a control rule $\alpha(t,\omega)$ and a stopping rule γ to maximize (24) where x_t evolves according to

(1) $$x_t = x_0 + \int_0^t \sigma(x_s,\alpha)dw_s + \int_0^t b(x_s,\alpha)ds.$$

Let $\tilde{w}(x)$ and $\mathcal{C}$ be the payoff function and the interval of the continuation region containing J for this problem. It follows from (28) and Krylov's characterization of the optimal solution to this problem

[4:1.4.5] that $\alpha(t,\omega) = 1$ for all t, ω. Thus $\tilde{w}(x) = w_1(x)$ and $\mathcal{C} = \mathcal{C}_1$.

<u>Step 1</u>. There is a point $\hat{x} \in J$ such that $w_1(\hat{x}) > w_0(\hat{x})$.

<u>Proof</u>: Suppose the contrary. Then for $x \in J$, $w(x) = w_0(x) = w_1(x)$ must satisfy

$$(29.0) \qquad a(x,0)w''(x) + b(x,0)w'(x) = rw(x),$$

and

$$(29.1) \qquad a(x,1)w''(x) + b(x,1)w'(x) = rw(x).$$

Subtract (29.0) from (29.1) to obtain

$$(a(x,1) - a(x,0)w''(x) + (b(x,1) - b(x,0))w'(x) = 0.$$

But all the terms on the left-hand side of this equation are strictly positive so it cannot hold. This contradiction proves 1.

<u>Step 2</u>. $w_1(x) > w_0(x)$ for $x \in \mathcal{C}_1$.

<u>Proof</u>: Note that since $\tilde{w}(x) = w_1(x)$ we must have

$$w_1(x) \geq w_0(x) \quad \text{for } x \in I$$

from which it follows that $\mathcal{C}_1 \supset \mathcal{C}_0$. We may as well assume that $x \in \mathcal{C}_0$ for if $x \notin \mathcal{C}_0$, Step 2 is true by definition.

Now fix $x \in \mathcal{C}_0$ and suppose without loss of generality that $x < \hat{x}$. Let $\underline{x}$ be a point in ${}_0$ which is less than x. For $i = 0,1$, let τ_i be the first exit time of x_t from $(\underline{x},\hat{x})$ where

$$x_t = x_0 + \int_0^t \sigma(x_s,i)\,dw_s + \int_0^t b(x_s,i)\,ds$$

Then it follows from the nature of the optimal strategy (Krylov [5:1.5.4]) and Bellman's Principle (Krylov [5:1.4.17]) that

(30) $$w_0(x) = E_x e^{-r\tau_0} w_0(x_{\tau_0})$$

$$= w_0(\underline{x})\ E_x[e^{-r\tau_0} \mid x_{\tau_0} = \underline{x}]$$

$$+ w_0(\hat{x})\ E_x[e^{-r\tau_0} \mid x_{\tau_0} = \hat{x}]$$

$$< w_1(\underline{x})\ E_x[e^{-r\tau_0} \mid x_{\tau_0} = \underline{x}]$$

$$+ w_1(\hat{x})\ E_x[e^{-r\tau_0} \mid x_{\tau_0} = \hat{x}].$$

However, this last quantity is just the expected reward which one would receive if one followed the strategy of setting $\alpha = 0$ until the first exit from the interval $(\underline{x},\hat{x})$. The expected reward obtained from following this strategy cannot be greater than $w_1(x)$, the expected reward from following the optimal strategy. Thus, $w_0(x) < w_1(x)$.

Step 3. Suppose y_0 is a free boundary for the problem with parameters $a(x,0)$, $b(x,0)$. Then y_0 is not a free boundary for the problem with parameters $a(x,1)$, $b(x,1)$. It follows that $y_1 > y_0$.

Proof: Recall that $a(x,1) = a(x,0)$ and $b(x,1) = b(x,0)$ for $x \in (d,y_0]$. Thus both $w_1(x)$ and $w_0(x)$ are solutions of

(31) $$a(x,0)U''(x) + b(x,0)U'(x) - rU(x) = 0$$

on $(d,y_0]$. It follows that $h(x) = w_1(x) - w_0(x)$ on (d,y_0) is also a solution of the homogeneous second-order equation (31). Suppose $y_0 = y_1$ is a free boundary for both problems. Then $w_1(y_0) = w_0(y_0) = y_0$ and $w_1'(y_1) = w_0'(y_0) = 1$; equivalently or

(32) $$h(y_0) = 0 \quad \text{and} \quad h'(y_0) = 0.$$

But the only solution to (31) satisfying (32) is the trivial solution. Thus $h(x) = 0$ for x $(d,y_0]$ or $w_1(x) = w_0(x)$ on $(d,y_0]$. This contradicts Step 2 and establishes Step 3.

A similar argument establishes that if L_0 is a free boundary for the problem with parameters $a(x,0)$ and $b(x,0)$, then $L_1 < L_0$ and this completes the proof of Proposition 5.

We may summarize Proposition 5 briefly as saying that if the value function is strictly convex in a neighborhood of x_0, more variance near x_0 (an increase in $a(x)$ in a neighborhood of x_0) increases value and cutting size. It is clear from the proof that if the value function is strictly concave, then a decrease in variance near x_0 has the same effect. If w is increasing, then an increase in the growth rate increases value.

<u>Corollary 3</u>. If y is a free boundary, $\frac{b(y)}{y} < r$

<u>Proof</u>: We know that if $\frac{b(x)}{x} > r$, $x \in \mathcal{C}$ unless $x = Q$ or S, for any optimal stopping problem. We can consider any stopping problem as

having been derived from another through an increase (or decrease) in variance which strictly expands the continuation region. Since $A = \{x : b(x)/x > r\}$ is contained in all continuation regions, every continuation region must strictly include A.

We now show that $w'(x) > 0$ so that increases in the growth rate always increase value.

Proposition 6. Let $w(x)$ be the payoff function of an optimal stopping problem with absorbing barriers Q and Z. Then, $w'(x) > 0$ for all $x \in (Q,Z]$.

Proof: Since $w'(x) = 1$ on $\mathcal{S}$ we need only show that $w'(x) > 0$ for $x \in \mathcal{C} = (L,y)$. Near L, $w'(x) \geq 1 > 0$ since $w(x)$ is increasing faster than x on the lower boundary of the continuation region. Let x_0 be the first $x > L$ such that $w'(x_0) = 0$. Then $w''(x) \leq 0$. But since $L[w] = 0$ on $\mathcal{C}$, $w''(x_0) = rw(x_0)/a(x_0) > 0$. This contradiction proves Proposition 6.

To ask whether increases in variance increase or decrease value is to ask whether $w(x)$ is concave or convex.

Proposition 7. There is always an interval in each continuation region in which $w(x)$ is concave.

Proof: Let $\mathcal{C} = (L,y)$ and consider $h(x) = w(x) - x$ on $[L,y]$. Then $h(L) = h(y) = 0$ but $h(x) > 0$ for $x \in \mathcal{C}$. Thus $h(x)$ must have a relative maximum on (c,y) so $h''(x) = w''(x)$ must be less than 0 in a neighborhood of this maximum.

Proposition 8.

(a) If $\hat{x}$ is a free boundary then $w''(\hat{x}) > 0$.

(b) If L is a coerced boundary and $b(L)/L > r$ then $w''(L) > 0$.

Proof:

(a) Consider again $h(x) = w(x) - x$. Then $h(x)$ satisfies

$$a(x)h''(x) + bh'(x) - rh(x) = rx - b(x) \tag{33}$$

but if y is a free boundary, $h(\hat{x}) = 0$ and $h'(\hat{x}) = 0$. Thus

$$a(\hat{x})h''(\hat{x}) = r\hat{x} - b(x)$$

which is positive by Corollary 3. Since $a(\hat{x}) > 0$ and $h''(\hat{x}) = w''(\hat{x})$, we have that $w''(\hat{x}) > 0$.

(b) If L is a coerced boundary $h(L) = 0$ and $w''(L) = h''(L) =$

$$\frac{rL - b(L)}{a(L)} - \frac{h'(L)}{a(L)} b(L) < 0.$$

Taken together, the two parts of Proposition 8 imply that if $b(x)/x$ is decreasing, increasing variance near the lower absorbing barrier decreases value while increasing variance near the free barrier increases value. If one is edging gingerly away from a precipice, one is not pleased if required to make the trip on roller skates. Near the free barrier the convexity of Ee^{-rT} in T dominates effects of increased variance on the probability of absorption. Propositions 7 and 8 together imply that the effects of increased variance are ambiguous. In any problem with a free boundary, the effects of increases in variance are ambiguous. Some local increases in variance increase value. Others decrease it. Perhaps the most useful condition for determining which region is which is given next in Corollary 4.

Consider a tree with growth rates $b(x)$ with a coerced lower boundary Q and free upper boundary y^*. Then for x near y^*, $w(x)$ is convex while for some x, $w(x)$ is concave. For if it were to be the case that (Q,y^*) can be split into two intervals, one near Q where $w(x)$ is concave and one near y^* where $w(x)$ is convex, it is necessary and sufficient that there be a single $\hat{x}$ such that $w''(\hat{x}) = 0$. A sufficient condition for this is that $w''(\hat{x}) = 0$ should imply that $w'''(\hat{x}) > 0$. Since $L[w] = 0$, $w''(x) = 0$ implies

$$w'''(\hat{x}) = (w'(\hat{x})/a(\hat{x}))\ (r - b'(\hat{x})).$$

Since $w'(\hat{x})/a(\hat{x}) > 0$, we have proved

<u>Corollary 4</u>. If $r - b'(x) > 0$ for all x, there is a Z such that the continuation region (Q,y^*) can be divided up into a region (Q,Z) where $w(x)$ is concave and a region (Z,y^*) where $w(x)$ is convex.

We conclude our exploration of comparative statics with the observation that increasing the lower absorbing barrier invariably decreases value and cutting size.

<u>Proposition 9</u>. Consider a problem with parameters $a(x)$ and $b(x)$ on the interval $[B_0,Z]$. Suppose $b(x)/x$ is decreasing so that there is a coerced boundary at B_0 and a free boundary at y_0. Consider another problem with higher absorbing barrier B_1. For $i = 0,1$, let

$w_i(x)$ and $\mathcal{C}_i = (B_i y_i)$ be the value function and continuation regions for the two problems. Then

$$w_0(x\ ) > w_1(x) \quad \text{for } x \in \mathcal{C}_1$$

and

$$y_1 < y_0.$$

<u>Proof</u>: If $w_1(x)$ and $\mathcal{C}_1$ is empty, the proposition is obviously true. It is obvious that $\mathcal{C}_0 \supset \mathcal{C}_1$ and $w_0(x) \geq w_1(x)$ for $x \in [B_1, Z]$. Consider $h(x) = w_0(x) - w_1(x)$. Then on $[B_1, y_1]$ both $w_0(x)$ and $w_1(x)$ satisfy $L[w] = 0$. Thus, $h(x)$ satisfies $L[w] = 0$ also. Since $h(x) \geq 0$ on (B_1, y_1) if there were an $x_0 \in (B_1, y_1)$ such that $h(x_0) = 0$, x_0 would be a relative minimum of $h(x)$ and we would also have $h'(x_0) = 0$. But this implies that $h(x)$ is the trivial solution to the second-order differential equation and that $h(x) = 0$ for all $x \in [B_1, y_1]$. However, $B_1 \in \mathcal{C}_0$ since if $B_1 > y_0$, $\mathcal{C}_1$ is empty, a contradiction. Thus, $w_0(B_1) > w_1(B_1) = B_1$ and $h(B_1) > 0$. This contradiction establishes that $h(x) > 0$ for $x \in [B, y_1]$.

We use a similar argument to show that $y_0 > y_1$. Suppose that $y_0 = y_1$. Then y_0 is a free boundary for both problems and $w_0(y_1) = y_1 = w_1(y_1)$ and $w'_0(y_1) = 1 = w'_1(y_1)$ so that $h(y_1) = 0$ and $h'(y_1) = 0$ which again implies $h(y) = 0$ on $[B_1, y_1]$. This contradiction completes the proof.

Appendix

Other Boundary Conditions[4]

In this appendix we develop heuristically the comparative statics of trees whose growth path is governed by a diffusion process, but which either have a boundary at infinity (Equation (20) of the text) or a reflecting barrier at Q. Again we assume that the optimal stopping rule is to wait until the tree reaches a given size y. As explained in the text, the valuation function is of the form

(A.1) $$w(x) = f(x)\frac{y}{f(y)}$$

where f(x) is a solution of

(A.2) $$L[f] = a(x)f''(x) + b(x)f'(x) - rf(x) = 0$$

satisfying

(A.3) $$\lim_{x \to \infty} |f(x)| < \infty$$

for a boundary at $-\infty$ or

(A.4) $$f'(Q) = 0$$

for a reflecting barrier at Q. Furthermore, since y is chosen to maximize $\frac{y}{f(y)}$, the optimal y must satisfy

(A.5) $$y = \frac{f'(y)}{f(y)}$$

We can write Equations (A.1) and (A.5) in a more useful form by considering

(A.6) $$g(x) = \log f(x).$$

Let

(A.7) $$h(x) = g'(x) = f(x)/f'(x).$$

Then (A.1) becomes

(A.8) $$w(x) = y \exp [g(x) - g(y)]$$

$$= y \exp - \int_x^y h(s)ds ,$$

while (A.5) is equivalent to

(A.9) $$h(y) = 1/y.$$

It is clear from (A.7) that a parameter change which decreases h(s) everywhere on the path from x to y will increase the value function w(x). It is also easy to check that second-order conditions for maximization require that at y, h(x) intersects 1/x from below. Thus a decrease in h( ) increases optimal cutting size as well as increasing value.

Further, h satisfies a simply and easily analyzed first-order differential equation. Substitute into (A.2) to obtain

(A.10) $$h'(x) = \frac{r}{a(x)} - \frac{b(x)}{a(x)} h(x) - h^2(x).$$

Note that since (A.10) is a first-order equation only one boundary condition is required to specify a solution. Condition (A.4) (reflecting barrier at Q) is equivalent to

(A.11) $h(Q) = 0$

while it can be shown that Condition (A.3) ($w(x)$ bounded as $x \rightarrow -\infty$) is equivalent to

(A.12) $$\lim_{Q \rightarrow \infty} |h(Q)| < \infty .$$

Thus, to find the value of a tree of size x, it is only necessary to solve (A.10), pick a particular solution, say $\tilde{h}$, by applying the boundary condition (A.11) or (A.12). Then integrate this $\tilde{h}$ from x until $\tilde{h}$ crosses the rectangular hyperbola at y, the optimal cutting size. The log of the value of the tree is log y plus the value of the integral.

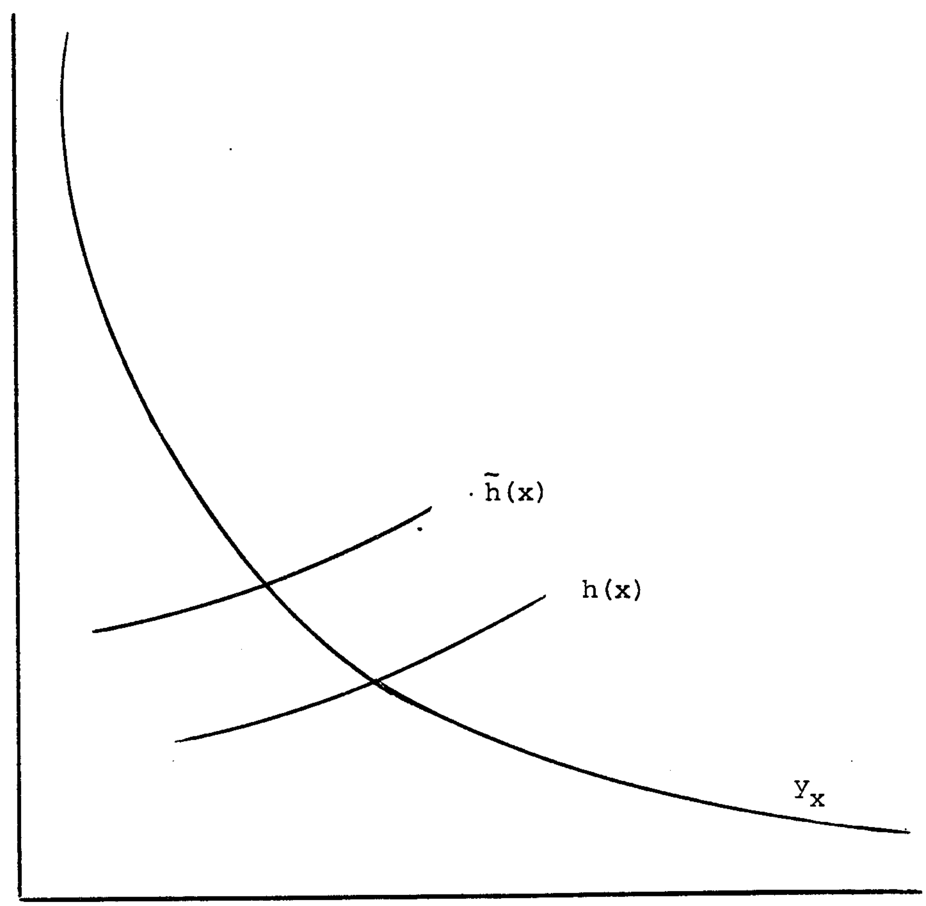
This suggests an easy way of getting comparative statics results. Suppose one tree -- call it the h tree -- grows according to h while another tree - the $\hat{h}$ tree -- grows according to $\hat{h}$, where $h(x) < \hat{h}(x)$. Then it is clear from (A.8) that for any arbitrary cutting size y, if both the h tree and the $\hat{h}$ tree are cut down at y, the h tree is worth more because, in an obvious notation,

$$H(x,y) = y \exp\left[- \int_x^y (h(s)ds \right] > y \exp\left[- \int_x^y \hat{h}(s)ds \right] = \hat{H}(x,y).$$

Furthermore, since it is easy to check that second-order conditions require that at the optimal cutting size y, h intersects the rectangular hyperbola 1/x from below, the optimal cutting size for the h tree is greater than the optimal cutting size for the $\hat{h}$ tree (See Figure A.1). Thus, we see that <u>any parameter change which uniformly decreases h( ) increases value and cutting size</u>.

Figure A.1

Determination of the Optimal Cutting Size



To apply this principle we need to analyze (A.10) a little more. It is straightforward to factor (A.10) into

$$h'(x) = (\lambda(x) - g(x))(g(x) - \mu(x))$$

where

$$\lambda(x) = \frac{-b(x) + (b^2(x) + 4ra(x))^{1/2}}{2a(x)} > 0$$

and

$$\mu(x) = \frac{-b(x) + (b^2(x) + 4ra(x))^{1/2}}{2a(x)\ .} < 0$$

Thus for $h(x) > \lambda$, $h'(x) < 0$ while for $\lambda(x) > g(x) > \mu(x)$ $h'(x) > 0$ and for $h(x) < \mu$, $h'(x) < 0$. The phase diagram for h is as in Figure A.2. Since $\lambda(x) > 0 > \mu(x)$, solutions of (A.10) which correspond to reflecting barriers are increasing. If $Q < \hat{Q}$, the solution which reflects at Q is everywhere below the solution which reflects at $\hat{Q}$. Increasing the reflecting barrier increases both value and cutting size.

To do comparative statics with respect to the parameters $\sigma^2(x)$ (= 2a(x)), b(x) and r, consider a local change of a parameter in a small interval Δ. Outside Δ the parameters are unchanged and the solutions are the same. Suppose we have a solution h(x). If the local change increases h(x) in Δ, then on the boundary of Δ it will link up with another solution $\tilde{h}(x)$ which is greater than h(x) to the right of Δ. Thus this local change will decrease value and cutting size. This is illustrated in Figure A. 3. The dashed line shows how the local change affects the path h(x) in Δ.

To see how local changes affect h, differentiate (A.10) to obtain

Figure A.2

Phase Diagram for h(x)

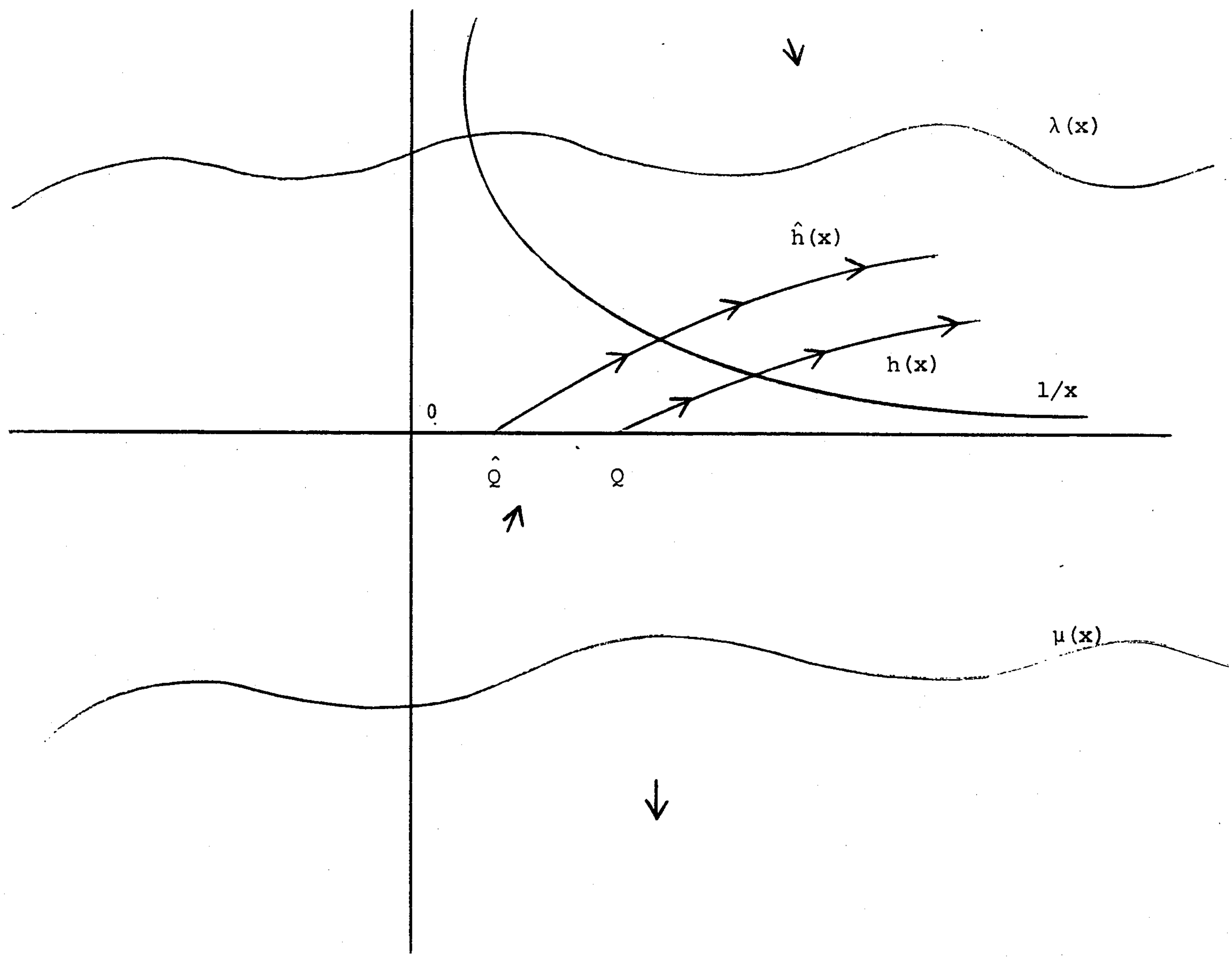
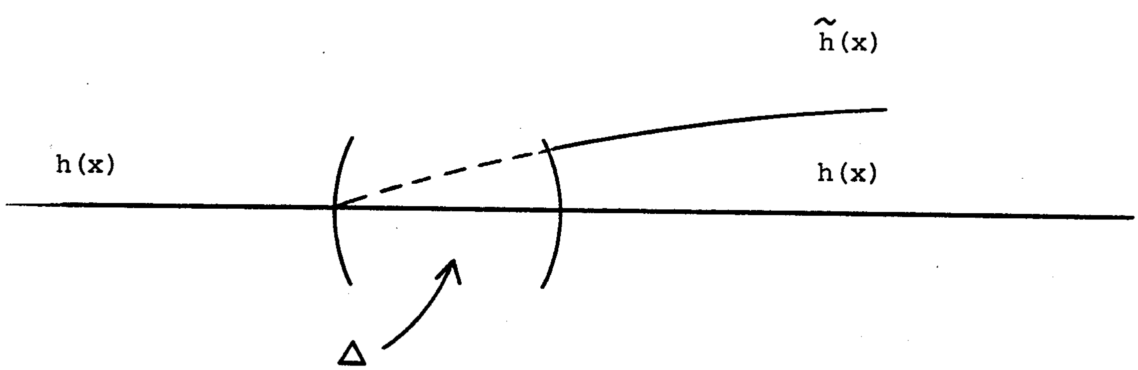


Figure A.3

Effects of a local parameter change



$$\frac{dh'}{db} = -\frac{h}{a} < 0$$

so that increases in the mean growth rate increase value and cutting time. Similarly

$$\frac{dh'}{dr} = b^{-1} > 0$$

so that increases in the discount rate have the opposite effect. To see the effect of increases in variance, note that

$$\frac{dh'}{da} = -\frac{(r - bh)}{a^2}$$

which is negative whenever $h(x) < r/b(x)$.

It is straightforward to show that solutions of (A.10) intersect $1/x$ from below if and only if they intersect at a point at which $1/x < r/b(x)$. Thus, increased variance near the optimal cutting size will increase value and cutting size. Similarly, increased variance near the reflecting barrier will also increase value and cutting size (as $r/b(x) > 0$). It is possible for trees with reflecting barriers to produce solutions of (A.10) which exceed $r/b(x)$ but it is not necessary that they do so. Thus it is quite possible that all local increases in variance will increase value and cutting size. This result contrasts with the case of absorbing barriers. As is shown in the text, for trees with absorbing barriers, some increases in variance decrease value and cutting size, others increase value and cutting size.

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Footnotes

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[1] See Karlin and Taylor [4, pages 162-169], and Wentzell [9, Chapter 11].

[2] We note here that there is an obvious generalization of Proposition 2 to non-stationary $\overline{X}$; this does not seem to be the case for Proposition 3.

[3] See Hartman [8, pages 350-62].

[4] We are extremely grateful to Jim Mirrlees who suggested the analysis given here.